

Area law for three-dimensional U(1) lattice gauge theory with the Wilson action at every coupling

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Abstract

We prove that three-dimensional U(1) lattice gauge theory with the Wilson action confines static charges at every coupling. More precisely, for each $\beta > 0$ there are $\sigma(\beta) > 0$ and $R_1(\beta) < \infty$ such that rectangular Wilson loops of sides $R, T \geq R_1(\beta)$ satisfy $\langle \cos \theta_C \rangle \leq \exp(-\sigma(\beta)RT + c(R + T))$, uniformly in the finite box, with an absolute constant c and an explicit lower bound on $\sigma(\beta)$. For the Villain action this is the theorem of Göpfert and Mack. For the Wilson action it is cited as known but has remained open. The proof rests on a comparison. The Wilson plaquette weight is a mixture of Villain weights whose coupling has a very light upper tail. Hence, by Ginibre's inequality, the Wilson loop at coupling β is at most an average, over a sparse Bernoulli set of frozen plaquettes, of the Villain loop at coupling $K\beta$, and frozen plaquettes are insulating walls for the monopole gas. A variational regularization (built on the smallest minimizer of a cut functional) freezes more plaquettes, which only increases the loop, and produces slit domains on which the multiscale analysis of Göpfert and Mack runs with uniform constants, with wall fields graded by their periods around the handles. On these domains the loop decays like the exponential of the minimal spanning area with frozen plaquettes free. A Peierls estimate shows that this area is at least $(1 - \eta)RT$ outside an event of probability at most $e^{-RT + O(R+T)}$, with $\eta \rightarrow 0$ as $K \rightarrow \infty$. We carry over the steps cited from Brydges–Federbush and Göpfert–Mack with explicit changes, and we prove in full a tree-graph counting step of Brydges whose printed deduction is incomplete. As corollaries, we obtain the area law at every coupling for all closed subgroups of U(n) containing the scalars, also with a Higgs field blind to the centre (as for U(2) with an adjoint Higgs field), and for the Georgi–Glashow model at infinite Higgs coupling.

Keywords lattice gauge theory, confinement, Wilson loop, area law, Coulomb gas, Debye screening, Villain action

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1 Introduction

1.1 The model and the main result

Three-dimensional U(1) lattice gauge theory is the simplest gauge theory in which static charges are expected to be confined at every coupling. By Polyakov's mechanism [Pol77, BMK77], magnetic monopoles form a plasma that screens the magnetic field at the Debye length and

squeezes the electric flux between two static charges into a tube of finite tension. G6pfert and Mack proved this for the Villain action [GM82]. In this paper we prove it for the Wilson action.

Definition 1.1 (Wilson and Villain measures; Wilson loop). A finite box $\Lambda \subset \mathbb{Z}^3$ is a cell complex with vertex set Λ , bonds (oriented edges) e and plaquettes p (free boundary conditions throughout); for $\theta \in (\mathbb{R}/2\pi\mathbb{Z})^{E(\Lambda)}$ and an oriented plaquette p , θ_p is the sum of θ_e around ∂p .

1. The *Wilson measure* at inverse coupling $\beta > 0$ and the *Villain weight* at coupling $\tau > 0$ are

$$\mu_{\Lambda,\beta}^W(d\theta) = \frac{1}{Z_{\Lambda,\beta}^W} \prod_p e^{\beta \cos \theta_p} \prod_e d\theta_e, \quad \tilde{v}_\tau(\theta) \propto \sum_{m \in \mathbb{Z}} e^{-\tau(\theta - 2\pi m)^2/2}; \quad (1.1)$$

the Villain measure $\mu_{\Lambda,\tau}^V$ is $\mu_{\Lambda,\beta}^W$ with $e^{\beta \cos \theta_p}$ replaced by $\tilde{v}_\tau(\theta_p)$.

2. For a rectangle $C = C_{R,T}$ of sides $R, T \in \mathbb{N}$ in a coordinate plane, an oriented closed lattice path, $\theta_C := \sum_{e \in C} \theta_e$, and the Wilson loop expectation is $\langle \cos \theta_C \rangle_{\Lambda,\beta}^W$.
3. The model has an *area law at coupling* β if there are $\sigma > 0$, $c < \infty$ and $R_1 < \infty$ such that $\langle \cos \theta_C \rangle_{\Lambda,\beta}^W \leq \exp(-\sigma RT + c(R + T))$ for every rectangle C with $\min(R, T) \geq R_1$ and every finite box $\Lambda \supseteq C$.

This area law, uniform in the box and with an explicit perimeter term, is stronger than positivity of the iterated-limit string tension α of [GM82, (1.7)], whose finite-loop estimate carries the loss term P_F of [GM82, (7.7)]. The area law passes to the infinite-volume limit, which exists (Theorem 2.1(c)).

Theorem 1.2 (Main theorem). *For every $\beta > 0$ there are $\sigma(\beta) > 0$ and $R_1(\beta) < \infty$ such that for every rectangle C of sides R, T with $\min(R, T) \geq R_1(\beta)$ and every finite box $\Lambda \supseteq C$,*

$$\langle \cos \theta_C \rangle_{\Lambda,\beta}^W \leq \exp(-\sigma(\beta)RT + c(R + T)), \quad (1.2)$$

where c is an absolute constant. Moreover, for every $K > K_* := 377.38$ there is $\beta_0(K) < \infty$ such that, for every $\beta \geq \beta_0(K)$, $\sigma(\beta)$ and $R_1(\beta)$ can be chosen, depending on K , so that (1.2) holds and

$$\sigma(\beta) \geq 1.35 \cdot 10^{-4} c_g c_L (K\beta)^{-1/2} e^{-2.49436 K\beta}, \quad (1.3)$$

where c_g, c_L are the fixed regularization constants of Section 8.2; for $\beta < \beta_0(K)$, (1.2) then holds with $\sigma(\beta)$ and $R_1(\beta)$ equal to the values so chosen at $\beta_0(K)$.

The first assertion follows from the second with one fixed $K > K_*$. With $c_g = 1/4$ and $c_L = 1.6 \cdot 10^{-19}$ (Section S1.1.1), (1.3) reads $\sigma(\beta) \geq 5.4 \cdot 10^{-24} (K\beta)^{-1/2} e^{-2.49436 K\beta}$ (Section S1.1.3).

Corollary 1.3 (Groups containing the scalars; Higgs fields blind to the centre). *Let $G \subseteq U(n)$ be a closed subgroup with $\{zI : |z| = 1\} \subseteq G$, in $d = 3$ with the Wilson action $\beta \operatorname{Re} \operatorname{Tr} U_p$ (free boundary conditions), with or without a Higgs field coupled through any action that depends on the links only through $\rho(U_e)$, ρ a representation of G trivial on the scalars. At every $\beta > 0$ and all Higgs couplings, for every rectangle C with $\min(R, T) \geq R_1(2n\beta)$ and every finite box $\Lambda \supseteq C$, $|\langle \operatorname{Tr} \prod_{e \in C} U_e \rangle_{G,\beta}| \leq 2n \exp(-\sigma(2n\beta)RT + c(R + T))$; in particular for $U(2)$ with an adjoint Higgs field.*

Corollary 1.4 (Georgi–Glashow model at infinite Higgs coupling). *The $SU(2)$ lattice gauge theory with the Wilson action $\beta \cdot \frac{1}{2} \operatorname{Tr} U_p$ and a unit adjoint Higgs field at hopping coupling $\kappa = \infty$, defined in each finite box as the limit $\kappa \rightarrow \infty$, is the $U(1)$ Wilson model at coupling β , and it obeys the area law (1.2) at every $\beta > 0$.*

1.2 Background

The Villain action. G6pfert and Mack [GM82] made Polyakov’s mechanism rigorous for the Villain action. By duality the model is a Coulomb gas of monopoles, which they control by an iterated Mayer expansion down to the Debye scale [GM81a] and a single-scale expansion built on the analysis of Debye screening of Brydges and Federbush [BF80, Bry78]. Their Theorem 1 is the large- β bound $\alpha \geq cm_D\beta^{-1}$ [GM82, (1.9)], and all couplings follow [GM82, Corollary 2] by “the 3-dimensional version of Guth’s inequality” [Gut80, Lemma 4.1]. The expansions use the Gaussian form of the dual weight, a choice made explicit in [GM82, Appendix A]: “Instead of this action [the Wilson action (A.4)] we choose the Villain action”.

