Equality cases and every-copy results for the Werner endpoint inequality

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Abstract

Whether every entangled state with a non-positive partial transpose (NPT) can be distilled is open. For the Werner states $\rho(d,\alpha)$, a criterion of Costa Rico reduces k-copy undistillability to an inequality for matrices of rank at most two: an alternating sum of the squared norms of their partial traces must be nonnegative. At $\alpha=-1/2$ we call it the endpoint inequality. It is known for two and three copies, and if it held for every k in some dimension $d\geq 3$, NPT bound entanglement would exist. For two copies we show that every equality case has rank two, with its range and that of its adjoint in products of two-dimensional subspaces. We also show that a first failure would produce, at some $\alpha>-1/2$, an equality case whose range has reduced states of rank at least three on every copy, unlike all known equality cases at $\alpha=-1/2$. For every k we prove the endpoint inequality on an explicit class of matrices whose ranges generically have such reduced states. For four copies we give a computer-assisted proof of it in every dimension for all matrices C such that the ranges of C and of its adjoint lie in one two-dimensional subspace, as they do for all normal matrices of rank two.

Keywords: entanglement distillation; NPT bound entanglement; Werner states; partial transpose; Schmidt rank; partial trace inequalities; semidefinite programming.

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1. Introduction

Entanglement distillation turns many copies of a mixed state into fewer, nearly pure, maximally entangled pairs. States with a positive partial transpose cannot be distilled [HHH98], and whether every state with a non-positive partial transpose (NPT) can be distilled has been open since the late 1990s. A state ρ is k-copy distillable exactly when some vector ψ of Schmidt rank at most two, across the cut between the parties' k-fold systems, has $\langle\psi|(\rho^{\otimes k})^{\Gamma}|\psi\rangle<0$ [DSS+00, DCLB00], and twirling reduces the question to the Werner states $\rho(d,\alpha)=(I+\alpha F)/(d^2+\alpha d)$, with F the swap. For $d\geq 3$ and $-1/2\leq\alpha<-1/d$ these states are NPT but cannot be distilled from one copy. DiVincenzo, Shor, Smolin, Terhal and Thapliyal [DSS+00] and Dür, Cirac, Lewenstein and Bruß [DCLB00] found NPT Werner states that are undistillable from any fixed number of copies, on ranges of α that depend on that number. Writing ψ as a matrix C of rank at most two, Costa Rico [CR25] turned k-copy undistillability into an inequality: $\rho(d,\alpha)$ is k-copy undistillable exactly when

$$q_k(t,C)=\sum_{J\subseteq\{1,\dots,k\}}(-t)^{|J|}\,\|\operatorname{Tr}_J C\|_F^2\;\geq\;0,\qquad t=-\alpha, \tag{1.1}$$

for every complex matrix C of rank at most two on $(\mathbb{C}^d)^{\otimes k}$, where Tr_J is the partial trace over the copies in J.

At the endpoint $t=1/2$ the inequality holds in every dimension for two copies, by four independent proofs [FGP26, FHPV26, SC26, BGH26], and for three copies, by a computer-assisted proof [Tom26a]. Below the endpoint, for each k, k-copy undistillability holds for all $\alpha\geq-\gamma_k$, with γ_k of order $1/k$ and, unlike the ranges of [DSS+00, DCLB00], independent of d. Qi, Gupur, Wu and Guo [QGWG24] proved the first such bound, and [BGH26] and [Tom26a] gave larger γ_k. These ranges contain NPT states exactly when $\gamma_k>1/d$. If the inequality held at the endpoint for all k in some dimension $d\geq 3$, the NPT state $\rho(d,-1/2)$ would be undistillable, and distillable entanglement would then be neither additive nor convex [SST01]. Such a state would also show that the partial transpose, which certifies entanglement, does not certify that maximally entangled pairs can be extracted from many copies. We leave that question open.

If the range of C lies in a product of subspaces of dimension at most two, one for each copy, then $q_k(t,C)\geq 0$ for every $t\leq 1/2$ (Lemma 2.6). In every known equality case at the endpoint, the range and co-range have a local support of dimension at most two at some copy, where the local support at a copy is the support of the one-copy reduction. A first failure would come with larger local supports. If the inequality holds for $k-1$ copies but fails for k, then for some $t<1/2$ the form $q_k(t,\cdot)$ is still nonnegative on matrices of rank at most two but vanishes at one of rank two whose range and co-range have the same local supports, of dimension at least three at every copy; conversely, such a matrix forces a failure (Section 6). Theorem A locates the equality cases for two copies, and Theorem B proves the inequality for every k on a class of matrices with larger local supports.

> **Theorem A** (Theorem 3.2 and Corollary 3.3). Let A and B be finite-dimensional, and let $C\neq 0$ be an operator on $A\otimes B$ of rank at most two with $q_2(1/2,C)=0$. Then C has rank two, its range lies in a product $W_A\otimes W_B$ of subspaces $W_A\subseteq A$, $W_B\subseteq B$ of dimension at most two, and its co-range lies in such a product $W_A'\otimes W_B'$. Conversely, every two-dimensional subspace of such a product is the co-range of such an operator.

Bharti, Gajjala and Haug had asked for a classification of the equality cases. Theorem A classifies their co-ranges and constrains their ranges. Examples of equality cases appear in [BGH26, SC26]. In dimension four the two-copy equality cases also describe the equality cases of the singular-value inequality of [PPHH10] for Kronecker sums $A \otimes I + I \otimes B$ with traceless A and B, and Theorem A adds a rank condition (Corollary 3.4). Liu and Chen had determined these equality cases for special classes [LC26].

To test the inequality beyond the known equality cases, fix orthonormal vectors $|0\rangle, |1\rangle, |2\rangle$ in $\mathbb{C}^d$, $d \geq 3$, and for $c \in \{0,1\}$ let $\mathcal{V}_c \subseteq (\mathbb{C}^d)^{\otimes k}$ be spanned by $|c\rangle^{\otimes k}$ and the k vectors obtained from it by replacing one factor with $|2\rangle$. For generic $f_0 \in \mathcal{V}_0$ and $f_1 \in \mathcal{V}_1$ the two-dimensional subspace $\operatorname{span}\{f_0, f_1\}$ has local supports of dimension three at every copy.

> **Theorem B** (Theorem 4.7). Let $k \geq 2$. For all $f_0 \in \mathcal{V}_0$ and $f_1 \in \mathcal{V}_1$, every operator C of rank at most two with $\operatorname{ran} C \subseteq \operatorname{span}\{f_0, f_1\}$, and arbitrary co-range, satisfies $q_k(1/2, C) \geq 0$.

These results first appeared in versions 2 and 3 of [Tom26a], whose proofs of Theorems 4.6 and 4.7 (there 10.6 and 10.7) did not cover $d > 3$ with an arbitrary co-range; Lemma 4.8 closes this case.

We also prove the inequality, under conditions on the co-range, for every operator of rank at most two with range in $\mathcal{V}_0 + \mathcal{V}_1$ (Section 4), near the generic equality cases with two-dimensional local supports, and near the product equality cases whenever it holds for $k-1$ copies (Section 5). We conjecture a lower bound on the endpoint minimum of q_k in terms of the one-copy reductions of the co-range, which would give the inequality for every k. We support it numerically and prove its two-copy case on one class of subspaces (Section 7).

Normal operators of rank two, among them the Hermitian and the positive semidefinite ones, have their range and co-range in one plane, that is, in one two-dimensional subspace. For three copies, Wu and Zou proved the endpoint inequality for these operators in every dimension [WZ26]. We prove the four-copy inequality for every operator whose range and co-range lie in one plane, whatever its local supports.

> **Theorem C** (Theorem 8.1). Let $d \geq 2$. Every operator C on $(\mathbb{C}^d)^{\otimes 4}$ such that $\operatorname{ran} C + \operatorname{ran} C^*$ lies in a plane satisfies $q_4(1/2, C) \geq 0$.

The proof is computer-assisted. It restricts the quartic moment relaxation used in [Tom26a] to these operators, where it involves the moments of a single vector and a one-parameter family of objectives, and one exact certificate covers the whole family in all large dimensions; smaller dimensions follow by embedding. At $d = 3$ we have also proved Theorem C in Lean 4 with Mathlib; its trusted base is stated in Appendix A.

Theorem A rests on the equality cases of the elementary proof of EP_2 by Song and Chen [SC26] and on a lemma on compound matrices. The other main tools are a lifting that turns equality cases into kernel vectors of a linear map on a doubled space (Section 2.2), second-order perturbation theory [Kat95] and the Morse–Bott lemma [BH04] near equality cases, and a copy-removal lemma built on the separable ball of Gurvits and Barnum [GB02] (Lemma 6.1). That lemma extends the copy-removal theorem of Wu and Zou [WZ26] from three copies at the endpoint to all k and all $t \leq 1/2$, given the inequality for one copy fewer. Apart from the computations marked † in Table 2 (Appendix A) and the results we cite, the proofs are by hand.

For $k \leq 3$ the endpoint statements of Section 4 also follow from the results for two and three copies. We know of no earlier statements of the other results. Section 2 collects the criterion, the facts about q_k that we use and the lifting; Section 8 proves Theorem C, Section 9 lists open questions, and Appendix B contains the longer proofs.

2. Preliminaries

2.1 The criterion and the form

In the criterion of Section 1 [DSS+00, DCLB00], ρ acts on $(\mathbb{C}^d)^{\otimes 2}$, the first factor held by one party and the second by the other, and Γ is the partial transpose on the second party. For $C \in L\big((\mathbb{C}^d)^{\otimes k}\big)$ and $t \in \mathbb{R}$, the form $q_k(t, C)$ is that of (1.1), with $\operatorname{Tr}_\emptyset C = C$.

Theorem 2.1 (Costa Rico [CR25, Theorem 1]). For $\alpha \in [-1, 1]$, the state $\rho(d, \alpha)$ is k-copy distillable if and only if there is $C \in L\big((\mathbb{C}^d)^{\otimes k}\big)$ of rank at most two with $q_k(-\alpha, C) < 0$.

For $t \in [0,1]$ let $\mathrm{EP}_k(t; d)$ be the statement that $q_k(t, C) \geq 0$ for every $C \in L\big((\mathbb{C}^d)^{\otimes k}\big)$ of rank at most two, and write $\mathrm{EP}_k(d) = \mathrm{EP}_k(1/2; d)$. Write $\Phi = \sum_{i,j} |ii\rangle\langle jj|$ for the unnormalised maximally entangled projector of one copy. With $\psi_C = \operatorname{vec} C$ one has $q_k(t, C) = \langle \psi_C | (I - t\Phi)^{\otimes k} | \psi_C \rangle$. Lemmas 2.2 and 2.3 and Corollary 2.4 are proved in [Tom26a, Section 2]; we repeat the short proofs.

Lemma 2.2. For product operators, $q_k\big(t, \bigotimes_m C_m\big) = \prod_m q_1(t, C_m)$, where $q_1(t, X) = \|X\|_F^2 - t\,|\operatorname{Tr} X|^2$.

Proof. The partial trace of a product over J is the product of the traces of the factors in J and the remaining factors. So the sum over J factorises. □

Lemma 2.3. Let $k \geq 1$ and $0 \leq t' \leq t < 1$. *(a)* If $q_k(t, C) \geq 0$ for every C of rank at most two on $(\mathbb{C}^D)^{\otimes k}$, then the same holds on $(\mathbb{C}^d)^{\otimes k}$ for every $d \leq D$. *(b)* If $q_k(t, C) \geq 0$ for every C of rank at most two on $(\mathbb{C}^d)^{\otimes k}$, then $q_j(t', C) \geq 0$ for every $j \leq k$ and every C of rank at most two on $(\mathbb{C}^d)^{\otimes j}$.

Proof. (a) For an isometry $\iota\colon \mathbb{C}^d \to \mathbb{C}^D$ the operator $C' = \iota^{\otimes k} C\, \iota^{*\otimes k}$ has the same rank as C, and $\operatorname{Tr}_J C' = \iota^{\otimes(k-|J|)} (\operatorname{Tr}_J C)\, \iota^{*\otimes(k-|J|)}$, so $q_k(t, C') = q_k(t, C)$.

(b) First let $t' = t$. For a unit vector u, the operator $C \otimes |u\rangle\langle u|^{\otimes(k-j)}$ has rank at most two, and the argument of Lemma 2.2 shows that its value of $q_k(t,\cdot)$ is $q_j(t, C)(1-t)^{k-j}$. Hence $q_j(t,\cdot) \geq 0$ for $j \leq k$. Now let $t' < t$. Write $I - t'\Phi = (t'/t)(I - t\Phi) + (1 - t'/t) I$ and expand the j-fold tensor power. The term with $I - t\Phi$ on a set S of copies contributes a positive multiple of $\langle \psi_C | (I - t\Phi)^{\otimes S} \otimes I | \psi_C \rangle = \sum_{x,y} q_{|S|}(t, C_{xy})$, where C_{xy} are the matrix elements of C over the copies outside S in a product basis. Each C_{xy} has rank at most two, so every term is nonnegative. □

Corollary 2.4. If $q_k(t, C) \geq 0$ for some $t \in [0, 1)$ and every C of rank at most two on $(\mathbb{C}^d)^{\otimes k}$, then $\rho(d, \alpha)$ is k-copy undistillable for every $\alpha \in [-t, 1]$.

Proof. For $-t \leq \alpha \leq 0$, Lemma 2.3(b) gives $q_k(-\alpha, \cdot) \geq 0$, and Theorem 2.1 applies. For $\alpha > 0$ every term of $q_k(-\alpha, C)$ is nonnegative. □

The following result of [Tom26a] has a computer-assisted proof.

Theorem 2.5 ([Tom26a, Theorem 1.2]). $\mathrm{EP}_3(d)$ holds for every d.

The following elementary fact covers operators whose range has local dimension at most two everywhere.

Lemma 2.6. If $\operatorname{ran} C \subseteq W_1 \otimes \cdots \otimes W_k$ with $\dim W_l \leq 2$ for every l, then $q_k(t, C) \geq 0$ for $0 \leq t \leq 1/2$.

Proof. Here $q_k(t, C) = \langle \psi_C | (I - t\Phi)^{\otimes k} | \psi_C \rangle$, the vector ψ_C lies in $\bigotimes_l (W_l \otimes \mathbb{C}^d)$, and on $W_l \otimes \mathbb{C}^d$ the compression of $I - t\Phi$ has eigenvalues in $\{1,\, 1 - t \dim W_l\} \subseteq [0, 1]$. □

2.2 Planes, zeros and a lifting

Fix $k \geq 1$ and $d \geq 2$, and put $H = (\mathbb{C}^d)^{\otimes k}$. Since $\operatorname{Tr}_J(C^*) = (\operatorname{Tr}_J C)^*$, we have $q_k(t, C^*) = q_k(t, C)$, so range and co-range play symmetric roles.

A *plane* is a two-dimensional subspace $V \subseteq H$. Put

$$m_k(V) = \min\{ q_k(1/2, C) :\ \|C\|_F = 1,\ \operatorname{ran} C^* \subseteq V \}.$$

Every operator of rank at most two has its co-range in some plane, so $\mathrm{EP}_k(d)$ holds if and only if $m_k(V) \geq 0$ for every plane V. We call V a *zero* if $m_k(V) = 0$. The *local support* of V at copy l is the support $S_l(V) \subseteq \mathbb{C}^d$ of the reduced operator of the orthogonal projector P_V on copy l. The plane V is *qubit-supported* if $\dim S_l(V) \leq 2$ for every l, that is, if $V \subseteq W_1 \otimes \cdots \otimes W_k$ with $\dim W_l \leq 2$.

Let $Q = \mathbb{C}^2$ with orthonormal basis q_1, q_2. We encode a pair $a, b \in H$ as $\psi = a \otimes q_1 + b \otimes q_2 \in H \otimes Q$, and a second pair f_1, f_2 as $\varphi = f_1 \otimes q_1 + f_2 \otimes q_2$. For $t \in \mathbb{R}$ let $D_t(X) = \operatorname{Tr}(X)\, I - tX$ on $L(\mathbb{C}^d)$, and let $E_t = D_t^{\otimes k} \otimes \mathrm{id}_Q$, a linear map on $L(H \otimes Q)$. Expanding $D_t^{\otimes k}$ over the set of copies on which $-tX$ is chosen gives

$$\langle \varphi | E_t(\psi\psi^*) | \varphi \rangle = q_k\big(t,\ f_1 a^* + f_2 b^*\big). \tag{2.1}$$

Hence $\mathrm{EP}_k(t; d)$ holds if and only if $E_t(\psi\psi^*) \succeq 0$ for every ψ. If (a, b) is an orthonormal basis of a plane V, then $\|f_1 a^* + f_2 b^*\|_F = \|\varphi\|$, and $m_k(V)$ is the smallest eigenvalue of $E_{1/2}(\psi\psi^*)$. We also use $E = 2^k E_{1/2} = D^{\otimes k} \otimes \mathrm{id}_Q$, with $D(X) = 2 \operatorname{Tr}(X)\, I - X$. The local supports of the plane spanned by a and b are the supports $S_l(\psi)$ of the reduced operators of $\psi\psi^*$ on single copies.

Lemma 2.7. *(a)* Every qubit-supported plane is a zero. *(b)* Let $k \geq 2$, let $U \subseteq \mathbb{C}^d$ be a plane and let $\chi \in (\mathbb{C}^d)^{\otimes(k-1)}$ be nonzero. If $\mathrm{EP}_{k-1}(d)$ holds, then the plane $U \otimes \chi$, with U at the first copy, is a zero.

Proof. (a) Let $V \subseteq W_1 \otimes \cdots \otimes W_k$ with $\dim W_l \leq 2$. By Lemma 2.6, applied to C^*, $q_k(1/2, C) \geq 0$ whenever $\operatorname{ran} C^* \subseteq V$, so $m_k(V) \geq 0$. By Lemma B.2 below, applied with two-dimensional local spaces $U_l \supseteq W_l$, the kernel of $E(\psi\psi^*)$ contains the nonzero vector $\Theta\psi$, where (a, b) is a basis of V. Writing $\Theta\psi = f_1 \otimes q_1 + f_2 \otimes q_2$, (2.1) gives $q_k(1/2, f_1 a^* + f_2 b^*) = 0$, and $f_1 a^* + f_2 b^* \neq 0$ because a and b are linearly independent. So $m_k(V) = 0$.

(b) If $\operatorname{ran} C^* \subseteq U \otimes \chi$, the local support of $\operatorname{ran} C^*$ at the first copy lies in U, so Lemma 6.1 below, with $t = 1/2$, gives $q_k(1/2, C) \geq 0$. Hence $m_k(U \otimes \chi) \geq 0$. For any nonzero χ', the operator $C = P_U \otimes |\chi'\rangle\langle\chi|$ has co-range $U \otimes \chi$, and by Lemma 2.2, $q_k(1/2, C) = q_1(1/2, P_U)\, q_{k-1}(1/2, |\chi'\rangle\langle\chi|) = 0$, since $q_1(1/2, P_U) = 2 - 4 \cdot \tfrac{1}{2} = 0$. □

A product plane $U \otimes \chi$ is qubit-supported only if every local support of χ has dimension at most two.

3. Two copies: the zero set

For two copies we write $H = A \otimes B$ with arbitrary finite-dimensional A and B, and $q_2(C) = q_2(1/2, C) = \|C\|_F^2 - \frac{1}{2}\big(\|\operatorname{Tr}_A C\|_F^2 + \|\operatorname{Tr}_B C\|_F^2\big) + \frac{1}{4}|\operatorname{Tr} C|^2$. The proof uses the equality cases of the elementary proof of EP_2 by Song and Chen [SC26], together with a lemma on compound matrices.

Notation. On $H \otimes H$ let F_A and F_B exchange the A-factors and the B-factors of the two copies of H. They commute, and $F_{AB} = F_A F_B$ is the full swap. Put $P_{--} = \frac{1}{4}(I - F_A)(I - F_B)$, $P_{++} = \frac{1}{4}(I + F_A)(I + F_B)$ and $P_{\mathrm{Sym}^2 H} = \frac{1}{2}(I + F_{AB})$, so that $\operatorname{ran} P_{\mathrm{Sym}^2 H} = \operatorname{ran} P_{++} \oplus \operatorname{ran} P_{--}$. For a subspace $R \subseteq H$, $P_{\mathrm{Sym}^2 R}$ denotes the projector onto the symmetric subspace of $R \otimes R$. For $u \in H$ with coefficient matrix $U \in M_{\dim A, \dim B}$ let $C_2(U) \in H \otimes H$ be the tensor of 2×2 minors, $C_2(U)_{(ij),(kl)} = U_{ij} U_{kl} - U_{kj} U_{il}$. With the unnormalised wedge $e \wedge e' = e \otimes e' - e' \otimes e$ one has $\operatorname{ran} P_{--} = \Lambda^2 A \otimes \Lambda^2 B$ and $P_{--}(u \otimes u) = \frac{1}{2} C_2(U)$. The map C_2 is quadratic, $C_2(X + Y) = C_2(X) + \Delta(X, Y) + C_2(Y)$ with Δ bilinear and symmetric;

$C_2(M)=0$ if and only if $\operatorname{rank} M\le 1$; and, read as the second compound matrix $\Lambda^2 M$, C_2 is multiplicative, $C_2(XY)=C_2(X)C_2(Y)$ (Cauchy–Binet), with image $\Lambda^2(\operatorname{col} M)$, which determines the column space of M when $\operatorname{rank} M\ge 2$; dually, $C_2(M)$ determines the row space.

Lemma 3.1. *Let $L\subseteq A\otimes B$ be a subspace of dimension at least two, and suppose that $C_2(M)\in\mathbb{C}z$ for every $M\in L$, for a fixed tensor z. Then either every element of L is a product vector, or $L\subseteq A_0\otimes B_0$ with $\dim A_0,\dim B_0\le 2$.*

Proof. Write $C_2(M)=\phi(M)z$ with ϕ a quadratic form on L. If $\phi=0$, every $M\in L$ has rank at most one. Otherwise $\{\phi\neq 0\}$ is dense in L, and all its elements have the same rank $r\ge 2$, since $\binom{r}{2}$ equals the rank of z. Suppose $r\ge 3$, and take M,M' with $\phi(M)\phi(M')\neq 0$. Then $C_2(M')=cC_2(M)$ with $c\neq 0$, so M and M' have the same column space and the same row space, and $G=M'M^{-1}$, defined on this r-dimensional column space, satisfies $C_2(G)=cI$. So $Gv\wedge Gw=c\,v\wedge w$ for all v,w, and G maps every two-dimensional subspace into itself. Since $r\ge 3$, every vector is then an eigenvector, so $G=\lambda I$ and $M'=\lambda M$. Thus the dense set $\{\phi\neq 0\}$ lies on the line $\mathbb{C}M$, which contradicts $\dim L\ge 2$. Hence $r=2$, and z is a multiple of $(a\wedge a')\otimes(b\wedge b')$. Every M with $\phi(M)\neq 0$ then has column space $A_0=\operatorname{span}\{a,a'\}$ and row space $B_0=\operatorname{span}\{b,b'\}$, and by density $L\subseteq A_0\otimes B_0$. □

We use three facts from [SC26]. Let C satisfy $\operatorname{ran} C^*\subseteq S$ for a plane S, let (e_1,e_2) be an orthonormal basis of S and put $x_i=Ce_i$, so that $C=x_1e_1^*+x_2e_2^*$. Put $w=e_1\otimes x_2-e_2\otimes x_1=(I\otimes C)\varepsilon_S$ with $\varepsilon_S=e_1\otimes e_2-e_2\otimes e_1$.

