

Undistillability of Werner states from three, four and five copies

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Abstract

Whether every state with a non-positive partial transpose is distillable reduces to the Werner states $\rho(d, \alpha) = (I + \alpha F)/(d^2 + \alpha d)$. The state $\rho(d, \alpha)$ is one-copy distillable exactly when $\alpha < -1/2$, and in July 2026 the same threshold was proved for two-copy distillability. We prove that it also holds for three copies: for every $d \geq 3$, $\rho(d, \alpha)$ is three-copy distillable if and only if $\alpha < -1/2$. For four copies we prove undistillability for every $\alpha \geq -79/160$ in every dimension, and for every $\alpha \geq -0.4979$ when $d = 3$. Five copies of the qutrit state are undistillable for every $\alpha \geq -0.47$. By a criterion of Costa Rico, each statement is equivalent to the nonnegativity of an explicit form on complex matrices of rank at most two. We certify these inequalities with quartic moment relaxations, to which we add one coupled Gram class. For three copies, exact rational certificates settle each dimension up to 16, and a single certificate whose entries are polynomials in $1/d$ settles every dimension $d \geq 9$. For four copies in every dimension, one limiting certificate applies to all large dimensions with an exactly bounded residual, and zero padding passes the result to smaller dimensions. The certificates are checked in exact and ball arithmetic by verifiers that are independent of the code that built them. At four copies the relaxation is not exact at the endpoint $\alpha = -1/2$, and the endpoint remains open for every number of copies from four on.

1. Introduction

Entanglement distillation converts many copies of a mixed state into fewer, nearly pure maximally entangled pairs by local operations and classical communication. A state with a positive partial transpose cannot be distilled [HHH98]. Whether the converse holds, that is, whether all states with a non-positive partial transpose (NPT) are distillable, has been open since the late 1990s [DSS+00, DCLB00]. Problem 5 in the list of Horodecki, Rudnicki and Życzkowski [HRZ22] asks whether the state $\rho(4, -1/2)$ defined below is two-copy distillable. Twirling reduces the general question to the Werner family [Wer89, HH99, DSS+00, DCLB00]

$$\rho(d, \alpha) = \frac{I + \alpha F}{d^2 + \alpha d}, \quad -1 \leq \alpha \leq 1,$$

where F is the swap on $\mathbb{C}^d \otimes \mathbb{C}^d$. The state $\rho(d, \alpha)$ is NPT exactly when $\alpha < -1/d$, and it is one-copy distillable exactly when $\alpha < -1/2$ [DSS+00, DCLB00]. For $d \geq 3$ the window $-1/2 \leq \alpha < -1/d$ consists of NPT states that are undistillable from a single copy. A state is undistillable if and only if it is k -copy undistillable for all k . So the existence of NPT bound entanglement in the Werner family is the question whether some state in this window stays k -copy undistillable for all k .

Two copies. In preprints posted in July 2026, Fu, Gao and Park [FGP26], Fraser, Huber, Pozsgay and Vona [FHPV26], Song and Chen [SC26] and Bharti, Gajjala and Haug [BGH26] independently proved that $\rho(d, \alpha)$ is two-copy undistillable for all $\alpha \geq -1/2$ and all d . Each proof establishes a sharp partial-trace inequality for matrices of rank at most two. These results rest on the criterion of Costa Rico [CR25], which turns k -copy undistillability of Werner states into the nonnegativity of a quartic form (Section 2).

Three copies. Wu and Zou [WZ26] proved three-copy nonnegativity at the endpoint $\alpha = -1/2$ for normal rank-two matrices in all dimensions and for further nonnormal sectors. An anonymous computer-assisted proof archived on Zenodo [Ano26] settles the qutrit case $d = 3$ for all rank-two matrices. It uses a quartic moment relaxation with an exact rational certificate. Wang [Wan26] proved that k -copy undistillability at local dimension m transfers to all $d \geq m$ at a smaller parameter. Applied to the qutrit certificate, this gives three-copy undistillability for $\alpha \geq -(d+5)/(5d+1)$. However, this bound tends to $-1/5$ as $d \rightarrow \infty$.

Main results.

Theorem 1.1. Let $d \geq 3$ and $\alpha \in [-1, 1]$. The state $\rho(d, \alpha)$ is three-copy distillable if and only if $\alpha < -1/2$.

For $d = 2$ the statement is classical, because $\rho(2, \alpha)$ is separable for $\alpha \geq -1/2$. Theorem 1.1 is equivalent to the following matrix inequality (Section 2).

Theorem 1.2. Let H_1, H_2, H_3 be finite-dimensional Hilbert spaces and let C be an operator on $H_1 \otimes H_2 \otimes H_3$ of rank at most two. Then

$$q_3(C) := \sum_{J \subseteq \{1,2,3\}} \left(-\frac{1}{2}\right)^{|J|} \|\text{Tr}_J C\|_F^2 \geq 0.$$

Moreover, the constant $1/2$ cannot be increased, and equality occurs. Let P be a rank-two projector on H_1 and let $u_i \perp v_i$ be unit vectors in H_i for $i = 2, 3$. The operator $C = P \otimes |u_2\rangle\langle v_2| \otimes |u_3\rangle\langle v_3|$ has rank two and $q_3(C) = 0$. Replacing $1/2$ by any $t > 1/2$ makes the same C negative (Lemma 2.2).

Four and five copies. For larger numbers of copies only weaker intervals were known. Bharti, Gajjala and Haug [BGH26] proved k -copy undistillability for $\alpha \geq -\gamma_k$ with γ_k of order $1/k$. The same relaxation method, used below the endpoint, gives much longer intervals.

Theorem 1.3. Let $d \geq 2$. The state $\rho(d, \alpha)$ is four-copy undistillable for every $\alpha \in [-79/160, 1]$.

Theorem 1.4. The qutrit state $\rho(3, \alpha)$ is four-copy undistillable for every $\alpha \in [-0.4979, 1]$, and five-copy undistillable for every $\alpha \in [-0.47, 1]$.

Since $79/160 > 1/3$, Theorem 1.3 gives, in every dimension $d \geq 3$, an interval of NPT Werner states that are four-copy undistillable. For qutrits, Theorem 1.4 leaves only $-1/2 \leq \alpha < -0.4979$ of the four-copy window open. At the endpoint itself the method stops: the complete four-replica relaxation is not exact at $k = 4$ and $\alpha = -1/2$ (Remark 8.1).

Method. We prove Theorem 1.2 for $H_i = \mathbb{C}^d$ and all d . Zero padding then gives the general case. We use the quartic moment relaxation of [Ano26]. Its variables are twirled fourth moments of a lifting of C to a pair of unit vectors, and its constraints are positive semidefinite (PSD) blocks and linear equalities that all physical moments satisfy. Numerically, this relaxation is exact for $d \leq 4$, but it is inconclusive at the 10^{-9} level for $d = 5$ and not exact for $d = 6$. We therefore add one more coupled Gram class, called gram2 below, and with it exact certificates exist in all dimensions. We prove that each constraint is valid (Theorem 3.5), so an exact certificate at dimension d proves Theorem 1.2 at d . Section 4 gives exact certificates for $3 \leq d \leq 16$. Section 5 gives one certificate for all $d \geq 9$. A loop-counting lemma shows that the reduced moment data are polynomials in d . A polynomial left-kernel witness shows that the certificate identity is solvable for every d , and a perturbation bound, uniform in $s = 1/d$, keeps the corrected multipliers positive semidefinite. Section 6 proves Theorem 1.3 with one limiting certificate for all large dimensions, and Section 7 proves Theorem 1.4 with exact certificates at $d = 3$. Section 8 discusses consequences and the endpoint.

Computer-verified steps are marked †. Each such step refers to a row of Table 1 in Appendix A.

2. Setting and reduction to the endpoint

Let ρ act on $(\mathbb{C}^d)^{\otimes 2}$, the first factor held by one party and the second by the other. The state ρ is k -copy distillable if and only if there is a vector ψ of Schmidt rank at most two, across the cut between the parties' k -fold systems, with $\langle \psi | (\rho^{\otimes k})^\Gamma | \psi \rangle < 0$ [DSS+00, DCLB00]. Here Γ is the partial transpose on the second party. A state is undistillable if it is k -copy undistillable for all k .