The Wilson action. In three dimensions the rigorous results are the area law at strong coupling [OS78], for all compact groups, and logarithmic bounds at all couplings, for $U(1)$ [GJ77] and for closed $G \subseteq U(n)$ containing the scalars [Cha26]. Fr6hlich’s comparison [Fr679] bounds the Wilson loop of such a group by a $U(1)$ loop with the Wilson action, “ $A' = -\beta' \sum_p \cos(\theta_{\partial p})$ ” [Fr679, (6)], at coupling $2n\beta$ [Fr679, (23), (24)]. Thus an area law at every coupling for groups with a central $U(1)$ requires the Wilson case of $U(1)$, not the Villain case. This case is stated as known in [CNS25, For26] (and a mass gap for all β in [Nis25]). Cao, Park and Sheffield write, for the Wilson action [CPS25, open problem (b)]: “When $d = 3$ and $N = 1$, the area law holds for all β , see [GM81]. Because $U(1)$ is the center of $U(N)$ for general N , this implies that the $U(N)$ area law holds for all β and all $N \geq 1$, see [Fr679].” (There [GM81] denotes [GM82].) Chatterjee [Cha26, v2] leaves open whether Fr6hlich’s comparison can be combined with the Villain result of [GM82] to give an area law for $U(n)$ with the Wilson action, “since doing so carefully would require additional technology and a careful comparison between formulations and limits.”

Related work. The mixture of Lemma 2.2 is used to prove lower bounds for the XY model [AHPS21, Lemma 9.6] and the annealed Villain model [DvEG26]. Here we use it for an upper bound. The same Bessel dual weights and their Gaussian mixture enter recent work on the critical planar XY model [Ope26, §2.1]. In four dimensions a Coulomb phase is proved for the Villain [Gut80, FS82] and the Wilson action [Kin86].

1.3 Strategy

The proof has four steps.

(i) *Comparison* (Section 2). By a theorem of Hartman and Watson [HW74], $e^{\beta \cos \theta} / I_0(\beta)$ is a mixture of Villain weights $\tilde{v}_{1/u}$ (Lemma 2.2), and Villain loops are nondecreasing in each coupling (Lemma 2.3). Raising the couplings below $\tau = K\beta$ to τ and the others to ∞ , and dominating the set of the latter by a Bernoulli set of density $\delta = e^{-c_\delta(K)\beta}$, $c_\delta(K)/K \rightarrow \infty$ (Lemmas 2.4 and 2.6), we bound the Wilson loop at β by the average over a Bernoulli set D of the Villain loop at τ with D frozen (Theorem 2.5), in one box Λ° of the size of the loop (Proposition 2.9), where frozen plaquettes are insulating walls (Proposition 2.10).

(ii) *Regularization* (Sections 3 and 4). A deterministic construction maps each D to a hole set and frozen sheets. Freezing more plaquettes only increases the loop. The hole set of its first phase is the smallest minimizer of a cut functional that penalizes blocks crowded with marks. Thus each cost bound is a comparison with an explicit competitor. A local seal removes thin pieces of blocks, and a second phase enforces non-fragmentation, which is not local. The output, a slit domain $G'(D)$, has uniform isoperimetry, cells of controlled size and shape with a Poincar6 inequality, and non-fragmentation, with constants independent of β , K ,

C, D (Theorems 3.20, 3.21).

(iii) *Deterministic area law* (Sections 6 and 7, Appendices A to C and D). On each output $G'(D)$ we run the analysis of [GM82, GM81a, BF80] with uniform constants, grading the outer partition functions by the periods of their walls around the handles of the insulating set. On each output the loop decays like the exponential of the min-cut area $A_{G'}(C)$, in which frozen plaquettes are free (Theorem 7.57). We carry over the steps used with explicit changes, and in Appendix A we prove in full a tree-graph counting step of [Bry78], whose printed deduction is incomplete.

(iv) *Min-cut lower tail* (Section 5). A Peierls estimate, which is the only place where the condition $K > K_*$ enters (Lemma 2.6), shows that $A_{G'}(C) \geq (1 - \eta_T)RT$ outside an event of probability at most $2^{4|C|+5}e^{-RT}$, with $\eta_T \rightarrow 0$ as $K \rightarrow \infty$ (Theorem 5.1).

In Section 8 we combine the four steps into the following chain: for $\beta \geq \beta_0(K)$, with $D \sim \text{Ber}(\delta)$ on the interior plaquettes of Λ° ,

$$\langle \cos \theta_C \rangle_{\Lambda, \beta}^W \leq \mathbb{E}_D g_{\Lambda^\circ, \tau}^{\text{fix}}(C; D) \quad (1.4)$$

$$\leq \mathbb{E}_D W_{G'(D)}(C) \quad (1.5)$$

$$\leq \mathbb{E}_D \exp(-\sigma_V A_{G'(D)}(C) + \sigma_V \eta_S RT + c_N(R + T)) \quad (1.6)$$

$$= \exp(-\sigma_V RT + \sigma_V \eta_S RT + c_N(R + T)) \mathbb{E}_D \exp(\sigma_V (RT - A_{G'(D)}(C))) \quad (1.7)$$

$$\leq \exp(-\sigma_V (1 - \eta_T - \eta_S)RT + c_N(R + T) + 4|C| + 4), \quad (1.8)$$

and Theorem 2.1 extends it to all β .

Two natural routes fail. First, the Wilson dual weight $I_n(\beta)/I_0(\beta)$, of order $e^{-|n|\log|n|(1+o(1))}$, is dominated by no Villain dual weight $e^{-n^2/(2t)}$. Second, on the carpet graphs of [CG25], where the Villain action is a limit of Wilson actions, the Wilson model is the limit with the auxiliary plaquettes frozen, and hence Ginibre's inequality bounds the Wilson loop from below, not from above.

1.4 Conventions

Interfaces. In Sections 2 and 8 we use the construction only through Theorem 3.20, namely through the fact that the output is a deterministic function of D with $D \subseteq \mathfrak{F}$ of (2.9). In Sections 6 and 7 we use the construction only through $A_{G'}(C)$ and through the inputs listed in Theorem 7.57, namely (R2), (R5) to (R10), (N3), Lemma 4.4, the frame of Section 3.1 and the coarse alignment of the output. In Section 5 we use the construction of Section 3 and the history bounds of Theorem 3.21.

Couplings. We write β for the Wilson coupling, $\tau = K\beta$ for the Villain coupling of the comparison, and $\beta_{\text{GM}} = 4\pi^2\tau$ for the coupling in the normalization of [GM82] (Sections 6, 7, Appendices A to C).

Constants and units. We fix the constants once, in the order of Section 8.2. Those fixed before K do not depend on β, τ, K, C or D , and c, C with or without indices denote such constants, except where a dependence is shown or stated (as for $c_\delta(K)$, which also depends on β , for c'_g and c_u , which depend on β through the rounding of L_c , and for c_{dep}). The letter C also denotes the loop. One site is one coarse cell of the loop-adapted grid in Sections 3 to 5, unless “fine” is written, and the lattice spacing is one in Sections 6 and 7 (Section 2.5).

Supplement and editions. Sections S1, S2, S3 of the supplementary material give the values of the constants with their arithmetic, full proofs of steps sketched here, and numerical checks,

on which no proof depends. Equation, section and page numbers of [GM82] and [GM81a] refer to the preprints [GM81c] (DESY 81-036) and [GM81b] (DESY 81-014). Equation numbers of [Frö79] refer to its preprint IHES/P/79/02, and lemma numbers of [Gut80] to its preprint CLNS 79/439.

2 From the Wilson action to a diluted Villain model

In this section we bound the Wilson loop by an average of Villain loops over a Bernoulli set of frozen plaquettes, and we show that frozen plaquettes are insulating walls.

