

AN UPPER BOUND FOR THE WEAK TYPE $(1, 1)$ CONSTANT OF THE CENTRED MAXIMAL OPERATOR OVER SQUARES

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ABSTRACT. Let c_2 be the best constant in the weak type $(1, 1)$ inequality for the centred Hardy–Littlewood maximal operator over axis-parallel squares in the plane. The covering argument gives $c_2 \leq 4$, and Lin proved $c_2 < 3.879$, with a proof formalized in Lean. We prove $c_2 \leq 3.59123$. The proof uses a comparison principle of Lin, in coordinates (u, v) in which squares become diamonds: if an even, nonnegative, integrable kernel K is at least 1 on the diamond $|u| + |v| \leq 1$ and $A_\alpha K \geq 0$ away from the origin, where $A_\alpha = -|\partial_u|^\alpha - |\partial_v|^\alpha$ and $0 < \alpha < 2$, then c_2 is at most half the integral of K . We prove this principle in detail, following Lin, and give closed forms for A_α applied to cubic B-splines and to $(|u| + |v|)^{-\alpha}$. With these, $A_\alpha K$ is explicit for kernels made of a truncation of $(|u| + |v|)^{-\alpha}$ and finitely many tensor products of cubic B-splines. We exhibit such a kernel for $\alpha = 47/40$ with B-spline spacing $1/32$, and two independent programs verify its properties in interval arithmetic. This improves the bound $c_2 < 3.615749$ that Lin obtained with a kernel of this type.

1. INTRODUCTION

For $f \in L^1(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ let

$$Mf(x) = \sup_{r>0} \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)| dy,$$

where $Q(x, r)$ is the closed square with centre x , sides parallel to the axes and side length $2r$, and $|E|$ is the Lebesgue measure of E . Let c_2 be the least constant such that

$$(1.1) \quad \lambda |\{x : Mf(x) \geq \lambda\}| \leq c_2 \|f\|_1 \quad (f \in L^1(\mathbb{R}^2), \lambda > 0).$$

The same constant is obtained with the strict level sets $\{Mf > \lambda\}$, since $\{Mf \geq \lambda\}$ is the decreasing intersection of the sets $\{Mf > \lambda'\}$, $\lambda' < \lambda$, which have finite measure.

The Vitali covering argument shows that $c_2 \leq 9$, and a refinement yields $c_2 \leq 4$ [13, Exercise 42]; both are stated there for balls, and the same arguments apply to squares. In dimension one the best constant is $(11 + \sqrt{61})/12$ [11]. No best constant is known in higher dimensions, and the constants for cubes tend to infinity with the dimension [2]. Aldaz’s lower bound $c_2 \geq 3/4 - \sqrt{2}/4 + \sqrt{6}/2 = 1.6211\dots$ [1] was improved to $c_2 \geq 1.6855\dots$ by Lin [8]. The constant c_2 is entry 47a of the list of optimization constants collected by Tao and others [14].

Lin [7] proved an upper bound for c_2 by comparison with a nonlocal operator, adapting the partial balayage method of Spector and Stockdale [12]. In coordinates (u, v) in which squares become diamonds, let $r = |u| + |v|$ and let $D = \{r \leq 1\}$ be the unit diamond. For $0 < \alpha < 2$ let $A_\alpha = -|\partial_u|^\alpha - |\partial_v|^\alpha$, the Fourier multiplier with symbol $-(|\xi_1|^\alpha + |\xi_2|^\alpha)$. Lin’s comparison principle states that $c_2 \leq \frac{1}{2} \int K$ for every even integrable kernel K with $K \geq \mathbf{1}_D$ (that is, $K \geq 0$, and $K \geq 1$ on D) and $A_\alpha K \geq 0$ on $\mathbb{R}^2 \setminus \{0\}$. With $\alpha = 1$ and the kernel $K = (R/r - 1)_+ / (R - 1) - \varepsilon(1 - r)_+^2$, $R = 3797/2000$, $\varepsilon = 159411/200000$, this yields $c_2 \leq R^2 / (R - 1) - \varepsilon/6 < 3.878624$, which Lin verified in Lean [8, theorem `CenteredMaximal.Cauchy.weakTypeConstant_two_le`].

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ratio]. Lin also reports a kernel for $\alpha = 6/5$, a truncated power $a(r^{-6/5} - R^{-6/5})_+$ plus a combination of cubic B-splines with spacing $1/16$, with $\frac{1}{2} \int K < 3.615749$ [7, 9]. As of [7], this bound had not been verified in Lean; [9] refers to exact certificates in an unpublished supplement. The coefficients, together with a Lean formalization of $c_2 < 452/125 = 3.616$, were briefly available in Lin's repository [10]. With these coefficients our first program (Section 5) confirms that Lin's kernel satisfies the hypotheses of Theorem 2.2 with $\frac{1}{2} \int K = 3.6157487\dots$, so $c_2 < 3.615749$.

Theorem 1.2. $c_2 \leq 3.59123$.

The proof applies the comparison principle to an explicit kernel in Lin's family,

$$(1.3) \quad K(u, v) = c \Psi(|u| + |v|) + S(u, v), \quad S(u, v) = \sum_{i,j} a_{ij} B\left(\frac{u}{h} - i\right) B\left(\frac{v}{h} - j\right),$$

where $\Psi(r) = (r^{-\alpha} - R^{-\alpha})_+$ and B is the centred cubic B-spline. We take $\alpha = 47/40$, $R = 33/20$ and $h = 1/32$ (Section 4).

Section 2 proves the comparison principle in detail, following [7], and writes out the extension to integrable functions (Lemma 2.5; see Remark 2.8). Section 3 computes $A_\alpha K$ in closed form. The one-dimensional fractional operator maps the cubic B-spline to a finite sum of powers $|s - k|^{3-\alpha}$ (Lemma 3.2), and A_α maps $r^{-\alpha}$ to a multiple of $r^{-2\alpha}$ (Lemma 3.4). The truncation of the power contributes one-dimensional integrals given by hypergeometric functions (Lemma 3.5). Section 4 describes the kernel and how it was found. Section 5 states the four inequalities that the proof needs and describes the two programs that verify them. Section 6 records how tight the certificate is and what the method yields for other parameters.