For $C \in L((\mathbb{C}^d)^{\otimes k})$ and $J \subseteq \{1, \dots, k\}$, let $\text{Tr}_J C$ denote the partial trace over the copies in J , with $\text{Tr}_\emptyset C = C$. For $t \in \mathbb{R}$ put

$$q_k(t, C) = \sum_{J \subseteq \{1, \dots, k\}} (-t)^{|J|} \|\text{Tr}_J C\|_F^2.$$

Theorem 2.1 (Costa Rico [CR25, Theorem 1]). For $\alpha \in [-1, 1]$, the state $\rho(d, \alpha)$ is k -copy distillable if and only if there is $C \in L((\mathbb{C}^d)^{\otimes k})$ of rank at most two with $q_k(-\alpha, C) < 0$.

The form $q_k(t, \cdot)$ depends only on the matrix C , and the local dimension merely restricts the space in which C lives. We write $q_3(C) = q_3(1/2, C)$, and we say that $\text{EP}_3(d)$ holds if $q_3(C) \geq 0$ for all $C \in L((\mathbb{C}^d)^{\otimes 3})$ of rank at most two. If $d' \leq d$, then $\text{EP}_3(d)$ implies $\text{EP}_3(d')$, because zero padding preserves the rank and all partial-trace norms.

Lemma 2.2. (a) For product operators, $q_k(t, \bigotimes_m C_m) = \prod_m q_1(t, C_m)$, where $q_1(t, X) = \|X\|_F^2 - t|\text{Tr } X|^2$. (b) If $\alpha < -1/2$, then $\rho(d, \alpha)$ is three-copy distillable. (c) If $\text{EP}_3(d)$ holds, then $\rho(d, \alpha)$ is three-copy undistillable for all $\alpha \in [-1/2, 1]$.

Proof. (a) The partial trace of a product over J is the product of the traces of the factors in J and the remaining factors. So the sum over J factorizes.

(b) Let P be a rank-two projector and let $e \perp e'$ be unit vectors in $\mathbb{C}^d$. Then $q_1(t, P) = 2 - 4t$ and $q_1(t, |e\rangle\langle e'|) = 1$. By (a), the rank-two operator $C = P \otimes |e\rangle\langle e'| \otimes |e\rangle\langle e'|$ has $q_3(t, C) = 2 - 4t$, which is negative for $t = -\alpha > 1/2$. Theorem 2.1 gives three-copy distillability.

(c) This is the case $k = 3$, $t = 1/2$ of Corollary 2.4 below. $\square$

Theorem 1.1 therefore follows from $\text{EP}_3(d)$ for all $d \geq 3$, which is Theorem 1.2 for equal local dimensions. Zero padding gives Theorem 1.2 for arbitrary H_1, H_2, H_3 . The operator in part (b) at $t = 1/2$ is the equality case of Theorem 1.2, and for $t > 1/2$ it shows that the constant $1/2$ is sharp.

For more than three copies we use the following monotonicity, in the dimension and in the parameter. Write $\Phi = \sum_{i,j} |ii\rangle\langle jj|$ for the unnormalized maximally entangled projector of one copy. With $\psi_C = \text{vec } C$ one has $q_k(t, C) = \langle \psi_C | (I - t\Phi)^{\otimes k} | \psi_C \rangle$.

Lemma 2.3. Let $k \geq 1$ and $0 \leq t' \leq t < 1$. (a) If $q_k(t, C) \geq 0$ for every C of rank at most two on $(\mathbb{C}^D)^{\otimes k}$, then the same holds on $(\mathbb{C}^d)^{\otimes k}$ for every $d \leq D$. (b) If $q_k(t, C) \geq 0$ for every C of rank at most two on $(\mathbb{C}^d)^{\otimes k}$, then $q_j(t', C) \geq 0$ for every $j \leq k$ and every C of rank at most two on $(\mathbb{C}^d)^{\otimes j}$.

Proof. (a) For an isometry $\iota: \mathbb{C}^d \rightarrow \mathbb{C}^D$ the operator $C' = \iota^{\otimes k} C \iota^{*\otimes k}$ has the same rank as C , and $\text{Tr}_J C' = \iota^{\otimes(k-|J|)} (\text{Tr}_J C) \iota^{*\otimes(k-|J|)}$, so $q_k(t, C') = q_k(t, C)$.

(b) First let $t' = t$. For a unit vector u , the operator $C \otimes |u\rangle\langle u|^{\otimes(k-j)}$ has rank at most two, and the argument of Lemma 2.2(a) shows that its value of $q_k(t, \cdot)$ is $q_j(t, C)(1-t)^{k-j}$. Hence $q_j(t, \cdot) \geq 0$ for $j \leq k$. Now let $t' < t$. Write $I - t'\Phi = (t'/t)(I - t\Phi) + (1 - t'/t)I$ and expand the j -fold tensor power. The term with $I - t\Phi$ on a set S of copies contributes a positive multiple of $\langle \psi_C | (I - t\Phi)^{\otimes S} \otimes I | \psi_C \rangle = \sum_{x,y} q_{|S|}(t, C_{xy})$, where C_{xy} are the matrix elements of C over the copies outside S in a product basis. Each C_{xy} has rank at most two, so every term is nonnegative. $\square$

Corollary 2.4. If $q_k(t, C) \geq 0$ for some $t \in [0, 1)$ and every C of rank at most two on $(\mathbb{C}^d)^{\otimes k}$, then $\rho(d, \alpha)$ is k -copy undistillable for every $\alpha \in [-t, 1]$.

Proof. For $-t \leq \alpha \leq 0$, Lemma 2.3(b) gives $q_k(-\alpha, \cdot) \geq 0$, and Theorem 2.1 applies. For $\alpha > 0$ every term of $q_k(-\alpha, C)$ is nonnegative. $\square$

3. The moment relaxation

We describe the relaxation for k copies, prove that each of its constraints is valid, and state the certificate conditions. Sections 4 and 5 use it with $k = 3$, and Sections 6 and 7 with $k = 4$ and $k = 5$. The moment families X_{31}, X_{22}, X_{13} , the lifting and the coupled Gram block are those of [Ano26], whereas gram2 and the classes used only for four copies are new. Our validity proof follows [Ano26, Lemmas 2–4] and replaces its basis-completeness computation [Ano26, Lemma 5] by a spanning argument.

3.1 Lifting

Let $k \geq 1$, $H = (\mathbb{C}^d)^{\otimes k}$, $Q = \mathbb{C}^2$ and $V = H \otimes Q$. Let $C \neq 0$ have rank at most two. Since $q_k(t, \cdot)$ is homogeneous of degree two, we may assume $\|C\|_F^2 = 1/2$. Choose an isometry $A: Q \rightarrow H$ with $\text{ran } C \subseteq \text{ran } A$, and put $B = 2C^*A$. Then $C = AB^*/2$, because AA^* fixes $\text{ran } C$, and $\|B\|_F^2 = 4\|C\|_F^2 = 2$. Put

$$a = \frac{1}{\sqrt{2}} \sum_{x,r} A_{xr} |x\rangle \otimes |r\rangle, \quad b = \frac{1}{\sqrt{2}} \sum_{x,r} B_{xr} |x\rangle \otimes |r\rangle \quad \text{in } V.$$

Then $\|a\| = \|b\| = 1$, $\text{Tr}_H |a\rangle\langle a| = I_Q/2$ because $A^*A = I$, and $C_{xy} = \sum_r a_{xr} \overline{b_{yr}}$. We call (a, b) a *physical pair* for C .

3.2 Moments and the source space

We work on four replicas $V^{\otimes 4}$. For a letter string $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) \in \{a, b\}^4$ put $Y_\ell = |\ell_1 \otimes \ell_2 \otimes \ell_3 \otimes \ell_4\rangle\langle \ell_1 \otimes \ell_2 \otimes \ell_3 \otimes \ell_4|$. The moment families are

$$Y_{31} = Y_{aaab}, \quad Y_{22} = Y_{aabb}, \quad Y_{13} = Y_{bbba},$$

and for four copies also $Y_{40} = Y_{aaaa}$ and $Y_{04} = Y_{bbbb}$. Let G be the group generated by the local unitaries $g = U_1 \otimes \cdots \otimes U_k \otimes R$ on V (with $U_i \in U(d)$ and $R \in U(2)$), the permutations of the k sites, and complex conjugation in the product basis. An element of G maps a physical pair for C to a physical pair for UCU^* , for a site permutation of C , or for $\overline{C}$, and each of these has the same value of $q_k(t, \cdot)$. The *physical moments* are

$$X_f = L \mathbb{E}_{g \in G} Y_f(ga, gb), \quad f \in \{40, 31, 22, 13, 04\}, \quad L = (2d^k)^4,$$

with the Haar measure on the unitaries and the uniform measure on the finite part. The positive constant L is the normalization used by the verifiers. It plays no role in validity, because each constraint is homogeneous, and the objective below carries the factor L^{-1} .