Complexes and fields. Let X be a finite 2-dimensional cell complex with oriented edges $E(X)$, oriented plaquettes $P(X)$ and incidence numbers $[p : e]$. For $\theta \in \mathbb{T}^{E(X)}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we set $\theta_p := \sum_e [p : e] \theta_e$, and for $n \in \mathbb{Z}^{P(X)}$, $(\partial n)_e := \sum_p n_p [p : e]$. An *integer 1-cycle* is a $C \in \mathbb{Z}^{E(X)}$ with zero boundary, $\theta_C := \sum_e C_e \theta_e$, and $Z_2(X) := \ker \partial$. The Villain weight of Definition 1.1, normalized by $\int \tilde{v}_t d\theta / 2\pi = 1$, is $\tilde{v}_t(\theta) = \sum_{n \in \mathbb{Z}} e^{-n^2/(2t)} e^{in\theta} = (2\pi t)^{1/2} \sum_{m \in \mathbb{Z}} e^{-t(\theta - 2\pi m)^2/2} > 0$ (Poisson summation), and $\tilde{v}_t \rightarrow 1$ uniformly as $t \rightarrow 0$. Let $\mu_{X,t}^V \propto \prod_p \tilde{v}_{t_p}(\theta_p) \prod_e d\theta_e$ for $t \in (0, \infty)^{P(X)}$, and $\mu_{X,\beta}^W \propto \prod_p e^{\beta \cos \theta_p} \prod_e d\theta_e$. For $\tau > 0$, $D \subseteq P(X)$ and $s > 0$ let $t_p^{\tau, D, s}$ be τ off D and s on D . Then the *frozen expectation* $g_{X,\tau}(C; D) := \lim_{s \rightarrow \infty} \langle \cos \theta_C \rangle_{X, t^{\tau, D, s}}^V$ exists by Lemma 2.3, and the plaquettes of D are *frozen*.

Ginibre's inequality. Let $\mu \propto \exp(\sum_A J_A \cos(A \cdot \phi)) d\phi$ on a torus $\mathbb{T}^m$, where A runs over a finite family of integer vectors and $J_A \geq 0$. Then for all integer vectors B, B'

$$\langle \cos(B \cdot \phi) \rangle_\mu \geq 0, \quad \langle \cos(B \cdot \phi) \cos(B' \cdot \phi) \rangle_\mu - \langle \cos(B \cdot \phi) \rangle_\mu \langle \cos(B' \cdot \phi) \rangle_\mu \geq 0, \quad (2.1)$$

by [Gin70, Propositions 2 and 3] applied to [Gin70, Example 4] (the torus, with S the set of functions $\cos(m \cdot \phi)$). In particular, $\partial \langle \cos(B \cdot \phi) \rangle_\mu / \partial J_A \geq 0$ for all integer vectors B .

2.1 Monotonicity

Theorem 2.1 (Monotonicity for the Wilson action). *Let C be an integer 1-cycle supported in a box $\Lambda \subset \mathbb{Z}^3$, in the $U(1)$ Wilson model with free boundary conditions. (a) $\langle e^{i\theta_C} \rangle_{\Lambda, \beta}^W = \langle \cos \theta_C \rangle_{\Lambda, \beta}^W \in [0, 1]$. (b) $\langle \cos \theta_C \rangle_{\Lambda, \beta}^W$ is nondecreasing in β , and nondecreasing in Λ under inclusion. (c) Along every increasing sequence of boxes Λ_k exhausting $\mathbb{Z}^3$ the limit $\langle \cos \theta_C \rangle_\beta^W := \lim_k \langle \cos \theta_C \rangle_{\Lambda_k, \beta}^W$ exists, equals the supremum of $\langle \cos \theta_C \rangle_{\Lambda, \beta}^W$ over all boxes Λ containing the support of C , and is nondecreasing in β .*

Proof. (a) and the monotonicity in β are (2.1) with $J_p = \beta$ (and $\theta \mapsto -\theta$). For boxes $\Lambda \subseteq \Lambda'$ the expectation on Λ' with coupling $s \in [0, \beta]$ on the plaquettes not in Λ is nondecreasing in s and equals the one on Λ at $s = 0$, where the edges outside Λ interact with nothing. (c) follows, since each box lies in Λ_k for large k .

2.2 The Hartman–Watson mixture and the annealed coupling field

Lemma 2.2 (Hartman–Watson mixture). *For every $\beta > 0$ there is a probability measure ω_β on $(0, \infty)$ such that, for all $\nu \geq 0$ and $\theta \in \mathbb{T}$,*

$$\int_{(0, \infty)} e^{-\nu^2 u/2} \omega_\beta(du) = \frac{I_\nu(\beta)}{I_0(\beta)}, \quad \frac{e^{\beta \cos \theta}}{I_0(\beta)} = \int_{(0, \infty)} \tilde{v}_{1/u}(\theta) \omega_\beta(du), \quad (2.2)$$

and $\omega_\beta((0, \varepsilon)) \leq e^{\nu^2 \varepsilon/2} I_\nu(\beta)/I_0(\beta)$ for all $\varepsilon > 0$ and $\nu \geq 0$.

Proof. By [HW74, Theorem 1.3], $I_\nu(\beta)/I_0(\beta) = \int_0^\infty e^{-\nu^2 t} W(dt, \beta)$ ($0 \leq \nu < \infty$) with a distribution function $W(\cdot, \beta)$ on $[0, \infty)$. We let ω_β be its image under $u = 2t$, which has no atom at 0 because $I_\nu(\beta) \leq (\beta/2)^\nu e^{\beta^2/4}/\Gamma(\nu+1) \rightarrow 0$ as $\nu \rightarrow \infty$. The second identity follows from the first by Fourier inversion and Fubini, as $\sum_n I_{|n|}(\beta)/I_0(\beta) = e^\beta/I_0(\beta) < \infty$ (for the XY model it is [AHPS21, Lemma 9.6]). The tail bound is the Chernoff bound $\mathbf{1}_{u < \varepsilon} \leq e^{\nu^2(\varepsilon - u)/2}$.

Lemma 2.3 (Villain monotonicity). *Let X be a finite 2-complex and C an integer 1-cycle. (a) For $t \in (0, \infty)^{P(X)}$, $\langle \sin \theta_C \rangle_{X,t}^V = 0$ and $\langle \cos \theta_C \rangle_{X,t}^V \geq 0$. (b) $\langle \cos \theta_C \rangle_{X,t}^V$ is nondecreasing in each t_p . (c) (a) and (b) hold when some plaquettes are removed (weight 1, the limit $t_p \rightarrow 0$); a removed plaquette gives a value no larger than any $t_p > 0$. (d) For fixed couplings off $D \subseteq P(X)$, the limit as the couplings on D tend to ∞ exists, lies in $[0, 1]$, and is nondecreasing in each remaining coupling and in D ; in particular $g_{X,\tau}(C; D) \in [0, 1]$ exists and is nondecreasing in D (for inclusion) and in τ .*

Proof. On a cube of $\mathbb{Z}^3$ with couplings in $(0, \infty)$, (b) is [CG25, Corollary 1.6] (and [Gut80, Lemma C.1], for plaquette-dependent couplings). We use the lemma only on boxes, and we adapt the cited results as follows. Removal, $t_p \rightarrow 0$, is dominated convergence in (2.6), whose terms decrease as $t_p \downarrow 0$ to those of the complex without p , which gives (c). A box is a cube with the outside plaquettes removed, and an integer 1-cycle C has $\theta_C = \theta_\ell$ for one closed path ℓ . Freezing, $t_p \rightarrow \infty$, is a monotone limit in $[0, 1]$, which gives (d), as $t^{\tau, D, s} \leq t^{\tau, D', s}$ for $D \subseteq D'$, $s \geq \tau$. For (a) we use $\theta \mapsto -\theta$, and the value with all plaquettes removed is $\int \cos \theta_C d\theta/(2\pi)^{|E|} \geq 0$. In Section S2.1 we prove the lemma on all finite 2-complexes.