2. THE COMPARISON PRINCIPLE

2.1. Diamonds. Let $T(u, v) = (u + v, u - v)$. Then $|\det T| = 2$ and $T(D) = [-1, 1]^2$, since $\max(|u + v|, |u - v|) = |u| + |v|$. For $g \in L^1(\mathbb{R}^2)$ let

$$M_D g(z) = \sup_{s>0} \frac{1}{2s^2} \int_{z+sD} |g(w)| dw.$$

If $g = f \circ T$, then $\int_{z+sD} |g| = \frac{1}{2} \int_{Q(Tz,s)} |f|$ and $|z + sD| = 2s^2$, so $M_D g = (Mf) \circ T$. Since also $|\{M_D g > \lambda\}| = \frac{1}{2} |\{Mf > \lambda\}|$ and $\|g\|_1 = \frac{1}{2} \|f\|_1$, the best weak type $(1, 1)$ constant of M_D is c_2 .

2.2. The operator. For $0 < \alpha < 2$ and a function f on $\mathbb{R}^2$ let

$$(2.1) \quad Af(x) = \sum_{j=1}^2 \int_{\mathbb{R}} [f(x + te_j) + f(x - te_j) - 2f(x)] \frac{dt}{|t|^{1+\alpha}},$$

where e_1, e_2 are the coordinate vectors. We write $\hat{g}(\xi) = \int g(x) e^{-ix \cdot \xi} dx$, so that $\int g \bar{w} = (2\pi)^{-2} \int \hat{g} \bar{\hat{w}}$ for $g, w \in L^2(\mathbb{R}^2)$, and $\psi(\xi) = |\xi_1|^\alpha + |\xi_2|^\alpha$.

For $\phi \in C_c^\infty(\mathbb{R}^2)$ the bracket in (2.1) is at most $\min(t^2 \|\nabla^2 \phi\|_\infty, 4\|\phi\|_\infty)$ in absolute value, so the integral converges absolutely and $A\phi$ is bounded and continuous. Outside $\text{supp } \phi$ only the terms $\phi(x \pm te_j)$ remain, so $A\phi$ vanishes outside the union of $\text{supp } \phi$ and two strips of bounded width parallel to the axes, and it is $O(|x|^{-1-\alpha})$ along each strip. Hence $A\phi \in L^1 \cap L^\infty$, and by Fubini's theorem $\widehat{A\phi} = -d_\alpha \psi \hat{\phi}$, where

$$d_\alpha = \int_{\mathbb{R}} \frac{2(1 - \cos t)}{|t|^{1+\alpha}} dt = \frac{4\Gamma(1-\alpha) \cos(\pi\alpha/2)}{\alpha} > 0$$

(with $d_1 = 2\pi$). Thus $A = d_\alpha A_\alpha$ on $C_c^\infty(\mathbb{R}^2)$, where A_α is the Fourier multiplier with symbol $-\psi$. The sign conditions below do not depend on this normalization.

For $v \in L^1(\mathbb{R}^2)$ let

$$\mathcal{E}(v, v) = \sum_{j=1}^2 \iint [v(x + te_j) - v(x)]^2 \frac{dt dx}{|t|^{1+\alpha}} \in [0, \infty].$$

By Plancherel's identity (in $[0, \infty]$) for each t and Tonelli's theorem, $\mathcal{E}(v, v) = (2\pi)^{-2} d_\alpha \int \psi |\hat{v}|^2$. Let H be the space of $v \in L^1(\mathbb{R}^2)$ with $\mathcal{E}(v, v) < \infty$. Since $\hat{v}$ is bounded and $\psi(\xi) \geq |\xi|^\alpha$, every $v \in H$ has $\hat{v} \in L^2$, so $H \subset L^1 \cap L^2$. On H we use the bilinear form $\mathcal{E}(v, w) = (2\pi)^{-2} d_\alpha \int \psi \hat{v} \overline{\hat{w}}$. Then $C_c^\infty(\mathbb{R}^2) \subset H$, and $\mathcal{E}(v, \phi) = -\int v A\phi$ for $v \in H$ and $\phi \in C_c^\infty(\mathbb{R}^2)$ by Plancherel's identity. Since $|a_+ - b_+| \leq |a - b|$, $v_+ \in H$ for every $v \in H$.

For $K \in L^1(\mathbb{R}^2)$ we write $AK \geq 0$ on $\mathbb{R}^2 \setminus \{0\}$ if $\int K A\phi \geq 0$ for every nonnegative $\phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$.

Theorem 2.2 (Lin). *Let $0 < \alpha < 2$ and let $K \in L^1(\mathbb{R}^2)$ be even, with $K \geq 0$ almost everywhere, $K \geq 1$ almost everywhere on D , and $AK \geq 0$ on $\mathbb{R}^2 \setminus \{0\}$. Then $c_2 \leq \frac{1}{2} \int K$.*

This is [7, Theorem 1], where the first two conditions are written $K \geq \mathbf{1}_D$; see also [9, Principle 2 and Corollary 3.1]. We follow the proof in [7], which treats the transfer step (Lemma 2.5 below) for $f \in C_c^\infty$; the extension to integrable f is outlined in [9] and formalized in [8], and we write it out. For $s > 0$ let $K_s(x) = s^{-2} K(x/s)$, and for $f \geq 0$ let $M_K f = \sup_{s>0} K_s * f$.