Write $V^{\otimes 4} = \bigotimes_l (\mathbb{C}^{n_l})^{\otimes 4}$ over the $k+1$ local factors $l \in \{1, \dots, k, Q\}$, with $n_l = d$ for the sites and $n_Q = 2$. For $p \in S_4$ let V_p permute the four replicas of one local factor. A *word* is $w = (w_1, \dots, w_k, w_Q) \in S_4^{k+1}$, and $V_w = \bigotimes_l V_{w_l}$. Let $G_f \subseteq S_4$ be the stabilizer of the letter string f . The *source space* $\mathcal{S}_f$ is the real span of the orbit sums of words under three operations, each applied to all components at once: (i) $w \mapsto gwh$ for $g, h \in G_f$; (ii) permutations of the k site components; (iii) $w \mapsto w^{-1}$.

Lemma 3.1. For each physical pair, $X_f \in \mathcal{S}_f$.

Proof. By Schur-Weyl duality the commutant of $\{U^{\otimes 4} : U \in U(n)\}$ on $(\mathbb{C}^n)^{\otimes 4}$ is spanned by the V_p , and the commutant of a tensor product of such groups is the tensor product of the commutants. So the Haar average lies in $\text{span}\{V_w\}$, and $X_f = \sum_w x_w V_w$ for some coefficients. Linear independence is not needed. The operator X_f is invariant under (i), because $V_\sigma Y_f = Y_f V_\sigma = Y_f$ for $\sigma \in G_f$ and V_σ commutes with $g^{\otimes 4}$. It is invariant under (ii) because of the site average. It is real and symmetric because of the conjugation average, and $V_w^\top = V_{w^{-1}}$ with each V_w real. Each operation permutes the coefficients of a representation. Averaging the real parts of the coefficients over the finite group generated by (i)–(iii) gives a representation with real coefficients that are constant on orbits. $\square$

3.3 Positive blocks

Lemma 3.2 (Gram classes). Let $F \subseteq \{1, 2, 3, 4\}$ and let $x^{(1)}, \dots, x^{(R)} \in \{a, b\}^4$ be letter strings with the same value of $s(x) = \#a(x|_{F^c}) - \#a(x|_F)$. Put $u_r = \bigotimes_i y_i^{(r)}$ with $y_i^{(r)} = x_i^{(r)}$ for $i \notin F$ and $y_i^{(r)} = \overline{x_i^{(r)}}$ for $i \in F$. (a) For all r, s there are a family $f \in \{40, 31, 22, 13, 04\}$ and a fixed bijection of the eight tensor indices, independent of (a, b) , that turns Y_f into $|u_r\rangle\langle u_s|$. We write this rearrangement as $R_{rs}(Y_f)$. It has the form $(V_p Y_f V_q^*)^{\Gamma_F}$ for some $p, q \in S_4$, where Γ_F transposes the replicas in F . (b) For each physical pair, the block operator $M(X) = [R_{rs}(X_f)]_{r,s=1}^R$ is positive semidefinite.

Proof. (a) An entry of $|u_r\rangle\langle u_s|$ is a product of eight vector components. Four of them are unconjugated: those of $x^{(r)}$ on F^c and of $x^{(s)}$ on F . The other four are conjugated. The number of letters a among the unconjugated components minus the number among the conjugated ones is $s(x^{(r)}) - s(x^{(s)}) = 0$. So both groups carry the same letter multiset, which is the multiset of exactly one family f . Matching the unconjugated components with the ket slots of Y_f and the conjugated ones with the bra slots reproduces the entry. When a letter repeats, all matchings give the same operator, because the matched factors are equal vectors.

(b) For a unitary $g \in G$, $u_r(ga, gb) = W_g u_r(a, b)$ with $W_g = \bigotimes_{i \notin F} g \otimes \bigotimes_{i \in F} \overline{g}$, the same unitary for all r . Hence $M(Y(ga, gb)) = (I_R \otimes W_g) |w\rangle\langle w| (I_R \otimes W_g)^*$ with $w = \bigoplus_r u_r(a, b)$. Complex conjugation gives $|\overline{w}\rangle\langle \overline{w}|$, and site permutations give permuted copies of w . Since M is linear, $M(X)$ is an average of positive semidefinite matrices. $\square$

The certificates use four kinds of Gram classes.

1. *Partial transposes* ($R = 1$): $X_f^{\Gamma_T} \succeq 0$ for whole-replica cuts T . The certificates for three and four copies use the cuts $\emptyset, \{4\}, \{1\}, \{1, 2\}$ for X_{31} and X_{13} , and $\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}$ for X_{22} .
2. *The coupled Gram block of [Ano26]*: $F = \{4\}$ and the rows $a \otimes a \otimes b \otimes \bar{a}$ and $b \otimes b \otimes a \otimes \bar{b}$. Its diagonal blocks come from X_{31} and X_{13} and its off-diagonal block from X_{22} .
3. *gram2*: $F = \{3, 4\}$ and the rows $u' = a \otimes a \otimes \bar{a} \otimes \bar{b}$ and $v'' = a \otimes b \otimes \bar{b} \otimes \bar{b}$. In index form, with replica indices $i_1 \dots i_4$ and $j_1 \dots j_4$,

$$A'(X)[i; j] = X_{31}[i_1 i_2 j_3 j_4; j_1 j_2 i_3 i_4], \quad B'(X)[i; j] = X_{22}[i_1 i_2 j_3 j_4; i_3 j_1 i_4 j_2], \quad D'(X)[i; j] = X_{13}[i_2 j_3 j_4 i_1; j_2 i_3 i_4 j_1],$$
and $\begin{bmatrix} A' & B' \\ B'^* & D' \end{bmatrix}$ is the Gram matrix of (u', v'') averaged over G . Its diagonal blocks are $X_{31}^{\Gamma_{34}}$ and a replica-permuted $X_{13}^{\Gamma_{34}}$.
4. *Four-copy classes*: the partial-transpose blocks of X_{40} and X_{04} , and three classes that use them: $F = \{4\}$ with rows $a \otimes a \otimes a \otimes \bar{a}$ and $a \otimes a \otimes b \otimes \bar{b}$; $F = \{4\}$ with rows $b \otimes b \otimes b \otimes \bar{b}$ and $a \otimes b \otimes b \otimes \bar{a}$; and $F = \{3, 4\}$ with rows $a \otimes a \otimes \bar{a} \otimes \bar{a}$, $b \otimes b \otimes \bar{b} \otimes \bar{b}$ and $a \otimes b \otimes \bar{a} \otimes \bar{b}$. They are used only in Section 7.

The relaxation of [Ano26] contains only the first two kinds. Numerically, its value is zero for $d = 3, 4$, zero to within 2×10^{-9} for $d = 5$, and -2.85×10^{-7} for $d = 6$ ($\dagger$, V19). The gram2 block is violated by the optimal pseudo-moment at $d = 6$. With all three kinds, exact certificates exist in all dimensions (Sections 4 and 5).

3.4 Equalities

Two kinds of linear equalities vanish on physical moments.

- *Stiefel contrasts*. For a replica r let $\text{Tr}_{H,r}$ trace out the H -part of replica r , and $\text{Tr}_{V,r}$ the whole replica r . Put $S_r(X) = \text{Tr}_{H,r} X - (I_Q/2)_r \otimes \text{Tr}_{V,r} X$. If f carries the letter a at replica r , then $S_r(X_f) = 0$, because $\text{Tr}_H |a\rangle\langle a| = I_Q/2$ and $\|a\| = 1$, and because S_r commutes with the action of G . The certificates use $r = 1$ for X_{31} , X_{22} and X_{40} , and $r = 4$ for X_{13} . (At a b -replica the contrast does not vanish.)
- *Common marginals*. $\text{Tr}_3 X_{31} = \text{Tr}_3 X_{22}$, with the surviving replicas in the order $(1, 2, 4)$, and $\text{Tr}_2 X_{22}$ with the order $(1, 3, 4)$ equals $\text{Tr}_2 X_{13}$ with the order $(4, 1, 3)$. Both identities hold because $\|a\| = \|b\| = 1$. With X_{40} and X_{04} there are two more, $\text{Tr}_4 X_{31} = \text{Tr}_4 X_{40}$ and $\text{Tr}_4 X_{13} = \text{Tr}_4 X_{04}$, for the same reason.