The annealed coupling field. Fix X and $\beta > 0$. Let $Z(t) := \int \prod_p \tilde{v}_{t_p}(\theta_p) d\theta/(2\pi)^{|E(X)|} > 0$ for $t \in (0, \infty)^{P(X)}$, and let μ_{ann} be the probability measure on $(0, \infty)^{P(X)}$ with density proportional to $Z(1/u)$, $1/u := (1/u_p)_p$, with respect to $\omega_\beta^{\otimes P(X)}$ (the normalization is finite and positive by (2.2) and Fubini). Put $\vartheta_J(u) := \sum_{n \in \mathbb{Z}} e^{-n^2 u/2}$ and $\delta_\beta(\varepsilon) := \int_{(0, \varepsilon)} \vartheta_J(u) \omega_\beta(du)$.

Lemma 2.4 (Annealed coupling field). *Let X be a finite 2-complex, $\beta > 0$ and $\varepsilon > 0$. (i) For every bounded measurable F on $\mathbb{T}^{E(X)}$, $\langle F \rangle_{X, \beta}^W = \int \langle F \rangle_{X, 1/u}^V \mu_{\text{ann}}(du)$. (ii) (Tilt bound.) For every $p \in P(X)$ and every $u_{-p} := (u_q)_{q \neq p}$, $\mu_{\text{ann}}(u_p < \varepsilon \mid u_{-p}) \leq \delta_\beta(\varepsilon)$. (iii) Under μ_{ann} the random set $D_\varepsilon(u) := \{p : u_p < \varepsilon\}$ is stochastically dominated by $\text{Ber}(\delta')^{\otimes P(X)}$ for every $\delta' \in [\min(\delta_\beta(\varepsilon), 1), 1]$: $\int F(D_\varepsilon(u)) \mu_{\text{ann}}(du) \leq \mathbb{E}_{D \sim \text{Ber}(\delta')^{\otimes P(X)}} F(D)$ for every nondecreasing function F on the subsets of $P(X)$. (iv) $\delta_\beta(\varepsilon) \leq \inf_{\nu \geq 0} e^{\nu^2 \varepsilon/2} \sum_{n \in \mathbb{Z}} I_{\sqrt{n^2 + \nu^2}}(\beta)/I_0(\beta)$.*

Proof. (i) We multiply the second identity of (2.2) over p , integrate against F and against 1, and use Fubini.

(ii) The tilt is bounded because, as a function of one coupling, Z is a combination of the functions $e^{-n^2 u_p/2}$ with nonnegative coefficients, each at most the constant coefficient. We fix p and u_{-p} . As μ_{ann} has density proportional to $Z(1/u)$ with respect to a product measure, the conditional law of u_p given u_{-p} has density proportional to $u_p \mapsto Z(1/u)$ with respect to ω_β . Expanding $\tilde{v}_{1/u_p}(\theta_p) = \sum_n e^{-n^2 u_p/2} e^{in\theta_p}$ (absolutely convergent), we obtain $Z(1/u) = \sum_{n \in \mathbb{Z}} e^{-n^2 u_p/2} A_n$, $A_n := \int e^{in\theta_p} \prod_{q \neq p} \tilde{v}_{1/u_q}(\theta_q) d\theta / (2\pi)^{|E|}$, with $A_0 > 0$ independent of u_p . Here $A_n/A_0 = \langle e^{in\theta_p} \rangle$ in the system with p removed. Since $n\theta_p = \theta_{C'}$ for the integer 1-cycle $C' = n \partial p$, Lemma 2.3(a),(c) shows that A_n is real and $0 \leq A_n \leq A_0$. Hence $1 \leq Z(1/u)/A_0 \leq \vartheta_J(u_p)$ (the lower bound from $n = 0$), and $\mu_{\text{ann}}(u_p < \varepsilon \mid u_{-p}) = \int_{(0,\varepsilon)} (Z/A_0) d\omega_\beta / \int_{(0,\infty)} (Z/A_0) d\omega_\beta \leq \int_{(0,\varepsilon)} \vartheta_J d\omega_\beta = \delta_\beta(\varepsilon)$.

(iii) Let $Y_p := \mathbf{1}[u_p \geq \varepsilon]$ and fix an order $s_1, s_2, \dots$ of $P(X)$. Since $Y_{s_1}, \dots, Y_{s_j}$ are functions of $u_{-s_{j+1}}$, (ii) and the tower property give $\mathbb{P}(Y_{s_{j+1}} = 1 \mid Y_{s_1}, \dots, Y_{s_j}) \geq 1 - \min(\delta_\beta(\varepsilon), 1)$. By [LSS97, Lemma 1.1] this yields the domination for $\delta' = \min(\delta_\beta(\varepsilon), 1)$, and $\mathbb{E}F(D)$ is nondecreasing in δ' (monotone coupling). (iv) is the Chernoff bound $\mathbf{1}_{u < \varepsilon} \leq e^{\nu^2(\varepsilon - u)/2}$ integrated against $\vartheta_J(u) \omega_\beta(du)$, with (2.2).

Theorem 2.5 (Wilson below diluted Villain). *For every finite 2-complex X , integer 1-cycle C , $\beta > 0$, $\varepsilon > 0$ and $\delta' \in [\min(\delta_\beta(\varepsilon), 1), 1]$,*

$$0 \leq \langle \cos \theta_C \rangle_{X,\beta}^W \leq \mathbb{E}_{D \sim \text{Ber}(\delta') \otimes P(X)} g_{X,1/\varepsilon}(C; D). \quad (2.3)$$

Proof. The Wilson model is a Villain model with random couplings $1/u_p$ (Lemma 2.4(i)), and by monotonicity we round all couplings up, the weak ones to $1/\varepsilon$ and the strong ones to ∞ . Positivity is (2.1). For fixed u and $D = D_\varepsilon(u)$ we have $1/u_p \leq 1/\varepsilon$ off D , and hence Lemma 2.3(b),(d) give $\langle \cos \theta_C \rangle_{X,1/u}^V \leq g_{X,1/\varepsilon}(C; D_\varepsilon(u))$. We integrate against $\mu_{\text{ann}}(du)$ (Lemma 2.4(i)) and apply Lemma 2.4(iii) to the nondecreasing function $g_{X,1/\varepsilon}(C; \cdot)$ (Lemma 2.3(d)).

2.3 The density of frozen plaquettes

From now on we take $\varepsilon = 1/(K\beta)$, so that the Villain coupling is $\tau = K\beta$, and set $\delta := \delta_\beta(1/(K\beta))$ and $c_\delta(K) := |\log \delta|/\beta$, a function of K and β (the notation shows only K). For $K > e/2$ let $a_* = a_*(K) > e/2$ be the solution of $a_* = K \log(2a_*)$ (unique on $(e/2, \infty)$, where $a \mapsto a/\log 2a$ increases), and put $c_\delta^{\text{sp}}(K) := a_*(\frac{1}{2} \log 2a_* - 1) + 1$.

Lemma 2.6 (Density of frozen plaquettes). *Let $K > e/2$ and $\beta \geq \pi^{-2}$. Then*

$$\log \delta \leq -\beta \left(c_\delta^{\text{sp}}(K) - \frac{1}{4a_*} \right) + \log((2a_*\beta + 5)\pi\sqrt{\beta}) + \frac{1}{2}, \quad (2.4)$$

so that $c_\delta(K) \geq c_\delta^{\text{sp}}(K) - 1/(4a_*) - O_K(\log \beta/\beta)$. The function $K \mapsto c_\delta^{\text{sp}}(K)/(2K)$ is increasing on $(e/2, \infty)$; it equals 14.9662 at $K = 377.3635$, and it is at least 14.9663 for $K \geq K_* = 377.38$. Consequently, for every $K > K_*$ there are $\epsilon_K > 0$ and $\beta_1(K)$ such that

$$\frac{c_\delta(K)}{2K} \geq 14.9662 + \epsilon_K \quad (\beta \geq \beta_1(K)). \quad (2.5)$$

The number 14.9662 is $6 \cdot 2.4943551 = 14.96613$ rounded up, where the Debye exponent $\pi^2 v_{\text{Cb}}(0) = 2.4943550872 \dots$ gives $\log(1/\tilde{m}) = \pi^2 v_{\text{Cb}}(0) \tau + O(\log \tau)$ (Definition 3.1),

with $v_{\text{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0,0) = 0.2527310\dots$, half of Watson's integral [Wat39] in the normalization of [Zuc11], whose closed form is [Zuc11, (3.6)] (Section S1.1.1). Thus (2.5) says $\delta^{1/2} \leq \tilde{m}^6 e^{-\epsilon_K \tau + O(\log \tau)}$: one mark, of probability at most $\delta^{1/2}$ (a plaquette marks at most two sites), outweighs the entropy $(\tilde{m}^{-1})^{6+o(1)}$ of a root and a part in the Peierls count of Lemma 5.12, which is the only place where we use $K > K_*$.