Lemma 2.3 ([7, Lemma 1]). *If $K \geq 0$ almost everywhere and $K \geq 1$ almost everywhere on D , then $M_D g \leq \frac{1}{2} M_K |g|$.*

Proof. $K_s \geq s^{-2} \mathbf{1}_{sD}$ almost everywhere and $D = -D$, so $\frac{1}{2s^2} \int_{z+sD} |g| = \frac{1}{2s^2} \int_{sD} |g(z-w)| dw \leq \frac{1}{2} (K_s * |g|)(z)$. $\square$

Lemma 2.4 ([7, Lemma 2]). *Let $0 \leq f \in C_c^\infty(\mathbb{R}^2)$ and $\mu > 0$. There are $U \in H$ with $U \geq 0$ and a measurable σ with $0 \leq \sigma \leq \mu$ such that $\int U A\phi = \int (\sigma - f)\phi$ for every $\phi \in C_c^\infty(\mathbb{R}^2)$, $\sigma = \mu$ almost everywhere on $\Omega = \{U > 0\}$, and $\int \sigma = \int f$. In particular $\mu|\Omega| \leq \|f\|_1$.*

We quote this lemma and outline Lin's proof; the Lean formalization contains a complete proof (Remark 2.8). Lin states the lemma for A_α , and dividing his solution by d_α yields the form above. His proof, in the present normalization, minimizes $\frac{1}{2} \mathcal{E}(v, v) + \mu \int v - \int f v$ over nonnegative $v \in H$. The functional is coercive by the fractional Sobolev inequality [4], and weak compactness yields a minimizer U . Truncation in the variational inequality leads to the Lewy–Stampacchia bound $-f \leq AU \leq \mu - f$ in the sense of distributions. The function $\sigma = f + AU$ then satisfies $0 \leq \sigma \leq \mu$. Testing the Euler–Lagrange identity with U itself shows that $\int U(\mu - \sigma) = 0$, so $\sigma = \mu$ on Ω . Finally, $\int \sigma = \int f$ follows by testing with $\chi(\cdot/\rho)$ and using $\|A[\chi(\cdot/\rho)]\|_\infty = \rho^{-\alpha} \|A\chi\|_\infty$.

Lemma 2.5 (after [7, Lemma 3]). *Under the hypotheses of Theorem 2.2, every bounded, compactly supported, measurable $f \geq 0$ and every $\lambda > 0$ satisfy $|\{M_K f > \lambda\}| \leq \|K\|_1 \|f\|_1 / \lambda$.*

Proof. First let $f \in C_c^\infty(\mathbb{R}^2)$. Put $\mu = \lambda / \|K\|_1$ and take U, σ, Ω from Lemma 2.4. Fix $s > 0$. The kernel K_s is even, nonnegative and integrable with $\|K_s\|_1 = \|K\|_1$, and $AK_s \geq 0$ on $\mathbb{R}^2 \setminus \{0\}$, because $\int K_s A\phi = s^{-\alpha} \int K A[\phi(s \cdot)]$.

For $\phi \in C_c^\infty(\mathbb{R}^2)$ let $\Lambda\phi = K_s * A\phi$, a bounded, continuous, integrable function. We show that Λ satisfies the positive maximum principle. Let first $\phi \geq 0$ with $\phi(0) = 0$. Choose $\eta \in C_c^\infty(\mathbb{R}^2)$ with $0 \leq \eta \leq 1$ and $\eta = 1$ near 0, and put $g_\varepsilon = \eta(\cdot/\varepsilon)\phi$. Since ϕ has a minimum at 0,

$\|g_\varepsilon\|_\infty \leq C\varepsilon^2$ and $\|\nabla^2 g_\varepsilon\|_\infty \leq C$, hence $\|Ag_\varepsilon\|_\infty \leq C\varepsilon^{2-\alpha}$. Since K_s is even,

$$\Lambda\phi(0) = \int K_s A\phi = \int K_s A(\phi - g_\varepsilon) + \int K_s Ag_\varepsilon \geq -\|K\|_1 C\varepsilon^{2-\alpha},$$

because $\phi - g_\varepsilon \geq 0$ vanishes near 0. Hence $\Lambda\phi(0) \geq 0$, and by translation invariance $\Lambda\phi(x_0) \geq 0$ whenever $\phi \geq 0$ and $\phi(x_0) = 0$. If ϕ attains a nonnegative global maximum m at x_0 , apply this to $m\chi(\cdot/\rho) - \phi$, where $\chi \in C_c^\infty$ satisfies $0 \leq \chi \leq 1$ and $\chi = 1$ on the unit ball, and let $\rho \rightarrow \infty$, using $\|\Lambda[\chi(\cdot/\rho)]\|_\infty \leq \|K\|_1 \rho^{-\alpha} \|A\chi\|_\infty$. We obtain $\Lambda\phi(x_0) \leq 0$.

By Courrège's theorem [3],

$$\Lambda\phi = -\gamma\phi + b \cdot \nabla\phi + \text{tr}(\Theta\nabla^2\phi) + \int (\phi(\cdot + z) - \phi - \mathbf{1}_{|z|<1} z \cdot \nabla\phi) d\nu(z)$$

with $\gamma \geq 0$, a positive semidefinite matrix Θ and a Lévy measure ν . Testing with $\chi(\cdot/\rho)$ shows that $\gamma = 0$. Since K_s and A are even, Λ commutes with $\phi \mapsto \phi(-\cdot)$; after averaging the representation with its reflection we may take $b = 0$ and ν symmetric. We compare the Fourier transforms of the two expressions for $\Lambda\phi$, which Fubini's theorem permits since $\int \min(1, |z|^2) d\nu(z) < \infty$, and obtain

$$(2.6) \quad \xi \cdot \Theta \xi + \int (1 - \cos z \cdot \xi) d\nu(z) = d_\alpha \psi(\xi) \hat{K}_s(\xi) \quad (\xi \in \mathbb{R}^2).$$

The right side is at most $d_\alpha \|K\|_1 \psi(\xi) = O(|\xi|^\alpha)$, and the left side is at least $\xi \cdot \Theta \xi$, so $\Theta = 0$ because $\alpha < 2$. For $v, w \in H$ let $\mathcal{E}_\Lambda(v, w) = (2\pi)^{-2} d_\alpha \int \psi \hat{K}_s \hat{v} \hat{w}$. By (2.6) with $\Theta = 0$, Plancherel's identity for each z , Tonelli's theorem and polarization,

$$\mathcal{E}_\Lambda(v, w) = \frac{1}{2} \iint [v(x+z) - v(x)] [w(x+z) - w(x)] dx d\nu(z),$$

and $0 \leq \mathcal{E}_\Lambda(v, v) \leq \|K\|_1 \mathcal{E}(v, v)$.