3.5 Objective

Lemma 3.3 (swap identity). Let (a, b) be a physical pair for C . For $S \subseteq \{1, \dots, k\}$, let F_S swap the sites in S between two copies of V , and let F_Q swap the two auxiliary factors. Then $\langle a \otimes b | F_Q F_S | a \otimes b \rangle = \|\text{Tr}_S C\|_F^2$.

Proof. Write $x = (x_S, x_{S^c})$ for an index of H . Put $x' = (y_S, x_{S^c})$ and $y' = (x_S, y_{S^c})$. Then

$$\langle a \otimes b | F_Q F_S | a \otimes b \rangle = \sum_{x, y, r, s} \overline{a_{xr}} \overline{b_{ys}} a_{x's} b_{y'r} = \sum_{x, y} \overline{C_{xy'}} C_{x'y}.$$

Summing over x_S and y_S turns this into $\sum_{x_{S^c}, y_{S^c}} |(\text{Tr}_S C)_{x_{S^c} y_{S^c}}|^2$. $\square$

For $t \in [0, 1]$ let $Z_t = F_Q \prod_{i=1}^k (I - t F_i) = \sum_S (-t)^{|S|} F_Q F_S$, acting on replicas 1 and 4, and define the objective $c_t(X) = L^{-1} \text{Tr} [(Z_{t,14} \otimes I_{23}) X_{31}]$. The replicas 2 and 3 of Y_{31} carry $|a\rangle\langle a|$, of trace one. Since Z_t commutes with $g \otimes g$, Lemma 3.3 gives $c_t(X) = q_k(t, C)$ for the physical moments of each physical pair for C . For $k = 3$ and $t = 1/2$ we write $c = c_{1/2}$. On words the objective has the closed form

$$c_t(V_w) = L^{-1} \prod_{l=1}^k \left(d^{\kappa(w_l)} - t d^{\kappa((14) \circ w_l)} \right) \cdot 2^{\kappa((14) \circ w_Q)}, \quad (3.1)$$

where $\kappa(p)$ is the number of cycles of p , so that $\text{Tr} V_p = n^{\kappa(p)}$ on $(\mathbb{C}^n)^{\otimes 4}$. Since $L = 16d^{4k}$, the objective of a word is d^{-4k} times a polynomial of degree at most $4k$ in d .

3.6 Certificates and validity

Definition 3.4. A certificate for (k, d, t) consists of the following data.

- Finitely many blocks $H_b(X) = K_b^* M_b(X) K_b$, where each M_b is an operator from Lemma 3.2 and each K_b is a fixed matrix.
- Real symmetric matrices $Z_b \succeq 0$.
- Finitely many equality functionals E_e , each a compression of a Stiefel contrast or a common-marginal identity of Section 3.4, with real coefficients λ_e .

The data must satisfy the identity

$$c_t(X) = \sum_b \text{Tr}(Z_b H_b(X)) + \sum_e \lambda_e E_e(X) \quad \text{for all } X \in \prod_f \mathcal{S}_f, \quad (3.2)$$

with the product over the families that the blocks use.

It suffices to check (3.2) on the orbit sums, which span the source space.

Theorem 3.5 (validity). If a certificate for (k, d, t) exists, then $q_k(t, C) \geq 0$ for every C of rank at most two on $(\mathbb{C}^d)^{\otimes k}$. For $k = 3$ and $t = 1/2$ this is $\text{EP}_3(d)$.

Proof. Let C have rank at most two, with $\|C\|_F^2 = 1/2$, and let X be the physical moments of a physical pair for C . By Lemma 3.1, X lies in the source space. By (3.2) and Section 3.5,

$$q_k(t, C) = c_t(X) = \sum_b \text{Tr}(Z_b H_b(X)) + \sum_e \lambda_e E_e(X).$$

Each $H_b(X) = K_b^* M_b(X) K_b$ is positive semidefinite by Lemma 3.2, so each trace is nonnegative. Each $E_e(X)$ vanishes by Section 3.4. Hence $q_k(t, C) \geq 0$. $\square$

The compressions K_b are arbitrary: their ranks and their symmetry adaptation affect how tight the relaxation is, but not its validity. Validity does, however, depend on two parts of the data: the equality functionals must have the form of Section 3.4 at a -replicas, and each partial-transpose block must transpose whole replicas. A verifier that reads these functionals from a certificate file must therefore check them. We audit them for each certificate file (†, V5).

For four and five copies we use certificates whose identity holds up to a small residual, which a positive multiple of the trace absorbs. Let $\tau(X) = L^{-1} \text{Tr} X_{31}$, so that $\tau(X) = 1$ for the physical moments. For each local factor l let $G_l = (n_l^{\kappa(p^{-1}q)})_{p,q \in S_4}$ be the Gram matrix of the replica permutations, let G_l^+ be its pseudo-inverse, and put $K = \prod_l \|G_l^+\|_\infty$, the product of row-sum norms.

Lemma 3.6 (slack certificates). Let blocks H_b and matrices $Z_b \succeq 0$, equality functionals E_e with coefficients λ_e , and a number ν be as in Definition 3.4, and define the residual functional

$$\rho(X) = c_t(X) - \nu \tau(X) - \sum_b \text{Tr}(Z_b H_b(X)) - \sum_e \lambda_e E_e(X).$$

For each family f let $R_f(o)$ be the value of ρ on the orbit sum o of words of f , and suppose that $\nu \geq LK \sum_f \sum_o |R_f(o)|$. Then $q_k(t, C) \geq 0$ for every C of rank at most two.

Proof. Let X be the physical moments of a physical pair for C . They are averaged over the group G , so the minimal-norm coefficients $x = G^+ g$ of X_f in the word basis, with $g_v = \text{Tr}(V_v^* X_f)$ and $G = \bigotimes_l G_l$, are constant on the orbits of Section 3.2: G and g are invariant under these operations, and so G^+ commutes with them. Since $X_f \succeq 0$, each $|g_v| \leq \text{Tr} X_f = L$, and the row-sum norm of $G^+ = \bigotimes_l G_l^+$ is K . Hence $|x_w| \leq KL$ and $|\rho(X)| \leq LK \sum_f \sum_o |R_f(o)| \leq \nu$. The blocks are positive semidefinite and the equalities vanish, as in the proof of Theorem 3.5, so $q_k(t, C) = c_t(X) \geq \nu + \rho(X) \geq 0$. $\square$

The verifiers check the inequality of Lemma 3.6 in exact rational arithmetic, with $|R_f(o)|$ computed modulo several primes and reconstructed by the Chinese remainder theorem.

3.7 Symmetry reduction

The certificates are written in a symmetry-adapted basis. On one local factor $(\mathbb{C}^n)^{\otimes 4}$ with a cut T , the operators commuting with the twirl are spanned by the partially transposed permutations V_p^Γ , $p \in S_4$. Their irreducible modules carry labels (f, λ, μ) , where f is a number of contracted pairs and λ, μ are partitions of the numbers of uncut and cut replicas minus f , with $\text{len } \lambda + \text{len } \mu \leq 4 - 2f$. The verifiers use, for each label, a basis of the multiplicity space of the form $D(d)C$, where the columns of $D(d)$ are explicit *diagram vectors* and C is a fixed matrix (Section 5.1). A *type* t is a Gram class together with one label for each site factor and for the auxiliary factor. Its reduced block is the compression

$K_t^* M_t(X) K_t$ by the tensor product of the module bases of these labels, followed by fixed integer congruences (stabilizer symmetrization and site-permutation sums). As a functional of the orbit coordinates of X , each entry of a reduced block is a sum of products of local matrix elements $\langle D_i | V_p^T | D_j \rangle$, one factor for each local factor. So the dimension d enters a certificate only through the local matrix elements of the k site factors, through the equality maps, which are compressions of partial traces, and through the objective (3.1). The auxiliary factor has dimension two for all d .