Proof. Roughly speaking, the heavy dual tail of the Wilson weight comes from ω_β near $u = 0$. Applying Lemma 2.4(iv) at its saddle point $\nu = a_*\beta$, with the series of I_μ , log-convexity of Γ , Stirling's inequality and $I_0(\beta) \geq e^{\beta-1/2}/(\pi\sqrt{\beta})$ for $\beta \geq \pi^{-2}$, we obtain (2.4). The computation, the monotonicity and the two values are in Section S1.1.2. For $K > K_*$, $\epsilon_K := \frac{1}{2}(c_\delta^{\text{sp}}(K)/(2K) - 14.9662) \geq 9 \cdot 10^{-5}$, since $c_\delta^{\text{sp}}(K)/(2K)$ is increasing and equals 14.96639 at K_* (Section S1.1.2). For these K , the value 14.9663 of the statement gives $\epsilon_K \geq 5 \cdot 10^{-5}$, which also suffices. Further, as $1/(8Ka_*) < 10^{-7}$ for $K \geq K_*$, (2.4) gives (2.5) once the $O_K(\log \beta/\beta)$ term is below $K\epsilon_K$.

2.4 The dual representation and the frozen limit

Let Λ' be a box, regarded as a closed cubical complex with cubes Λ'_3 , plaquettes Λ'_2 and boundary plaquettes $\partial_2 \Lambda'$, and let $\partial : \mathbb{Z}^{\Lambda'_3} \rightarrow \mathbb{Z}^{\Lambda'_2}$ be the boundary map, cubes oriented outward. Since the complex is contractible, ∂ is injective, $Z_2(\Lambda') = \partial \mathbb{Z}^{\Lambda'_3}$, and each integer 1-cycle C in Λ' bounds an integer 2-chain n_0 . Dually, cubes are sites, an interior plaquette is a bond between its two cubes, and a boundary plaquette is a bond to an exterior ground site, where $k = 0$.

Lemma 2.7 (Dual representation). *Let $t \in (0, \infty)^{\Lambda'_2}$, let C be an integer 1-cycle in Λ' , and let n_0 be any integer 2-chain of Λ' with $\partial n_0 = C$. Then*

$$\langle \cos \theta_C \rangle_{\Lambda', t}^V = \frac{\sum_{k \in \mathbb{Z}^{\Lambda'_3}} \prod_p e^{-(n_0 + \partial k)_p^2 / (2t_p)}}{\sum_{k \in \mathbb{Z}^{\Lambda'_3}} \prod_p e^{-(\partial k)_p^2 / (2t_p)}}. \quad (2.6)$$

Proof. By the expansion $\prod_p \tilde{v}_{t_p}(\theta_p) = \sum_{n \in \mathbb{Z}^{\Lambda'_2}} \prod_p e^{-n_p^2 / (2t_p)} e^{i(\partial n) \cdot \theta}$ (absolutely convergent, even coefficients), $\langle e^{i\theta_C} \rangle = \langle \cos \theta_C \rangle$ is the ratio of the sums over $\partial n = C$ and over $\partial n = 0$, and $n = n_0 + \partial k$, $k \in \mathbb{Z}^{\Lambda'_3}$, parametrizes the first set bijectively for each n_0 ($Z_2(\Lambda') = \partial \mathbb{Z}^{\Lambda'_3}$, ∂ injective). This matters when frozen plaquettes surround an edge of C .

We now fix $\tau > 0$ and a set $F \subseteq \Lambda'_2$ of plaquettes to be frozen, and let t_p^s be τ for $p \notin F$ and s for $p \in F$. Let $\mathcal{K}_F$ be the set of components not containing the ground of the *unfrozen graph* (with the cubes and the ground as vertices and the plaquettes not in F as bonds). A cube with all faces in F is such a component. Fix $c_{\mathfrak{K}} \in \mathfrak{K}$ and put $W := \{\sum_{\mathfrak{K} \in \mathcal{K}_F} a_{\mathfrak{K}} \mathbf{1}_{\mathfrak{K}} : a \in \mathbb{Z}^{\mathcal{K}_F}\}$ and $W' := \{k \in \mathbb{Z}^{\Lambda'_3} : k(c_{\mathfrak{K}}) = 0 \text{ for all } \mathfrak{K} \in \mathcal{K}_F\}$. Then $\mathbb{Z}^{\Lambda'_3} = W \oplus W'$, and $(\partial w)_p = 0$ for $w \in W$, $p \notin F$.

Lemma 2.8 (Frozen limit and zero modes). *In this setting, for every integer 1-cycle C in Λ' and every integer 2-chain n_0 of Λ' with $\partial n_0 = C$,*

$$\lim_{s \rightarrow \infty} \langle \cos \theta_C \rangle_{\Lambda', t^s}^V = \frac{\sum_{k \in W'} \exp\left(-\frac{1}{2\tau} \sum_{p \notin F} (n_0 + \partial k)_p^2\right)}{\sum_{k \in W'} \exp\left(-\frac{1}{2\tau} \sum_{p \notin F} (\partial k)_p^2\right)}. \quad (2.7)$$

The quadratic form $k \mapsto \sum_{p \notin F} (\partial k)_p^2$ is positive definite on $W' \otimes \mathbb{R}$, so both sums converge; the summands are invariant under W , so the sums run over $\mathbb{Z}^{\Lambda'_3}/W$ (k modulo one constant on each component in $\mathcal{K}_F$) and do not depend on the choice of W' . In particular every cube all of whose faces are frozen drops out.

Proof. Freezing deletes dual bonds. The only issue is the soft directions W , one zero mode for each component in $\mathcal{K}_F$ (isolated cubes included, such as the cubes outside Λ' in Proposition 2.9), along which only the frozen weights act, at strength $1/s$. Poisson summation over $w \in W$ ($k = w' + w$; positive definite as ∂ is injective) gives a prefactor that cancels in the ratio times a factor in $[0, 2]$ tending to 1. A w' with $\mathbf{1}_{F^c} \partial w' = 0$ is constant on components and vanishes on that of the ground (positive definiteness), and dominated convergence gives (2.7) (details in Section S2.1).

2.5 One box with insulating walls

For a box Λ' , $D \subseteq \Lambda'_2 \setminus \partial_2 \Lambda'$ and $\tau > 0$, $g_{\Lambda', \tau}^{\text{fix}}(C; D) := g_{\Lambda', \tau}(C; D \cup \partial_2 \Lambda')$ is the Villain expectation on Λ' with the plaquettes of D and the boundary plaquettes frozen. We write $C \subset \Lambda' \setminus \partial \Lambda'$ if all edges of C lie in Λ' and none lies in the boundary of Λ' .