Since $\sigma - f \in L^1 \cap L^\infty$, the function $\Lambda U := K_s * (\sigma - f)$ is bounded and lies in L^2 . By the equation of Lemma 2.4 and Plancherel's identity, the tempered, locally integrable function $g = \hat{\sigma} - \hat{f} + d_\alpha \psi \hat{U}$ satisfies $\int g \hat{\phi} = 0$ for every $\phi \in C_c^\infty$. The functions $\hat{\phi}$ are dense in the Schwartz space, so $g = 0$, that is, $\hat{\sigma} - \hat{f} = -d_\alpha \psi \hat{U}$. Hence $\widehat{\Lambda U} = -d_\alpha \psi \hat{K}_s \hat{U}$; likewise $\widehat{\Lambda \phi} = -d_\alpha \psi \hat{K}_s \hat{\phi}$ for $\phi \in C_c^\infty$. Therefore, by Plancherel's identity,

$$(2.7) \quad -\int (\Lambda U) w = \mathcal{E}_\Lambda(U, w), \quad -\int (\Lambda \phi) w = \mathcal{E}_\Lambda(\phi, w) \quad (w \in H).$$

Let $0 \leq \phi \in C_c^\infty$, $\varepsilon > 0$ and $W = (\varepsilon\phi - U)_+ \in H$. Applying the pointwise inequality $(a-b)(a_+ - b_+) \geq (a_+ - b_+)^2$ inside the double integral for $\mathcal{E}_\Lambda$, we find $\mathcal{E}_\Lambda(U - \varepsilon\phi, W) \leq -\mathcal{E}_\Lambda(W, W) \leq 0$. Combined with (2.7), this yields

$$-\int (\Lambda U) \frac{W}{\varepsilon} = \frac{1}{\varepsilon} \mathcal{E}_\Lambda(U, W) \leq \mathcal{E}_\Lambda(\phi, W) = -\int (\Lambda \phi) W \leq \varepsilon \|\Lambda \phi\|_\infty \int \phi.$$

As $\varepsilon \rightarrow 0$, $W/\varepsilon \rightarrow \phi \mathbf{1}_{\{U=0\}}$ with $0 \leq W/\varepsilon \leq \phi$, so $\int_{\{U=0\}} (\Lambda U) \phi \geq 0$. Hence $\Lambda U \geq 0$ almost everywhere on $\{U = 0\}$, and there

$$K_s * f = K_s * \sigma - \Lambda U \leq K_s * \sigma \leq \mu \|K_s\|_1 = \lambda.$$

The second inequality uses $K_s \geq 0$ and $\sigma \leq \mu$.

For each x , $s \mapsto K_s * f(x)$ is continuous, since $s \mapsto K_s$ is continuous in L^1 . Taking the inequality for all rational s outside a common null set, we get $M_K f \leq \lambda$ almost everywhere on $\{U = 0\}$, so $|\{M_K f > \lambda\}| \leq |\Omega| \leq \|f\|_1/\mu = \|K\|_1 \|f\|_1/\lambda$.

Now let $f \geq 0$ be bounded, measurable and compactly supported, and let $f_\varepsilon = f * \rho_\varepsilon$ for a mollifier ρ_ε . Then $\|f_\varepsilon\|_1 = \|f\|_1$. Each $K_s * f$ is continuous, being the convolution of an integrable function with a bounded one, so $K_s * f_\varepsilon = (K_s * f) * \rho_\varepsilon \rightarrow K_s * f$ everywhere. Hence

$M_K f \leq \liminf_{\varepsilon \rightarrow 0} M_K f_\varepsilon$, every point of $\{M_K f > \lambda\}$ lies in $\{M_K f_\varepsilon > \lambda\}$ for all small ε , and $|\{M_K f > \lambda\}| \leq \liminf_{\varepsilon} |\{M_K f_\varepsilon > \lambda\}| \leq \|K\|_1 \|f\|_1 / \lambda$. $\square$

Proof of Theorem 2.2. Let $g \in L^1(\mathbb{R}^2)$ and $f_n = \min(|g|, n) \mathbf{1}_{\{|x| \leq n\}}$. Since $K \geq 0$, $K_s * f_n \uparrow K_s * |g|$ for each s by monotone convergence, hence $M_K f_n \uparrow M_K |g|$ and $\{M_K |g| > \lambda\} = \bigcup_n \{M_K f_n > \lambda\}$. By Lemma 2.5, $|\{M_K |g| > \lambda\}| \leq \|K\|_1 \|g\|_1 / \lambda$. By Lemma 2.3, $|\{M_D g > \lambda\}| \leq |\{M_K |g| > 2\lambda\}| \leq \frac{1}{2} \|K\|_1 \|g\|_1 / \lambda$, so the weak type constant of M_D is at most $\frac{1}{2} \|K\|_1 = \frac{1}{2} \int K$. It equals c_2 . $\square$

Remark 2.8. The Lean formalization [8] proves Lemma 2.4 for bounded, measurable, compactly supported $f \geq 0$ and every $0 < \alpha < 2$ (`exists_obstacle_solution`; all theorem names in this remark are in the namespace `CenteredMaximal`), and the passage from such f to integrable f (`isKernelWeakTypeBound_of_pointwise`). Its version of the transfer step (`convolution_le_of_eq_zero`) assumes that K has compact support and that AK is a finite measure minus a multiple of the point mass at 0. The kernel (1.3) with $\alpha > 1$ is outside that class, because $Ar^{-\alpha}$ is not integrable near the origin (Lemma 3.4).