4. Three copies: exact certificates for $3 \leq d \leq 16$

Construction. Each zero C of q_3 yields physical moments X with $c(X) = 0$. Identity (3.2) then forces $\text{Tr}(Z_t H_t(X)) = 0$ for each t , hence $Z_t H_t(X) = 0$, so every certificate vanishes on the ranges of the blocks at the zeros of q_3 . Such zeros are plentiful. Examples include the operators $P \otimes |u\rangle\langle v|$ with P a rank-two projector on one site, and many rank-two operators supported on two-dimensional subspaces of the three sites. On the latter, $q_3 \geq 0$ holds because $\rho(2, -1/2)$ is separable, and q_3 has many zeros. We compute the ranges exactly from finitely many zero modes with Gaussian-integer entries and restrict each multiplier to the orthogonal complement. On this face we solve the relaxation numerically with a primal-dual interior-point method. We round the solution to rationals and then solve exactly for 272 equality coefficients and 2362 Gram entries, so that (3.2) holds on all 4675 orbit sums (1142, 2391 and 1142 for the three families). For all d in this range other than $d = 3$ the linear system has rank 2634. Positive semidefiniteness of each multiplier is checked exactly, through the sign pattern of the characteristic polynomial.

Two constructions and two verifiers. We build the certificates for $d \geq 5$ twice, with independent code.

- Construction I covers $5 \leq d \leq 16$. Each certificate has 1342 PSD multipliers, including coupled Gram and gram2 blocks, and 272 equality coefficients. Verifier I evaluates the literal permutation operators, partial transposes, partial traces and the Gram index rules of Section 3.3, and checks (3.2) on all orbit sums.
- Construction II covers $d \in \{5, \dots, 10, 12, 16\}$. Each certificate has 1347 PSD multipliers and 272 equality coefficients. Verifier II derives the entry operator of each block literally from its row patterns, by the matching in the proof of Lemma 3.2(a).
- For $d = 3$ and $d = 4$ we use certificates with 715 and 1237 PSD multipliers, respectively, and 272 equality coefficients, checked by Verifier I. The $d = 3$ certificate reproduces the result of [Ano26], whose certificate has 716 multipliers.

Neither verifier uses the construction code, and Verifier II re-derives each block entry from its row patterns, independently of the index tables of Verifier I. Their literal maps agree with dense implementations of the index rules, partial traces and objective ($\dagger$, V6), and deliberately perturbed certificates fail both.

Proposition 4.1. $\text{EP}_3(d)$ holds for all $d \leq 16$.

Proof. The certificate at $d = 16$ passes both verifiers ($\dagger$, V3, V4), and its equality data and cuts pass the audit ($\dagger$, V5). By Theorem 3.5, $\text{EP}_3(16)$ holds, and $\text{EP}_3(d)$ for $d \leq 16$ follows by zero padding. $\square$

The certificates at the smaller dimensions ($\dagger$, V1-V4) are not needed for Proposition 4.1. Several of them enter the second construction of Section 5.4. The verification package contains compact certificates for $d \leq 8$ ($\dagger$, V17), which together with Theorem 5.5 cover every dimension.

5. Three copies: a certificate uniform in $d \geq 9$

We now prove $\text{EP}_3(d)$ for all $d \geq 9$ with one certificate whose entries are polynomials in $s = 1/d$. Section 5.1 shows that the reduced data are polynomials in d . Section 5.2 fixes the face and the scaling. Section 5.3 proves a perturbation lemma and constructs the certificate. Section 5.4 describes a second, independent construction for $d \geq 17$.

5.1 Polynomial local data

Fix $m \in \{3, 4\}$ replicas of one site factor $\mathbb{C}^d$ and a cut T of transposed replicas. Fix a label (f, λ, μ) as in Section 3.7. A *diagram vector* of this label is

$$D(d) = \sum_{j \in [d]^f} e_{x(j)},$$

where the index string $x(j)$ is built as follows. The f summation indices $j_1, \dots, j_f$ sit on f contraction pairs, each consisting of one uncut and one cut replica. The remaining uncut replicas carry fixed values from $\{0, 1, \dots, \text{len } \lambda - 1\}$, and the remaining cut replicas carry fixed values from $\{d - 1, d - 2, \dots, d - \text{len } \mu\}$, in a pattern determined by the vector.

Lemma 5.1 (loop counting). Let $d \geq 4$. For all $p \in S_m$ and all diagram vectors D_i, D_j of one label,

$$\langle D_i(d) | V_p^\Gamma | D_j(d) \rangle = \varepsilon d^\kappa, \quad \varepsilon \in \{0, 1\}, \quad 0 \leq \kappa \leq 2f,$$

with ε and κ independent of d .

Proof. In the computational basis, a matrix element of V_p^Γ is a product of Kronecker deltas over a perfect matching of the $2m$ index positions, namely the walled Brauer diagram of p . Add an edge for each contraction pair of D_i (on the row positions) and of D_j (on the column positions). In the resulting graph, positions on a pair have degree two and positions with a fixed value have degree one. Hence each component is a cycle through summed positions or a path between two fixed positions. Summing over the free indices, each cycle contributes a factor d , and each path contributes the Kronecker delta of its two endpoint values. A delta between two values from $\{0, 1, 2, 3\}$, or between two values of the form $d - 1 - r$, does not depend on d . A delta between $t \in \{0, \dots, \text{len } \lambda - 1\}$ and $d - 1 - r$ with $r < \text{len } \mu$ equals one only if $d = t + r + 1$. But $t + r \leq \text{len } \lambda + \text{len } \mu - 2 \leq 2$, so this would need $d \leq 3$. Each cycle alternates between diagram edges and pair edges, and distinct cycles are disjoint, so the number κ of cycles is at most the number $2f$ of pair edges. $\square$

The verifier computes, at each dimension d , module bases $J(d)$ for the labels and the local matrices $\text{FT}(d) = J(d)^\top V_p^\Gamma J(d)$. These bases have the form $J(d) = D(d)C$ with a matrix C that does not depend on d , for $d = 4, \dots, 10$ ($\dagger$, V7).

Corollary 5.2. Let $\text{FT}_{\text{poly}}(d)$ be the interpolant of degree at most four through $\text{FT}(4), \dots, \text{FT}(8)$. Then $\text{FT}_{\text{poly}}(d) = (D(d)C)^\top V_p^\Gamma (D(d)C)$ for all $d \geq 4$, and each label block of FT_{poly} has degree at most $2f$. For each $d \geq 4$, the relaxation assembled from $\text{FT}_{\text{poly}}(d)$ is the relaxation of Section 3 with the local compressions $D(d)C$, so Theorem 3.5 applies to it.

Proof. By Lemma 5.1, $(D(d)C)^\top V_p^\Gamma (D(d)C)$ is a polynomial in d of degree at most $2f \leq 4$. It agrees with $\text{FT}(d)$ at the five points $d = 4, \dots, 8$, so it equals FT_{poly} . The remaining data are of the same kind for all d . The three-replica data used by the equality maps have $f = 0$, so they do not depend on d , and the partial-trace maps and the objective (3.1) are explicit. Theorem 3.5 allows arbitrary compressions. $\square$

As a check, the polynomial data equal the verifier's literal data at each $d \in \{6, 9, 10, \dots, 26, 30, 31, 50\}$ ($\dagger$, V8). Certificates built from the polynomial data at $d = 17$ and $d = 23$ pass Verifier I unchanged ($\dagger$, V20).

5.2 Face, scaling and degrees

We fix a face of the relaxation: each multiplier has the form $Z_t = N_t \hat{Z}_t N_t^\top$ for a fixed matrix N_t , and the certificate of Section 5.3 uses 1347 multiplier types. For a type t let $F_t \leq 6$ be the total number of contracted pairs in its three site labels. We rescale each multiplier as $\hat{Z}_t = d^{2F_t} Z_t$, followed by one fixed congruence, and each equality coefficient by d^3 . Then identity (3.2) restricted to the face becomes a linear system

$$A(s)x = c(s), \quad s = 1/d,$$

with 4675 rows, one for each orbit sum. Each column of $A(s)$ and the right-hand side $c(s)$ are polynomials in s of degree at most 12. This follows from Corollary 5.2 and (3.1). Before rescaling, a block column of type t is a polynomial in d of degree at most $2F_t$, an equality column has degree at most three, and the objective is d^{-12} times a polynomial of degree at most 12. The verifier checks the inputs of these degree bounds from the diagram data ($\dagger$, V13).