Proposition 2.9 (One box). *Let C be an integer 1-cycle, Λ' a box with $C \subset \Lambda' \setminus \partial \Lambda'$, $K, \beta > 0$, $\tau = K\beta$, δ as in Section 2.3 and $\delta' := \min(\delta, 1)$. For every finite box $\Lambda \supseteq C$, $\langle \cos \theta_C \rangle_{\Lambda, \beta}^W \leq \mathbb{E}_{D \sim \text{Ber}(\delta')^{\otimes (\Lambda'_2 \setminus \partial_2 \Lambda')}} g_{\Lambda', \tau}^{\text{fix}}(C; D)$.*

Proof. We use volume monotonicity and then freeze everything outside Λ' . Let $\Lambda'' \supseteq \Lambda \cup \Lambda'$ be a box and $F_{\text{out}} := \Lambda''_2 \setminus (\Lambda'_2 \setminus \partial_2 \Lambda')$. Then Theorem 2.1(b), Theorem 2.5 on $X = \Lambda''$ ($\varepsilon = 1/(K\beta)$) and Lemma 2.3(d) give $\langle \cos \theta_C \rangle_{\Lambda, \beta}^W \leq \mathbb{E}_{D'' \sim \text{Ber}(\delta')^{\otimes \Lambda''_2}} g_{\Lambda'', \tau}(C; D \cup F_{\text{out}})$, $D'' \sim \text{Ber}(\delta')^{\otimes \Lambda''_2}$, $D := D'' \cap (\Lambda'_2 \setminus \partial_2 \Lambda')$. By Lemma 2.8 on Λ'' and on Λ' (n_0 in Λ'), the right side is $g_{\Lambda', \tau}^{\text{fix}}(C; D)$, as the unfrozen plaquettes are the same, the cubes outside Λ' are isolated, and the ground is not reached.

Slit graphs. Let Λ' be a box, $\mathfrak{F} \subseteq \Lambda'_2$ with $\partial_2 \Lambda' \subseteq \mathfrak{F}$, and $\mathbb{X} \subseteq \Lambda'_3$ a set of cubes all of whose faces lie in $\mathfrak{F}$. The *slit graph* $G' = (V, E)$ has $V := \Lambda'_3 \setminus \mathbb{X}$ and $E := E(V) \setminus S$, with $E(V)$ the nearest-neighbour pairs in V and S those whose common plaquette lies in $\mathfrak{F}$. Conversely, each triple $(\Lambda', \mathbb{X}, S)$, S a set of bonds with both ends in $\Lambda'_3 \setminus \mathbb{X}$, arises in this way, with $\mathfrak{F} := F(\mathbb{X}) \cup F_S \cup \partial_2 \Lambda'$ ($F(\mathbb{X})$ the faces of the cubes of $\mathbb{X}$, F_S the plaquettes dual to S). The phrase “every slit graph” means every such triple. As E omits the bonds of S , a slit graph is in general not an induced subgraph of $\mathbb{Z}^3$. Its components are denoted $\mathfrak{K}$ (an isolated vertex is a component), its vertices have degree at most 6, and $d_{G'}(x, y) \geq \|x - y\|_1$, with $d_{G'} = \infty$ across components. We orient a bond $b = (x, y) \in E$ and its plaquette p_b compatibly (p_b as in ∂x). For an integer 2-chain Ξ of Λ' with $\partial \Xi = C$ let $\xi_b \in \mathbb{Z}$ be the coefficient of p_b in Ξ (for the flat rectangle spanned by C , the signed crossing number, in $\{0, \pm 1\}$), and for $k \in \mathbb{Z}$ put

$$Z_{G'}(k, \Xi) := \sum_{[m]} \exp\left(-\frac{1}{2\tau} \sum_{b=(x,y) \in E} (m(y) - m(x) - k\xi_b)^2\right), \quad W_{G'}(C) := \frac{Z_{G'}(1, \Xi)}{Z_{G'}(0, \Xi)}, \quad (2.8)$$

where $[m]$ runs over the classes of $m \in \mathbb{Z}^V$ modulo functions constant on each component.

Coarse and fine. In Sections 3 to 5 the hole set $\mathbb{X}$ is a set of coarse sites and the frozen set S a set of coarse bonds (Definition 3.7), with the coarse slit graph $(\Lambda^\circ \setminus \mathbb{X}, E(\Lambda^\circ \setminus \mathbb{X}) \setminus S)$. A coarse face carries between r_0^2 and $\bar{r}_0^2$ fine plaquettes (Definition 3.4). On the fine lattice the hole cubes are the fine cubes in sites of $\mathbb{X}$, $F(\mathbb{X})$ is the set of their faces (Definition 3.6), F_S the set of fine plaquettes in the coarse faces dual to the bonds of S , and

$$\mathfrak{F} := F(\mathbb{X}) \cup F_S \cup \partial_2 \Lambda^\circ. \quad (2.9)$$

For the output $(\mathbb{X}, S) = (\mathbb{X}(D), S(D))$ of Theorem 3.20 the fine slit graph of this $\mathfrak{F}$ is the output $G' = G'(D)$. Its vertices are the fine cubes of Λ° outside the holes, and a bond at a vertex is missing from E exactly when it leads into a hole, is frozen (dual to F_S), or leaves Λ° .

Proposition 2.10 (Frozen plaquettes are insulating walls). *Let Λ' , $\mathfrak{F}$, $\mathbb{X}$ and G' be as above, $\tau > 0$, and C an integer 1-cycle in Λ' . For every $D \subseteq \mathfrak{F} \setminus \partial_2 \Lambda'$, $g_{\Lambda', \tau}^{\text{fix}}(C; D) \leq g_{\Lambda', \tau}^{\text{fix}}(C; \mathfrak{F} \setminus \partial_2 \Lambda') = W_{G'}(C)$, and $W_{G'}(C)$ depends neither on the choice of Ξ nor on the choice of $\mathbb{X}$.*

Proof. Freezing raises the loop (Lemma 2.3(d) gives the inequality) and deletes a dual bond. Hence holes, sheets, cuts and box walls are admissible. For the identity we apply Lemma 2.8 with $F = \mathfrak{F}$, $n_0 = \Xi$. Here no component reaches the ground, cubes with all faces in $\mathfrak{F}$ (those of $\mathbb{X}$ among them) are isolated and drop out, and $(\Xi + \partial k)_{p_b} = \xi_b + k(x) - k(y)$ gives $Z_{G'}(1, \Xi)/Z_{G'}(0, \Xi)$ with $m = k|_V$. A change $\Xi' = \Xi + \partial U$ is absorbed by $m \mapsto m - U|_V$. On outputs, $D \subseteq \mathfrak{F}(D)$ for all D (Theorem 3.20), and hence the inequality applies to every D .

Definition 2.11 (Min-cut area). For $\mathfrak{F}$ as in (2.9), the *min-cut area* of C in G' is $A_{G'}(C) := \min\{\#(\text{supp } \Sigma \setminus \mathfrak{F}) : \Sigma \text{ an integer 2-chain of } \Lambda^\circ \text{ with } \partial \Sigma = C\}$.

3 The regularization: construction and main properties

In this section we define a deterministic map $D \mapsto G'(D)$ from sets D of frozen plaquettes to slit domains and state the results that later sections use, Theorems 3.20 (terminal state) and 3.21 (history). We prove both in Section 4. Throughout Sections 3 to 5 we fix $K > K_*$, $\beta \geq \beta_0(K)$, $\tau = K\beta$ and the constants fixed before K (Section 8, with values in Section S1). The loop $C = \partial \Xi$ bounds the rectangle $\Xi := [r_1, r_1 + R] \times [t_1, t_1 + T] \times \{z_0\}$ ($r_1, t_1, z_0 \in \mathbb{Z}$, fine units), with $\min(R, T) \geq R_1$. We treat other planes by permuting the axes. All statements hold for arbitrary $D \subseteq \Lambda_2^\circ \setminus \partial_2 \Lambda^\circ$. Units are coarse (one site is one coarse cell of the grid) and distances are coarse ℓ^∞ distances unless “fine” or “Euclidean” is written. We write $Q_\infty(u, r) := \{x : |x - u|_\infty \leq r\}$.

The design. The Phase-1 hole set is the smallest minimizer of a cut functional (Definition 3.11). We pay its cost by comparison with competitors that replace a region, at a bounded cost per mark (Section 4.3). Fills add whole blocks and are triggered by a count of marks. Penalizing defective interfaces or weak links instead fails by cascades, since freezing the bonds of one defect can create defects of the same kind in other blocks (cells are defined through windows), whereas filling blocks changes no bond inside other blocks. We enforce (NF_*) by a rule of Phase 2 (the cut rule), not by a term of the functional, since the deletion competitor of the payment can create a fragmented split. A local seal fills the thin pieces of blocks that minimality does not exclude (Definition 3.16).