3. THE OPERATOR ON THE KERNEL

Throughout this section $1 < \alpha < 2$. The kernel (1.3) is continuous on $\mathbb{R}^2 \setminus \{0\}$, and, for $c > 0$, $K(x) \rightarrow \infty$ as $x \rightarrow 0$. We first show that the pointwise values of AK determine the sign condition of Theorem 2.2.

Lemma 3.1. *Let K be given by (1.3) with finitely many nonzero a_{ij} . For every $x = (u, v)$ with $uv \neq 0$ and $|u| + |v| \neq R$ the integral (2.1) for $AK(x)$ converges absolutely, and $\int K A\phi = \int (AK)\phi$ for every $\phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$.*

Proof. Let $G(x) = \sum_j \int |K(x + te_j) + K(x - te_j) - 2K(x)| |t|^{-1-\alpha} dt \in [0, \infty]$. At a point x as in the statement, K is C^2 near x and bounded on the two lines $x + \mathbb{R}e_j$, which do not pass through the origin, so $G(x) < \infty$. We show that G is locally integrable on $\mathbb{R}^2 \setminus \{0\}$. The part $|t| \geq 1$, integrated over a compact set E , is at most $C(\|K\|_1 + \int_E |K|)$, since $K \in L^1$. For $|t| < 1$, the spline part S is C^2 with bounded second derivatives, so its second differences are $O(t^2)$. The function $\Psi(|u| + |v|)$ is C^2 off the axes, the curve $r = R$ and the origin. Near a point at distance δ from the axes or from $r = R$ and away from the origin, its second difference along e_j is $O(t^2)$ for $|t| < \delta$ and $O(|t|)$ for $|t| \geq \delta$ (segments that pass near the origin are treated next), so its contribution to G is $O(\delta^{1-\alpha})$, which is integrable because $\alpha < 2$. Finally, if the line $x + \mathbb{R}e_j$ passes at distance δ from the origin, the singularity $(|\tau| + \delta)^{-\alpha}$ of Ψ on it contributes $O(\delta^{1-\alpha} |x|^{-1-\alpha})$, again integrable near the axes. Hence $G \in L^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\})$.

For $\varepsilon > 0$ let A_ε denote (2.1) restricted to $|t| > \varepsilon$. By Fubini's theorem and the substitutions $x \mapsto x \mp te_j$, $\int K A_\varepsilon \phi = \int (A_\varepsilon K) \phi$. As $\varepsilon \rightarrow 0$ the left side tends to $\int K A\phi$ by dominated convergence, since the bracket for ϕ is $O(\min(t^2, 1))$ and $K \in L^1$. The right side tends to $\int (AK)\phi$ by dominated convergence with majorant $G|\phi|$. $\square$

3.1. B-splines. Let $B(x) = \frac{1}{12} \sum_{k=-2}^2 w_k |x - k|^3$ with $(w_{-2}, \dots, w_2) = (1, -4, 6, -4, 1)$. This is the centred cubic B-spline: B is C^2 , supported in $[-2, 2]$, with $B(x) = (4 - 6x^2 + 3|x|^3)/6$ for $|x| \leq 1$, $B(x) = (2 - |x|)^3/6$ for $1 \leq |x| \leq 2$, and $\int B = 1$. For a function b on $\mathbb{R}$ let $L_\alpha b(s) = \int [b(s+t) + b(s-t) - 2b(s)] |t|^{-1-\alpha} dt$.

Lemma 3.2. $L_\alpha B(s) = \kappa \sum_{k=-2}^2 w_k |s - k|^{3-\alpha}$ with $\kappa = -2/(\alpha(1-\alpha)(2-\alpha)(3-\alpha))$.

Proof. With $\hat{b}(\xi) = \int b(x) e^{-ix\xi} dx$ we have $\hat{B}(\xi) = (2 - 2\cos\xi)^2/\xi^4$ and $(2 - 2\cos\xi)^2 = \sum_k w_k e^{-ik\xi}$, and $\widehat{L_\alpha B}(\xi) = -d_\alpha |\xi|^\alpha \hat{B}(\xi)$ with d_α as in Section 2. For $1 < q < 2$ the Fourier

transform of the tempered distribution $|x|^q$ is the homogeneous distribution $c_q|\xi|^{-1-q}$, $c_q = -2\Gamma(1+q)\sin(\pi q/2)$, defined by analytic continuation in q . Its product with the smooth function $\sum_k w_k e^{-ik\xi}$, which vanishes to order 4 at 0, is the locally integrable function $c_q|\xi|^{-1-q}(2-2\cos\xi)^2$, because the pairing with this product converges absolutely for $-1-q > -5$. Hence, with $q = 3-\alpha$, the Fourier transform of $\sum_k w_k|x-k|^{3-\alpha}$ is $c_{3-\alpha}|\xi|^{\alpha-4}(2-2\cos\xi)^2$, and $\widehat{L_\alpha B}$ is $-d_\alpha/c_{3-\alpha}$ times it. Both $L_\alpha B$ and $\sum_k w_k|x-k|^{3-\alpha}$ are continuous and bounded, so they agree up to this factor. Finally $-d_\alpha/c_{3-\alpha} = -2\Gamma(1-\alpha)/(\alpha\Gamma(4-\alpha)) = \kappa$. $\square$

Consequently, since only finitely many a_{ij} are nonzero,

$$(3.3) \quad AS(u, v) = h^{-\alpha} \kappa \sum_m \left[\left| \frac{u}{h} - m \right|^{3-\alpha} P_m\left(\frac{v}{h}\right) + \left| \frac{v}{h} - m \right|^{3-\alpha} P'_m\left(\frac{u}{h}\right) \right],$$

where $P_m(y) = \sum_j (\sum_k w_k a_{m-k,j}) B(y-j)$ and $P'_m(y) = \sum_i (\sum_k w_k a_{i,m-k}) B(y-i)$.