- (P) [SC26, Proposition 2] $q_2(C)=t_1+\beta$, where $t_1=2\|P_{--}(e_1\otimes x_1)\|^2+\|P_{--}(e_1\otimes x_2+e_2\otimes x_1)\|^2+2\|P_{--}(e_2\otimes x_2)\|^2$ and $\beta=\|P_{++}w\|^2-\frac12\|P_{\mathrm{Sym}^2 S}w\|^2$.
- (L3) [SC26, Lemma 3] Let $z\in\operatorname{ran}P_{--}$, so that its coefficient matrix Z (in a product basis of $H\otimes H$) is symmetric, with singular values $s_1\ge s_2\ge\dots$, and let P be a rank-two projector on H. Then $\langle z|(P\otimes I+I\otimes P)z\rangle=2\operatorname{Tr}(PZZ^*)\le 2(s_1^2+s_2^2)\le\|z\|^2$. The first inequality is Ky Fan's. For the second, take a Takagi decomposition $z=\sum_p s_p\,u_p\otimes u_p$ with orthonormal u_p and coefficient matrices U_p [HJ13], put $z_2=s_1u_1\otimes u_1+s_2u_2\otimes u_2$ and $a_p=\operatorname{Tr}\big((U_p^*U_p)^2\big)$. Then $s_1^2+s_2^2=\langle P_{--}z_2|z\rangle\le\|P_{--}z_2\|\,\|z\|$ and $\|P_{--}z_2\|^2=\frac12\|z_2\|^2-\frac14\langle z_2|(F_A+F_B)z_2\rangle$, where $\frac12\langle z_2|(F_A+F_B)z_2\rangle=s_1^2a_1+s_2^2a_2+2s_1s_2\operatorname{Re}\operatorname{Tr}\big((U_1^*U_2)^2\big)\ge\big(s_1a_1^{1/2}-s_2a_2^{1/2}\big)^2$, because $|\operatorname{Tr}((U_1^*U_2)^2)|\le\operatorname{Tr}(U_1U_1^*U_2U_2^*)\le(a_1a_2)^{1/2}$; the last step is the Cauchy–Schwarz inequality.
- (L4) [SC26, Lemma 4] For $\xi\in S\otimes H$, $\|P_{\mathrm{Sym}^2 S}\xi\|^2\le 2\|P_{++}\xi\|^2$. In the proof, $\xi_0=P_{\mathrm{Sym}^2 H}\xi=y+u$ with $y\in\mathrm{Sym}^2 S$ and $u\in(S\otimes S^\perp)\oplus(S^\perp\otimes S)$; with $z=P_{--}\xi_0$ and $N=P_S\otimes I+I\otimes P_S$ one has $N(y/2+u)=\xi_0$, hence $\|z\|^2=\langle z|N(y/2+u)\rangle\le\langle z|Nz\rangle^{1/2}\big(\|y\|^2/2+\|u\|^2\big)^{1/2}\le\|z\|\big(\|y\|^2/2+\|u\|^2\big)^{1/2}$, by the Cauchy–Schwarz inequality for the form N and then (L3); finally $\|P_{++}\xi\|^2=\|\xi_0\|^2-\|z\|^2$.

Theorem 3.2. *Let $V\subseteq A\otimes B$ be a plane and let $C\neq 0$ satisfy $\operatorname{ran}C^*\subseteq V$ and $q_2(C)=0$. Then $V\subseteq A_0\otimes B_0$ for some subspaces $A_0\subseteq A$ and $B_0\subseteq B$ of dimension at most two.*

Proof. Put $S=V$ and use the notation above. By (P) and (L4) with $\xi=w$, both $t_1=0$ and $\beta=0$.

Step 1. Since $P_{--}F_{AB}=P_{--}$, polarising t_1 in $s=\lambda_1e_1+\lambda_2e_2$ shows that $t_1=0$ if and only if $P_{--}(s\otimes Cs)=0$ for every $s\in S$.

Step 2. With $\xi_0=y+u$, z and N as in (L4) for $\xi=w$, we have $\beta=\|y\|^2/2+\|u\|^2-\|z\|^2$, so $\beta=0$ is equality in the chain of (L4). Either $z=0$, and then $y=u=0$; or $z\neq 0$, and then $\langle z|Nz\rangle=\|z\|^2$, which is equality in (L3) with $P=P_S$, and the Cauchy–Schwarz step is tight, so $N^{1/2}(y/2+u)$ is a multiple of $N^{1/2}z$. Comparing $\langle z|Nz\rangle$ with $\langle z|\xi_0\rangle=\|z\|^2$ shows that the multiple is one, so $Nz=\xi_0$. In terms of C, with $C_S=P_SC$ and $C_\perp=P_{S^\perp}C$,

$$\xi_0=\tfrac12(I\otimes C-C\otimes I)\varepsilon_S,\qquad y=\tfrac12(I\otimes C_S-C_S\otimes I)\varepsilon_S,\qquad u=\tfrac12(I\otimes C_\perp-C_\perp\otimes I)\varepsilon_S,$$

so $u=0$ if and only if $\operatorname{ran}C\subseteq S$.

Case $z=0$. Then $\xi_0=0$, so $C_\perp=0$, and comparing coefficients of $y=0$ in $\mathrm{Sym}^2 S$ gives $C=cP_S$ with $c\neq 0$. Step 1 now gives $P_{--}(s\otimes s)=0$ for $s\in S$. Since $\|P_{--}(s\otimes s)\|^2=\frac12\big(\|s\|^4-\operatorname{Tr}\big((M_s^*M_s)^2\big)\big)$, with M_s the coefficient matrix of s, every vector of S is a product vector. A two-dimensional space of product vectors has the form $a\otimes B_0$ or $A_0\otimes b$, and in both cases $V\subseteq A_0\otimes B_0$.

Case $z\neq 0$. Equality in (L3) forces equality in each step of its proof, for every Takagi decomposition of z: (a) $s_1=s_2=:s>0$. Indeed $s_2>0$, since otherwise $z=s_1u_1\otimes u_1$ would satisfy $\|z\|=\|P_{--}z\|\le\|z\|/\sqrt{2}$; equality in the Cauchy–Schwarz step forces $U_2U_2^*=U_1U_1^*$, so $a_1=a_2$, and then $s_1a_1^{1/2}=s_2a_2^{1/2}$ gives $s_1=s_2$. (b) $S\subseteq E_s$, the eigenspace of ZZ^* for s^2; this is equality in Ky Fan's inequality, since $ZZ^*\preceq s^2I$ and $\operatorname{Tr}(P_SZZ^*)=2s^2$. (c) $z=2P_{--}z_2=s\big(C_2(U_1)+C_2(U_2)\big)$ for every pair (u_1,u_2) of Takagi vectors with value s; this is equality in $\langle P_{--}z_2|z\rangle\le\|P_{--}z_2\|\,\|z\|$, with $\|P_{--}z_2\|^2=\frac12\|z_2\|^2$ and $\|z\|^2=2\|z_2\|^2$.

On E_s the map $Jv=s^{-1}Z\bar v$ is an antiunitary involution: $Z\bar Z(Z\bar v)=Z\,\overline{Z\bar Z}v=s^2Z\bar v$, $J^2=s^{-2}Z\bar Z=\mathrm{id}$ on E_s, and $\langle Jv|Jw\rangle=\langle w|v\rangle$. The Takagi vectors with value s are the unit vectors fixed by J, and every J-fixed orthonormal family in E_s extends to a Takagi decomposition of z. So (c) holds for every J-fixed orthonormal pair in E_s.

Sub-case $e=\dim E_s\ge 3$. Fix a J-fixed orthonormal basis $f_1,\dots,f_e$ of E_s with coefficient matrices F_j. For $j\neq j'$ choose j'' different from both; (c) for the pairs $(f_j,f_{j''})$ and $(f_{j'},f_{j''})$ gives $C_2(F_j)=C_2(F_{j'})$, hence $C_2(F_j)=z/(2s)$ for all j. Every real unit vector $\alpha\in\mathbb{R}^e$ gives a J-fixed unit vector $\sum_j\alpha_jf_j$, which extends to a J-fixed orthonormal basis, so also $C_2\big(\sum_j\alpha_jF_j\big)=z/(2s)$. The quadratic map $\alpha\mapsto C_2\big(\sum_j\alpha_jF_j\big)$ on $\mathbb{R}^e$ therefore equals $\big(\sum_j\alpha_j^2\big)z/(2s)$, which forces $\Delta(F_j,F_{j'})=0$ for $j\neq j'$. By complex bilinearity, $C_2\big(\sum_j c_jF_j\big)=\big(\sum_j c_j^2\big)z/(2s)$ for all $c\in\mathbb{C}^e$. So $C_2(M)\in\mathbb{C}z$ for every $M\in E_s$, and E_s does not consist of product vectors since $C_2(F_1)\neq 0$. Lemma 3.1 gives $S\subseteq E_s\subseteq A_0\otimes B_0$.

Sub-case $\dim E_s = 2$. Then $S = E_s = \operatorname{span}\{u_1, u_2\}$, the other Takagi vectors are orthogonal to S, and z has no component in $(S \otimes S^\perp) \oplus (S^\perp \otimes S)$. Comparing these components in $Nz = \xi_0$ gives $u = 0$, so $C = C_S \neq 0$ and $z = P_{--}y$. In $\operatorname{Sym}^2 S$, with the basis $(e_1 \otimes e_1,\ e_2 \otimes e_2,\ e_1 \otimes e_2 + e_2 \otimes e_1)$ and $C_S = [c_{ij}]$, the vectors $T(s) = s \otimes Cs + Cs \otimes s$ are spanned by

$$T_{11} = (2c_{11}, 0, c_{21}), \qquad T_{22} = (0, 2c_{22}, c_{12}), \qquad T_{12} = (2c_{12}, 2c_{21}, c_{11} + c_{22}),$$

with $\det[T_{11}, T_{22}, T_{12}] = 4 \operatorname{Tr}(C_S) \det(C_S)$, and $y = \big(c_{12}, -c_{21}, (c_{22} - c_{11})/2\big)$. By Step 1, P_{--} kills every $T(s)$. If $\operatorname{Tr}(C_S)\det(C_S) \neq 0$, then $P_{--}(\operatorname{Sym}^2 S) = 0$ and $z = P_{--}y = 0$, which is excluded. If $\det(C_S) = 0$, then in an orthonormal basis of S in which $C_S = \begin{pmatrix} \tau & c \\ 0 & 0 \end{pmatrix}$, the span of the $T(s)$ contains every vector with vanishing second coordinate, because $C_S \neq 0$; so does y, hence again $z = 0$. So $\operatorname{Tr}(C_S) = 0 \neq \det(C_S)$. Then the $T(s)$ span the kernel of $\ell = (-c_{21}, c_{12}, 2c_{11})$, and $\ell \cdot y = 2\det(C_S) \neq 0$, so $\operatorname{Sym}^2 S = \ker \ell \oplus \mathbb{C}y$ and $P_{--}(\operatorname{Sym}^2 S) = \mathbb{C}z$. Therefore $C_2(M_s) = 2P_{--}(s \otimes s) \in \mathbb{C}z$ for every $s \in S$, and these tensors span $\mathbb{C}z \neq 0$. Lemma 3.1 gives $S \subseteq A_0 \otimes B_0$. $\square$

Corollary 3.3. *A plane $V \subseteq A \otimes B$ is a zero of $q_2(1/2, \cdot)$ if and only if it is qubit-supported. If $q_2(C) = 0$ for some $C \neq 0$ of rank at most two, then C has rank two and both $\operatorname{ran} C$ and $\operatorname{ran} C^*$ are qubit-supported planes.*

Proof. The first statement is Lemma 2.7(a) together with Theorem 3.2. For the second, apply Theorem 3.2 to C and to C^*. An operator $C = |f\rangle\langle a|$ of rank one satisfies $q_2(1/2, C) \geq \frac{1}{4}\|C\|_F^2$: since $\psi_C = f \otimes \bar{a}$ is a product vector, $q_2(1/2, C) = \langle f \otimes a | X^\Gamma | f \otimes a \rangle$, where X^Γ is the partial transpose of $X = (I - \frac{1}{2}\Phi)^{\otimes 2}$ on the column factor. It equals $(I - \frac{1}{2}F)^{\otimes 2} \succeq \frac{1}{4}I$, with F the swap at each copy. $\square$

This classifies the equality cases of the two-copy partial-trace inequality at the level of planes.

The same equality analysis describes the equality cases of a singular-value inequality for Kronecker sums. Pankowski, Piani, Horodecki and Horodecki [PPHH10] reduced two-copy distillability of $\rho(4, \alpha)$ to a bound on $\sigma_1(X)^2 + \sigma_2(X)^2$ for $X = A \otimes I + I \otimes B$, and Song and Chen derived the bound from EP_2 [SC26, Corollary 6]: for $d \geq 4$ and traceless $A, B \in M_d$,

$$\sigma_1(X)^2 + \sigma_2(X)^2 \leq \frac{3d - 4}{d}\big(\|A\|_F^2 + \|B\|_F^2\big). \tag{3.1}$$

Its proof combines $(\sigma_1^2 + \sigma_2^2)^{1/2} = \max\{|\operatorname{Tr}(C^*X)| : \operatorname{rank} C \leq 2,\ \|C\|_F = 1\}$, the identity $\operatorname{Tr}(C^*X) = \langle (\operatorname{Tr}_B C)_0 | A \rangle + \langle (\operatorname{Tr}_A C)_0 | B \rangle$ with $Y_0 = Y - \frac{\operatorname{Tr} Y}{d} I$, and the bound $\|(\operatorname{Tr}_A C)_0\|_F^2 + \|(\operatorname{Tr}_B C)_0\|_F^2 = \|\operatorname{Tr}_A C\|_F^2 + \|\operatorname{Tr}_B C\|_F^2 - \frac{2}{d}|\operatorname{Tr} C|^2 \leq 2\|C\|_F^2 + \big(\frac{1}{2} - \frac{2}{d}\big)|\operatorname{Tr} C|^2$, which is EP_2. At $d = 4$ the last coefficient vanishes, and Liu and Chen [LC26] determined the equality cases of (3.1) for seven special classes of pairs (A, B). Corollary 3.4 describes all of them through the two-copy equality cases.

Corollary 3.4. *Let $A, B \in M_4$ be traceless and not both zero, and put $X = A \otimes I + I \otimes B$. Then $\sigma_1(X)^2 + \sigma_2(X)^2 = 2\big(\|A\|_F^2 + \|B\|_F^2\big)$ if and only if $(A, B) = \kappa\big((\operatorname{Tr}_B C)_0, (\operatorname{Tr}_A C)_0\big)$ for some $\kappa \neq 0$ and some $C \neq 0$ of rank at most two with $q_2(C) = 0$. For every such pair there is $c \in \mathbb{C}$ with $\operatorname{rank}(A + cI) \leq 2$ and $\operatorname{rank}(B + cI) \leq 2$.*

Proof. Let equality hold, and take C of rank at most two with $\|C\|_F = 1$ and $|\operatorname{Tr}(C^*X)|^2 = \sigma_1(X)^2 + \sigma_2(X)^2$. In

$$|\operatorname{Tr}(C^*X)| \leq \|(\operatorname{Tr}_B C)_0\|_F \|A\|_F + \|(\operatorname{Tr}_A C)_0\|_F \|B\|_F \leq \big(\|(\operatorname{Tr}_B C)_0\|_F^2 + \|(\operatorname{Tr}_A C)_0\|_F^2\big)^{1/2}\big(\|A\|_F^2 + \|B\|_F^2\big)^{1/2} \leq \sqrt{2}\,\big(\|A\|_F^2 + \|B\|_F^2\big)^{1/2}$$

every step is then an equality. The first two give $(A, B) = \kappa\big((\operatorname{Tr}_B C)_0, (\operatorname{Tr}_A C)_0\big)$, and the last one, at $d = 4$, is $q_2(C) = 0$. Conversely, let $q_2(C) = 0$ with $\|C\|_F = 1$, and put $A = (\operatorname{Tr}_B C)_0$ and $B = (\operatorname{Tr}_A C)_0$. Then $\operatorname{Tr}(C^*X) = \|A\|_F^2 + \|B\|_F^2 = \|\operatorname{Tr}_A C\|_F^2 + \|\operatorname{Tr}_B C\|_F^2 - \frac{1}{2}|\operatorname{Tr} C|^2 = 2$, so $\sigma_1(X)^2 + \sigma_2(X)^2 \geq 4 = 2\big(\|A\|_F^2 + \|B\|_F^2\big)$, and (3.1) gives equality; scaling by κ does not change the ratio. Finally, by Corollary 3.3 the range of C lies in $A_1 \otimes B_1$ with $\dim A_1, \dim B_1 \leq 2$, so $\operatorname{Tr}_B C$ and $\operatorname{Tr}_A C$ have rank at most two, and $c = \kappa \operatorname{Tr}(C)/4$ satisfies $A + cI = \kappa \operatorname{Tr}_B C$ and $B + cI = \kappa \operatorname{Tr}_A C$. $\square$

The rank condition is necessary but not sufficient: $A = \operatorname{diag}(1, -1, 0, 0)$ and $B = 0$ satisfy it, yet $\sigma_1(X)^2 + \sigma_2(X)^2 = 2$ is half of $2(\|A\|_F^2 + \|B\|_F^2)$. In each of the seven classes of [LC26, Theorem 7], the pairs that attain equality satisfy the rank condition, as one checks by hand.

4. Any number of copies: a structured class

This section proves $q_k(1/2, C) \geq 0$ for every k and every operator of rank at most two whose range lies in the span of the two families of Section 1, first under conditions on the co-range and then, in Theorem 4.7, for an arbitrary co-range when the range is spanned by one vector of each family.

4.1 Setting and statements

Throughout we write $r = t/(1 - t)$. Let $d \geq 3$. Fix orthonormal vectors $|0\rangle, |1\rangle, |2\rangle \in \mathbb{C}^d$ and let P_{01} be the orthogonal projector onto $\operatorname{span}\{|0\rangle, |1\rangle\}$. For a string $x \in \{0, 1, 2\}^k$ write $|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_k\rangle$, and let $h(x, y)$ be the Hamming distance. For a copy j and $c \in \{0, 1\}$ let c^k be the constant string, let $2_j c^{k-1}$ be the string with 2 at copy j and c elsewhere, and let $c_j \bar{c}^{k-1}$ be the string with c at copy j and $\bar{c} = 1 - c$ elsewhere. We call the strings $c_j \bar{c}^{k-1}$ *single-flip strings*.

The value $q_k(t, C)$ is unchanged under $C \mapsto UCU^*$ for every product unitary $U = U_1 \otimes \cdots \otimes U_k$, because $\operatorname{Tr}_J(UCU^*)$ equals the corresponding conjugate of $\operatorname{Tr}_J C$. Hence everything below holds for arbitrary orthonormal triples at

each copy. Since q_k is also invariant under isometric embeddings of the local spaces (proof of Lemma 2.3(a)), we may assume $d = 3$ whenever the co-range lies in $\mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}^{\otimes k}$; for Theorems 4.6 and 4.7 see Lemma 4.8.

The class. Let

$$R = \{0^k,\ 1^k\} \cup \{2_j 0^{k-1},\ 2_j 1^{k-1} : 1 \le j \le k\}, \qquad \mathcal{V} = \mathrm{span}\{|x\rangle : x \in R\}.$$

So $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1$, where $\mathcal{V}_c = \mathrm{span}\{|c^k\rangle, |2_j c^{k-1}\rangle : 1 \le j \le k\}$. As in Section 1, $\mathcal{V}_c$ is spanned by the *backbone* $|c^k\rangle$ and the *aligned tails* $|2_j c^{k-1}\rangle$. The two backbones are locally orthogonal at every copy, but the two families share the level $|2\rangle$.

For a string x, its *pattern* $P(x) \subseteq \{1, \dots, k\}$ is the set of copies where x has the digit 2. Let Π_P be the orthogonal projector onto $\mathrm{span}\{|x\rangle : P(x) = P\}$. If $\mathrm{ran}\, C \subseteq \mathcal{V}$, then $C = \Pi_\emptyset C + \sum_{j=1}^{k} \Pi_{\{j\}} C$. Here $\Pi_\emptyset C$ carries the backbone rows, and $\Pi_{\{j\}} C$ carries the tail rows at copy j.

For the co-range we use three nested spaces:

$$\mathcal{V} \ \subseteq\ A^\natural := \mathrm{span}\{|0^k\rangle, |1^k\rangle\} + E \ \subseteq\ A^\flat,$$

where E is the span of the strings with exactly one digit 2. The space $A^\flat$ is the span of the strings with at most one digit 2, with the single-flip strings removed. There are $2k$ single-flip strings for $k \ge 3$, and two, namely 01 and 10, for $k = 2$.

> **Theorem 4.1.** Let $k \ge 2$, $0 < t \le 1/2$, and let C be an operator of rank at most two with $\mathrm{ran}\, C \subseteq \mathcal{V}$ and $\mathrm{ran}\, C^* \subseteq A^\natural$. Then
>
> $$q_k(t, C) \ \ge\ (1 - r) \sum_{P \in \{\emptyset, \{1\}, \dots, \{k\}\}} q_k(t, \Pi_P C).$$
>
> Each term on the right is nonnegative. In particular $q_k(1/2, C) \ge 0$.

> **Theorem 4.2.** Let $k \ge 2$ and let C be an operator of rank at most two with $\mathrm{ran}\, C \subseteq \mathcal{V}$ and $\mathrm{ran}\, C^* \subseteq A^\flat$. Then $q_k(1/2, C) \ge 0$.

For $t < 1/2$, Theorem 4.1 says more than nonnegativity. The cross terms between different row blocks never cancel more than the fraction r of $\sum_P q_k(t, \Pi_P C)$, however many tails are active, and this margin does not depend on k. The constant $1 - r$ is attained for every k (Section 4.5). For operators with range and co-range in $\mathcal{V}$, the proof also gives a quantitative form for every rank (Remark 4.4).

Since $\Pi_\emptyset C$ has range in $\mathrm{span}\{|0^k\rangle, |1^k\rangle\} \subseteq \mathrm{span}\{|0\rangle, |1\rangle\}^{\otimes k}$ and $\Pi_{\{j\}} C$ has range in $|2\rangle \otimes \mathrm{span}\{|0^{k-1}\rangle, |1^{k-1}\rangle\}$, each $q_k(t, \Pi_P C)$ is nonnegative by Lemma 2.6.