5.3 The certificate

Lemma 5.3. Let A be a real $m \times n$ matrix and $c \in \mathbb{R}^m$ with c in the range of A . Let R be a set of $\rho = \text{rank } A$ rows and B an $n \times \rho$ matrix with $\theta = \|I - A_R B\| < 1$. For $\tilde{x} \in \mathbb{R}^n$ put $r = c - A\tilde{x}$ and $\delta = B(A_R B)^{-1}r_R$. Then $A(\tilde{x} + \delta) = c$, and

$$\|\delta\| \leq \|B r_R\| + \frac{\|B\| \theta \|r_R\|}{1 - \theta}.$$

Proof. Since $\theta < 1$, the matrix $A_R B$ is invertible, so $\text{rank } A_R = \rho = \text{rank } A$. Hence the rows R span the row space, and $A = K A_R$ for some K . The residual r lies in the range of A , so $r = A y$ for some y . Then $A\delta = K A_R \delta = K r_R = K A_R y = A y = r$. For the bound, expand $(A_R B)^{-1} = \sum_{i \geq 0} (I - A_R B)^i$. $\square$

Proposition 5.4. For every integer $d \geq 9$ the system $A(1/d)x = c(1/d)$ has a rational solution whose multipliers are positive semidefinite.

Proof. We work with $s \in [0, 1/9]$. Each block is written as $\hat{Z}_t = T_t D_t T_t^\top$ with a fixed integer matrix T_t , and corrections are applied to D_t .

- *Base certificate.* Each $D_t(s)$ is a polynomial of degree at most $12 - 2F_t$, written in the Bernstein basis of $[0, 1/9]$, and each equality coefficient is a polynomial of degree nine. Every Bernstein coefficient block C of $\hat{Z}_t$ satisfies $C \succeq m T_t T_t^\top$ with $m = 2.712241 \times 10^{-6}$ (†, V14). The Bernstein basis is nonnegative and sums to one, so $\hat{Z}_t(s) \succeq m T_t T_t^\top$ on $[0, 1/9]$.
- *First correction.* An explicit polynomial correction of degree 20 with dyadic coefficients changes each D_t by at most $\varepsilon_1 = 1.8771 \times 10^{-9}$ in operator norm on $[0, 1/9]$. After it, the residual $r'(s)$ of the identity has degree at most 32. Its exact values at 38 nodes, which determine it, are at most 3.934×10^{-19} in absolute value (†, V14).
- *Consistency.* Let S be a fixed set of 2634 coordinates and A_S the corresponding columns of A . A matrix $W(s)$ of 1934 constant vectors and 107 integral polynomial vectors of degree at most five satisfies $W^\top A_S \equiv 0$ and $W^\top r' \equiv 0$ as polynomial identities, and $\text{rank } W(1/9) = 2041 = 4675 - 2634$ (†, V15). Hence $\text{rank } A_S(s) \leq 2634$ for all but finitely many s , and then for all s , since the rank of a polynomial matrix never exceeds its generic rank. The bound $\theta < 1$ below gives $\text{rank } A_S(s) \geq 2634$. Thus the rank is 2634 on all of $[0, 1/9]$. At all but finitely many s the columns of $W(s)$ span the left kernel of $A_S(s)$, so $r'(s)$ lies in the range of $A_S(s)$. At constant rank the projector onto the range is continuous in s , and therefore $r'(s)$ lies in the range of $A_S(s)$ for every $s \in [0, 1/9]$.
- *Second correction.* We apply Lemma 5.3 to $A_S(s)$ and $r'(s)$, with a fixed set R of 2634 rows and with $B(s)$ interpolated from approximate inverses at 21 Chebyshev points. Midpoint-radius bounds at the nodes and the Lebesgue constant 3.249 of the interpolation give $\theta \leq 1.2234 \times 10^{-5}$ for all $s \in [0, 1/9]$. The resulting correction changes each D_t by at most 5.1592×10^{-9} in operator norm (†, V16).

Since $\varepsilon_1 + 5.1592 \times 10^{-9} \leq 7.0362 \times 10^{-9} < m$, each corrected block satisfies $\hat{Z}_t(s) \succeq (m - 7.0362 \times 10^{-9}) T_t T_t^\top \succeq 0$ on $[0, 1/9]$. At $s = 1/d$ all data are rational, so the solution is rational. $\square$

Theorem 5.5. $\text{EP}_3(d)$ holds for every $d \geq 9$.

Proof. By Proposition 5.4 the relaxation assembled from the polynomial data at dimension d has an exact certificate. By Corollary 5.2 this is a certificate in the sense of Definition 3.4, with compressions $D(d)C$. Theorem 3.5 gives $\text{EP}_3(d)$. $\square$

Proof of Theorems 1.1 and 1.2. Proposition 4.1 and Theorem 5.5 give $\text{EP}_3(d)$ for all $d \geq 2$. Lemma 2.2 gives Theorem 1.1, and zero padding gives Theorem 1.2. $\square$

5.4 A second construction for $d \geq 17$

A second certificate, built with independent code on the face computed at $d = 6$ (1342 multiplier types), covers all $d \geq 17$. Its starting point interpolates the analytic centres of the exact certificates at 16 dimensions between 17 and 7061. Its consistency comes from an explicit left kernel instead of a witness $W(s)$. The kernel consists of the 1934 combinations of orbit sums that vanish as operators, together with 107 vectors obtained from exact zero modes of q_3 and their tangents through the Weingarten function of S_4 , whose denominators divide $d^2(d^2 - 1)(d^2 - 4)(d^2 - 9)$ [CS06]. These vectors annihilate the objective, because the corresponding polynomial identity has degree at most 24 and

vanishes at the 30 dimensions that carry exact certificates on the same face (†, V9-V11). Lemma 5.3, with a priori floating-point error bounds, then gives a correction of at most 8.1×10^{-3} against a margin of 0.5495, and at most 7.45×10^{-2} when every rounding term is multiplied by ten (†, V12, V18).

6. Four copies in every dimension

We prove Theorem 1.3. By Lemma 2.3(a) and Corollary 2.4 it suffices to show $q_4(t_0, C) \geq 0$, with $t_0 = 79/160$, for every C of rank at most two on $(\mathbb{C}^n)^{\otimes 4}$ and every $n \geq n_0 = 1.25 \times 10^9$: a statement at one dimension passes to all smaller ones by zero padding. We use the relaxation of Section 3 with $k = 4$, consisting of the three families X_{31}, X_{22}, X_{13} , the partial-transpose blocks, the coupled Gram block and gram2, and one certificate in the slack form of Lemma 3.6 for all $n \geq n_0$.

Polynomial residual. The site data of the blocks are the polynomials FT_{poly} of Corollary 5.2, since the four replica slots of one site do not depend on k . The objective and trace factors of a site are $\sum_j s^j ([\kappa = 4 - j] - t[\kappa((14) \circ p) = 4 - j])$ and $\sum_j s^j [\kappa = 4 - j]$ in $s = 1/n$, and the scaled partial traces are linear in s . Hence, for fixed multipliers Q_b , coefficients λ_e and slack ν , the residual of a word w has the form

$$\rho^{(n)}(V_w) = R_0(w) + s R_1(w) + r_2(w; s), \quad |r_2(w; s)| \leq s^2 B_2(w) \quad (0 \leq s \leq 1),$$

where $B_2(w)$ is the absolute coefficient sum of the part of degree at least two. With $S_0 = \sum_o |R_0(o)|$, $S_1 = \sum_o |R_1(o)|$ and $S_2 = \sum_w B_2(w)$, summed over all families, the condition of Lemma 3.6 at dimension n follows from

$$\nu \geq LK(n)(S_0 + s S_1 + s^2 S_2).$$

The constant satisfies $LK(n) = 16(n^4 \|G_n^{-1}\|_\infty)^4 \|G_2^+\|_\infty$ with $\|G_2^+\|_\infty = 31/180$. A Neumann series in $G_n = n^4(I + E_n)$, where $\|E_n\|_\infty \leq 6/n + 11/n^2 + 6/n^3$ by the cycle counts of S_4 , gives $LK(n) \leq 16 \cdot (31/180)(1 - 6/n - 11/n^2 - 6/n^3)^{-4}$ for $n \geq 12$.