3.1 Scales, the box, the grid

Definition 3.1 (Scales and units). Let $\tilde{m} \in [m_D/2, 2m_D]$ be the screening mass of Section 6 (fine units). The scales are: the fine side $r_0 \geq 1$ of a coarse cell, which is at most $\bar{r}_0 := 2r_0 - 1$; the block scale L_c , the largest even integer $\leq c_L \tilde{m}^{-1}/r_0$ (coarse), and $L := r_0 L_c$ (fine); $\rho_\lambda := L_c/12$ and the hull penalty $\lambda := 1/\rho_\lambda = 12/L_c$; $V_* := (1 - \eta_w)^3 \rho_\lambda^3/2$ with $\eta_w := 1/10$; the window bound $k_P := 304L_c$; the removal scale $\ell := \ell_D/r_0$ with $\ell_D := L[\tilde{m}^{-1}/L]$; and $R_1 := (2r_\square + 5)L$ (fine) with $r_\square := (k_P + 3)/L_c \leq 305$.

The lower bounds on τ used here ($L_c \geq 4$, $\tilde{m}L \geq c_L/2$) are in Section S1.2. Since $c_L \leq 1/308$, we have

$$\ell \geq k_P + 4L_c = 308L_c, \quad \tilde{m}L \leq c_L \leq \frac{1}{2}. \quad (3.1)$$

Definition 3.2 (The box Λ°). Put $m := \max(R, T)$, $M_\Lambda := \max(3m, r_0(\ell + 3))$ (fine), M'_Λ the even integer in $\{M_\Lambda, M_\Lambda + 1\}$, and $o = (o_1, o_2, o_3) := (r_1 + R/2, t_1 + T/2, z_0)$, the centre of Ξ . Then $\Lambda^\circ := \prod_{\mu=1}^3 [a_\mu, b_\mu]$ with $a_\mu := \lfloor o_\mu \rfloor - M'_\Lambda/2$ and $b_\mu := a_\mu + M'_\Lambda$, a cube whose faces are lattice planes; Λ_2° is its set of plaquettes and $\partial_2 \Lambda^\circ$ the set of those contained in $\partial \Lambda^\circ$.

Lemma 3.3 (Properties of Λ°). (a) Λ° and its grid $\mathfrak{G}(C)$ depend only on C and the constants. (b) Consecutive special planes are at fine distance $\geq L$. (c) Every coarse side of Λ° is $\geq \ell \geq k_P + 4L_c$, and Λ° is a union of blocks and of removal cells. (d) The coarse sides n_μ of Λ° satisfy $\max_\mu n_\mu \leq 2 \min_\mu n_\mu$. (e) Λ° contains the fine ℓ^∞ -neighbourhood of radius m of Ξ , so $\text{dist}(C, \partial \Lambda^\circ) \geq \max(R, T)$, $C \subset \Lambda^\circ \setminus \partial \Lambda^\circ$, and the fine side of Λ° is $\geq 3 \max(R, T)$.

Proof. Parts (a) to (d) are proved in Section S2.2. For (e) we use $M'_\Lambda \geq 3m$ and $o_\mu - \frac{1}{2} \leq \lfloor o_\mu \rfloor \leq o_\mu$. They give $r_1 - a_1 \geq (M'_\Lambda - R)/2 \geq m$, and the integer $b_1 - (r_1 + R)$ is at least $o_1 - \frac{1}{2} + M'_\Lambda/2 - r_1 - R = (M'_\Lambda - R - 1)/2 \geq m - \frac{1}{2}$, hence at least m . Direction 2 is the same with T , and $z_0 - a_3 = b_3 - z_0 = M'_\Lambda/2 \geq m$.

Definition 3.4 (The loop-adapted grid $\mathfrak{G}(C)$). The grid $\mathfrak{G}(C)$ is a function of C and the constants (construction in Section S2.2). Its *special planes* are the planes of C and the faces of Λ° . Its *sites* are boxes of fine sides in $[r_0, \bar{r}_0]$ forming, combinatorially, a box of $\mathbb{Z}^3$, with nearest-neighbour bonds E ; the coarse face dual to a bond is the common face of its two sites. *Blocks* are the boxes between consecutive *block planes* of the three directions; a *block face* F' is the common face of two adjacent blocks, and $|F'|$ is the number of bonds across it; the *block distance* of two blocks is the ℓ^∞ distance of their indices. For a *removal cell* Δ , $|\partial \Delta|$ is the number of bonds of $\mathbb{Z}^3$ with exactly one end in Δ , bonds leaving Λ° included. The *window* $\hat{Q}(Q)$ of a block Q is the union of the blocks at block distance at most 75 from Q ; $\mathcal{Q}$ is the set of windows. For a block face F' between blocks Q, Q' , in the plane P' , the *tube zone* $Z(F')$ is the set of sites of $Q \cup Q'$ in the $L_c/2$ layers nearest to P' on each side, and the *tube* τ_b of a bond b across F' is the segment of its line formed by its L_c sites in $Z(F')$. Later proofs use only the following properties (proof in Section S2.2; sizes in Table S1.2). (G1) Consecutive parallel block planes are at coarse distance in $[L_c, 2L_c - 1]$. Blocks have coarse sides in $[L_c, 2L_c - 1]$, fine sides in $[L, 2L - 1]$, volume in $[L_c^3, 8L_c^3]$, and $|F'| \in [L_c^2, (2L_c - 1)^2]$. (G2) Every block is a box with six faces, each a block face or contained in $\partial \Lambda^\circ$; a line in direction μ crosses only block planes perpendicular to e_μ ; every special plane is a block plane; Λ° and every removal cell are unions of blocks. (G3) Removal cells tile Λ° , have coarse sides in $[\ell, 2\ell + 2L_c - 3] \subseteq [\ell, 3\ell]$, and $|\partial \Delta| \in [6\ell^2, 54\ell^2]$. (G4) Every window $\hat{Q}(Q)$ is a box of coarse sides less than k_P that contains $Q_\infty(u, 75L_c) \cap \Lambda^\circ$ for every site $u \in Q$; $Q \cup Q' \subseteq \hat{Q}(Q)$ for every block Q' adjacent to Q ; and

every site lies in at most 151^3 windows. (G5) $Z(F') \subseteq Q \cup Q'$; tube zones of distinct parallel block faces are disjoint; a tube crosses no block plane other than P' .

Lemma 3.5 (Alignment). *By (G2) the planes of C are block planes; so, with $P := \{x_3 = z_0\}$: (A1) every plaquette of Ξ lies in the block plane P , and no bond with both ends in one block crosses Ξ ; (A2) every block face lies in Ξ or contains no plaquette of Ξ ; (A3) every block lies on one side of P , and C lies on lines of block planes. Every plaquette of Ξ is interior to Λ° (Lemma 3.3(e)).*

Definition 3.6 (Marks). For a set Y of sites, $F(Y)$ is the set of fine plaquettes that are faces of fine cubes contained in sites of Y . A site is *marked* iff one of its fine cubes has a face in D ; $\mathbb{M} = \mathbb{M}(D)$ is the set of marked sites, and $N(P) := |P|$ for $P \subseteq \mathbb{M}$. Then $D \subseteq F(\mathbb{M})$, and a plaquette of D marks at most two sites, since it is a face of at most two fine cubes.

3.2 States, functionals and steps

For a set Y of sites, we write $E(Y)$ for the set of bonds with both ends in Y , ∂Y for the set of bonds of $E = E(\Lambda^\circ)$ with exactly one end in Y , and $V(A)$ for the set of endpoints of a bond set A . A bond *touches* Y if it has an end in Y . Bonds leaving Λ° are not in E , that is, the faces of the box are free.