4.2 Sectors

Write $C = \sum_{x,y} C_{xy} |x\rangle\langle y|$ in the product basis. From $q_k(t, C) = \langle \psi_C | (I - t\Phi)^{\otimes k} | \psi_C \rangle$ with $\psi_C = \sum C_{xy} |x\rangle|y\rangle$,

$$q_k(t, C) = \sum_{x,y,x',y'} \overline{C_{xy}}\, C_{x'y'} \prod_{l=1}^{k} \left[\delta_{x_l x'_l} \delta_{y_l y'_l} - t\, \delta_{x_l y_l} \delta_{x'_l y'_l} \right]. \tag{4.1}$$

The product vanishes unless, at every copy l, either $(x_l, y_l) = (x'_l, y'_l)$, or $x_l = y_l$ and $x'_l = y'_l$. For an entry (x, y) let $O = \{l : x_l \ne y_l\}$ and $N = O^c$. Two entries interact only if they have the same set O and agree on O. We call the class of entries with given (O, x_O, y_O) a *sector*. Then $q_k(t, C)$ is the sum over sectors of a Hermitian form in the entries of that sector. On N both entries of a pair are diagonal, $x_N = y_N =: z$ and $x'_N = y'_N =: z'$, and the form has coefficients

$$\prod_{l \in N} \left(\delta_{z_l z'_l} - t \right) = (1 - t)^{|N| - h(z,z')} (-t)^{h(z,z')}.$$

There are two cases.

- An entry that is alone in its sector contributes $(1 - t)^{k - h(x,y)} |C_{xy}|^2$.
- The diagonal entries $v_x = C_{xx}$ form one sector ($O = \emptyset$), with form $(1 - t)^k \sum_{x,x'} (-r)^{h(x,x')}\, \overline{v_x} v_{x'}$.

The same decomposition applies to each $\Pi_P C$, which keeps the entries of C in the rows with pattern P.

> **Lemma 4.3.** Let $\mathrm{ran}\, C \subseteq \mathcal{V}$, and let $C_{xy} = 0$ unless the column string y has at most one digit 2 and is not a single-flip string.
>
> (a) If $x \ne y$ and $y \in R$, the entry (x, y) is alone in its sector.
>
> (b) Every sector with at least two entries, other than the diagonal sector, has the form
>
> $$\sigma = \{(c^k,\ \bar{c}^{\,O} c^N)\} \cup \{(2_i c^{k-1},\ \bar{c}^{\,O} 2_i c^{N \setminus \{i\}}) : i \in N\}$$
>
> for some $c \in \{0, 1\}$ and a partition $O \cup N$ of the copies with $O, N \ne \emptyset$. Here the first member, the *backbone entry*, is present only if its column $\bar{c}^{\,O} c^N$ is an allowed column. Such a column is a mixed string without the digit 2. It is never allowed in $A^\natural$, and in $A^\flat$ it is allowed exactly when $|O| \ge 2$ and $|N| \ge 2$.

Proof. Let (x, y) be an off-diagonal entry with $x \in R$, let $O \ne \emptyset$ be its set of differing copies, and let $(x', y') \ne (x, y)$ be a second entry of the same sector. Then $x' \in R$, $x'_O = x_O$, $y'_O = y_O$ and $x'_N = y'_N \ne x_N$.

Suppose first that x has its digit 2 at a copy $j \in O$. Then x' also has 2 at j, so $x' = 2_j c'^{k-1}$ and $x = 2_j c^{k-1}$ with $c' \neq c$. If O contained another copy l, then $x'_l = x_l$ would force $c' = c$. So $O = \{j\}$ and $y_j \in \{0,1\}$. If $y_j = c$, then $y' = c_j \bar{c}^{k-1}$. If $y_j = \bar{c}$, then $y = \bar{c}_j c^{k-1}$. In both cases one of the two entries has a single-flip column and therefore vanishes. So (x, y) is absent or alone.

Suppose next that x has no digit 2 on O. Then $x_O = c^O$, where $x = c^k$ or $x = 2_i c^{k-1}$ with $i \in N$, and $y_l \neq c$ for $l \in O$. If y had a 2 on O, then $y_N = x_N$ could not contain a 2, so $x = c^k$. The partner x' would then need $x'_N \neq c^N$, so x'_N would contain a 2, and y' would have two digits 2. Hence $y_O = \bar{c}^O$. The entries of the sector are then exactly the members of σ listed in (b). These are the backbone entry $(c^k, \bar{c}^O c^N)$ and the tail entries $(2_i c^{k-1}, \bar{c}^O 2_i c^{N\setminus\{i\}})$ for $i \in N$, all of which have at most one 2 in the column. The column $\bar{c}^O c^N$ is a single-flip string exactly when $|O| = 1$ or $|N| = 1$, which gives the last assertion of (b).

For (a), let $y \in R$ in this second case. If $N = \emptyset$, the sector consists of the single entry $(c^k, \bar{c}^k)$. Otherwise, a column in R with $y_O = \bar{c}^O$ and $y_N = x_N$ forces $x = 2_i c^{k-1}$, $N = \{i\}$ and $y = 2_i \bar{c}^{k-1}$. Its only partner is the backbone entry, whose column $\bar{c}^{k-1} c_i$ is single-flip and excluded. $\square$

With the coefficients above, a sector of type σ with backbone entry p_0 and tail entries $(p_i)_{i \in N}$ has the form

$$(1-t)^{|N|}\Big[\,|p_0 - r\textstyle\sum_i p_i|^2 + (1-r^2)\sum_i |p_i|^2\,\Big], \tag{4.2}$$

and the same expression with $p_0 = 0$ if the backbone entry is absent. The coefficient between p_0 and p_i is $(1-t)^{|N|-1}(-t)$, and between p_i and $p_{i'}$ with $i \neq i'$ it is $(1-t)^{|N|-2}t^2$.

4.3 The slack

Put

$$\mathrm{sl}(C) := q_k(t, C) - (1-r)\sum_P q_k(t, \Pi_P C).$$

Each entry lies in exactly one row block $\Pi_P C$, and the sector decomposition holds blockwise. So $\mathrm{sl}(C)$ is a sum over sectors.

- A lone entry contributes $r(1-t)^{k-h(x,y)}|C_{xy}|^2 \geq 0$.
- In a sector σ without its backbone entry, the tails lie in different row blocks. By (4.2) the contribution is $(1-t)^{|N|}\big[r(1-r)\sum_i |p_i|^2 + r^2|\sum_i p_i|^2\big] \geq 0$.
- The diagonal sector contributes $(1-t)^k Q(v)$, where $v = (C_{xx})_{x \in R}$ and

$$Q(v) = r\|v\|^2 + \sum_{x \neq x'} \tilde{F}_{xx'}\,\overline{v_x} v_{x'}, \qquad \tilde{F}_{xx'} = \begin{cases} (-r)^{h(x,x')} & P(x) \neq P(x'), \\ r(-r)^{h(x,x')} & P(x) = P(x'). \end{cases}$$

Under the hypotheses of Theorem 4.1, Lemma 4.3 shows that every off-diagonal entry with row and column in R is alone, and that all other multi-entry sectors lack their backbone entry. Therefore

$$\frac{\mathrm{sl}(C)}{(1-t)^k} \;\geq\; r \sum_{x \neq y \in R} (1-t)^{-h(x,y)}|C_{xy}|^2 + Q(v). \tag{4.3}$$

4.4 Proof of Theorem 4.1

Let $n_x = |v_x|$ for $x \in R$.

Frame shares. Since $\operatorname{rank} C \leq 2$, we can write $C = \sum_{i=1}^{2} |\xi^i\rangle\langle\eta^i|$ with $\xi^1, \xi^2 \in \mathcal{V}$ orthonormal. If the rank is smaller, some η^i vanish. For $x \in R$ and every column string y put $\xi_x = (\langle x|\xi^1\rangle, \langle x|\xi^2\rangle)$ and $\eta_y = (\langle y|\eta^1\rangle, \langle y|\eta^2\rangle)$ in $\mathbb{C}^2$. Then $C_{xy} = \langle \eta_y, \xi_x\rangle$, and orthonormality gives $\sum_{x \in R} \xi_x \xi_x^* = I_2$. Hence the shares $u_x := \|\xi_x\|^2$ satisfy

$$\sum_{x \in R} u_x = 2.$$

This is the only place where the rank enters. For a column $y \in R$ we get $\sum_{x \in R} |C_{xy}|^2 = \|\eta_y\|^2$ and $n_y = |\langle \eta_y, \xi_y\rangle| \leq \|\eta_y\|\sqrt{u_y}$.

The off-diagonal part. Since $h(x,y) \geq 1$ for $x \neq y$,

$$\sum_{x \neq y \in R} (1-t)^{-h(x,y)}|C_{xy}|^2 \;\geq\; \frac{1}{1-t}\sum_{y \in R}\big(\|\eta_y\|^2 - n_y^2\big) \;\geq\; \frac{1}{1-t}\Big[\frac{\big(\sum_y n_y\big)^2}{2} - \sum_y n_y^2\Big].$$

For the last step, $\sum_y \|\eta_y\|^2 \geq \sum_y n_y^2/u_y \geq (\sum_y n_y)^2 / \sum_y u_y$ by Cauchy–Schwarz. Terms with $u_y = 0$ have $n_y = 0$.

The diagonal part. We have $Q(v) \geq r\sum n_x^2 - \sum_{x \neq x'} |\tilde{F}_{xx'}| n_x n_{x'}$. Put $\omega = 1/(2(1-t))$ and $\zeta = \omega - r = (1-2t)/(2(1-t)) \geq 0$. Inserting both bounds into (4.3),

$$\frac{\mathrm{sl}(C)}{(1-t)^k} \;\geq\; r\Big[\zeta \sum_x n_x^2 + \sum_{x \neq x'} \Big(\omega - \frac{|\tilde{F}_{xx'}|}{r}\Big) n_x n_{x'}\Big]. \tag{4.4}$$

Here $|\tilde{F}_{xx'}|/r = r^{h(x,x')-1}$ for different patterns and $r^{h(x,x')}$ for equal patterns; this gives the following coefficients.

- $\{c^k, 2_j c^{k-1}\}$: different patterns at distance 1, so the coefficient is $\omega - 1 = -\zeta$.

- $\{2_i c^{k-1}, 2_j c^{k-1}\}$ with $i \neq j$: different patterns at distance 2, so the coefficient is $\omega - r = \zeta$.
- Pairs across the two families. The pair $\{0^k, 1^k\}$ has distance k and equal patterns, and $\{2_j 0^{k-1}, 2_j 1^{k-1}\}$ has distance $k-1$ and equal patterns. All other cross pairs have distance k and different patterns. So $|\tilde{F}|/r \leq r^{k-1} \leq r$, and the coefficient is at least ζ (recall $k \geq 2$).

Collecting the terms of each family,

$$\frac{\mathrm{sl}(C)}{(1-t)^k} \;\geq\; r\zeta\Big[\Big(n_{0^k} - \sum_j n_{2_j 0^{k-1}}\Big)^2 + \Big(n_{1^k} - \sum_j n_{2_j 1^{k-1}}\Big)^2\Big] + 2r\zeta \sum_{\text{cross pairs}} n_x n_{x'} \;\geq\; 0.$$

This proves Theorem 4.1. □

Remark 4.4 (a rank-free form). Let $\operatorname{ran} C$ and $\operatorname{ran} C^*$ lie in $\mathcal{V}$, with C of any rank. Then every off-diagonal entry is alone (Lemma 4.3(a)), and the frame shares can be avoided. The slack is exactly

$$\frac{\mathrm{sl}(C)}{(1-t)^k} = r \sum_{x \neq y \in R} (1-t)^{-h(x,y)} |C_{xy}|^2 + Q(v),$$

and $\sum_{x \neq y} |C_{xy}|^2 = \|C\|_F^2 - \|v\|^2$ gives

$$\frac{\mathrm{sl}(C)}{(1-t)^k} \;\geq\; r\omega\Big[2\|C\|_F^2 - \Big(\sum_x n_x\Big)^2\Big] + \Big[Q(v) - 2r\omega\|v\|^2 + r\omega\Big(\sum_x n_x\Big)^2\Big],$$

where $r/(1-t) = 2r\omega$. The second bracket is at least $r\big[\zeta \sum_x n_x^2 + \sum_{x \neq x'} (\omega - |\tilde{F}_{xx'}|/r) n_x n_{x'}\big] \geq 0$ by the coefficient table of Section 4.4. Since $r\omega(1-t)^k = \frac{r}{2}(1-t)^{k-1}$, this gives

$$\mathrm{sl}(C) \;\geq\; \frac{r(1-t)^{k-1}}{2}\Big[2\|C\|_F^2 - \Big(\sum_{x \in R} |C_{xx}|\Big)^2\Big] \;\geq\; \frac{r(1-t)^{k-1}}{2}\big(2\|C\|_F^2 - \|C\|_1^2\big),$$

where $\|\cdot\|_1$ is the trace norm. For rank two the right side is $\frac{r}{2}(1-t)^{k-1}(s_1 - s_2)^2$, with $s_1 \geq s_2$ the singular values of C. At $t = 1/2$ this reads $q_k(1/2, C) \geq 2^{-k}(s_1 - s_2)^2$. The constant is sharp. The operator $|0^k\rangle\langle 0^k| + |2_1 0^{k-1}\rangle\langle 2_1 0^{k-1}| + w|0^k\rangle\langle 2_1 0^{k-1}| + \bar{w}|2_1 0^{k-1}\rangle\langle 0^k|$ with $|w| \leq 1$ attains equality.

4.5 Proof of Theorem 4.2, examples, and one copy

Proof of Theorem 4.2. At $t = 1/2$ we have $r = 1$, so $\mathrm{sl}(C) = q_k(1/2, C)$. Lemma 4.3 applies to columns in $A^\flat$. Lone entries contribute $2^{h(x,y)-k}|C_{xy}|^2 \geq 0$. A sector of type σ contributes, by (4.2) at $r = 1$, the amount $2^{-|N|}|p_0 - \sum_i p_i|^2 \geq 0$, with $p_0 = 0$ if its backbone entry is absent. Off-diagonal entries with row and column in R are alone. So (4.3) holds at $t = 1/2$, and the rest of Section 4.4 applies verbatim. □

Equality. For $C = |0^k\rangle\langle 0^k| + |2_1 0^{k-1}\rangle\langle 2_1 0^{k-1}|$ one has $q_k(t, C) = 2(1-2t)(1-t)^{k-1}$ and $\sum_P q_k(t, \Pi_P C) = 2(1-t)^k$. So Theorem 4.1 holds with equality for every k and every $t \leq 1/2$, and $q_k(1/2, C) = 0$.

Asymptotic equality with all tails active. Let $\chi = -(-1)^k$, let $\gamma \neq 0$ and $L > 0$, and put

$$x_1 = |0^k\rangle + \tfrac{\gamma}{L}\sum_j |2_j 0^{k-1}\rangle, \qquad x_2 = |1^k\rangle - \tfrac{\gamma}{L}\sum_j |2_j 1^{k-1}\rangle,$$

$$y_1 = L|0^k\rangle + \gamma\sum_j |2_j 0^{k-1}\rangle, \qquad y_2 = \chi L|1^k\rangle - \chi\gamma\sum_j |2_j 1^{k-1}\rangle.$$

Then $C = |x_1\rangle\langle y_1| + |x_2\rangle\langle y_2|$ has rank two and range and co-range in $\mathcal{V}$, and all k tail blocks $\Pi_{\{j\}} C$ are nonzero. The backbone block $\Pi_\emptyset C \Pi_\emptyset = L\big(|0^k\rangle\langle 0^k| + \chi|1^k\rangle\langle 1^k|\big)$ satisfies $q_k(1/2, \Pi_\emptyset C \Pi_\emptyset) = 0$, because $q_k(1/2, |0^k\rangle\langle 0^k| + \chi|1^k\rangle\langle 1^k|) = 2^{1-k} + 2\chi(-1/2)^k = 0$. As $L \to \infty$, $q_k(1/2, C) = O(L^{-2})$ while $\sum_P q_k(1/2, \Pi_P C)$ stays bounded away from 0, so the ratio $q_k(1/2, C)/\sum_P q_k(1/2, \Pi_P C)$ tends to 0. So at $t = 1/2$ Theorems 4.1 and 4.2 are tight also when all k tails are active, and the argument of Section 4.4 is asymptotically tight along this family, with shares $u_{0^k}, u_{1^k} \to 1$. The coefficients γ in x_i and y_i must be equal. With coefficient γ_y in y_1, y_2 and γ_x in x_1, x_2 one gets $2^k q_k(1/2, C) \to 4k(\gamma_y - \gamma_x)^2$.

Rank three. The operator $C_3 = |0^k\rangle\langle 0^k| + (-1)^{k+1}|1^k\rangle\langle 1^k| + |2_1 0^{k-1}\rangle\langle 2_1 0^{k-1}|$ has rank three, with range and co-range in $\mathcal{V}$. Only its diagonal sector is nonzero, so by Section 4.2, $2^k q_k(1/2, C_3) = 3 - 2 + 4(-1)^k(-1)^{k+1} = -3$. Thus Theorems 4.1 and 4.2 fail for rank three, for every k.

One copy. For $k = 1$ we have $\mathcal{V} = A^\natural = \mathbb{C}^3$, and the analogue of Theorem 4.1 states $q_1(t, C) \geq (1-r)\big(q_1(t, P_{01}C) + q_1(t, |2\rangle\langle 2|C)\big)$ for every C of rank at most two on $\mathbb{C}^3$. A short proof follows. Write $C = XY^*$ with $X, Y \colon \mathbb{C}^2 \to \mathbb{C}^3$ and $\operatorname{ran} Y = \operatorname{ran} C^*$, and put $H = Y^*Y$. Let X_{01}, Y_{01} and x_2, y_2 be the rows at $|0\rangle, |1\rangle$ and at $|2\rangle$. Then

$$q_1(P_{01}C) = \mathrm{Tr}(X_{01} H X_{01}^*) - t|\tau|^2, \qquad q_1(|2\rangle\langle 2|C) = x_2 H x_2^* - t|g|^2,$$

with $\tau = \mathrm{Tr}(Y_{01}^* X_{01})$ and $g = x_2 y_2^*$, and the cross term of $q_1(C)$ is $-2t\,\mathrm{Re}(\bar{\tau} g)$. By Cauchy–Schwarz, $|\tau|^2 \leq u_0 \,\mathrm{Tr}(X_{01} H X_{01}^*)$ and $|g|^2 \leq u_1 x_2 H x_2^*$, where $u_0 = \mathrm{Tr}(Y_{01} H^{-1} Y_{01}^*)$, $u_1 = y_2 H^{-1} y_2^*$ and $u_0 + u_1 = \mathrm{Tr}(H^{-1} Y^* Y) = 2$. (If C has rank one, take X, Y with one column. Then $u_0 + u_1 = 1$, and the bound below only improves, since $u/(1-tu)$ increases.) Since $u_i \leq 2$ and $t \leq 1/2$, we have $1 - tu_i \geq 0$. If $1 - tu_i = 0$, then $t = 1/2$, the other share vanishes, and so does the cross term. Otherwise

$$\frac{t^2|\tau|^2|g|^2}{q_1(P_{01}C)\, q_1(|2\rangle\langle 2|C)} \leq \frac{t^2 u_0 u_1}{(1-tu_0)(1-tu_1)} \leq r^2.$$

With $u_0 = 2 - u_1$, the last inequality is equivalent to $(1-2t)(1-u_1)^2 \geq 0$. The claim follows from $2\sqrt{ab} \leq a + b$.

4.6 Arbitrary co-ranges

We keep the notation of Sections 4.1–4.5 and work at $t = 1/2$. This subsection replaces the condition on the co-range in Theorem 4.2 by the vanishing of C on a small set of strings (Theorem 4.6) and then proves Theorem 4.7. Write $R_c = \{c^k\} \cup \{2_j c^{k-1} : 1 \leq j \leq k\}$, so that $R = R_0 \cup R_1$ and $\mathcal{V}_c = \operatorname{span}\{|x\rangle : x \in R_c\}$.

Sectors for arbitrary columns. The sector decomposition of Section 4.2 holds for every C. For rows in R and arbitrary columns the sectors are as follows. For a word θ on a set O of copies and a string z on its complement N, we write θz for the string that equals θ on O and z on N.

Lemma 4.5. Let $k \geq 2$ and $\operatorname{ran} C \subseteq \mathcal{V}$. Every sector with at least two entries, other than the diagonal sector, is one of the following, where $O \cup N$ is a partition of the copies with $O, N \neq \emptyset$:

(i) $\sigma^{\times}_{j,c} = \{(2_j 0^{k-1},\, c_j 0^{k-1}),\ (2_j 1^{k-1},\, c_j 1^{k-1})\}$ for a copy j and $c \in \{0,1\}$; here $O = \{j\}$;

(ii) $\sigma_{c,O,\theta} = \{(c^k,\, \theta\, c^N)\} \cup \{(2_i c^{k-1},\, \theta\, 2_i c^{N\setminus\{i\}}) : i \in N\}$ for $c \in \{0,1\}$ and a word $\theta \in \{\bar{c}, 2\}^O$.

For $\theta = \bar{c}^O$ these are the sectors σ of Lemma 4.3(b). At $t = 1/2$ the forms of these sectors, multiplied by 2^k, are $2\,|p + (-1)^{k-1} p'|^2$ for $\sigma^{\times}_{j,c}$, where p and p' are the entries in the rows $2_j 0^{k-1}$ and $2_j 1^{k-1}$, and $2^{|O|}\,|p_0 - \sum_i p_i|^2$ for $\sigma_{c,O,\theta}$, where p_0 is the entry in row c^k and p_i the entry in row $2_i c^{k-1}$. In particular they are positive semidefinite.

Proof. Two distinct entries (x, y), (x', y') of a non-diagonal sector satisfy $x_O = x'_O$, $y_O = y'_O$ and $x_N = y_N \neq x'_N = y'_N$, so $N \neq \emptyset$ and $x \neq x'$. Strings of the two families agree only at copies where both read 2, and each string of R has at most one digit 2. So if the rows lie in different families, then $O = \{j\}$, the rows are $2_j 0^{k-1}$ and $2_j 1^{k-1}$, the column digit at j is some $c \in \{0,1\}$, and $y_N = 0^N$, $y'_N = 1^N$. This is $\sigma^{\times}_{j,c}$. If the rows lie in the same family R_c, then $x_O = c^O$, since a digit 2 of x on O would also be a digit 2 of x' and would force $x = x'$. The column then has $y_O = \theta \in \{\bar{c}, 2\}^O$, and every member is $(c^k, \theta\, c^N)$ or $(2_i c^{k-1}, \theta\, 2_i c^{N\setminus\{i\}})$ with $i \in N$. This is $\sigma_{c,O,\theta}$. The forms follow from the coefficients in Section 4.2 with $t = 1/2$, using $h(0^N, 1^N) = k - 1$ for $\sigma^{\times}_{j,c}$, and $h(c^N, 2_i c^{N\setminus\{i\}}) = 1$ and $h(2_i c^{N\setminus\{i\}}, 2_{i'} c^{N\setminus\{i'\}}) = 2$ for $i \neq i'$. □

Among these sectors, those that contain an entry with column in R are $\sigma^{\times}_{j,0}$ and $\sigma^{\times}_{j,1}$, and the sectors

$$\sigma_{c,\{1,\ldots,k\}\setminus\{j\},\bar{c}^{k-1}} = \{(2_j c^{k-1},\, 2_j \bar{c}^{k-1}),\ (c^k,\, c_j \bar{c}^{k-1})\}, \qquad \sigma_{c,\{i\},2} = \{(c^k,\, 2_i c^{k-1})\} \cup \{(2_j c^{k-1},\, 2_i 2_j c^{k-2}) : j \neq i\},$$

for $c \in \{0,1\}$ and all copies i, j. In each of them, the other entries have columns among the *partner strings*

$$\mathcal{P} = \{c_j \bar{c}^{k-1}\} \cup \{2_i 2_j c^{k-2} : i < j,\ c \in \{0,1\}\},$$

the single-flip strings together with the strings that carry two digits 2 and agree with 0^k or 1^k elsewhere. There are $k^2 + k$ partner strings for $k \geq 3$, and three, namely 01, 10 and 22, for $k = 2$.