The certificate. We obtain (Q_b, λ, ν) from the limiting relaxation at $s = 0$, solved with the largest possible slack $\nu = 10^{-4}$ at t_0 . Each multiplier is stored as $Q_b = M_b M_b^\top$ with an explicit rational matrix M_b , so it is positive semidefinite by construction. The relaxation at $s = 0$ is strictly feasible for $t \leq 79/160$ and has negative value at $t = 0.495$ (†, V26). So, within this method, 79/160 is within 0.00125 of the best constant for all dimensions.

Proposition 6.1. For the certificate above, $S_0 = 8.800 \times 10^{-7}$, $S_1 = 8221.29$ and $S_2 \leq 8.09 \times 10^9$. With $s_0 = 1/n_0 = 8 \times 10^{-10}$ and $LK(n_0) \leq 2.7555556$,

$$\nu - LK(n_0)(S_0 + s_0 S_1 + s_0^2 S_2) \geq 7.94 \times 10^{-5} > 0.$$

Proof. S_0 and S_1 are computed exactly, modulo 18 primes: 17 for the reconstruction by the Chinese remainder theorem and one as a check. S_2 is bounded by the full absolute shadow of the quadratic remainder (†, V21). The bound on $LK(n_0)$ is the Neumann bound above, evaluated exactly. $\square$

Proof of Theorem 1.3. For $n \geq n_0$ the right-hand side $LK(n)(S_0 + s S_1 + s^2 S_2)$ is decreasing in n , so the condition of Lemma 3.6 holds at every $n \geq n_0$ by Proposition 6.1. Hence $q_4(79/160, \cdot) \geq 0$ on rank-two operators at every $n \geq n_0$. Lemma 2.3(a) passes this to every dimension, and Corollary 2.4 gives four-copy undistillability for every $\alpha \in [-79/160, 1]$. $\square$

As end-to-end checks, the literal objective and trace functionals reproduce q_4 and the trace on physical product states at $n = 4, 50, 1000$ and 1.25×10^9 , the equality functionals vanish there, and each block term of the certificate is nonnegative (†, V22).

7. Four and five qutrit copies

We prove Theorem 1.4. By Corollary 2.4 it suffices to show

$$q_4(4979/10000, C) \geq 0 \quad \text{on } (\mathbb{C}^3)^{\otimes 4}, \quad q_5(47/100, C) \geq 0 \quad \text{on } (\mathbb{C}^3)^{\otimes 5},$$

for every C of rank at most two. Both follow from exact certificates in the slack form of Lemma 3.6, built at $d = 3$ for the relaxation of Section 3 with $k = 4$ and $k = 5$.

Four copies. We use the complete four-replica relaxation: the five families $X_{40}, X_{31}, X_{22}, X_{13}, X_{04}$, all Gram classes of Section 3.3, and the equalities of Section 3.4. The certificate at $t = 4979/10000$ has 4893 multiplier blocks. The verifier checks each of them for positive semidefiniteness exactly, through the characteristic polynomial, and computes the residual sums modulo 14 primes. The slack is $\nu = 1.148 \times 10^{-5}$, against $LK \sum_f \sum_o |R_f(o)| = 1.86 \times 10^{-12}$, with $L = (2 \cdot 3^4)^4$ and $K = (131/1440)^4 (31/180)$ (†, V23). The verification package also contains certificates at five smaller values of t (†, V23). Within this relaxation the threshold is about 0.49793. The smaller relaxation without X_{40}, X_{04} and their Gram classes reaches about 0.49701 (†, V26).

Five copies. Here the coupled Gram class alone suffices, together with the equalities of Section 3.4. The certificate at $t = 47/100$ has 495 multiplier blocks. Its slack is $\nu = 2.006 \times 10^{-3}$, against $LK \sum_f \sum_o |R_f(o)| = 3.51 \times 10^{-9}$, with $L = (2 \cdot 3^5)^4$ and $K = (131/1440)^5 (31/180)$ (†, V24).

Proposition 7.1. For every C of rank at most two, $q_4(4979/10000, C) \geq 0$ on $(\mathbb{C}^3)^{\otimes 4}$ and $q_5(47/100, C) \geq 0$ on $(\mathbb{C}^3)^{\otimes 5}$.

Proof. The certificates satisfy the hypothesis of Lemma 3.6, and their blocks are of the kinds covered by Lemma 3.2 (†, V23–V25). $\square$

Theorem 1.4 follows from Proposition 7.1 and Corollary 2.4.

8. Consequences and scope

Copy thresholds. For $k = 1$ the threshold $\alpha = -1/2$ is classical [DSS+00, DCLB00], for $k = 2$ it is the theorem of [FGP26, FHPV26, SC26, BGH26], and for $k = 3$ it is Theorem 1.1. Taken together, for all $d \geq 3$ and $k \leq 3$ the state $\rho(d, \alpha)$ is k -copy distillable if and only if $\alpha < -1/2$. A second and a third copy add nothing beyond the first, in every dimension. For four and five copies the thresholds are not settled, and Theorems 1.3 and 1.4 bound them.

Dimension lifting. Wang's theorem [Wan26] transfers k -copy undistillability at local dimension m to all $d \geq m$, at the parameter $\Lambda_{m,d}(1/2) \rightarrow (m-2)/(2m-1)$ as $d \rightarrow \infty$. For $k = 3$, Theorem 1.1 makes this transfer unnecessary, because it holds at the endpoint in all dimensions. For $k = 4$ the limiting certificate of Section 6 gives far more than the transfer: lifting the qutrit result of Theorem 1.4 gives, as $d \rightarrow \infty$, only the parameter $4937/25021 \approx 0.197$, whereas Theorem 1.3 gives $79/160$ in every dimension.

Four and more copies. At the endpoint the question remains open. For four copies the open range is $-1/2 \leq \alpha < -79/160$ in general and $-1/2 \leq \alpha < -0.4979$ for qutrits, and for five qutrit copies it is $-1/2 \leq \alpha < -0.47$. For five or more copies in other dimensions, intervals of order $1/k$ are known [BGH26]. A negative answer at some k would give the first Werner state that is NPT and undistillable from one copy, yet distillable from finitely many copies. On the other hand, a positive answer for all k at some d would prove that NPT bound entanglement exists.

Remark 8.1 (the endpoint for four copies). At the endpoint $t = 1/2$ the four-copy relaxation is not exact for $d = 3$. Numerically, its value is -2.614247×10^{-3} with the three families of Section 3, -1.867209×10^{-3} with gram2 added, and -1.307412×10^{-3} for the complete four-replica relaxation (†, V19). The two-replica profile $(\mathbb{E} \|\text{Tr}_J C\|_F^2 / \|C\|_F^2)_{|J|=0, \dots, 4}$ of the optimal pseudo-moment is within 1.03×10^{-3} of the profile of a physical zero $P \otimes |u\rangle\langle v|$, and local minimization of $q_4(1/2, \cdot)$ started near this point finds no negative value. The relaxation is therefore not exact at $k = 4$, and a six-replica relaxation, with an estimated 10^9 orbit coordinates at $d = 3$, is out of reach. So the method of this paper does not settle four copies at the endpoint.

Data and code availability

The certificates and standalone verifiers accompany this paper as four verification packages. The first covers three copies. It contains the exact certificates for $d = 3, \dots, 8$, the certificate of Proposition 5.4 for $d \geq 9$, and verifiers that import only numpy, python-flint and the Python standard library. One command runs all its checks in about 33 minutes with eight worker processes and less than 6 GB of memory. The second contains the certificate of Section 6 with the exact computation of its residual sums. The third and fourth contain the qutrit certificates of Section 7, for four copies at

six values of t and for five copies at two values of t , together with their verifiers. Each package README lists the files, their SHA-256 checksums and the expected output.

References

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Appendix A. Computations

Table 1 lists the computations that the proofs use. "Exact" means rational or integer arithmetic. Run times are wall-clock times on one machine. Rows V13-V17 are the checks of the verification package described under Data and code availability.

Table 1. Computer-verified steps.