3.2. Powers of the diamond norm.

Lemma 3.4. *Let $r = |u| + |v|$ and*

$$C_\alpha = -\frac{8\Gamma(1-\alpha)\Gamma(2\alpha)\cos^2(\pi\alpha/2)}{\Gamma(1+\alpha)} > 0.$$

Then $\int r^{-\alpha} A\phi = C_\alpha \int r^{-2\alpha} \phi$ for every $\phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$. For every (u, v) with $uv \neq 0$ the integral (2.1) for $Ar^{-\alpha}$ converges absolutely and equals $C_\alpha r^{-2\alpha}$.

Proof. For $0 < \operatorname{Re} s < 2$ the function $\widehat{f_s} = r^{-s}$ is locally integrable and tempered, and $r^{-s} = \Gamma(s)^{-1} \int_0^\infty \lambda^{s-1} e^{-\lambda|u|} e^{-\lambda|v|} d\lambda$. Since $e^{-\lambda|x|} = 2\lambda/(\lambda^2 + \xi^2)$, Fubini's theorem shows that $\hat{f}_s = m_s$, where

$$m_s(\xi) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{4\lambda^{s+1} d\lambda}{(\lambda^2 + \xi_1^2)(\lambda^2 + \xi_2^2)} = k_s \frac{|\xi_1|^s - |\xi_2|^s}{\xi_1^2 - \xi_2^2}, \quad k_s = \frac{2\pi}{\Gamma(s)\sin(\pi s/2)},$$

by partial fractions. Let $\phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ be real. Since $f_\alpha \in L^1 + L^2$ and $A\phi \in L^1 \cap L^\infty$ with $\widehat{A\phi} = -d_\alpha \psi \hat{\phi} \in L^1$, Parseval's identity and $(|\xi_1|^\alpha + |\xi_2|^\alpha)(|\xi_1|^\alpha - |\xi_2|^\alpha) = |\xi_1|^{2\alpha} - |\xi_2|^{2\alpha}$ give

$$\int f_\alpha A\phi = -\frac{d_\alpha k_\alpha}{(2\pi)^2} \int n(\xi) \overline{\hat{\phi}(\xi)} d\xi, \quad n(\xi) = \frac{|\xi_1|^{2\alpha} - |\xi_2|^{2\alpha}}{\xi_1^2 - \xi_2^2}.$$

The function $F(s) = \int r^{-s} \phi$ is entire. The function $G(s) = (2\pi)^{-2} \int m_s \overline{\hat{\phi}}$ is analytic on $0 < \operatorname{Re} s < 4$, apart from a possible pole of k_s at $s = 2$. It is removable, because $(|\xi_1|^s - |\xi_2|^s)/(\xi_1^2 - \xi_2^2)$ equals 1 at $s = 2$ and $\int \hat{\phi} = (2\pi)^2 \phi(0) = 0$. Since $F = G$ on $0 < \operatorname{Re} s < 2$, they agree on $0 < \operatorname{Re} s < 4$, and at $s = 2\alpha \in (2, 4)$ we get $\int r^{-2\alpha} \phi = k_{2\alpha} (2\pi)^{-2} \int n \overline{\hat{\phi}}$. Hence $\int r^{-\alpha} A\phi = C_\alpha \int r^{-2\alpha} \phi$ with $C_\alpha = -d_\alpha k_\alpha / k_{2\alpha}$, which simplifies to the stated formula. For $1 < \alpha < 2$, $\Gamma(1-\alpha) < 0$, so $C_\alpha > 0$.

Let $x = (u, v)$ with $uv \neq 0$. Then $r^{-\alpha}$ is smooth near x and at most $\min(|u|, |v|)^{-\alpha}$ on the lines $x + \mathbb{R}e_j$, so (2.1) converges absolutely, locally uniformly in x , and $Ar^{-\alpha}$ is continuous on $\{uv \neq 0\}$. For $\phi \in C_c^\infty(\{uv \neq 0\})$ the argument in the second paragraph of the proof of Lemma 3.1 shows that $\int r^{-\alpha} A\phi = \int (Ar^{-\alpha})\phi$; on the left, dominated convergence applies because the bracket for ϕ vanishes unless x lies in $\operatorname{supp} \phi$ or in one of two strips that avoid a neighbourhood of the origin, where $r^{-\alpha}$ is bounded. Hence $Ar^{-\alpha} = C_\alpha r^{-2\alpha}$ on $\{uv \neq 0\}$. $\square$

The constants of Lemmas 3.2 and 3.4 are $\kappa = 6.4600705\dots$ and $C_\alpha = 4.2391860\dots$ for $\alpha = 47/40$.

3.3. The truncated power. Let $R > 0$ and $\Psi(r) = (r^{-\alpha} - R^{-\alpha})_+$. By the symmetries of A it suffices to consider $u, v \geq 0$.

Lemma 3.5. *For $u, v > 0$ with $r = u + v < R$,*

$$A\Psi(|u| + |v|) = C_\alpha r^{-2\alpha} + 4I(R - r) + 2I(R + u - v) + 2I(R + v - u),$$

where, for $0 < d < \infty$,

$$I(d) = \int_0^\infty [R^{-\alpha} - (R + s)^{-\alpha}](d + s)^{-1-\alpha} ds = \frac{R^{-\alpha} d^{-\alpha}}{\alpha} - \frac{d^{-2\alpha}}{2\alpha} {}_2F_1\left(\alpha, 2\alpha; 2\alpha + 1; -\frac{R - d}{d}\right).$$