> **Theorem 4.6.** Let $k \geq 2$ and let C be an operator of rank at most two with $\operatorname{ran} C \subseteq \mathcal{V}$ and $C|y\rangle = 0$ for every $y \in \mathcal{P}$. Then $q_k(1/2, C) \geq 0$.

In terms of the co-range, the hypothesis asks that $\operatorname{ran} C^*$ lie in the span of the strings outside $\mathcal{P}$. Since $A^\flat$ is orthogonal to every partner string, Theorem 4.6 contains Theorem 4.2, and it strictly extends Theorem 4.2 for $k \geq 3$. For $k = 2$ both theorems allow the same six of the nine column strings. For $k = 3$ Theorem 4.6 allows 15 of the 27 column strings, against 14 in Theorem 4.2, and for $k = 4$ it allows 61 of 81, against 40. The allowed strings include strings with two or more digits 2 other than the $2_i 2_j c^{k-2}$, for instance 2^k.

Proof. By Lemma 4.5 and the list above, once the partner columns vanish, an entry with row and column in R is either a diagonal entry or alone in its sector. Every sector that contains no entry with column in R has a nonnegative form: a lone entry trivially, and the other sectors by Lemma 4.5. Dropping these sectors gives

$$2^k q_k(1/2, C) \;\geq\; \sum_{x \neq y \in R} 2^{h(x,y)} |C_{xy}|^2 + \sum_{x,x' \in R} (-1)^{h(x,x')}\, \overline{v_x} v_{x'},$$

which is 2^k times the right side of (4.3) at $t = 1/2$. The argument of Section 4.4 applies verbatim. It uses only rows in R (frame shares, $\sum_x u_x = 2$) and holds for an arbitrary co-range. □

Block-diagonal planes. Now let

$$f_0 = \alpha_0 |0^k\rangle + \sum_j \alpha_j |2_j 0^{k-1}\rangle \in \mathcal{V}_0, \qquad f_1 = \beta_0 |1^k\rangle + \sum_j \beta_j |2_j 1^{k-1}\rangle \in \mathcal{V}_1.$$

> **Theorem 4.7.** Let $k \geq 2$. For all $f_0 \in \mathcal{V}_0$ and $f_1 \in \mathcal{V}_1$, every operator C of rank at most two with $\operatorname{ran} C \subseteq \operatorname{span}\{f_0, f_1\}$, and arbitrary co-range, satisfies $q_k(1/2, C) \geq 0$.

This class contains the family of Section 4.5 with all tails active. The proof (Appendix B.4) has four steps. Writing $C = |f_0\rangle\langle g_0| + |f_1\rangle\langle g_1|$ turns the claim into positive semidefiniteness of a 2×2 block operator built from f_0 and f_1, whose diagonal blocks are at least $\|f_c\|^2 I$. Its off-diagonal block acts between two subspaces of dimension $2k + 1$, so Schur complements reduce the condition to these coordinates. Two minimisations then bound the resulting quadratic form from below by a form G with one variable for each copy, and merging two copies into one never increases G; after all copies are merged, G is a sum of squares.

Lemma 4.8. *Let $0 \leq t \leq 1/2$, let $W = \operatorname{span}\{|0\rangle, |1\rangle, |2\rangle\} \subseteq \mathbb{C}^d$, let Π be the orthogonal projector onto $W^{\otimes k}$, and let $\operatorname{ran} C \subseteq \mathcal{V}$, with any co-range. Then $q_k(t, C) \geq q_k(t, C\Pi)$.*

The operator $C\Pi$ has rank at most that of C, range contained in that of C, co-range in $W^{\otimes k}$, and the same entries C_{xy} for $x, y \in \{0,1,2\}^k$. So it suffices to prove Theorems 4.6 and 4.7 for $d = 3$. For Theorem 4.6 the sector argument also applies directly when $d > 3$: a sector whose columns have a digit ≥ 3 is one of the sectors of Lemma 4.5 with such digits in the column word on O, and its form is the same.

Proof. For $S \subseteq \{1, \dots, k\}$ let Π_S be the orthogonal projector onto the product of $W^\perp$ at the copies in S and W at the other copies; these projectors sum to I, and $\Pi_\emptyset = \Pi$. The partial trace over a copy in S annihilates $C\Pi_S$, because it pairs the range of C, which lies in W at that copy, with the co-range of Π_S, which lies in $W^\perp$ there. So $\operatorname{Tr}_J C = \sum_{S \cap J = \emptyset} \operatorname{Tr}_J(C\Pi_S)$. For $S \cap J = \emptyset$ the co-range of $\operatorname{Tr}_J(C\Pi_S)$ lies in the product of $W^\perp$ at the copies in S and W at the copies outside $S \cup J$, and these spaces are mutually orthogonal for different S. Hence

$$q_k(t, C) = \sum_S \sum_{J \cap S = \emptyset} (-t)^{|J|} \, \| \operatorname{Tr}_J(C\Pi_S) \|_F^2 .$$

The term $S = \emptyset$ is $q_k(t, C\Pi)$. Let $S \neq \emptyset$ and write $C\Pi_S = \sum_{a,b} C_{ab} \otimes |a\rangle\langle b|$, where $|a\rangle$ and $|b\rangle$ run over product bases of the copies in S that consist, at each copy, of $|0\rangle, |1\rangle, |2\rangle$ and an orthonormal basis of $W^\perp$, and C_{ab} acts on the other copies. The term of S equals $\sum_{a,b} q_{k-|S|}(t, C_{ab})$, with $q_0(t, c) = |c|^2$. The range of C_{ab} consists of the vectors $\langle a|_S v$, $v \in \mathcal{V}$, obtained by pairing the copies in S with $\langle a|$, and it is zero unless a is a string in $\{0,1,2\}^S$. If $a = c^S$ with $c \in \{0,1\}$, it lies in $\operatorname{span}\{|c\rangle, |2\rangle\}^{\otimes(k-|S|)}$. If a has the digit 2 at one copy i and the digit c at the other copies of S, it lies in $\operatorname{span}\{|c^{k-|S|}\rangle\}$, or in $\operatorname{span}\{|0^{k-1}\rangle, |1^{k-1}\rangle\}$ when $S = \{i\}$. For every other string it is zero. In each case Lemma 2.6 gives $q_{k-|S|}(t, C_{ab}) \geq 0$. □

Remark 4.9 (no margin with an arbitrary co-range). For $t < 1/2$ the margin of Theorem 4.1 does not extend to operators with range in $\mathcal{V}$ and an arbitrary co-range. Let $x = \sum_j |2_j 0^{k-1}\rangle$ and $y = \big((|0\rangle + |2\rangle)/\sqrt{2}\big)^{\otimes k}$, and choose v to minimise $q_k(t, C)$ for $C = |0^k\rangle\langle v| + |x\rangle\langle y|$. Then, by a direct computation, $q_k(t, C) / \sum_P q_k(t, \Pi_P C)$ tends to 0 like $1/k$.

5. Local positivity near the zeros

For every k, the known zeros include the qubit-supported planes and, when $\mathrm{EP}_{k-1}(d)$ holds, the product planes $U \otimes \chi$ (Lemma 2.7). Near both families the endpoint inequality holds. We state the results here and prove them in Appendix B.

Let $Z \subseteq H \otimes Q$ be the set of vectors ψ with $\dim S_l(\psi) \leq 2$ for every l. A vector $\psi_0 \in Z$ whose local supports all have dimension two is *generic* if its coefficient tensor in suitable orthonormal bases of the local supports, an element of $(\mathbb{C}^2)^{\otimes k} \otimes Q$, satisfies three nondegeneracy conditions (Definition B.1). Generic vectors are dense among the vectors of Z whose local supports all have dimension two (Appendix B.1).

Theorem 5.1. *Let $k \geq 1$, $d \geq 2$, and let $\psi_0 \in Z$ be generic. There are a neighbourhood N of ψ_0 in $H \otimes Q$ and $c > 0$ such that*

$$\lambda_{\min} E_{1/2}(\psi\psi^*) \;\geq\; c \cdot \operatorname{dist}(\psi, Z)^2 \qquad \text{for all } \psi \in N .$$

In particular $q_k\big(1/2,\, f_1 a^* + f_2 b^*\big) \geq 0$ for all $f_1, f_2 \in H$ whenever $a \otimes q_1 + b \otimes q_2 \in N$.

Only the pair (a, b), which spans the co-range, has to be close to a generic zero; the range vectors are arbitrary. Theorem 5.1 uses no positivity for fewer copies. The constant c and the neighbourhood depend on ψ_0.

Theorem 5.2. *Let $k \geq 2$ and $d \geq 2$, and suppose that $\mathrm{EP}_{k-1}(d)$ holds. Let $U \subseteq \mathbb{C}^d$ be a plane with orthonormal basis e_0, e_1, let $0 \neq \chi \in (\mathbb{C}^d)^{\otimes(k-1)}$, and let $\psi_0 = (e_0 \otimes q_1 + e_1 \otimes q_2) \otimes \chi$, with U at the first copy and the factors reordered. There is a neighbourhood N of ψ_0 in $H \otimes Q$ with $E_{1/2}(\psi\psi^*) \succeq 0$ for all $\psi \in N$.*

Every basis of the plane $U \otimes \chi$ is obtained from ψ_0 by an invertible linear map on Q, which preserves positivity of $E_{1/2}(\psi\psi^*)$, so Theorem 5.2 covers every basis of every such plane. Since $\mathrm{EP}_3(d)$ holds for every d (Theorem 2.5), Theorem 5.2 is unconditional for $k = 4$: four-copy positivity at the endpoint holds near every product plane, in every dimension. At product zeros the kernel of $E_{1/2}(\psi_0\psi_0^*)$ has dimension d^{k-1}, so the Morse–Bott argument behind Theorem 5.1 does not apply; the proof combines a second-order bound with a splitting at the qubit copy.

6. A first zero below the endpoint

We first extend the qubit-copy argument to all $t \leq 1/2$, with an explicit margin. For an operator C and a copy i, let P_i and Q_i be the orthogonal projectors onto the local supports at copy i of $\operatorname{ran} C$ and of $\operatorname{ran} C^*$, and put $\tau_i = \operatorname{Tr}(P_i Q_i) \leq \min(\operatorname{rank} P_i, \operatorname{rank} Q_i)$.

Lemma 6.1. *Let $k \geq 2$ and $0 \leq t \leq 1/2$, and suppose that $\mathrm{EP}_{k-1}(t; d)$ holds. Let C be an operator of rank at most two on $(\mathbb{C}^d)^{\otimes k}$ with $\tau_i \leq 2$ at some copy i. Then*

$$q_k(t, C) \;\geq\; (1 - 2t) \sum_{x,y} q_{k-1}(t, C_{xy}) \;\geq\; 0,$$

where $C_{xy} = (\langle x| \otimes I)\, C\, (|y\rangle \otimes I)$, with x and y running over orthonormal bases of $\operatorname{ran} P_i$ and $\operatorname{ran} Q_i$ at copy i.

Proof. Recall $q_k(t, C) = \langle \psi_C | (I - t\Phi)^{\otimes k} | \psi_C \rangle$ with $\psi_C = \operatorname{vec} C$ (Section 2). Since $C = (P_i \otimes I) C (Q_i \otimes I)$ and $\operatorname{vec}(PXQ) = (P \otimes Q^{\mathsf{T}}) \operatorname{vec} X$, the vector ψ_C lies in $K \otimes (\text{other copies})$, where $K = \operatorname{ran} P_i \otimes \operatorname{ran} \bar{Q}_i$. With $\Phi = |\Omega\rangle\langle\Omega|$ and $\omega = (P_i \otimes \bar{Q}_i)\Omega$, the compression of $I - t\Phi$ to K is

$$G = I_K - t|\omega\rangle\langle\omega| = (1-2t)\, I_K + 2t\Big(I_K - \tfrac{1}{2}|\omega\rangle\langle\omega|\Big), \qquad \|\omega\|^2 = \mathrm{Tr}(P_i Q_i) = \tau_i.$$

Since $\big\|\frac{1}{2}|\omega\rangle\langle\omega|\big\|_2 = \tau_i/2 \le 1$, the operator $I_K - \frac{1}{2}|\omega\rangle\langle\omega|$ is separable across $\mathrm{ran}\, P_i \otimes \mathrm{ran}\, \bar{Q}_i$ by the theorem of Gurvits and Barnum [GB02]: it equals $\sum_\alpha |u_\alpha\rangle\langle u_\alpha| \otimes |v_\alpha\rangle\langle v_\alpha|$. Let $W = (I - t\Phi)^{\otimes(k-1)}$ on the other copies. Then

$$q_k(t, C) = (1-2t)\langle\psi_C|I_K \otimes W|\psi_C\rangle + 2t \sum_\alpha q_{k-1}(t, C_\alpha), \qquad C_\alpha = (\langle u_\alpha| \otimes I)\, C\, (|\bar{v}_\alpha\rangle \otimes I),$$

and $\langle\psi_C|I_K \otimes W|\psi_C\rangle = \sum_{x,y} q_{k-1}(t, C_{xy})$. All the operators C_α and C_{xy} have rank at most two. □

For $k = 3$ and $t = 1/2$ this is [WZ26, Theorem 7.2]. Below the endpoint, positivity for fewer copies comes with a margin.

Lemma 6.2. *Let $j \ge 1$ and $0 \le t < s < 1$, and suppose that $\mathrm{EP}_j(s; d)$ holds. Then $q_j(t, C) \ge (1 - t/s)^j \|C\|_F^2$ for every C of rank at most two on $(\mathbb{C}^d)^{\otimes j}$.*

Proof. In the proof of Lemma 2.3(b), with (t', t) there replaced by (t, s), every term of the expansion is nonnegative, and the term in which I is chosen at every copy equals $(1 - t/s)^j \|C\|_F^2$. □

If $\mathrm{EP}_{k-1}(d)$ holds and $t < 1/2$, Lemmas 2.3(b), 6.1 and 6.2 (with $s = 1/2$) give

$$q_k(t, C) \;\ge\; (1-2t)^k \|C\|_F^2 \qquad \text{whenever } \tau_i \le 2 \text{ at some copy } i. \tag{6.1}$$

Theorem 6.3. *Let $k \ge 2$ and $d \ge 2$, and suppose that $\mathrm{EP}_{k-1}(d)$ holds. (a) $\mathrm{EP}_k(d)$ fails if and only if there are $t \in (0, 1/2)$ and $C \ne 0$ of rank at most two such that $\mathrm{EP}_k(t; d)$ holds and $q_k(t, C) = 0$. In that case $t = t^* := \max\{s \in [0, 1/2] : \mathrm{EP}_k(s; d)\}$. (b) Let t and C be as in (a). Then C has rank two; at every copy the local supports of $\mathrm{ran}\, C$ and of $\mathrm{ran}\, C^*$ coincide and have dimension at least three; and $\frac{d}{ds} q_k(s, C)\big|_{s=t} < 0$.*

Proof. We first prove (b) for any $t \in (0, 1/2)$ and $C \ne 0$ of rank at most two with $\mathrm{EP}_k(t; d)$ and $q_k(t, C) = 0$.

Rank. For $C = |f\rangle\langle a|$ the argument in the proof of Corollary 3.3 gives $q_k(t, C) \ge (1-t)^k \|C\|_F^2 > 0$. So C has rank two.

Supports. Let (a, b) be an orthonormal basis of $\mathrm{ran}\, C^*$, write $C = f_1 a^* + f_2 b^*$, and put $\psi = a \otimes q_1 + b \otimes q_2$ and $\varphi = f_1 \otimes q_1 + f_2 \otimes q_2$. By (2.1), $E_t(\psi\psi^*) \succeq 0$ and $\langle\varphi|E_t(\psi\psi^*)|\varphi\rangle = 0$, so $\varphi \in \ker E_t(\psi\psi^*)$. Let $U_l = S_l(\psi)$ be the local supports of $\mathrm{ran}\, C^*$, and for $T \subseteq \{1, \dots, k\}$ let $H_T = \big(\bigotimes_{l \in T} U_l^\perp \otimes \bigotimes_{l \notin T} U_l\big) \otimes Q$. For X supported on U_l, $D_t(X) = \big[\mathrm{Tr}(X) P_{U_l} - tX\big] + \mathrm{Tr}(X) P_{U_l^\perp}$, and both terms preserve U_l and $U_l^\perp$. Since $\psi \in H_\emptyset$, it follows that $E_t(\psi\psi^*) = \bigoplus_T A_T$ with A_T acting on H_T. Let $T \ne \emptyset$ and $\Phi' \in H_T$; write $\Phi' = \sum_z z \otimes \Phi'_z$ over an orthonormal basis (z) of $\bigotimes_{l \in T} U_l^\perp$, and $\psi_x = (\langle x| \otimes I)\psi$ over an orthonormal basis (x) of $\bigotimes_{l \in T} U_l$. Then, by (2.1) on the $k - |T|$ copies outside T,

$$\langle\Phi'|A_T|\Phi'\rangle = \sum_{z,x} q_{k-|T|}(t, C_{zx}),$$

where C_{zx} is the operator of rank at most two that (2.1) attaches to the pair (Φ'_z, ψ_x). By Lemmas 2.3(b) and 6.2 this is at least $(1-2t)^{k-|T|} \sum_{z,x} \|C_{zx}\|_F^2 = (1-2t)^{k-|T|} \|\Phi'\|^2$, where the last equality uses that a, b are orthonormal. So $A_T \succ 0$ for $T \ne \emptyset$, and $\varphi \in H_\emptyset$, that is, $f_1, f_2 \in \bigotimes_l U_l$. Hence the local supports of $\mathrm{ran}\, C$ lie in those of $\mathrm{ran}\, C^*$, and applying the same argument to C^* gives equality. Then $P_i = Q_i$ is the projector onto U_i and $\tau_i = \dim U_i$, and (6.1) shows $\dim U_i \ge 3$.

Derivative. At each copy write $I - s\Phi = (1 - s/t)I + (s/t)(I - t\Phi)$ and expand: $q_k(s, C) = \sum_S (1 - s/t)^{k-|S|} (s/t)^{|S|} R_S$, where $R_S = \langle\psi_C|(I - t\Phi)^{\otimes S} \otimes I|\psi_C\rangle$ is the sum of $q_{|S|}(t, \cdot)$ over the matrix elements of C at the copies outside S, and $R_{\{1,\dots,k\}} = q_k(t, C) = 0$. Hence $\frac{d}{ds} q_k(s, C)\big|_{s=t} = -\frac{1}{t} \sum_i R_{\{1,\dots,k\}\setminus\{i\}}$, and by Lemma 6.2 each $R_{\{1,\dots,k\}\setminus\{i\}} \ge (1-2t)^{k-1} \|C\|_F^2$. So the derivative is at most $-\frac{k}{t}(1-2t)^{k-1}\|C\|_F^2 < 0$.

(a) Suppose $\mathrm{EP}_k(d)$ fails. Let $\mu(s)$ be the minimum of $q_k(s, C)$ over operators of rank at most two with $\|C\|_F = 1$; it is continuous in s, and $\mu(0) = 1$. The set $\{s \in [0, 1/2] : \mu(s) \ge 0\}$ is closed, it is an interval by Lemma 2.3(b), it contains a neighbourhood of 0, and it does not contain $1/2$. So $0 < t^* < 1/2$, and $\mu(t^*) = 0$, because $\mu(t^*) > 0$ would extend the interval beyond t^*. A minimiser at t^* is the required C. Conversely, if $t \in (0, 1/2)$ and C are as in (a), the derivative in (b) gives $q_k(s, C) < 0$ for some $s \in (t, 1/2)$ arbitrarily close to t, so $\mathrm{EP}_k(s; d)$ fails for these s. By Lemma 2.3(b), $\mathrm{EP}_k(d)$ fails, and $t = t^*$. □

Remark 6.4. *(i)* For $d = 2$ part (b) is impossible, so Theorem 6.3 gives $\mathrm{EP}_k(2)$ for every k by induction; this is the separability of $\rho(2, \alpha)$ for $\alpha \ge -1/2$. *(ii)* By Theorem 2.5, Theorem 6.3 applies to $k = 4$ without hypotheses. *(iii)* Given $\mathrm{EP}_{k-1}(d)$, the absence of such first zeros is equivalent to $\mathrm{EP}_k(d)$. Such a zero is a global minimiser of $q_k(t^*, \cdot)$, and the proof shows that its blocks A_T for $T \ne \emptyset$ are positive definite. The zeros of Section 5 are not of this kind, and below the endpoint they are not zeros at all, by (6.1).

7. A quantitative conjecture

For a plane V and a copy l, let $\nu_3^{(l)}(V)$ be the third largest eigenvalue of the reduced operator of P_V on copy l, which has trace two, with $\nu_3^{(l)}(V) = 0$ if its rank is at most two. So $\min_l \nu_3^{(l)}(V) = 0$ exactly when some local support of V has dimension at most two.

Conjecture 7.1. *For all $k \ge 2$, $d \ge 2$ and every plane $V \subseteq (\mathbb{C}^d)^{\otimes k}$,*

$$m_k(V) \;\ge\; 2^{-k} \min_l \nu_3^{(l)}(V).$$

Conjecture 7.1 implies $\mathrm{EP}_k(d)$ for every k and d, and it implies that every zero has a copy at which its local support has dimension at most two. The constant 2^{-k} is suggested by numerical evidence: constrained minimisation of $m_k(V)$ subject to $\nu_3^{(l)}(V) \geq 2\delta$ at every copy gives, at $d = 3$, the minima of Table 1.

Table 1. Constrained minima of $m_k(V)$ at $d = 3$ under $\nu_3^{(l)}(V) \geq 2\delta$ at every copy, in units of $2^{1-k}\delta$.

k	$\delta = 10^{-3}$	$\delta = 0.02$	$\delta = 0.05$	$\delta = 0.10$
3	1.0003	1.007	1.022	1.05
4		1.011	1.028	1.058
5			1.036	1.073

In more than 150 exploratory local minimisations (Appendix A) of the ratio $2^k m_k(V)/\min_l \nu_3^{(l)}(V)$ for $k = 3, 4, 5$ and $d \leq 5$, no value below one occurred. The infimum one is approached near the product zeros of Theorem 5.2 and near a family of zeros of the form $\xi \otimes \chi$ with ξ on two copies. For two copies the infimum of the ratio appears to be $4 - 2\sqrt{2} \approx 1.17$. On one class of planes we can prove this.