ID	Statement checked	Method	Time	Used in
V1	Certificate at $d = 3$ (format of Construction I): 715 PSD multipliers, 272 equality coefficients; (3.2) on all 4675 orbit sums; each multiplier PSD	Verifier I, exact	814 s	Sec. 4
V2	Certificate at $d = 4$: 1237 PSD multipliers, 272 equality coefficients; same checks	Verifier I, exact	1668 s	Sec. 4
V3	Construction I, $5 \leq d \leq 16$: 1342 PSD multipliers, 272 equality coefficients; same checks	Verifier I, exact	1758–2977 s per d	Prop. 4.1
V4	Construction II, $d \in \{5, \dots, 10, 12, 16\}$: 1347 PSD multipliers, 272 equality coefficients; same checks	Verifier II, exact	229–1525 s per d	Prop. 4.1

ID	Statement checked	Method	Time	Used in
V5	Data audit of 22 certificate files: equality functionals are Stiefel contrasts at a -replicas or common marginals; each partial-transpose block transposes whole replicas	exact	seconds	Sec. 3.6, Prop. 4.1
V6	Semantics of both verifiers: index rules of Lemma 3.2 (15 row-pattern sets, 16 cuts, coupled Gram, gram2), partial traces, Stiefel contrasts, objective (3.1), swap identity; literal maps against dense implementations, $d = 3, \dots, 8$	floating point and exact; max. deviation 3.4×10^{-13}	minutes	Sec. 4
V7	$J(d) = D(d)C$ with C independent of d , for $d = 4, \dots, 10$; Lemma 5.1 by brute force: 171,720 matrix elements, $d = 4, \dots, 9$, no mismatch	exact	minutes	Sec. 5.1
V8	$\text{FT}_{\text{poly}}(d)$ equals the verifier's literal local data for $d \in \{6, 9, \dots, 26, 30, 31, 50\}$	exact	minutes	Sec. 5.1
V9	Columns of $A(s)$ and $c(s)$ have degree ≤ 12 ; finite differences of order 13 vanish	exact	minutes	Sec. 5.4
V10	The 1934 combinations of orbit sums that vanish as operators annihilate all columns, including the Gram and gram2 column maps, and the objective; the trace vectors of the 107 zero modes and tangents have rank 107; q_3 vanishes at the 31 zero modes	exact	minutes	Sec. 5.4
V11	Exact certificates on the face computed at $d = 6$, with strictly positive definite reduced multipliers, at 30 dimensions: $d = 6, 9 \leq d \leq 28$, and $d \in \{31, 38, 48, 64, 93, 150, 288, 790, 7061\}$	exact identity on all rows; exact Sylvester test, or a dyadic rounding of the multiplier within 5.3×10^{-15} in Frobenius norm followed by an exact Sylvester test of the rounding minus a multiple of the identity	hours	Sec. 5.4
V12	Second construction on $[0, 1/17]$: margin 0.5495 of the interpolated starting point (Bernstein coefficients); $\theta \leq 1.25 \times 10^{-4}$; residual $\leq 2.85 \times 10^{-8}$; block correction $\leq 8.1 \times 10^{-3}$, and $\leq 7.45 \times 10^{-2}$ with every rounding term multiplied by ten	floating point with a priori error bounds	minutes	Sec. 5.4
V13	Diagram data of the package: 41 labels with standard diagram keys and integer C ; $2F_i$ scaling for all 1347 types; column degree ≤ 12 , residual degree ≤ 32 ; multiplier classes and equality groups are those of Lemma 3.2 and Section 3.4	exact	seconds	Sec. 5.2
V14	Base certificate: all 13,233 Bernstein coefficient blocks $\geq mI$, $m = 2.712241 \times 10^{-6}$; first correction $\leq \varepsilon_1 = 1.8771 \times 10^{-9}$; residual r' exact at 38 nodes, $\max r' = 3.934 \times 10^{-19}$	exact	152 s	Prop. 5.4
V15	Witness $W(s)$ (1934 + 107 vectors): $W^T A_s \equiv 0$ at 18 nodes modulo 15 primes below 2^{26} , sufficient by a height bound; $W^T r' \equiv 0$ exactly at 38 nodes; rank $W(1/9) = 2041$	exact, multi-modular	1081 s	Prop. 5.4
V16	Second correction: $\theta \leq 1.2234 \times 10^{-5}$ (Lebesgue constant 3.249); block change $\leq 5.1592 \times 10^{-9}$; total $7.0362 \times 10^{-9} < m$	midpoint-radius arithmetic	162 s	Prop. 5.4
V17	Package certificates for $d = 3, \dots, 8$: (3.2) exact on all 4675 orbit sums, every one of the 1347 multipliers PSD by its exact characteristic polynomial; whole package in 33.1 min with eight workers	exact	3.0–5.2 min per d	Sec. 4, Data availability
V18	Lebesgue constants of the interpolation grids: rigorous upper bounds 3.4895 (28 points) and 3.0645 (16 points); true values 3.084 and 2.728	grid plus Markov inequality; 40-digit check	seconds	Sec. 5.4
V19	Relaxation values (numerical, interior-point method): relaxation of [Ano26] at $k = 3$: 0 at $d = 3, 4$, in $[-1.7 \times 10^{-9}, 3.4 \times 10^{-11}]$ at $d = 5$, -2.85×10^{-7} at $d = 6$; four-replica relaxation at $k = 4$, $d = 3$: -2.614247×10^{-3} , with gram2 -1.867209×10^{-3} , complete -1.307412×10^{-3}	floating point, primal and dual objectives agree to 10^{-9}	hours	Sec. 3.3, Remark 8.1
V20	Certificates built from the polynomial data at $d = 17$ and $d = 23$ pass Verifier I	Verifier I, exact	3877 s, 6392 s	Sec. 5.1

ID	Statement checked	Method	Time	Used in
V21	Theorem 1.3, residual sums of the certificate at $t = 79/160$: $S_0 = 8.799734 \times 10^{-7}$ and $S_1 = 8221.293$ exactly (modulo 18 primes, 17 for the reconstruction and one as a check), $S_2 \leq 8.0883 \times 10^9$ (absolute shadow), $LK(n_0) \leq 2.7555556$; margin 7.94×10^{-5}	exact, multi-modular	about 14 min per job, 54 jobs run concurrently	Sec. 6
V22	End-to-end checks of the certificate of Section 6 on physical product states at $n = 4, 50, 1000, 1.25 \times 10^9$: objective and trace reproduce q_4 and $\text{Tr } X$ (relative error $\leq 10^{-14}$), equalities vanish ($\leq 2.6 \times 10^{-13}$), all 3206 single-block terms nonnegative	floating point	minutes	Sec. 6
V23	Four qutrit copies: six certificates, at $t = 49/100, 99/200, 62/125, 497/1000$ (3816 blocks) and $t = 2489/5000, 4979/10000$ (4893 blocks); every block PSD by its exact characteristic polynomial; residual sums modulo 13 or 14 primes; slack condition of Lemma 3.6, at $t = 4979/10000$ with $\nu = 1.148 \times 10^{-5}$ and $LK \sum R = 1.86 \times 10^{-12}$	exact	4.9–9.5 min each (24 processes)	Sec. 7
V24	Five qutrit copies: two certificates, at $t = 9/20$ and $47/100$, with 495 blocks each; 13 primes; $\nu/(LK \sum R) = 1.88 \times 10^3$ and 5.72×10^5	exact	5.8–7.8 min each (48 processes)	Sec. 7
V25	Independent audit of V23 and V24: explicit Gram vectors for every block type, literal index rules ($\leq 2.9 \times 10^{-14}$), the stored equality groups, the constants K and L , re-runs of both verifiers, and a Monte Carlo test of the identity on physical points	exact and floating point	hours	Sec. 7
V26	Relaxation values (numerical, interior-point method): four copies, $s = 0$: strictly feasible for $t \leq 79/160$, negative at $t = 0.495$; four qutrit copies: without X_{40}, X_{04} , $+7.55 \times 10^{-6}$ at $t = 0.497$ and -2.84×10^{-4} at $t = 0.4975$; complete, $+1.60 \times 10^{-5}$ at $t = 0.4979$ and -4.16×10^{-5} at $t = 0.498$	floating point	hours	Sec. 6, Sec. 7