Proof. Write $\Psi(\rho) = \rho^{-\alpha} - R^{-\alpha} + \tau(\rho)$ with $\tau(\rho) = (R^{-\alpha} - \rho^{-\alpha})_+$. Constants are annihilated by A , so by Lemma 3.4, $A\Psi = C_\alpha r^{-2\alpha} + A[\tau(|u| + |v|)]$. Since τ vanishes on $\{\rho \leq R\}$, which contains $x = (u, v)$, we have $A[\tau](x) = \sum_j \int \tau(|x + te_j|_1) 2|t|^{-1-\alpha} dt$, where $|\cdot|_1$ is the diamond norm, after combining $\pm t$. On the line $x + te_1$ we have $|x + te_1|_1 = r + t$ for $t \geq 0$ and $= -t - u + v$ for $t \leq -u$. Thus $\tau > 0$ exactly for $t > R - r$ and for $t < -(R + u - v)$, and the two pieces contribute $2I(R - r)$ and $2I(R + u - v)$. The line $x + te_2$ contributes $2I(R - r)$ and $2I(R + v - u)$. The closed form follows from $\int_0^\infty (R + s)^{-\alpha} (d + s)^{-1-\alpha} ds = d^{-2\alpha} (2\alpha)^{-1} {}_2F_1(\alpha, 2\alpha; 2\alpha + 1; 1 - R/d)$. $\square$

Lemma 3.6. *For $u, v > 0$ with $u + v > R$,*

$$A\Psi(|u| + |v|) = 2J(u, v) + 2J(v, u), \quad J(x, y) = \int_{y-R}^{R-y} \Psi(|t| + y) |x - t|^{-1-\alpha} dt \quad (y < R),$$

with $J(x, y) = 0$ for $y \geq R$. On $\{u, v > 0, u + v > R\}$ the function $A\Psi$ is nonnegative and nonincreasing in u and in v .

Proof. Ψ vanishes at x , so $A\Psi(x) = \sum_j \int \Psi(|x + te_j|_1) 2|t|^{-1-\alpha} dt$. On the line $x + te_1$, parametrized by $s = u + t$, the integrand is $\Psi(|s| + v) |s - u|^{-1-\alpha}$, supported in $|s| < R - v < u$. As u increases, $|s - u|^{-1-\alpha}$ decreases for every s in the support. On the line $x + te_2$ the integrand is $\Psi(u + |v + t|) |t|^{-1-\alpha}$, which is nonincreasing in u . The same holds with u and v exchanged. $\square$

By Lemmas 3.2–3.6, $AK = cA\Psi + AS$ is explicit off the axes and off the curve $r = R$: a finite sum of powers $|u/h - m|^{3-\alpha}$ and $|v/h - m|^{3-\alpha}$ times piecewise cubic polynomials, plus $cA\Psi$, given by elementary and hypergeometric functions inside $r < R$ and by one-dimensional integrals of elementary functions outside.

4. THE KERNEL

The kernel (1.3) has the parameters $\alpha = 47/40$, $R = 33/20$, $h = 1/32$ and $c = 1977601376167/1099511627776 (= 1.7986179\dots)$. The coefficients satisfy $a_{ij} = a_{|i|, |j|} = a_{ji}$ and $a_{ij} = 0$ unless $\max(|i|, |j|) \leq 70$. There are 1,181 nonzero values a_{ij} with $0 \leq j \leq i$, all dyadic rationals with denominators dividing 2^{40} , and they are listed in the certificate file. Thus K has the symmetries of the square, and S vanishes outside $\max(|u|, |v|) < 9/4$. Since $\int B = 1$ and $\int_{\mathbb{R}^2} \Psi(|u| + |v|) du dv = \int_0^R (\rho^{-\alpha} - R^{-\alpha}) 4\rho d\rho$,

$$(4.1) \quad \int K = c \frac{2\alpha R^{2-\alpha}}{2-\alpha} + h^2 \sum_{i,j} a_{ij}.$$

For our kernel, $\frac{1}{2} \int K = 3.5912243275064523\dots$, of which the singular part contributes $3.8721047\dots$ and the spline part $-0.2808804\dots$; the spline part is negative near the origin, where it offsets part of the singular term.

The kernel was found by linear programming. For fixed α , R and h , the conditions $K \geq 0$, $K \geq 1$ on D and $AK \geq 0$ are linear in (c, a_{ij}) , and so is $\int K$; minimizing $\int K$ subject to these conditions at finitely many points is a linear program, which we solved with HiGHS [5].

Constraints imposed at finitely many points are not enough, because the optimal solutions have $AK = 0$ on large sets and turn negative between the sample points. We therefore added violated constraints from successively finer grids, required AK to exceed a small margin, and refined the grid around the points where the verifier of Section 5 failed. The coefficients were then rounded to the dyadic grid. Only the certificate enters the proof, so the search programs need not be trusted.

5. VERIFICATION

By Theorem 2.2 and Lemma 3.1, Theorem 1.2 follows from four statements about the kernel of Section 4:

- (C1) $K \geq 0$ on $\mathbb{R}^2 \setminus \{0\}$;
- (C2) $K \geq 1$ on $D \setminus \{0\}$;
- (C3) $\frac{1}{2} \int K \leq 3.59123$;
- (C4) $AK(x) \geq 0$ for almost every $x \neq 0$.

K is even, and $K \in L^1$ because $\alpha < 2$ and S has compact support. By the symmetries of K and of A it suffices to verify (C1), (C2) and (C4) on $\{0 \leq v \leq u\}$. Where K vanishes, (C4) follows from (C1), because then $AK(x) = \sum_j \int [K(x + te_j) + K(x - te_j)]|t|^{-1-\alpha} dt$.

(C3) is the evaluation of (4.1) in ball arithmetic. For (C1) and (C2), on each cell of the grid $h\mathbb{Z}^2$ in $\{0 \leq v \leq u\}$ the spline part is a polynomial of degree 3 in each variable, and the least of its Bernstein coefficients, computed exactly, is a lower bound. The singular part $c\Psi(u+v)$ is convex and nonincreasing in $r = u+v$ on $u, v \geq 0$, so on a cell it is bounded below by its value at the largest r , by a tangent line in r , or by a Taylor polynomial with remainder; these bounds are polynomials in (u, v) and are added to the Bernstein net. Cells cut by the line $u+v=1$ are split, and failing cells are subdivided.