Theorem 7.2. *Let $d \geq 2$, let (a_i) and (b_i) be orthonormal bases of $\mathbb{C}^d$, let $\alpha, \beta \in \mathbb{C}^d$ be orthonormal, and let $V = \operatorname{span}\{\sum_i \alpha_i a_i \otimes b_i,\ \sum_i \beta_i a_i \otimes b_i\}$. Then both reduced operators of P_V have the spectrum $(|\alpha_i|^2 + |\beta_i|^2)_i$, and with $\nu_3(V)$ its third largest element,*

$$m_2(V) \ \geq\ \Big(1 - \frac{1}{\sqrt{2}}\Big)\,\nu_3(V) = \frac{4 - 2\sqrt{2}}{4}\,\nu_3(V).$$

If $\nu_3(V) > 0$, equality holds exactly when this spectrum is $(1, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$, for instance for $V = \operatorname{span}\{|00\rangle, \Omega\}$ with $\Omega = (|11\rangle + |22\rangle)/\sqrt{2}$, where $C_* = |00\rangle\langle 00| - (\sqrt{2} - 1)|\Omega\rangle\langle\Omega|$ attains $m_2(V)$.

Every spectrum with largest element at most one occurs in this class, so $4 - 2\sqrt{2}$ is the exact infimum of the ratio of Conjecture 7.1 over the class. At $d = 3$ the class is a two-parameter family inside the twelve-dimensional space of planes modulo local unitaries. The proof is in Appendix B.3.

8. Four copies: common-plane operators

An operator C on $H = (\mathbb{C}^d)^{\otimes k}$ is a *common-plane* operator if $\operatorname{ran} C + \operatorname{ran} C^*$ lies in a plane.

Theorem 8.1. *Let $d \geq 2$. Every common-plane operator C on $(\mathbb{C}^d)^{\otimes 4}$ satisfies $q_4(1/2, C) \geq 0$.*

The proof uses the quartic moment relaxation of [Ano26], in the form of [Tom26a, Section 3], restricted to common-plane operators (†, D1). We first state this relaxation.

The relaxation. Let $Q = \mathbb{C}^2$ with the basis q_1, q_2 of Section 2.2, and let $\hat{a} \in H \otimes Q$ satisfy $\operatorname{Tr}_H |\hat{a}\rangle\langle\hat{a}| = I_Q/2$. We call the four tensor factors of $(H \otimes Q)^{\otimes 4}$ replicas, and we let X, the averaged fourth moment of $\hat{a}$, be the average of $|\hat{a}'\rangle\langle\hat{a}'|^{\otimes 4}$ over the vectors $\hat{a}'$ obtained from $\hat{a}$ by local unitaries on the copies of $\mathbb{C}^d$, diagonal unitaries on Q, permutations of the copies and complex conjugation in a product basis whose part on Q is q_1, q_2. As in the proof of [Tom26a, Lemma 3.1], X is a real linear combination of tensor products of permutations of the four replicas, one on each copy of $\mathbb{C}^d$, with a matrix unit $|u\rangle\langle v|$ on $Q^{\otimes 4}$ in which each basis vector of Q occurs as often in u as in v; let y be its coordinates in the orbit sums of these operators. The operator X satisfies the Stiefel contrasts of [Tom26a, Section 3.4]: its partial trace over the H-part of a replica equals $I_Q/2$, on the factor Q of that replica, tensored with its partial trace over the whole replica. We write these linear identities $\mathcal{E}y = 0$. It also satisfies $\mathcal{F}_b(y) \succeq 0$ for finitely many blocks $\mathcal{F}_b(y) = K_b^* \mathcal{X}_b(y) K_b$, where $\mathcal{X}_b(y)$ is the operator with coordinates y or its partial transpose on one replica and K_b is a fixed matrix [Tom26a, Lemma 3.2 and Definition 3.4]. The relaxation consists of all vectors y with $\mathcal{E}y = 0$ and $\mathcal{F}_b(y) \succeq 0$ for every b. A linear functional c is nonnegative on it if

$$c = \mathcal{E}^{\mathsf{T}}\lambda + \sum_b \mathcal{F}_b^*(Z_b)$$

for a vector λ and positive semidefinite matrices Z_b, where $\mathcal{F}_b^*$ is the adjoint of $y \mapsto \mathcal{F}_b(y)$, because then $c \cdot y = \sum_b \operatorname{Tr}\big(Z_b \mathcal{F}_b(y)\big) \geq 0$. We call $(\lambda, (Z_b))$ a certificate for c [Tom26a, Definition 3.4].

Proof. Write a common-plane operator as $C = C_1 + iC_2$ with $C_1 = (C + C^*)/2$ and $C_2 = (C - C^*)/2i$. Both are Hermitian with ranges in the plane V that contains $\operatorname{ran} C + \operatorname{ran} C^*$. Their partial traces are Hermitian, so $\|\operatorname{Tr}_J C\|_F^2 = \|\operatorname{Tr}_J C_1\|_F^2 + \|\operatorname{Tr}_J C_2\|_F^2$ for every J, and

$$q_k(t, C) = q_k(t, C_1) + q_k(t, C_2). \tag{8.1}$$

It therefore suffices to treat Hermitian C with $\operatorname{ran} C \subseteq V$. Choose an isometry $A\colon Q \to H$ with range V such that $A^* C A$ is diagonal. After scaling,

$$C = C_\theta = \tfrac{1}{2} A M_\theta A^*, \qquad M_\theta = \sqrt{2}\,\operatorname{diag}(\cos\theta, \sin\theta), \qquad \|C_\theta\|_F^2 = \tfrac{1}{2},$$

for some angle θ.

The slice. For an operator of rank at most two, [Tom26a, Section 3.1] uses the vectors $\hat{a} = \operatorname{vec}(A)/\sqrt{2}$ and $\hat{b} = \operatorname{vec}(2C^*A)/\sqrt{2}$ in $H \otimes Q$. Here $\operatorname{Tr}_H |\hat{a}\rangle\langle\hat{a}| = I_Q/2$ because A is an isometry, and $2C_\theta^* A = A M_\theta$ gives $\hat{b} = (I \otimes M_\theta)\hat{a}$. Diagonal unitaries on Q commute with M_θ, whereas general unitaries on Q, over which [Tom26a, Section 3.2] also averages, do not. So every averaged fourth moment of the pair $(\hat{a}, \hat{b})$ is a linear image of the moment X of $\hat{a}$ defined

above, and by [Tom26a, Section 3.5] $q_4(1/2, C_\theta) = c_\theta \cdot y$ for a vector c_θ whose entries are quadratic forms in $(\cos\theta, \sin\theta)$. It suffices to find a certificate for every c_θ.

One certificate for all θ. Write $c_\theta = \cos^2\theta\, c_{11} + \sin^2\theta\, c_{22} + 2\sin\theta\cos\theta\, c_{12}$. We find vectors $\lambda_1, \lambda_2, \lambda_3$ and positive semidefinite matrices $Z_b = \begin{pmatrix} Z_b^{11} & Z_b^{12} \\ Z_b^{21} & Z_b^{22} \end{pmatrix}$, with blocks of the size of $\mathcal{F}_b(y)$, such that

$$c_{11} = \mathcal{E}^{\mathsf{T}}\lambda_1 + \sum_b \mathcal{F}_b^*(Z_b^{11}), \qquad c_{22} = \mathcal{E}^{\mathsf{T}}\lambda_2 + \sum_b \mathcal{F}_b^*(Z_b^{22}), \qquad 2c_{12} = \mathcal{E}^{\mathsf{T}}\lambda_3 + \sum_b \mathcal{F}_b^*(Z_b^{12} + Z_b^{21}).$$

With $m_\theta = (\cos\theta, \sin\theta)^{\mathsf{T}} \otimes I$ and $\lambda(\theta) = \cos^2\theta\,\lambda_1 + \sin^2\theta\,\lambda_2 + \sin\theta\cos\theta\,\lambda_3$ these identities give $c_\theta = \mathcal{E}^{\mathsf{T}}\lambda(\theta) + \sum_b \mathcal{F}_b^*(m_\theta^{\mathsf{T}} Z_b m_\theta)$, a certificate for c_θ for every θ.

All dimensions. Following [Tom26a, Section 5], we use at every $d \geq 4$ the symmetry-adapted bases of [Tom26a, Section 5.1], in which the local matrix elements are polynomials in d, and put $s = 1/d$. After each block and each equality is scaled by a power of d, the data of the relaxation are polynomials in s, by [Tom26a, Lemma 5.1, Corollary 5.2 and (3.1)]. We write each multiplier as $Z_b = N_b \hat{Z}_b N_b^{\mathsf{T}}$ with fixed matrices N_b, chosen as described after the proof. The certificate identities then form a linear system $\mathcal{A}(s)x = c(s)$ for the entries x of $\lambda_1, \lambda_2, \lambda_3$ and of the $\hat{Z}_b$. As in the proof of [Tom26a, Proposition 5.4], exact computations give a solution with positive definite $\hat{Z}_b$ for every $s \in (0, 2^{-72}]$, that is, for every $d \geq 2^{72}$: a solution at $s = 0$ with positive definite $\hat{Z}_b$; 119 polynomial vectors that are orthogonal to $c(s)$ and span the left kernel of $\mathcal{A}(s)$ for these s, so that the system stays solvable; and bounds as in [Tom26a, Lemma 5.3] on the correction to the solution at $s = 0$, which keep every $\hat{Z}_b$ positive definite. For $d < 2^{72}$ the theorem follows by the embedding in the proof of Lemma 2.3(a), which maps common-plane operators to common-plane operators and preserves q_4. □

The matrices N_b make the numerical problem smaller. Their columns span the orthogonal complement of two kinds of directions: those that Lemma 8.2 below shows to be annihilated by the block at the averaged moment of every $\hat{a}$ with $\mathrm{Tr}_H |\hat{a}\rangle\langle\hat{a}| = I_Q/2$, and the ranges of the blocks at the moments of zeros of q_4, on which every certificate vanishes, as in [Tom26a, Section 4]. Validity does not depend on this choice, since $Z_b = N_b \hat{Z}_b N_b^{\mathsf{T}}$ is positive semidefinite on the whole block whenever $\hat{Z}_b$ is.

Lemma 8.2. *Let $\hat{a} \in H \otimes Q$ satisfy $\mathrm{Tr}_H |\hat{a}\rangle\langle\hat{a}| = I_Q/2$, let X be its averaged fourth moment, and let T be a set of replicas. Let $r \notin T$ and $r' \in T$ be replicas, let $\Omega_{rr'} = \sum_x |x\rangle \otimes |x\rangle$, summed over the product basis of H in which the partial transposes and complex conjugation are taken, on the H-parts of replicas r and r', and let $\eta \in Q \otimes Q$, read as a 2×2 matrix, have trace zero. Then X^{Γ_T}, the partial transpose on the replicas in T, annihilates $\Omega_{rr'} \otimes \eta \otimes w$, with η on the factors Q of replicas r and r', for every vector w on the other replicas.*

Proof. The moment X averages $|\hat{a}'\rangle\langle\hat{a}'|^{\otimes 4}$ over vectors $\hat{a}'$ obtained from $\hat{a}$ by local unitaries, diagonal unitaries on Q, permutations of the copies and complex conjugation, and each $\hat{a}'$ again satisfies $\mathrm{Tr}_H |\hat{a}'\rangle\langle\hat{a}'| = I_Q/2$. For one such vector, $(|\hat{a}'\rangle\langle\hat{a}'|^{\otimes 4})^{\Gamma_T} = |\psi\rangle\langle\psi|$, where ψ has $\overline{\hat{a}'}$ on the replicas in T and $\hat{a}'$ on the others. Writing $\hat{a}' = \sum_{x,q} a_{xq} |x\rangle|q\rangle$, the factor of $\langle\psi|\,\Omega_{rr'} \otimes \eta \otimes w\rangle$ from replicas r and r' is $\sum_{q,q'} \eta_{qq'} \sum_x \bar{a}_{xq} a_{xq'} = \frac{1}{2}\,\mathrm{Tr}\,\eta = 0$. □

Numerically, the equalities alone force every block to annihilate these directions (33 in all at $d = 3$).

Verification. A separate exact certificate at $d = 3$ proves that case without the embedding (†, D2), and for $d = 3$ we have also proved Theorem 8.1 in Lean 4 with Mathlib, with that certificate as data (D4). Appendix A describes how both certificates were checked and what the formal proof relies on.

Remark 8.3. For four copies the relaxation restricted to common-plane operators proves the inequality, and numerically its value is zero for three copies as well. Over all operators of rank at most two, the four-copy relaxation of [Tom26a], computed in floating point, has a negative value at $d = 3$ [Tom26a, Remark 10.1]; the restriction to common-plane operators is what lets it succeed. For five copies, at $d = 3$ and for the operators $C = P_V/2$ with V a plane, the restricted relaxation computed in floating point has the value -1.80×10^{-3}, and -3.98×10^{-4} when the partial transposes on two replicas are added. So at $d = 3$ the method of this section gives nothing for five copies, and we do not know whether the inequality holds for common-plane operators when $k \geq 5$.

9. Discussion and open problems

The endpoint inequality is open from four copies on, in every dimension $d \geq 3$. For four copies, Theorem 8.1 covers the common-plane operators.

Besides the family of Section 4, several classes remain open. If the co-range may lie in the span of all strings with at most one digit 2, Theorems 4.1, 4.2 and 4.6 no longer apply: through the single-flip columns an off-diagonal entry in $R \times R$ can share a sector with a second entry, so it is no longer a lone entry. A proof for this larger class must be sharp on the operator $(|1\rangle\langle 1| + |2\rangle\langle 2|)_j \otimes |1^{k-1}\rangle\langle 0^{k-1}|$, which uses such columns and attains the ratio $1 - r$ for every t and k. Backbones that are not locally orthogonal, an arbitrary backbone with tails aligned to one product vector, and the class whose range and co-range both lie in the span of the strings with at most one digit 2 are not covered either. In local minimisations from many starting points (Appendix A) the bound of Theorem 4.1 holds on all these classes for small k; on the first one it holds for $k \leq 5$ and $t \in \{0.40, 0.45, 0.49\}$, and so does $q_k(1/2, \cdot) \geq 0$. With range in $\mathcal{V}$ and an arbitrary co-range, Theorems 4.6 and 4.7 and Lemma 2.6 leave the operators of rank two whose range is not spanned by two vectors of $\mathcal{V}_0 \cup \mathcal{V}_1$ and whose co-range is not orthogonal to all partner strings; for them the inequality holds numerically for $k \leq 4$, and for $t < 1/2$ see Remark 4.9.

We do not know whether for $k \geq 3$ every zero has a copy at which its local support has dimension at most two. In unconstrained numerical minimisations of m_k for $k = 3, 4, 5$, every zero reached has such a copy. Theorems 5.1 and 5.2 say nothing about the non-generic zeros with two-dimensional local supports or about planes near zeros of other types.

Conjecture 7.1 is unproved for every $k \geq 2$. Even for two copies we know no linear lower bound $m_2(V) \geq c\,\nu_3(V)$ with $c > 0$ on all planes; by Corollary 3.3 and compactness, such a bound in a fixed dimension would follow from lower bounds near the planes with two-dimensional local supports that are uniform over them.

Data and code availability

The computations and the Lean proof accompany this paper as three packages, `release_endpoint`, `release_commonplane` and `release_lean_commonplane`, in its Zenodo record, doi:10.5281/zenodo.23992370; Appendix A describes them.

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Appendix A. Computations

Table 2 lists the computations that the proofs use, marked † in the text, one independent check (D3) and a formal proof (D4). "Exact" means rational or integer arithmetic. The package `release_endpoint`, for Sections 3–7, reproduces rows C1–C3, checks the remaining steps of the same proofs in floating point, and adds a few independent checks. The package `release_commonplane`, for Section 8, contains the relaxation data, the certificates and the programs that built them and that run the checks of rows D1–D3. Both certificates of Section 8 were checked twice in exact arithmetic, by the programs that accompany them and by independent code (row D3). The independent code rebuilds the relaxation data from their definitions, at enough points to determine the polynomial data, and finds them equal to the data used, up to a change of basis at $d = 3$ and, for the equality rows, up to scalar factors; it uses the same compression matrices, which do not affect validity [Tom26a, Section 3.6]. The package

`release_lean_commonplane` contains the formal proof of row D4 and a build script. It proves $q_4(1/2, C) \geq 0$ for every common-plane operator C on $(\mathbb{C}^3)^{\otimes 4}$, in two equivalent forms: $C = UcU^*$ with $U\colon \mathbb{C}^2 \to (\mathbb{C}^3)^{\otimes 4}$ an isometry and c any 2×2 matrix, and $PCP = C$ with P the orthogonal projection onto a plane. Lean checks every property of the certificate that the proof uses: 68 data checks are evaluated by compiled code (`native_decide`) and enter the proof as 68 axioms, and everything else is checked by Lean's kernel with only the standard axioms. The package's documentation states the theorems and lists the trusted base of the compiled checks. The scripts of `release_endpoint` and `release_commonplane` are written in Python 3. Those of `release_endpoint` use SymPy, NumPy and SciPy; those of `release_commonplane` use NumPy, SciPy, SymPy and python-flint, and the numerical solves that produced the certificates also use CVXOPT. The numerical evidence quoted in Sections 7 and 9 is exploratory and comes from local minimisations from many random starting points. The package `release_endpoint` reproduces one instance of it, for Conjecture 7.1 at $k = 3$ and $d = 3$: the ratio of Section 7 on 200 random planes, and a constrained minimisation as in Table 1, at $\delta = 0.05$, by sequential quadratic programming (the SLSQP routine of SciPy) from four random starts. It does not contain the other runs. This evidence and the floating-point computations quoted in Section 8 are not used in any proof.

Table 2. Computer-verified steps of the proofs.

ID	Statement checked	Method	Used in
C1	Theorem 4.7: the five rational identities marked (†) in Steps 3 and 4 of its proof	exact symbolic	App. B.4
C2	Theorem 5.1, condition (G2) of Definition B.1 is nonempty: for $\psi_0 = (3-5i, -5-4i, -4-2i, -3-i, -4+i, 3, 4-3i, 1-4i)$ on two copies, with either copy transverse, the form of Lemma B.7 annihilates the tilts, has rank two, and its compression to their orthogonal complement has positive leading minors	exact (Gaussian rationals)	Sec. 5, Lemma B.7
C3	Theorem 7.2: the two slacks at $m = \frac{1}{4}$ and the threshold $m_0 = 0.12832\ldots$ of $L' > R_1'$	symbolic	Sec. 7, App. B.3
D1	Theorem 8.1 for $d \geq 2^{72}$: the relaxation data as polynomials in $s = 1/d$ (blocks and equality rows by exact interpolation from exact builds at $d = 4, \ldots, 13$, the objective from its exact values at $d = 4, \ldots, 20$; all checked at $d = 15$); the solution at $s = 0$, with all 514 nonzero $\hat{Z}_b$ positive definite; 119 polynomial vectors of degree 8 in the left kernel of $\mathcal{A}(s)$, orthogonal to $c(s)$; the bounds of [Tom26a, Lemma 5.3] for $0 < s \leq 2^{-72}$	exact	Sec. 8
D2	Theorem 8.1 for $d = 3$, directly: an exact certificate at $d = 3$, with all 302 nonzero $\hat{Z}_b$ positive definite	exact	Sec. 8
D3	Independent check of D1 and D2: the relaxation data rebuilt from the definitions at $d = 3$, at $s = 0$ and at $d = 14, 20, 22, \ldots, 32$, and the objective at $d = 4, \ldots, 20$; the certificates checked against them; the bounds of [Tom26a, Lemma 5.3] re-derived	exact	Sec. 8
D4	Theorem 8.1 for $d = 3$ in Lean 4 with Mathlib, with the certificate of D2 as data	Lean 4	Sec. 8

Appendix B. Proofs of Theorems 5.1, 5.2, 7.2 and 4.7

Throughout, $E = D^{\otimes k} \otimes \mathrm{id}_Q$ with $D = 2\mathcal{T} - \mathrm{id}$ and $\mathcal{T}(X) = \mathrm{Tr}(X)\, I$, so that $E = 2^k E_{1/2}$ and, by (2.1),

$$\langle \varphi | E(\psi\psi^*) | \varphi \rangle = 2^k\, q_k\big(1/2,\, f_1 a^* + f_2 b^*\big) \qquad \text{for } \psi = a \otimes q_1 + b \otimes q_2,\ \varphi = f_1 \otimes q_1 + f_2 \otimes q_2. \tag{B.1}$$

For a copy m, E' denotes the same map on the other $k-1$ copies and Q. The *Q-rank* of ψ is $\dim \mathrm{span}\{a, b\}$. For $G \in GL(Q)$ one has $E\big((I \otimes G)\psi\psi^*(I \otimes G)^*\big) = (I \otimes G)E(\psi\psi^*)(I \otimes G)^*$, so positivity and kernels depend only on the plane spanned by a and b.

B.1 Generic qubit-supported zeros

Let $f(\psi) = \lambda_{\min} E(\psi\psi^*)$. Fix $\psi_0 \in Z$ with $\dim S_m(\psi_0) = 2$ for every copy m, put $U_m = S_m(\psi_0)$, and fix orthonormal bases (e_0, e_1) of the U_m. For $T \subseteq \{1, \ldots, k\}$ let $H_T = \big(\bigotimes_{m \in T} U_m^\perp \otimes \bigotimes_{m \notin T} U_m\big) \otimes Q$, so that $H \otimes Q = \bigoplus_T H_T$ and $\psi_0 \in H_\emptyset$, and write $\eta = \sum_T \eta_T$ accordingly. Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices in the bases (e_0, e_1) and (q_1, q_2), $Y = \sigma_2$, and $\Theta = Y^{\otimes(k+1)}\mathcal{K}$, with $\mathcal{K}$ complex conjugation in the product basis. Θ is an antiunitary map of $H_\emptyset$ onto itself; for vectors on fewer copies, Θ or Θ' denotes the same map on those copies. Put $\kappa = \Theta\psi_0/\|\psi_0\|$, and let $D_2 = \sum_{a=1}^3 \mathrm{Ad}_{\sigma_a}$ on $L(U_m)$, so that $D_2(X) = 2\,\mathrm{Tr}(X) P_{U_m} - X$. The *tilts* at copy m are the vectors $(X \otimes I)\psi_0$ with $X \in \mathrm{Hom}(U_m, U_m^\perp)$; they form a subspace $\mathrm{Tilt}_m \subseteq H_{\{m\}}$ of complex dimension $2(d-2)$.

Definition B.1. For $u \in U_i$ write $\psi \mapsto \langle u|_i \psi$ for the compression at copy i, and let $\mathcal{F}_i = \{e_0, e_1, (e_0 + \alpha e_1)/\sqrt{2} : \alpha^4 = 1\}$. The vector ψ_0 is *generic* if, for some choice of the bases (e_0, e_1), which we use from now on: (G1) ψ_0, and every vector obtained from it by compressing a proper subset of copies with vectors from the $\mathcal{F}_i$, has Q-rank two and satisfies $\ker E(\cdot\cdot^*) = \mathrm{span}\{\Theta\,\cdot\}$; (G2) for every two-copy vector arising from ψ_0 by such compressions, and each of its two copies taken as the transverse copy, the form of Lemma B.7 in its base case has rank two; (G3) for some choice of the stripped copy l in each compression step of the proof of Lemma B.7, the four vectors $\langle c|_l \langle b|_m(\cdot)$, $b, c \in \{0, 1\}$, are linearly independent.

Each condition constrains only the coefficient tensor of ψ_0 in the chosen bases, an element of $(\mathbb{C}^2)^{\otimes k} \otimes Q$, and is Zariski-open in it, in the real-algebraic sense for (G2) because κ involves complex conjugation. Each is nonempty: (G1) by Lemma B.8, since the compressions are linear surjections onto qubit spaces on fewer copies; (G2) by an exact computation (†, C2); and (G3) because the compressions are linear surjections and, on $j \geq 3$ copies, the four vectors are the components of a tensor in $(\mathbb{C}^2)^{\otimes j} \otimes Q$ along the bases at the copies l and m, which can be any four vectors of a space of dimension $2^{j-1} \geq 4$. Bases can be chosen to depend smoothly on planes near the U_m, and then

$\psi \mapsto$ (local supports, coefficient tensor) is a local diffeomorphism onto a product. Hence, among the vectors of Z whose local supports all have dimension two, the non-generic ones form a closed set of measure zero.

Lemma B.2 (block structure). *Let $U_1, \dots, U_k \subseteq \mathbb{C}^d$ be any two-dimensional subspaces, define H_T, Θ and D_2 from them and from orthonormal bases of them as above, and let $\psi \in H_\emptyset$. Then $E(\psi\psi^*) = \bigoplus_T A_T$ with A_T acting on H_T,*

$$A_T = 2^{|T|}\big(D_2^{\otimes(k-|T|)} \otimes \mathrm{id}_Q\big)(\mathrm{Tr}_T\, \psi\psi^*) \otimes P_T^\perp, \qquad P_T^\perp = \bigotimes_{m \in T} P_{U_m^\perp}.$$

In particular $E(\psi\psi^*) \succeq 0$, so $f \geq 0$ on Z, and $\Theta\psi \in \ker A_\emptyset$, so $f = 0$ on Z.

Proof. For X supported on U_m, $D(X) = D_2(X) + 2\,\mathrm{Tr}(X) P_{U_m^\perp}$. Expanding the tensor product and collecting the copies where the second term is chosen gives the decomposition. Each block is the image of the positive operator $\mathrm{Tr}_T\, \psi\psi^*$ under a completely positive map. For the kernel, $A_\emptyset = \sum_{\alpha \in \{1,2,3\}^k} (\sigma_\alpha \otimes I)\psi\psi^*(\sigma_\alpha \otimes I)$. Since $\sigma_a Y$ is symmetric for $a = 1,2,3$ and Y is antisymmetric, the matrix $\big(\bigotimes_m \sigma_{\alpha_m} Y\big) \otimes Y$ is antisymmetric, so $\langle \Theta\psi, (\sigma_\alpha \otimes I)\psi\rangle = \psi^{\mathsf{T}}\big((\bigotimes_m \sigma_{\alpha_m} Y) \otimes Y\big)\psi = 0$ for every α. $\square$

Lemma B.3 (six-term identity). *Fix a copy i and let $\psi, \varphi \in H \otimes Q$ have their copy-i components in U_i. Let (u_j, v_j), $j = 1, \dots, 6$, be the pairs $\big((e_0 + \alpha e_1)/\sqrt{2}, (e_0 - \alpha e_1)/\sqrt{2}\big)$ for $\alpha^4 = 1$, together with (e_0, e_1) and (e_1, e_0). Then*

$$\langle \varphi | E(\psi\psi^*) | \varphi \rangle = \sum_{j=1}^{6} \langle \varphi_{v_j} | E'(\psi_{u_j} \psi_{u_j}^*) | \varphi_{v_j} \rangle, \qquad \psi_u = \langle u|_i \psi,\ \varphi_v = \langle v|_i \varphi.$$

Moreover $\sum_j v_j v_j^* = 3P_{U_i}$, and each v_j is a unimodular multiple of $Y\bar{u}_j$.

Proof. By (B.1) the identity is equivalent to $q_k(1/2, C) = \frac{1}{2}\sum_j q_{k-1}(1/2, \langle v_j | C | u_j \rangle_i)$ for C supported on U_i at copy i on both sides. Split the sum defining q_k according to whether $i \in J$. Each term pairs $\|X\|_F^2$ with $-\frac{1}{2}\|\mathrm{Tr}_i\, X\|_F^2$ for $X = \mathrm{Tr}_J\, C$, $i \notin J$, an operator on $U_i \otimes K$. With $X_{ab} = \langle e_a | X | e_b \rangle$,

$$\|X\|^2 - \tfrac{1}{2}\|\mathrm{Tr}_i\, X\|^2 = \|X_{01}\|^2 + \|X_{10}\|^2 + \tfrac{1}{2}\|X_{00} - X_{11}\|^2 = \tfrac{1}{2}\sum_{\alpha^4=1} \|\langle u_\alpha | X | v_\alpha \rangle\|^2 + \tfrac{1}{2}\|X_{01}\|^2 + \tfrac{1}{2}\|X_{10}\|^2,$$

because $\langle u_\alpha | X | v_\alpha \rangle = \frac{1}{2}(X_{00} - X_{11} - \alpha X_{01} + \bar{\alpha} X_{10})$ and the fourth roots of unity cancel the cross terms. Compressions at copy i commute with Tr_J. The frame relation is a direct computation, and $Y\bar{u}_\alpha = -i\bar{\alpha} v_\alpha$, $Y\bar{e}_0 = i e_1$, $Y\bar{e}_1 = -i e_0$. $\square$

Second-order form. Put $A = E(\psi_0 \psi_0^*)$. If $\ker A = \mathrm{span}\{\Theta\psi_0\}$, which is part of (G1), then 0 is a simple eigenvalue of $A \succeq 0$ with unit eigenvector $\kappa \in H_\emptyset$. For $\eta \in H \otimes Q$ put $X_\eta = E(\psi_0 \eta^* + \eta \psi_0^*)$ and

$$S(\eta) = \langle \kappa | E(\eta\eta^*) | \kappa \rangle - \langle X_\eta \kappa | A^+ | X_\eta \kappa \rangle.$$

Since the eigenvalue is simple, $f(\psi_0 + \varepsilon\eta)$ is real-analytic in ε, its first-order term is $\langle \kappa | X_\eta | \kappa \rangle$ and its second-order coefficient is $S(\eta)$ [Kat95].

Lemma B.4 (first order and decoupling). *If $\ker A = \mathrm{span}\{\Theta\psi_0\}$, then $\langle \kappa | X_\eta | \kappa \rangle = 0$ for every η, and $S(\eta) = \sum_T S(\eta_T)$ with $S(\eta_\emptyset) = 0$.*

Proof. Let $T \neq T'$ and let m lie in their symmetric difference, say $m \in T \setminus T'$. At copy m the factor of $E(\eta_T \eta_{T'}^*)$ is $D(yx^*) = -yx^*$ with $y \in U_m^\perp$ and $x \in U_m$, whose range lies in $U_m^\perp$, so $\langle \kappa | E(\eta_T \eta_{T'}^*) | \kappa \rangle = 0$ because $\kappa \in H_\emptyset$. For $T \neq \emptyset$, $E(\psi_0 \eta_T^*)$ annihilates $H_\emptyset$, because at $m \in T$ its factor $-xy^*$ kills U_m; and $E(\eta_T \psi_0^*)$ maps $H_\emptyset$ into H_T, because at $m \in T$ it maps U_m into $U_m^\perp$ and at $m \notin T$ the factor $D(xx'^*) = 2\langle x', x\rangle I - xx'^*$ preserves U_m. Hence $X_{\eta_T}\kappa \in H_T$ for $T \neq \emptyset$. For $T = \emptyset$ the decomposition of Lemma B.2 holds for every operator on $H_\emptyset$, so $X_{\eta_\emptyset}$ is block-diagonal and $X_{\eta_\emptyset}\kappa \in H_\emptyset$. By Lemma B.2, A and A^+ are block-diagonal, so all cross terms of S vanish, and $\langle \kappa | X_{\eta_T} | \kappa \rangle = 0$ for $T \neq \emptyset$. Finally $\psi_0 + \varepsilon\eta_\emptyset \in Z$ for every ε, so f vanishes along this line, which gives $\langle \kappa | X_{\eta_\emptyset} | \kappa \rangle = 0$ and $S(\eta_\emptyset) = 0$. $\square$

Lemma B.5 (second-order nonnegativity). *If ψ_0 is generic, then $S(\eta_T) \geq 0$ for every T and $\eta_T \in H_T$.*

Proof. Fix $T \neq \emptyset$ with $|T| = j$. We prove by induction on the number n of copies outside T that $f(\psi_0 + \varepsilon\eta_T) \geq -o(\varepsilon^2)$ for every generic ψ_0 on $j + n$ copies and every $\eta_T \in H_T$.

Base, $n = 0$. Every copy is transverse. In this paragraph $\kappa = \Theta\psi_0$ is not normalised. Write $\psi_0 = \sum_{a \in \{0,1\}^j} |a\rangle \otimes \psi_a$ with $\psi_a \in Q$, and put $R = \sum_a \psi_a \psi_a^*$, which is positive definite by (G1). At each copy $D(yy'^*)$ acts on U_m as $2\langle y', y\rangle P_{U_m}$, so $\langle \kappa | E(\eta\eta^*) | \kappa \rangle = 2^j\,\mathrm{Tr}(R_\kappa \tau_\eta)$ with τ_η the reduced operator of $\eta\eta^*$ on Q and $R_\kappa = Y\bar{R}Y = \det(R)\, R^{-1}$ that of $\kappa\kappa^*$. Further $X_\eta \kappa = (-1)^j \langle \psi_0, \kappa \rangle \eta$, and A acts on $H_{\{1,\dots,j\}}$ as $2^j R \otimes P^\perp$. Hence

$$\|\kappa\|^2 S(\eta) = \mathrm{Tr}(R^{-1}\tau_\eta) \cdot \big[2^j \det R - 2^{-j} |\langle \psi_0, \Theta\psi_0 \rangle|^2\big]. \tag{B.2}$$

Since $Y^{\otimes j}$ maps $|a\rangle$ to a unimodular multiple of $|\bar{a}\rangle$, the bitwise complement, $|\langle \psi_0, \Theta\psi_0 \rangle| \leq \sum_a |\det[\psi_{\bar{a}}, \psi_a]|$. By the Cauchy–Schwarz inequality over the 2^{j-1} pairs $\{a, \bar{a}\}$ and the Cauchy–Binet formula $\det R = \sum_{a<b} |\det[\psi_a, \psi_b]|^2$,

$$|\langle \psi_0, \Theta\psi_0 \rangle|^2 \leq 4 \cdot 2^{j-1} \sum_{\text{pairs}} |\det[\psi_a, \psi_{\bar{a}}]|^2 \leq 2^{j+1} \det R \leq 4^j \det R.$$

So $S(\eta) \geq 0$, which gives the claim by Lemma B.4.

Step, $n \geq 1$. Choose a copy $i \notin T$ and put $\psi_u = \langle u|_i(\psi_0 + \varepsilon\eta_T)$ for $u \in U_i$. For $u \in \mathcal{F}_i$ this is the generic zero $\langle u|_i \psi_0$ on $j + n - 1$ copies plus a perturbation of pattern T: compressions of $\langle u|_i \psi_0$ are compressions of ψ_0, so (G1)–(G3) pass to it. Its simple kernel also forces local supports of dimension two at the remaining copies: otherwise it would be $e \otimes \varphi'$ with $e \in U_m$ a unit vector, with $A_\emptyset$ block $(2P_{U_m} - ee^*) \otimes A'_\emptyset$ and $A'_\emptyset \Theta\varphi' = 0$, and its kernel would contain the two-

dimensional space $U_m \otimes \operatorname{span}\{\Theta\varphi'\}$. So the induction hypothesis gives $\lambda_{\min} E'(\psi_u \psi_u^*) \geq -o(\varepsilon^2)$ for $u \in \mathcal{F}_i$. The copy-i components of $\psi_0 + \varepsilon\eta_T$ lie in U_i, so by the computation of Lemma B.2 at copy i alone, $E\big((\psi_0 + \varepsilon\eta_T)(\cdot)^*\big)$ is block-diagonal for $U_i \oplus U_i^\perp$ at copy i, with $U_i^\perp$ block $2E'\big(\sum_{x \in \{e_0, e_1\}} \psi_x \psi_x^*\big) \otimes I$, whose smallest eigenvalue is at least $2\sum_x \lambda_{\min} E'(\psi_x \psi_x^*) \geq -o(\varepsilon^2)$. On the U_i block Lemma B.3 gives $\langle\varphi|E|\varphi\rangle \geq 3\min\big(0, \min_j \lambda_{\min} E'(\psi_{u_j}\psi_{u_j}^*)\big)\|\varphi\|^2 \geq -o(\varepsilon^2)\|\varphi\|^2$. Hence $f(\psi_0 + \varepsilon\eta_T) \geq -o(\varepsilon^2)$. By Lemma B.4 the first-order term vanishes, so $f(\psi_0 + \varepsilon\eta_T) = \varepsilon^2 S(\eta_T) + O(\varepsilon^3)$ and $S(\eta_T) \geq 0$. □

Lemma B.6 (strictness for patterns of two or more copies). *If ψ_0 is generic and $|T| \geq 2$, then $S(\eta_T) > 0$ for every nonzero $\eta_T \in H_T$.*

Proof. We use the same induction. In the base case, (B.2) and the chain of inequalities give $\|\kappa\|^2 S(\eta) \geq (2^j - 2)\det R \operatorname{Tr}(R^{-1}\tau_\eta) > 0$ for $j \geq 2$. In the step, let φ_ε be the unit bottom eigenvector of $E\big((\psi_0 + \varepsilon\eta_T)(\cdot)^*\big)$; by (G1) it is analytic, with $\varphi_0 = \kappa$ in the U_i block. At $\varepsilon = 0$ the $U_i^\perp$ block is $2E'(\psi_{e_0}\psi_{e_0}^* + \psi_{e_1}\psi_{e_1}^*) \otimes I$. By Lemma B.2 and (G1) the two summands are positive with kernels $\operatorname{span}\{\Theta'\psi_{e_0}\}$ and $\operatorname{span}\{\Theta'\psi_{e_1}\}$, which meet only in 0 because $S_i(\psi_0)$ is two-dimensional. So the $U_i^\perp$ block is positive definite at $\varepsilon = 0$, φ_ε stays in the U_i block, and Lemma B.3 gives

$$f(\psi_0 + \varepsilon\eta_T) = \langle\varphi_\varepsilon|E|\varphi_\varepsilon\rangle \geq \sum_j \lambda_{\min} E'(\psi_{u_j}\psi_{u_j}^*)\, \|(\varphi_\varepsilon)_{v_j}\|^2.$$

Each compressed family has a simple kernel by (G1), so $\lambda_{\min} E'(\psi_{u_j}\psi_{u_j}^*) = \varepsilon^2 S_j + O(\varepsilon^3)$, with S_j the second-order form of the compressed zero at $\langle u_j|_i \eta_T$. Hence $S(\eta_T) \geq \sum_j S_j \|\kappa_{v_j}\|^2$. By Lemma B.3, $\kappa_{v_j} = \langle v_j|_i \kappa$ is a unimodular multiple of $\Theta'(\langle u_j|_i\psi_0)/\|\psi_0\|$, so $\|\kappa_{v_j}\| > 0$. The u_j span U_i, so some $\langle u_j|_i\eta_T$ is nonzero; for that j, $S_j > 0$ by induction, since its pattern still has at least two copies. The other S_j are nonnegative by Lemma B.5. □

Lemma B.7 (single-copy patterns). *If ψ_0 is generic, then the restriction of S to $H_{\{m\}}$ is positive semidefinite with kernel exactly* Tilt_m.

Proof. For $X \in \operatorname{Hom}(U_m, U_m^\perp)$ the line $\psi_0 + \varepsilon(X \otimes I)\psi_0$ stays in Z, since its local support at copy m is the rotated plane $(I + \varepsilon X)U_m$; so $S = 0$ on tilts, and $S \geq 0$ by Lemma B.5.

Reduction to $d = 3$. Write $\eta = \sum_y y \otimes \zeta_y$ over an orthonormal basis (y) of $U_m^\perp$. Since $D(yy'^*)$ acts on U_m as $2\langle y', y\rangle P_{U_m}$, and A acts on $H_{\{m\}}$ as $2E'(\operatorname{Tr}_m \psi_0\psi_0^*) \otimes P_{U_m^\perp}$, the form S is the orthogonal direct sum of $d-2$ copies of the case $d = 3$, and so is the tilt space. So take $d = 3$ and $\eta = e_2 \otimes \zeta$, with tilts $\zeta \in \operatorname{span}\{\psi_b : b = 0, 1\}$, $\psi_b = \langle b|_m \psi_0$.

Base, $k = 2$. Let l be the other copy and $\kappa_b = \langle b|_m \kappa$. The form S on $\zeta \in U_l \otimes Q$ is the 4×4 Hermitian matrix

$$S = 2E'(\tau_0) - \tfrac{1}{2} W^* E'(\sigma_0)^{-1} W, \qquad \sigma_0 = \operatorname{Tr}_m \psi_0\psi_0^*, \quad \tau_0 = \operatorname{Tr}_m \kappa\kappa^*, \quad W\zeta = \sum_b E'(\zeta\psi_b^*)\kappa_b,$$

where $E'(\sigma_0)^{-1}$ inverts the restriction of $E'(\sigma_0)$ to $U_l \otimes Q$, which is invertible by (G1) as in the proof of Lemma B.6. Its rank is two by (G2), so its kernel is exactly the tilt span.

Step, $k \geq 3$. Choose the copy $l \neq m$ given by (G3). As in the proof of Lemma B.6, $S(\eta) \geq \sum_j S_j \|\kappa_{v_j}\|^2$ with all $\|\kappa_{v_j}\| > 0$, where S_j is the form on $k-1$ copies at the compressed zero $\langle u_j|_l\psi_0$. Suppose that $\eta = e_2 \otimes \zeta$ is not a tilt, put $\psi_{bc} = \langle c|_l\langle b|_m\psi_0$, and suppose that the compressions by e_0, e_1 and $(e_0 + e_1)/\sqrt{2}$ are all tilts: $\langle e_0|_l\zeta = \sum_b x_b\psi_{b0}$, $\langle e_1|_l\zeta = \sum_b y_b\psi_{b1}$ and $(\langle e_0|_l + \langle e_1|_l)\zeta = \sum_b z_b(\psi_{b0} + \psi_{b1})$. By (G3) the ψ_{bc} are linearly independent, so $x = z = y$ and $\zeta = \sum_b x_b\psi_b$ is a tilt, a contradiction. So some compression is not a tilt of its generic compressed zero, its S_j is positive by induction, the other S_j are nonnegative, and $S(\eta) > 0$. Since $S \geq 0$ vanishes on its kernel, $S(t + x) = S(x)$ for tilts t, so S is positive definite on the complement of Tilt_m. □

Lemma B.8 (generic zeros exist). *Let $j \geq 1$ and $\psi \in (\mathbb{C}^2)^{\otimes j} \otimes Q$, viewed in $(\mathbb{C}^d)^{\otimes j} \otimes Q$ with $U_m = \mathbb{C}^2 \subseteq \mathbb{C}^d$ at every copy. Call ψ good if it has Q-rank two, $\ker A_\emptyset = \operatorname{span}\{\Theta\psi\}$, and for every $T \neq \emptyset$ the first factor $\big(D_2^{\otimes(j-|T|)} \otimes \mathrm{id}_Q\big)(\operatorname{Tr}_T \psi\psi^*)$ of A_T is nonsingular. The good vectors form a nonempty Zariski-open subset of $(\mathbb{C}^2)^{\otimes j} \otimes Q$, and a good ψ has $\ker E(\psi\psi^*) = \operatorname{span}\{\Theta\psi\}$ for every $d \geq 2$.*

Proof. Each condition says that a matrix depending linearly on ψ or on $\psi\psi^*$ has the largest possible rank; for $A_\emptyset$ this rank is $2^{j+1} - 1$ by Lemma B.2. So each condition is Zariski-open, and since the space is irreducible it suffices to show that each is nonempty. For the Q-rank this is clear. Below, Lemma B.3 is used with $d = 2$, where $E = A_\emptyset$ and $U_i = \mathbb{C}^2$.

The condition on $A_\emptyset$. For $j = 1$, Q-rank two means that the 2×2 matrix Ψ of ψ is invertible; then $\sigma_1\Psi, \sigma_2\Psi, \sigma_3\Psi$ are linearly independent, so $A_\emptyset$ has rank three. For $j \geq 2$ fix a copy i and write $\psi = e_0 \otimes x + e_1 \otimes y$. Suppose that x and y are linearly independent and that x, y and $(x + y)/\sqrt{2}$ satisfy the condition on $j - 1$ copies. Let $A_\emptyset\varphi = 0$. By Lemma B.3, $\langle\varphi|A_\emptyset|\varphi\rangle$ is a sum of six terms $\langle\varphi_v|A'_\emptyset(\psi_u)|\varphi_v\rangle$, each nonnegative by Lemma B.2, so each vanishes and $\varphi_v \in \ker A'_\emptyset(\psi_u)$ for every pair. The pairs (e_0, e_1), (e_1, e_0) and $\big((e_0 + e_1)/\sqrt{2}, (e_0 - e_1)/\sqrt{2}\big)$ give $\varphi_{e_1} = \gamma\Theta'x$, $\varphi_{e_0} = \beta\Theta'y$ and $\beta\Theta'y - \gamma\Theta'x \in \operatorname{span}\{\Theta'x + \Theta'y\}$. Since Θ' is antiunitary, $\Theta'x$ and $\Theta'y$ are linearly independent, so $\beta = -\gamma$, and $\ker A_\emptyset$ is at most one-dimensional, hence equal to $\operatorname{span}\{\Theta\psi\}$ by Lemma B.2. The pairs (x, y) with these properties form a finite intersection of Zariski-open sets, each nonempty by induction on j, because $(x, y) \mapsto (x + y)/\sqrt{2}$ is a linear surjection.