For (C4), the cells of $h\mathbb{Z}^2$ in $\{0 \leq v \leq u\}$ on which K is not identically zero are divided into sub-cells, uniformly in the first program and more finely toward the knot lines in the second. On a sub-cell, AS is given by (3.3). Each power $|u/h - m|^{3-\alpha}$ is replaced by a Taylor polynomial with an explicit remainder; for the two knot lines that bound the knot cell, near which the power is not smooth, it is bounded by its chord together with a tangent line or an explicit error term. The term $cA\Psi$ is bounded below either by a Taylor model with remainder or by termwise bounds at the corners of the sub-cell, which use that $C_\alpha r^{-2\alpha}$ and I are decreasing and, outside the curve $r = R$, Lemma 3.6. At some points the two terms $cA\Psi$ and AS are large and nearly cancel; there only a joint polynomial model of their sum, bounded through its Bernstein coefficients, proves positivity. The closed forms of Section 3 hold at every point of a sub-cell that lies off the axes and off the curve $r = R$; on sub-cells that meet the curve $r = R$ only the monotone bounds are used. A sub-cell passes if the resulting lower bound is nonnegative; otherwise it is quadrisected. Near the origin the term $cC_\alpha r^{-2\alpha} \geq 0$ of Lemma 3.5 dominates, and the same bounds apply to the sub-cells that contain the origin.

Two programs, written independently from a common definition of the kernel and of claims (C1)–(C4), carry out these checks with the Arb library [6] through python-flint, and both accept the certificate. The second program represents the spline part with exact rational Bernstein nets and uses polynomial models of degree at most (3, 3) for AK . It was written from the definitions and the closed forms of Section 3 with their proofs, without using the code of the first program, which was written together with the search.¹ Both programs rely on Arb's evaluation of ${}_2F_1$, also

¹As a check, the second program rejects four perturbed certificates: three in which one coefficient class is changed so that $K < 0$ at a known point, $K < 1$ at a known point of D , or $AK < 0$ at a known point, and one that claims a bound below $\frac{1}{2} \int K$. The enclosures used by each program, and the checks of its input, are documented in the repository listed under Data availability.

at arguments below -1 . Rigorous quadrature of the definition (2.1) agrees with the closed forms of Section 3 at several sample points, including the smallest values of AK found by sampling and by local search. At those points, this checks Arb's evaluation of ${}_2F_1$. Table 1 summarizes the runs.

TABLE 1. Verification of claims (C1)–(C4) for the kernel of Section 4. Times are wall-clock times with 16 processes on a shared 384-thread x86-64 server.

	first program	second program
(C3) $\frac{1}{2} \int K$	3.5912243275064523...	3.5912243275064523...
(C1), (C2) knot cells	2,628	1,368 (C1), 272 (C2)
least lower bound for K on D	–	1.0000993
(C4) cells	219,732 and 415,236	443,754 in 1,368 knot cells
least lower bound for AK	$1.3 \cdot 10^{-8}$	$3.6 \cdot 10^{-5}$
time	68 s	33 s

First program: two runs, with coarser and finer subdivisions; the (C4) counts include the cells that were split, and the time is that of the first run. Second program: the (C4) count is the number of sub-cells that passed, and the least lower bound for AK comes from a run that tries Taylor models on every sub-cell.

Proof of Theorem 1.2. The kernel of Section 4 satisfies (C1)–(C4). It is even and integrable. By (C4) and Lemma 3.1, $AK \geq 0$ on $\mathbb{R}^2 \setminus \{0\}$ in the sense of Section 2. Theorem 2.2 now yields $c_2 \leq \frac{1}{2} \int K \leq 3.59123$. $\square$

6. REMARKS

Point evaluations of AK on 33×33 grids in all knot cells of $\{0 \leq v \leq u\}$ on which K is not identically zero give a least value of $1.2 \cdot 10^{-3}$, so the margins of the certificate are small. On 65×65 grids in the knot cells $[ih, (i+1)h] \times [jh, (j+1)h]$ with $1.6 \leq (i+j)h \leq 2.35$ the least value is $1.1 \cdot 10^{-3}$, at (1.5415, 0.1328), and a local search from there finds $AK = 2.8 \cdot 10^{-4}$ at (1.5417, 0.1326), in a dip of width about 2^{-12} just outside the curve $r = R$; quadrature of the definition (2.1) confirms this value. The linear program imposed positive margins on AK only at its sample points, and AK is smaller between them. The lower bounds in Table 1 certify the positivity of AK ; they do not certify these sampled values.

To see the role of the exponent, note that for $\alpha = 1$ the function r^{-1} is a multiple of the Green function of A , and the kernel $(R/r - 1)_+/(R - 1)$ has $\frac{1}{2} \int K = R^2/(R - 1)$, which is minimal for $R = 2$, where it equals the covering bound 4. For $1 < \alpha < 2$, Lemma 3.4 shows that $r^{-\alpha}$ is strictly subharmonic for A off the origin, which the spline part can trade against $\int K$. For a singular part $r^{-\beta}$ with $1 < \beta < 2$, at points (u, v) with small $|v|$ the kink of $r^{-\beta}$ across the axis contributes a negative term of order $|v|^{1-\alpha}$ to $A r^{-\beta}$, and the lines passing near the origin contribute a positive term of order $|v|^{1-\beta}$. For $\beta = \alpha$ the two leading terms cancel, consistent with Lemma 3.4; for $\beta < \alpha$, $A r^{-\beta}$ tends to $-\infty$ at the axes. Both of Lin's kernels and ours use $\beta = \alpha$. In our searches with $h = 1/16$, the linear programs gave the smallest values for α between 1.15 and 1.2 and R near 1.65. For Lin's parameters $\alpha = 6/5$, $R = 7/4$ and $h = 1/16$ they yield about 3.599 with constraints at finitely many points; this is the value of a relaxation rather than a bound, and it is consistent with Lin's reported 3.615749.

DATA AVAILABILITY

The certificate, both verification programs, the search programs and the logs are archived at doi:10.5281/zenodo.23522057. They are also available in the repository <https://github.com/tejasvi/square-maximal-weak-type>.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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