The conditions on A_T. Put $\psi_a = \langle a|_T\psi$ for $a \in \{0, 1\}^T$. The first factor of A_T is the sum of the positive operators $A'_\emptyset(\psi_a)$ on the copies outside T, so it is nonsingular if and only if the kernels $\ker A'_\emptyset(\psi_a)$ meet only in 0. For $T = \{1, \ldots, j\}$ these kernels are the orthogonal complements of the ψ_a in Q, which meet only in 0 if and only if ψ has Q-rank two. Otherwise choose $i \notin T$ and write $\psi = e_0 \otimes x + e_1 \otimes y$ at copy i. If φ lies in every $\ker A'_\emptyset(\psi_a)$, then Lemma B.3 at copy i,

with the pairs (e_0,e_1) and (e_1,e_0), puts φ_{e_1} in every $\ker A''_\emptyset(\langle a|_T x)$ and φ_{e_0} in every $\ker A''_\emptyset(\langle a|_T y)$, where $A''_\emptyset$ acts on the copies outside $T\cup\{i\}$. So if x and y satisfy the condition on $j-1$ copies, then $\varphi=0$. Induction on $j-|T|$ shows that each condition is nonempty. □

Proof of Theorem 5.1. By (G1) the bottom eigenvalue of $E(\psi\psi^*)$ is simple near ψ_0, so f is real-analytic there. By lower semicontinuity of rank, every ψ near ψ_0 has local supports of dimension at least two, so near ψ_0 the set Z consists of the vectors in $\big(\bigotimes_m U'_m\big)\otimes Q$ for planes U'_m near U_m. On the vector bundle over a product of Grassmannians with fibre $\big(\bigotimes_m U'_m\big)\otimes Q$, the map $(U',\psi)\mapsto\psi$ is an injective immersion on the open set where every local support has dimension two, with continuous inverse $\psi\mapsto\big((S_m(\psi))_m,\psi\big)$. So near ψ_0, Z is an embedded submanifold of real dimension $2^{k+2}+4k(d-2)$, with tangent space $H_\emptyset\oplus\bigoplus_m \mathrm{Tilt}_m$ at ψ_0. The function f vanishes on Z by Lemma B.2, and its differential vanishes along Z near ψ_0: the simple-kernel condition is open, so for $z\in Z$ near ψ_0 the kernel of $E(zz^*)$ is $\mathrm{span}\{\Theta z\}$, with Θ built from bases of the local supports of z, and Lemma B.4 applies at z. The Hessian of f at ψ_0 is $2S$, and by Lemmas B.4–B.7 it is positive definite on the normal space $\big(H_\emptyset\oplus\bigoplus_m \mathrm{Tilt}_m\big)^\perp$: Lemma B.6 covers the patterns with $|T|\ge 2$ and Lemma B.7 the single copies modulo tilts. By continuity the normal Hessian stays positive definite along Z near ψ_0. The Morse–Bott lemma [BH04], that is, a Taylor expansion of f in normal coordinates of a tubular neighbourhood with a uniform cubic remainder, gives $f(z+\nu)\ge\frac{c}{2}|\nu|^2$ for $z\in Z$ near ψ_0 and small normal ν. Since $\mathrm{dist}(z+\nu,Z)\le|\nu|$ and $E=2^kE_{1/2}$, this proves the theorem. □

B.2 Product zeros

Let $k\ge 2$ and let ϑ be the transposition. We use the following facts.

- $L_{\mathrm{co}}(X)=\mathrm{Tr}(X)\,I-X^{\mathsf T}$ is completely positive, since its Choi matrix is $I-F$, twice the projector onto the antisymmetric subspace. Hence $D\circ\vartheta=\mathcal T+L_{\mathrm{co}}$ is completely positive, and so is $(D\circ\vartheta)^{\otimes n}-\mathcal T^{\otimes n}$ for every n, as a sum of tensor products of completely positive maps.
- With $D'=D^{\otimes(k-1)}$ on the copies other than a fixed copy m (without Q), $D'(\chi\chi^*)\succeq\|\chi\|^2 I$. Indeed D commutes with ϑ, so $D'(\chi\chi^*)=(D\circ\vartheta)^{\otimes(k-1)}(\bar\chi\bar\chi^*)\succeq\mathcal T^{\otimes(k-1)}(\bar\chi\bar\chi^*)=\|\chi\|^2 I$.

If $\mathrm{EP}_{k-1}(d)$ holds and ψ has a copy at which its local support has dimension at most two, then $E(\psi\psi^*)\succeq 0$, by Lemma 6.1 with $t=1/2$ and (B.1).

Fix the copy m, written first, a plane $U\subseteq\mathbb{C}^d$ with orthonormal basis e_0,e_1, and put $\Phi_U=e_0\otimes q_1+e_1\otimes q_2\in\mathbb{C}^d\otimes Q$. For $0\ne\chi\in H'=(\mathbb{C}^d)^{\otimes(k-1)}$ let $\psi_0=\Phi_U\otimes\chi$, with the factors reordered. Every $\eta\in H\otimes Q$ is uniquely

$$\eta=\sum_{\mu=0}^{3}(\sigma_\mu\Phi_U)\otimes\beta_\mu+\sum_{y,q}(y\otimes q)\otimes\gamma_{yq},\qquad \beta_\mu,\gamma_{yq}\in H',$$

with $\sigma_0=I$ and $\sigma_1,\sigma_2,\sigma_3$ the Pauli matrices on U acting at copy m, and y running over an orthonormal basis of $U^\perp$. For $x\in H'$ put $x_\perp=x-\frac{\langle\chi,x\rangle}{\|\chi\|^2}\chi$, $R_x=D'(xx^*)$, $N_x=D'(\chi x^*)$ and $M_x=D'(x\chi^*)=N_x^*$. The tilts at ψ_0 are the vectors $(X\otimes I)\psi_0$ with $X\in\mathrm{Hom}(U,U^\perp)$; they span $\{(y\otimes q)\otimes\chi\}$, so a vector with all $\beta_\mu=0$ is orthogonal to the tilts if and only if every γ_{yq} is orthogonal to χ.

Lemma B.9. *(a) $A_0=E(\psi_0\psi_0^*)=(2I-\Phi_U\Phi_U^*)\otimes R_\chi$ with $R_\chi\succeq\|\chi\|^2 I$. Hence $K=\ker A_0=\mathrm{span}\{\Phi_U\}\otimes H'$ has dimension d^{k-1}, and $A_0=2I\otimes R_\chi$ on $K^\perp$. (b) $P_K X_\eta P_K=0$ for every η, where $X_\eta=E(\psi_0\eta^*+\eta\psi_0^*)$. (c) The reduced second-order operator $S(\eta)=P_K\big[E(\eta\eta^*)-X_\eta A_0^+X_\eta\big]P_K$, written on H' through the unit vectors $\kappa=\Phi_U\otimes h/\sqrt{2}$ with $\|h\|=1$, is*

$$S(\eta)=2\sum_{a=1}^{3}\mathrm{Sch}(\chi,\beta_a)+\sum_{y,q}\mathrm{Sch}'(\chi,\gamma_{yq}),$$

with $\mathrm{Sch}(\chi,x)=R_x-N_x^*R_\chi^{-1}N_x$ and $\mathrm{Sch}'(\chi,x)=R_x-M_x^*R_\chi^{-1}M_x$. Both depend on x only through $x_\perp$. (d) $\mathrm{Sch}'(\chi,x)\succeq\|x_\perp\|^2 I$, without further assumptions. (e) If $\mathrm{EP}_{k-1}(d)$ holds, then $\mathrm{Sch}(\chi,x)\succeq 0$.

Proof. (a) Since ψ_0 is a product across (copy m, Q) and the other copies, $E(\psi_0\psi_0^*)=(D\otimes\mathrm{id}_Q)(\Phi_U\Phi_U^*)\otimes D'(\chi\chi^*)$. Here $(D\otimes\mathrm{id}_Q)(Y)=2I\otimes\mathrm{Tr}_m Y-Y$ and $\mathrm{Tr}_m\,\Phi_U\Phi_U^*=I_Q$, so $(D\otimes\mathrm{id}_Q)(\Phi_U\Phi_U^*)=2I-\Phi_U\Phi_U^*$, with eigenvalue 0 on Φ_U ($\|\Phi_U\|^2=2$) and 2 on its orthogonal complement. The bound on R_χ is the second fact above.

(b), (c) For Y on $\mathbb{C}^d\otimes Q$ one has $\langle\Phi_U|(D\otimes\mathrm{id}_Q)(Y)|\Phi_U\rangle=2\,\mathrm{Tr}\,Y-\langle\Phi_U|Y|\Phi_U\rangle$. For $Y=\Phi_U\alpha^*$ or $\alpha\Phi_U^*$ this vanishes, and since $E(\psi_0(\alpha\otimes\beta)^*)=(D\otimes\mathrm{id})(\Phi_U\alpha^*)\otimes N_\beta$, this gives (b). For $E(\eta\eta^*)$: $Y=(\sigma_\mu\Phi_U)(\sigma_\nu\Phi_U)^*$ gives $4\delta_{\mu\nu}-4\delta_{\mu0}\delta_{\nu0}$, $Y=(y\otimes q)(y'\otimes q')^*$ gives $2\delta_{yy'}\delta_{qq'}$, and mixed terms give 0; so $\langle\Phi_U\otimes h|E(\eta\eta^*)|\Phi_U\otimes h\rangle=4\sum_a\langle h|R_{\beta_a}|h\rangle+2\sum\langle h|R_{\gamma_{yq}}|h\rangle$. For X_η: $(D\otimes\mathrm{id})(\Phi_U(\sigma_a\Phi_U)^*)\Phi_U=2\sigma_a\Phi_U$ for $a=1,2,3$ and 0 for $\mu=0$; $(D\otimes\mathrm{id})((\sigma_\mu\Phi_U)\Phi_U^*)\Phi_U=0$; $(D\otimes\mathrm{id})(\Phi_U(y\otimes q)^*)\Phi_U=0$; and $(D\otimes\mathrm{id})((y\otimes q)\Phi_U^*)\Phi_U=-2(y\otimes q)$. Hence $X_\eta(\Phi_U\otimes h)=2\sum_a(\sigma_a\Phi_U)\otimes N_{\beta_a}h-2\sum_{y,q}(y\otimes q)\otimes M_{\gamma_{yq}}h$, which lies in $K^\perp$, where $A_0^+=\frac12 I\otimes R_\chi^{-1}$, with mutually orthogonal components. So $\langle X_\eta(\Phi_U\otimes h)|A_0^+|X_\eta(\Phi_U\otimes h)\rangle=4\sum_a\langle h|N_{\beta_a}^*R_\chi^{-1}N_{\beta_a}|h\rangle+2\sum\langle h|M^*R_\chi^{-1}M|h\rangle$, and dividing by $\|\Phi_U\otimes h\|^2=2$ gives (c). For the last claim, $\mathrm{Sch}(\chi,x)$ is the Schur complement over the first block of $(D'\otimes\mathrm{id}_2)(ww^*)$ with $w=|0\rangle\otimes\chi+|1\rangle\otimes x$; replacing x by $x-c\chi$ is a congruence by a lower-triangular matrix on the two-dimensional ancilla, which leaves this Schur complement unchanged. The same holds for Sch', with the transposed block matrix of (d).

(d) $\mathrm{Sch}'(\chi,x)$ is the Schur complement of $B_x=\begin{pmatrix}R_\chi & M_x\\ M_x^* & R_x\end{pmatrix}$. With w as above, B_x is the full transpose of $(\mathrm{id}_2\otimes(D\circ\vartheta)^{\otimes(k-1)})(ww^*)$, because D commutes with ϑ. By the first fact, $B_x\succeq\Gamma^{\mathsf T}\otimes I$, where Γ is the Gram matrix of

(χ, x). Schur complements over a positive definite block are monotone in the Loewner order, since $v^*(M/M_{11})v = \min_u (u,v)^* M (u,v)$. This gives $\mathrm{Sch}'(\chi,x) \succeq \big(\|x\|^2 - |\langle \chi, x\rangle|^2/\|\chi\|^2\big) I = \|x_\perp\|^2 I$.

(e) $\mathrm{Sch}(\chi,x)$ is the Schur complement of $(D' \otimes \mathrm{id}_2)(ww^*)$, which is positive semidefinite by $\mathrm{EP}_{k-1}(d)$ and (B.1), since w has Q-rank at most two. $\square$

The eigenvalues of $A_0 + \varepsilon X_\eta + \varepsilon^2 E(\eta\eta^*)$ that emanate from the d^{k-1}-fold eigenvalue 0 are $\varepsilon^2 \operatorname{eig} S(\eta) + O(\varepsilon^3)$, by reduction to the kernel with a vanishing first-order compression [Kat95], and the others stay at least $2\|\chi\|^2 + O(\varepsilon)$. So, given $\mathrm{EP}_{k-1}(d)$, $\lambda_{\min} E\big((\psi_0 + \varepsilon\eta)(\cdot)^*\big) \geq -O(\varepsilon^3)$ in every direction. To control a full neighbourhood we split at the qubit copy.

Lemma B.10 (splitting at a qubit copy). *Suppose that $\mathrm{EP}_{k-1}(d)$ holds. Let ψ_0 have a copy m whose local support U has dimension two, let $A_0 = E(\psi_0 \psi_0^*)$, which is positive semidefinite, and let $K = \ker A_0$. Suppose that (i) the $U^\perp$ block of A_0 at copy m is positive definite, so that K lies in the U block; and (ii) there is $c > 0$ with $\langle \kappa | E(\eta\eta^*) - X_\eta A_0^+ X_\eta | \kappa\rangle \geq c\|\eta\|^2\|\kappa\|^2$ for all $\kappa \in K$ and all η whose copy-m component lies in $U^\perp$ and which are orthogonal to the tilts at ψ_0. Then $E(\psi\psi^*) \succeq 0$ for every ψ in a neighbourhood of ψ_0.*

Proof. The reduced operator $\rho_m(\psi)$ of $\psi\psi^*$ on copy m has rank two at ψ_0, so for ψ near ψ_0 the span U' of its top two eigenvectors is well defined, depends real-analytically on ψ and is close to U; identify U' with U by a smooth unitary. Put $\psi_\parallel = (P_{U'} \otimes I)\psi$ and $\psi_\perp = (P_{U'^\perp} \otimes I)\psi$. Since U' is invariant under $\rho_m(\psi)$, $\langle \psi_\perp | (X \otimes I) | \psi_\parallel \rangle = \mathrm{Tr}_m \left[X P_{U'} \rho_m(\psi) P_{U'^\perp} \right] = 0$, so $\psi_\perp$ is orthogonal to the tilts at $\psi_\parallel$. At copy m, $D(xx'^*)$ is block-diagonal for $U' \oplus U'^\perp$ when x and x' lie in the same summand, and $D(xy^*) = -xy^*$ is block-off-diagonal for $x \in U'$ and $y \in U'^\perp$. Hence, exactly,

$$E(\psi\psi^*) = \begin{pmatrix} A_\parallel + E_{11} & X_{12} \\ X_{21} & B_\parallel + E_{22} \end{pmatrix},$$

where $\mathrm{diag}(A_\parallel, B_\parallel) = E(\psi_\parallel \psi_\parallel^*)$, the X's come from $E(\psi_\parallel \psi_\perp^* + \psi_\perp \psi_\parallel^*)$, and $\mathrm{diag}(E_{11}, E_{22}) = E(\psi_\perp \psi_\perp^*)$. By (i) and continuity, $B_\parallel + E_{22} \succeq b > 0$ near ψ_0, so $E(\psi\psi^*) \succeq 0$ if and only if the Schur complement $A_\parallel + \mathcal{S}(\psi_\perp) + O(\|\psi_\perp\|^4)$ is positive semidefinite, where $\mathcal{S}(\psi_\perp) = E_{11} - X_{12} B_\parallel^{-1} X_{21}$ is quadratic in $\psi_\perp$. Now $A_\parallel \succeq 0$ by the fact stated before Lemma B.9, since $\psi_\parallel$ has local support of dimension at most two at copy m. Near ψ_0, $A_\parallel$ has a cluster of small eigenvalues with spectral subspace $K_\parallel$ close to K, and $A_\parallel \succeq g > 0$ on the rest of the U' block. At $\psi_\parallel = \psi_0$, $P_K \mathcal{S} P_K$ is the operator in (ii); the tilt spaces vary continuously because $X \mapsto (X \otimes I)\psi_\parallel$ is injective, so by continuity and compactness $P_{K_\parallel} \mathcal{S} P_{K_\parallel} \succeq \frac{c}{2}\|\psi_\perp\|^2$ near ψ_0. Since $A_\parallel$ is block-diagonal for $K_\parallel \oplus K_\parallel^\perp$ and the couplings are $O(\|\psi_\perp\|^2)$ against g on $K_\parallel^\perp$, the final Schur complement is at least $a + \frac{c}{2}\|\psi_\perp\|^2 - O(\|\psi_\perp\|^4)$ with $a \succeq 0$, which is positive semidefinite near ψ_0. $\square$

Proof of Theorem 5.2. By the $GL(Q)$ covariance it suffices to treat $\psi_0 = \Phi_U \otimes \chi$. Condition (i) holds by Lemma B.9(a): the $U^\perp$ block of A_0 is $2I \otimes R_\chi \succeq 2\|\chi\|^2 I$. Condition (ii) holds with $c = 1$: an η with copy-m component in $U^\perp$ that is orthogonal to the tilts has all $\beta_\mu = 0$ and every γ_{yq} orthogonal to χ, so $\|\eta\|^2 = \sum \|\gamma_{yq}\|^2$, and for $\kappa = \Phi_U \otimes h/\sqrt{2}$, Lemma B.9(c), (d) give $\langle \kappa | E(\eta\eta^*) - X_\eta A_0^+ X_\eta | \kappa \rangle = \sum_{y,q} \langle h | \mathrm{Sch}'(\chi, \gamma_{yq}) | h \rangle \geq \|\eta\|^2 \|\kappa\|^2$. Lemma B.10 applies, and $E = 2^k E_{1/2}$. $\square$

B.3 Proof of Theorem 7.2

In this proof we normalise the lifting differently: $\psi = (x_1 \otimes q_1 + x_2 \otimes q_2)/\sqrt{2}$ for an orthonormal basis (x_1, x_2) of V, so that $\mathrm{Tr}_{12}\, \psi\psi^* = \frac{1}{2} I_Q$, and $E = D \otimes D \otimes \mathrm{id}_Q$. By (B.1), $\lambda_{\min} E(\psi\psi^*) = 2m_2(V)$. With $\rho_{1Q} = \mathrm{Tr}_2\, \psi\psi^*$ and $\rho_{2Q} = \mathrm{Tr}_1\, \psi\psi^*$,

$$E(\psi\psi^*) = 2I - 2\rho_{1Q} \otimes I_2 - 2I_1 \otimes \rho_{2Q} + \psi\psi^*. \tag{B.3}$$

Let $w_i = (\alpha_i, \beta_i)^{\mathsf{T}} \in \mathbb{C}^2$ and $\nu_i = \|w_i\|^2/2$, so that the reduced operators of P_V have spectrum $(2\nu_i)_i$; write $\nu_{(1)} \geq \nu_{(2)} \geq \dots$ and $m = \nu_{(3)}$, so that $\nu_3(V) = 2m$. We prove $\lambda_{\min} E(\psi\psi^*) \geq cm$ with $c = 4 - 2\sqrt{2}$, which is the claim. If $m = 0$, at most two of the w_i are nonzero, so V is qubit-supported, and $m_2(V) = 0$ by Lemma 2.7(a), with equality in the claim; so let $m > 0$.

(1) *Block form.* Since $\psi = 2^{-1/2} \sum_i a_i \otimes b_i \otimes w_i$, we have $\rho_{1Q} = \frac{1}{2}\sum_i |a_i\rangle\langle a_i| \otimes w_i w_i^*$ and $\rho_{2Q} = \frac{1}{2}\sum_j |b_j\rangle\langle b_j| \otimes w_j w_j^*$. By (B.3), in the decomposition $\bigoplus_{i,j} a_i \otimes b_j \otimes Q$,

$$E(\psi\psi^*) = \bigoplus_{i,j} G_{ij} + \psi\psi^*, \qquad G_{ij} = 2I_2 - w_i w_i^* - w_j w_j^*.$$

(2) $\nu_i \leq \frac{1}{2}$, because $\sum_i w_i w_i^* = I_2$ by orthonormality of α and β, so $w_i w_i^* \preceq I_2$. (3) For $i \neq j$ the block is not touched by $\psi\psi^*$, and $G_{ij} \succeq (2 - 2\nu_i - 2\nu_j) I \succeq 2(1 - \nu_{(1)} - \nu_{(2)}) I \succeq 2mI \succeq cmI$. (4) For $i = j$, $G_{ii} = 2I - 2 w_i w_i^*$ equals 2 on $a_i \otimes b_i \otimes w_i^\perp$, which is orthogonal to ψ. On $\mathcal{S} = \mathrm{span}\{a_i \otimes b_i \otimes \hat{w}_i : w_i \neq 0\}$, with $\hat{w}_i = w_i/\|w_i\|$, which contains ψ with $\langle a_i \otimes b_i \otimes \hat{w}_i, \psi \rangle = \sqrt{\nu_i}$, the operator restricts to $E_{\mathcal{S}} = \mathrm{diag}(2 - 4\nu_i) + uu^{\mathsf{T}}$ with $u_i = \sqrt{\nu_i}$. (5) *One inequality.* Let $D'' = \mathrm{diag}(2 - 4\nu_i - cm)$. For every i except the index (1) of the largest ν, $D''_i \geq 2 - 4\nu_{(2)} - cm \geq (2 - c)m > 0$, since $2\nu_{(2)} \leq \nu_{(1)} + \nu_{(2)} \leq 1 - m$. If $D''_{(1)} \geq 0$, then $E_{\mathcal{S}} - cm = D'' + uu^{\mathsf{T}} \succeq 0$. Otherwise D'' has exactly one negative entry; by interlacing all eigenvalues of $D'' + uu^{\mathsf{T}}$ but the lowest are positive, and $\det(D'' + uu^{\mathsf{T}}) = \det(D'')\big(1 + u^{\mathsf{T}} D''^{-1} u\big)$ with $\det D'' < 0$. Hence $E_{\mathcal{S}} \succeq cm$ if and only if $1 + \sum_i \nu_i / D''_i \leq 0$, that is, with $\nu_1 = \nu_{(1)}$,

$$\frac{\nu_1}{4\nu_1 + cm - 2} \;\geq\; 1 + \sum_{i \neq (1)} \frac{\nu_i}{2 - 4\nu_i - cm}. \tag{B.4}$$

(6) *Proof of (B.4).* Put $\delta = 4\nu_1 + cm - 2$, which lies in $(0, cm]$ by (2); and $\nu_{(2)} \geq m$ gives $\nu_1 \leq 1 - 2m$, that is, $\delta \leq 2 - (8 - c)m$; in particular $m < 2/(8 - c) < 0.293$. The function $h(x) = x/(2 - 4x - cm)$ is convex and increasing with $h(0) = 0$, so $h(x)/x$ increases. Since $\nu_{(i)} \leq m$ for $i \geq 3$, the right side of (B.4) is at most

$R(\nu_{(2)}) = 1 + h(\nu_{(2)}) + (1-\nu_1-\nu_{(2)})\,h(m)/m$, which is convex in $\nu_{(2)}$ on $[m, 1-\nu_1-m]$ and hence at most $\max(R_{\mathrm{i}}, R_{\mathrm{ii}})$, where

$$R_{\mathrm{i}} = 1 + \frac{1-\nu_1}{2-(4+c)m}, \qquad R_{\mathrm{ii}} = 1 + \frac{1-\nu_1-m}{\delta+(4-2c)m} + \frac{m}{2-(4+c)m}$$

correspond to $\nu_{(2)} = m$ and $\nu_{(2)} = 1-\nu_1-m$. The left side is $L = \frac{1}{4} + \frac{2-cm}{4\delta}$. Both $L - R_{\mathrm{i}}$ and $L - R_{\mathrm{ii}}$ decrease in δ: their derivatives are $-\frac{2-cm}{4\delta^2} + \frac{1}{4(2-(4+c)m)}$, which is negative because $\delta^2 \le c^2m^2 < (2-cm)(2-(4+c)m)$ for $m < 0.364$, and $-\frac{2-cm}{4\delta^2} + \frac{2-cm}{4(\delta+(4-2c)m)^2}$, which is negative because $4-2c > 0$. So it suffices to take $\delta = \min\big(cm,\, 2-(8-c)m\big)$.

- If $m \le \frac{1}{4}$, then $\delta = cm$ and $\nu_1 = \frac{1}{2}$. Here $L \ge R_{\mathrm{i}}$ is equivalent to $\frac{1}{2cm} \ge 1 + \frac{1}{4-2(4+c)m}$, and $L \ge R_{\mathrm{ii}}$ to $\frac{1}{c} \ge 2m + \frac{1-2m}{4-c} + \frac{2m^2}{2-(4+c)m}$. In the first, the left side decreases and the right side increases in m; in the second, the right side increases in m. At $m = \frac{1}{4}$ the slacks, as functions of c, are $\frac{c^2-8c+8}{c(4-c)}$ and half of it, and $c^2-8c+8 = 0$ at $c = 4-2\sqrt{2}$. So both hold, with equality exactly at $m = \frac{1}{4}$.
- If $\frac{1}{4} < m < \frac{2}{8-c}$, then $\delta = 2-(8-c)m$, $\nu_1 = 1-2m$ and $\nu_{(2)} = \nu_{(3)} = m$, so $R_{\mathrm{i}} = R_{\mathrm{ii}} = 1 + \frac{2m}{2-(4+c)m}$ and $L = \frac{1-2m}{2-(8-c)m}$. Both equal $1 + \frac{1}{\sqrt{2}}$ at $m = \frac{1}{4}$. Here $L' = \frac{4-c}{(2-(8-c)m)^2}$ and $R_{\mathrm{i}}' = \frac{4}{(2-(4+c)m)^2}$, both denominators are positive on the interval, and taking square roots shows that $L' > R_{\mathrm{i}}'$ holds exactly when $m > m_0 = \frac{4-2\sqrt{4-c}}{2(8-c)-(4+c)\sqrt{4-c}} = 0.12832\ldots$ (checked symbolically, †, C3). Since $m_0 < \frac{1}{4}$, $L \ge R_{\mathrm{i}}$.

Hence (B.4) holds and $E_{\mathcal{S}} \succeq cm$; with (3) and (4), $E(\psi\psi^*) \succeq cmI$. Equality requires $m = \frac{1}{4}$, $\nu_1 = \frac{1}{2}$, $\nu_{(2)} = \nu_{(3)} = \frac{1}{4}$ and the remaining ν_i zero. There $E_{\mathcal{S}} = \mathrm{diag}(0,1,1) + uu^{\mathsf{T}}$ with $u = (1/\sqrt{2}, 1/2, 1/2)$, whose smallest eigenvalue $1 - 1/\sqrt{2}$ is the smaller root of $2\ell^2 - 4\ell + 1 = 0$. Finally, let $\nu_{(1)} \le \frac{1}{2}$ and $\sum_i \nu_i = 1$. Then $(2\nu_i)_i$ is majorised by $(1,1,0,\ldots,0)$, so by the Schur–Horn theorem it is the diagonal of some rank-two projector; taking its range as $\mathrm{span}\{\alpha,\beta\}$ shows that all such spectra occur in the class. □

B.4 Proof of Theorem 4.7

By Lemma 4.8 it suffices to treat $d = 3$. The identities marked (†) are rational identities in the displayed variables, polynomial after clearing denominators; a symbolic check verifies all of them (†, C1).

Step 1: operator form. Let $\mathcal{L}_t(X) = \mathrm{Tr}(X)I - tX$ act on each copy. Expanding in the product basis and comparing with (4.1),

$$q_k(t, C) = \sum_{c,c'} \langle g_c | \mathcal{L}_t^{\otimes k}(|f_c\rangle\langle f_{c'}|) | g_{c'}\rangle \qquad \text{for } C = \sum_c |f_c\rangle\langle g_c|.$$

Every operator with range in $\mathrm{span}\{f_0, f_1\}$ has the form $|f_0\rangle\langle g_0| + |f_1\rangle\langle g_1|$. Put $\mathcal{K} = 2\mathcal{L}_{1/2}$ and $A_{cc'} = \mathcal{K}^{\otimes k}(|f_c\rangle\langle f_{c'}|)$ for $c, c' \in \{0,1\}$, so that $A_{01} = A_{10}^*$. Theorem 4.7 says that $\left[\begin{smallmatrix} A_{00} & A_{01} \\ A_{10} & A_{11}\end{smallmatrix}\right] \ge 0$. Per copy, $\mathcal{K} = \mathrm{Tr}(\cdot)I + \mathcal{R}$ with $\mathcal{R}(X) = \mathrm{Tr}(X)I - X$. The map $\mathcal{R}$ is co-completely positive, because its partially transposed Choi matrix $I - F$ is positive semidefinite, and tensor products of such maps are positive. Expanding the tensor power, every term is positive semidefinite on $|f\rangle\langle f|$, and the term without $\mathcal{R}$ is $\|f\|^2 I$. Hence

$$A_{cc} \;\ge\; \|f_c\|^2\, I. \tag{B.5}$$

The condition is closed in (f_0, f_1). If the tails of f_0 and f_1 sit at disjoint copies ($\alpha_j\beta_j = 0$ for all j), then $A_{10} = \varepsilon |f_1\rangle\langle f_0|$ with $\varepsilon = (-1)^k$ (Step 2), and (B.5) gives $|\langle g_1|A_{10}|g_0\rangle| \le \|f_0\|\|g_0\|\|f_1\|\|g_1\| \le (\langle g_0|A_{00}|g_0\rangle\langle g_1|A_{11}|g_1\rangle)^{1/2}$, which is the claim. In general, by closedness we may assume $\alpha_0, \beta_0 \ne 0$. The product unitaries $\bigotimes_l \mathrm{diag}(1, e^{i\phi_1^l}, e^{i\phi_2^l})$ preserve $\mathcal{V}_0$, $\mathcal{V}_1$ and q_k. They multiply α_j/α_0 by $e^{i\phi_2^j}$ and β_j/β_0 by $e^{i(\phi_2^j - \phi_1^j)}$. Rescaling f_c is absorbed by g_c. Hence we may assume that all $\alpha_0, \alpha_j, \beta_0, \beta_j$ are real and nonnegative, that $\|f_0\| = \|f_1\| = 1$, and, by closedness again, that all of them are positive. Put $a_j = \alpha_j^2$, $b_j = \beta_j^2$, $w_j = a_j + b_j$, $\bar{a} = \sum_j a_j = 1 - \alpha_0^2$, $\bar{b} = \sum_j b_j = 1 - \beta_0^2$ and $\kappa = \sum_j a_j b_j$. Then $0 < \bar{a}, \bar{b} < 1$.

Step 2: reduction to few coordinates. Per copy, $\mathcal{K}(|x\rangle\langle x'|) = 2\delta_{xx'}I - |x\rangle\langle x'|$. The strings of R_0 and R_1 differ at every copy, except that $2_j 0^{k-1}$ and $2_j 1^{k-1}$ agree at copy j. Therefore, with I_j the identity at copy j,

$$A_{10} = \varepsilon\Big[|f_1\rangle\langle f_0| - 2\sum_j \alpha_j\beta_j\, I_j \otimes |1^{k-1}\rangle\langle 0^{k-1}|\Big].$$

So, at $d = 3$, A_{10} vanishes on the orthogonal complement of $T_0 = \mathrm{span}\{|0^k\rangle, |2_j 0^{k-1}\rangle, |1_j 0^{k-1}\rangle : 1 \le j \le k\}$ and maps into $T_1 = \mathrm{span}\{|1^k\rangle, |2_j 1^{k-1}\rangle, |0_j 1^{k-1}\rangle : 1 \le j \le k\}$. Minimising $\langle g_0|A_{00}|g_0\rangle$ over the components of g_0 orthogonal to T_0 replaces A_{00} by the Schur complement $\tilde{A}_{00} = \big((A_{00}^{-1})_{T_0T_0}\big)^{-1}$, and likewise for A_{11}. The condition becomes $\left[\begin{smallmatrix} \tilde{A}_{00} & A_{01} \\ A_{10} & \tilde{A}_{11}\end{smallmatrix}\right] \ge 0$ on $T_0 \oplus T_1$.

To bound $\tilde{A}_{00}$ from below, note that every factor of $\mathcal{K}^{\otimes k}(|x\rangle\langle x'|)$ with $x, x' \in R_0$ maps $\mathrm{span}\{|0\rangle, |2\rangle\}$ into itself and has $|1\rangle$ as an eigenvector. So A_{00} is block diagonal with respect to the set of copies at which a basis string has the digit 1. The subspace T_0 meets only the block of strings without the digit 1, in its strings with at most one digit 2, and, for each j, the block of strings whose only digit 1 is at copy j, in its string $1_j 0^{k-1}$. Hence $\tilde{A}_{00}$ is the direct sum of the Schur complements of these blocks onto these strings.

Index the strings in $\{0,2\}^k$ by their patterns P, and call $|P|$ the level of such a string. Let $\partial_j = |0\rangle\langle 2|$ act at copy j, so that it lowers a digit 2 at copy j to 0, and put $\partial = \sum_j \alpha_j \partial_j$. A direct expansion shows that the block without the digit 1 is

$$\mathsf{M} := (\alpha_0 - \partial)^*\, \mathsf{W}\, (\alpha_0 - \partial) + 2\mathsf{WD}, \qquad \mathsf{W} = \mathrm{diag}\left(2^{|P|}\right), \quad \mathsf{D} = \mathrm{diag}\Big(\sum_{j \notin P} a_j\Big),$$

and that the block whose only digit 1 is at copy j is $2\big[\mathsf{M}^{(j)} + a_j \mathsf{W}\big]$, where $\mathsf{M}^{(j)}$ and W are the same operators for the $k-1$ copies other than j.

Lemma B.11. Let M be defined as above, on any number of copies, from $\alpha_0 > 0$ and $\alpha_j \geq 0$, and put $\Delta = \operatorname{diag}\big(\sum_{j\in P} a_j + 3\sum_{j\notin P} a_j\big)$. Let $\nu \geq 0$. For every level m, the level-m block S_m of the Schur complement of $\mathsf{M} + \nu\mathsf{W}$ onto the levels $\leq m$ satisfies

$$\mathsf{S}_m \;\geq\; (\alpha_0^2 + \nu)\, 2^m + 2^{m-1}\Delta_m,$$

where Δ_m is the level-m block of Δ.

Proof. The operator M is positive definite, since $\alpha_0 - \partial$ is invertible (∂ lowers the level), so all Schur complements exist. For a level-preserving operator X let X_m be its level-m block, and let $\partial_{m,m+1}$ be the block of ∂ from level $m+1$ to level m. The operator M couples only adjacent levels, with $\mathsf{M}_{m,m+1} = -\alpha_0 2^m \partial_{m,m+1}$ and $\mathsf{M}_{mm} = \alpha_0^2 2^m + 2^{m-1}(\partial^*\partial)_m + 2^{m+1}\mathsf{D}_m$. Hence, below the top level, $\mathsf{S}_m = (\mathsf{M} + \nu\mathsf{W})_{mm} - \mathsf{M}_{m,m+1}\mathsf{S}_{m+1}^{-1}\mathsf{M}_{m+1,m}$. At the top level, where all digits are 2, D and $\partial\partial^*$ vanish, so $\mathsf{S}_m = (\alpha_0^2+\nu)2^m + 2^{m-1}(\partial^*\partial)_m$ with $(\partial^*\partial)_m = \sum_j a_j = \Delta_m$. If the claim holds at level $m+1$, then $\mathsf{S}_{m+1} \geq (\alpha_0^2+\nu)2^{m+1}$ because $\Delta \geq 0$, so

$$\mathsf{M}_{m,m+1}\mathsf{S}_{m+1}^{-1}\mathsf{M}_{m+1,m} \;\leq\; 2^{m-1}\frac{\alpha_0^2}{\alpha_0^2+\nu}\,(\partial\partial^*)_m \;\leq\; 2^{m-1}(\partial\partial^*)_m,$$

and therefore $\mathsf{S}_m \geq (\alpha_0^2+\nu)2^m + 2^{m-1}\big([\partial^*,\partial] + 4\mathsf{D}\big)_m$. Finally, ∂_i and ∂_j^* commute for $i \neq j$, and $[\partial_j^*, \partial_j] = |2\rangle\langle 2| - |0\rangle\langle 0|$ at copy j, so $[\partial^*,\partial] + 4\mathsf{D} = \Delta$. $\square$

In the coordinates 0^k; $2_10^{k-1}, \dots, 2_k0^{k-1}$; $1_10^{k-1}, \dots, 1_k0^{k-1}$ of T_0, Lemma B.11 gives $\tilde{A}_{00} \geq \underline{A}_{00}$, where

$$\underline{A}_{00} = \begin{pmatrix} \alpha_0^2 + 2\bar{a} & -\alpha_0\vec{\alpha}^{\top} & 0 \\ -\alpha_0\vec{\alpha} & \operatorname{diag}(2\alpha_0^2 + 3\bar{a} - 2a_j) & 0 \\ 0 & 0 & \operatorname{diag}(2\alpha_0^2 + 3\bar{a} - a_j) \end{pmatrix}, \qquad \vec{\alpha} = (\alpha_1, \dots, \alpha_k)^{\top}.$$

Indeed, since M couples only adjacent levels, its Schur complement onto the levels ≤ 1 is $\left[\begin{smallmatrix} \alpha_0^2+2\bar{a} & -\alpha_0\vec{\alpha}^{\top} \\ -\alpha_0\vec{\alpha} & \mathsf{S}_1 \end{smallmatrix}\right]$, and $\mathsf{S}_1 \geq 2\alpha_0^2 + \Delta_1 = \operatorname{diag}(2\alpha_0^2 + 3\bar{a} - 2a_j)$ by Lemma B.11 with $\nu = 0$. For the block $2[\mathsf{M}^{(j)} + a_j\mathsf{W}]$, Lemma B.11 at level 0, applied to $\mathsf{M}^{(j)}$ with $\nu = a_j$, bounds the Schur complement onto 1_j0^{k-1} from below by $2\big[\alpha_0^2 + a_j + \tfrac{3}{2}(\bar{a} - a_j)\big] = 2\alpha_0^2 + 3\bar{a} - a_j$. The bound $\underline{A}_{11}$ is the mirror, with β, $\bar{b}$ and b_j. Hence Theorem 4.7 follows from $\left[\begin{smallmatrix} \underline{A}_{00} & \tilde{A}_{01} \\ \tilde{A}_{10} & \underline{A}_{11} \end{smallmatrix}\right] \geq 0$.

Step 3: a scalar form. All entries are now real, and replacing g_1 by εg_1 removes ε. Let $\lambda_0, \lambda_j, \lambda_j'$ be the coordinates on T_0 at 0^k, 2_j0^{k-1}, 1_j0^{k-1}, and μ_0, μ_j, μ_j' those on T_1 at 1^k, 2_j1^{k-1}, 0_j1^{k-1}. Since $2\alpha_0^2 + 3\bar{a} - a_j = 2 + \bar{a} - a_j$, the quadratic form of the last matrix is

$$\Big(\alpha\cdot\lambda + \beta\cdot\mu\Big)^2 + \sum_j \Big[2(\alpha_0\lambda_j - \alpha_j\lambda_0)^2 + (3\bar{a} - 2a_j)\lambda_j^2\Big] - \Big(\sum_j \alpha_j\lambda_j\Big)^2 + (\text{mirror in } \mu)$$
$$-4\sum_j \alpha_j\beta_j\lambda_j\mu_j + \sum_j (2 + \bar{a} - a_j)\lambda_j'^2 + \sum_j (2 + \bar{b} - b_j)\mu_j'^2 - 4\mu_0\sum_j \alpha_j\beta_j\lambda_j' - 4\lambda_0\sum_j \alpha_j\beta_j\mu_j',$$

with $\alpha\cdot\lambda = \alpha_0\lambda_0 + \sum_j \alpha_j\lambda_j$ and $\beta\cdot\mu = \beta_0\mu_0 + \sum_j \beta_j\mu_j$. We use $(\sum_j \alpha_j\lambda_j)^2 \leq \bar{a}\sum_j \lambda_j^2$ and its mirror, minimise over λ_j', μ_j', and put $\hat{\lambda}_0 = \alpha_0\lambda_0$, $\hat{\mu}_0 = \beta_0\mu_0$, $\hat{\lambda}_j = \alpha_j\lambda_j$, $\hat{\mu}_j = \beta_j\mu_j$ and $\tau_j = \hat{\lambda}_j + \hat{\mu}_j$. Since $2 + \bar{a} - a_j \geq 2$ and $2 + \bar{b} - b_j \geq 2$, the form is at least

$$\Big(\hat{\lambda}_0 + \hat{\mu}_0 + \sum_j \tau_j\Big)^2 + \sum_j \Big[\frac{2\hat{\lambda}_j^2}{a_j} + \frac{2\hat{\mu}_j^2}{b_j} - 4\hat{\lambda}_0\hat{\lambda}_j - 4\hat{\mu}_0\hat{\mu}_j - 2\tau_j^2\Big] + \frac{2(\bar{a} - \kappa)}{1 - \bar{a}}\hat{\lambda}_0^2 + \frac{2(\bar{b} - \kappa)}{1 - \bar{b}}\hat{\mu}_0^2.$$

Put $e_j = a_j\hat{\lambda}_0 + b_j\hat{\mu}_0$. For fixed τ_j,

$$\min_{\hat{\lambda}_j + \hat{\mu}_j = \tau_j} \Big[\frac{2\hat{\lambda}_j^2}{a_j} + \frac{2\hat{\mu}_j^2}{b_j} - 4\hat{\lambda}_0\hat{\lambda}_j - 4\hat{\mu}_0\hat{\mu}_j\Big] = \frac{2(\tau_j - e_j)^2}{w_j} - 2\big(a_j\hat{\lambda}_0^2 + b_j\hat{\mu}_0^2\big) \quad (\dagger).$$

Hence the form is at least

$$G := \Big(\hat{\lambda}_0 + \hat{\mu}_0 + \sum_j \tau_j\Big)^2 + 2\sum_j \Big[\frac{(\tau_j - e_j)^2}{w_j} - \tau_j^2\Big] + \frac{2(\bar{a}^2 - \kappa)}{1 - \bar{a}}\hat{\lambda}_0^2 + \frac{2(\bar{b}^2 - \kappa)}{1 - \bar{b}}\hat{\mu}_0^2.$$

Step 4: merging copies. Regard G as a function of the data (a_l, b_l) of the copies and of the variables $\hat{\lambda}_0$, $\hat{\mu}_0$, τ_l. Merge two copies $i \neq j$ into one copy with data $(a_i + a_j,\ b_i + b_j)$ and variable $\tau_i + \tau_j$. Then $\bar{a}$ and $\bar{b}$ do not change, κ grows by $\kappa_{ij} := a_ib_j + a_jb_i$, and with $\tilde{\tau}_l = (\tau_l - e_l)/w_l$,

$$G_{\text{before}} - G_{\text{after}} = \frac{2w_iw_j}{w_i + w_j}(\tilde{\tau}_i - \tilde{\tau}_j)^2 + 4\tau_i\tau_j + 2\kappa_{ij}\Big(\frac{\hat{\lambda}_0^2}{1 - \bar{a}} + \frac{\hat{\mu}_0^2}{1 - \bar{b}}\Big) \quad (\dagger).$$

Write $\tau_l = w_l\tilde{\tau}_l + e_l$. The part quadratic in $(\tilde{\tau}_i, \tilde{\tau}_j)$ is $2w_iw_j\big[(\tilde{\tau}_i - \tilde{\tau}_j)^2/(w_i + w_j) + 2\tilde{\tau}_i\tilde{\tau}_j\big]$, with eigenvalues $2w_iw_j$ and $2w_iw_j\big(2/(w_i + w_j) - 1\big)$. Both are positive, since $w_i + w_j \leq \bar{a} + \bar{b} < 2$. The minimum of the difference over $(\tilde{\tau}_i, \tilde{\tau}_j)$ is

$$2\kappa_{ij}\Big(\frac{\hat{\lambda}_0^2}{1 - \bar{a}} + \frac{\hat{\mu}_0^2}{1 - \bar{b}}\Big) - \frac{2(a_ib_j - a_jb_i)^2(\hat{\lambda}_0 - \hat{\mu}_0)^2}{w_iw_j(2 - w_i - w_j)} \quad (\dagger),$$

using $w_i e_j - w_j e_i = (a_i b_j - a_j b_i)(\hat\mu_0 - \hat\lambda_0)$. By Cauchy–Schwarz, $\frac{\hat\lambda_0^2}{1-\bar a} + \frac{\hat\mu_0^2}{1-\bar b} \geq \frac{(\hat\lambda_0-\hat\mu_0)^2}{2-\bar a-\bar b} \geq \frac{(\hat\lambda_0-\hat\mu_0)^2}{2-w_i-w_j}$. The difference is therefore at least $2(\hat\lambda_0-\hat\mu_0)^2\big[\kappa_{ij} w_i w_j - (a_i b_j - a_j b_i)^2\big] / \big(w_i w_j (2 - w_i - w_j)\big)$. This is nonnegative because

$$\kappa_{ij} w_i w_j - (a_i b_j - a_j b_i)^2 = a_i^2 a_j b_j + a_i a_j^2 b_i + 4 a_i a_j b_i b_j + a_i b_i b_j^2 + a_j b_i^2 b_j \quad (\dagger).$$

So merging never increases G. Merging all copies leaves one copy with data $(\bar a, \bar b)$, so that $w = \bar a + \bar b \in (0,2)$ and $\kappa = \bar a \bar b$. For it, with τ its variable,

$$G = \frac{2-w}{w}\Big(\tau + \frac{(\bar b - \bar a)(\hat\lambda_0 - \hat\mu_0)}{2-w}\Big)^2 + \frac{2(1-\bar a\bar b)\big((1-\bar b)\hat\lambda_0 + (1-\bar a)\hat\mu_0\big)^2}{(2-w)(1-\bar a)(1-\bar b)} \;\geq\; 0 \quad (\dagger).$$

Hence the form of Step 3 is nonnegative, and Theorem 4.7 follows. □