Definition 3.7 (States). A *state* is a triple $\sigma = (\mathbb{X}, \mathbb{U}, S)$: the *holes* $\mathbb{X} \subseteq \Lambda^\circ$, the *filled blocks* $\mathbb{U} \subseteq \mathbb{X}$, a union of blocks, and the *frozen bonds* $S \subseteq E(\Lambda^\circ \setminus \mathbb{X})$, each crossing a block face and each of one *kind*: $S = S_1 \sqcup S_{\text{cut}}$, seal bonds and cut bonds (Definitions 3.16 and 3.18). A block is *surviving* if it is not contained in $\mathbb{U}$. A frozen bond that $\mathbb{X}$ grows to touch leaves S ; it is *converted*, and from then on it is an ordinary hole face. The *slit graph* is $G'(\sigma) := (\Lambda^\circ \setminus \mathbb{X}, E(\Lambda^\circ \setminus \mathbb{X}) \setminus S)$, and $\mathcal{R}(\sigma) := S \cup \{b \in E : b \text{ touches } \mathbb{X}\}$ is the set of bonds missing from it. Along a history (a finite sequence of states produced by the steps below; Definition 4.1), the *base* is $B_t := \mathbb{M} \cup \mathbb{U}_t$, and the *ever-frozen set* $\bar{S}_t$ is the set of bonds frozen at some time $\leq t$ that do not touch B_t , converted bonds included, each keeping its kind.

Definition 3.8 (Hull functional). Let $S \subseteq E$ carry weights $w_b \in [0, 2]$; for $X' \subseteq \Lambda^\circ$,

$$F_\lambda^S(X') := |\partial X'| + \sum_{b \in S \cap E(\Lambda^\circ \setminus X')} w_b + \lambda |X'|. \quad (3.2)$$

Every frozen bond has the *hull weight* $w_b := 2 - \eta_w = \frac{19}{10}$.

Lemma 3.9 (Min-cut structure of the hull). *Let $S \subseteq E$ carry weights $w_b \in [0, 2]$. (a) F_λ^S is the sum of the cut function of a graph on Λ° with nonnegative capacities and of a modular function; in particular it is submodular. (b) For every $Y \subseteq \Lambda^\circ$, the minimizers of F_λ^S over $\{X' : Y \subseteq X' \subseteq \Lambda^\circ\}$ are closed under union and intersection; the smallest, $H_\lambda^S(Y)$, is the source side of a minimum s - t cut in a network with nonnegative capacities. Bonds of S that touch Y may be added to S or removed from it, with any weights, without changing $H_\lambda^S(Y)$. (c) H_λ^S is extensive and idempotent: $Y \subseteq H_\lambda^S(Y) = H_\lambda^S(H_\lambda^S(Y))$. (d) Let S, S' carry weights that agree on $S \cap S'$. If $Y \subseteq Y'$ and $\mathcal{R}(Y, S) \subseteq \mathcal{R}(Y', S')$, where $\mathcal{R}(Y, S) := S \cup \{b : b \text{ touches } Y\}$, then $H_\lambda^S(Y) \subseteq H_\lambda^{S'}(Y')$. In particular H_λ^S is a closure operator for fixed S and monotone in S .*

Proof. For (a), we write a frozen bond of weight $w \leq 2$ as a modular term plus a cut term of capacity $\frac{2-w}{2} \geq 0$. Weights above 2 would give negative capacities, which is why frozen bonds weigh at most 2. Indeed, with $c_b(X') := \mathbf{1}[\text{exactly one end of } b = \{u, v\} \text{ in } X']$, the three cases give $w_b \mathbf{1}[u, v \notin X'] + c_b(X') = w_b - \frac{w_b}{2}(\mathbf{1}[u \in X'] + \mathbf{1}[v \in X']) + \frac{2-w_b}{2}c_b(X')$ for $b \in S$, hence

$$F_\lambda^S(X') = \sum_{b \in E} \kappa_b c_b(X') + \sum_{x \in X'} \mu(x) + \sum_{b \in S} w_b, \quad (3.3)$$

with $\kappa_b := 1$ for $b \notin S$, $\kappa_b := (2 - w_b)/2$ for $b \in S$, and $\mu(x) := \lambda - \frac{1}{2} \sum_{b \in S, b \ni x} w_b$. This proves (a). Parts (b) and (c) are the lattice and closure properties of the minimizers of a submodular function on $\{X' \supseteq Y\}$. The network has capacities κ_b on the bonds, $\mu(x)^\pm$ to the sink or from the source, and ∞ from the source to Y . On $\{X' \supseteq Y\}$, a bond of S touching Y contributes 1 or 0 whatever its weight. (d) By (b) we may add the bonds of $S \setminus S'$, which touch Y' , to S' with their S -weights, so that $S \subseteq S'$. Then $g := F_\lambda^{S'} - F_\lambda^S = \sum_{b \in S' \setminus S} w_b \mathbf{1}[\text{no end of } b \text{ in } X']$ is non-increasing. With $M := H_\lambda^S(Y)$ and $M' := H_\lambda^{S'}(Y')$, submodularity gives $F_\lambda^S(M \cap M') + F_\lambda^{S'}(M \cup M') \leq F_\lambda^S(M) + F_\lambda^S(M') + g(M \cup M') \leq F_\lambda^S(M) + F_\lambda^{S'}(M')$. Each term on the left is at least the corresponding one on the right. Hence $M \cap M' \supseteq Y$ is a minimizer, and $M \subseteq M'$.

Definition 3.10 (Hull step). In a history, the *hull step* at time t replaces the hole set by $H_\lambda^{\bar{S}_t}(B_t)$, the hull from the base computed with the ever-frozen set and the hull weight, and converts the frozen bonds that touch it. A time t (or its state) is *hull-closed* if $\mathbb{X}_t = H_\lambda^{\bar{S}_t}(B_t)$.

Definition 3.11 (Forced blocks; the regularization functional). For a block Q let $3Q$ be the union of the at most 27 blocks at block distance at most 1 from Q , and $N(3Q) := |\mathbb{M} \cap 3Q|$. The block Q is *forced* if $N(3Q) \geq \vartheta L_c^2$, where $\vartheta := 3 \cdot 10^{-5}$ is the *forcing threshold*; $\mathfrak{B}_F = \mathfrak{B}_F(D)$ is the set of forced blocks. Put $\lambda_U := 2c_s/\ell$ and $\mu_B := 25$. For a set $\bar{S}$ of bonds that cross block faces, each with the hull weight, and for pairs $(\mathbb{X}, \mathbb{U})$ with $\mathbb{M} \subseteq \mathbb{X} \subseteq \Lambda^\circ$ and $\mathbb{U} \subseteq \mathbb{X}$ a union of blocks,

$$\mathcal{E}^{\bar{S}}(\mathbb{X}, \mathbb{U}) := F_\lambda^{\bar{S}}(\mathbb{X}) - (\lambda - \lambda_U)|\mathbb{U}| + \mu_B L_c^2 \#\{Q \in \mathfrak{B}_F : Q \not\subseteq \mathbb{U}\}. \quad (3.4)$$

Filling a block Q costs at most $|\partial Q| + \lambda_U|Q| < 24.6L_c^2 < \mu_B L_c^2$. Hence forced blocks are filled (Lemma 4.9), at a cost of at most $27 \cdot 24.6/\vartheta$ per mark. Filling a removal cell Δ costs at most w_* , and therefore a minimizer fills each cell whose insulation exceeds w_* (Theorem 4.17). The arithmetic behind these bounds is in Section S1.2.

Lemma 3.12 (Min-cut structure of the regularization functional). *Let $\bar{S}$ be as in Definition 3.11. (a) $\mathcal{E}^{\bar{S}}$ is, up to a constant, the cut function of an s - t network with nonnegative capacities; its minimizers are closed under componentwise union and intersection, and the smallest one, $(\mathbb{X}^*, \mathbb{U}^*)$, is the source side of a minimum cut, a deterministic function of $\mathbb{M}$ and $\bar{S}$. (b) $\mathbb{X}^* = H_\lambda^{\bar{S}}(\mathbb{M} \cup \mathbb{U}^*)$. (c) The smallest minimizer is componentwise nondecreasing in $\bar{S}$.*

Proof. (a) We add to the network of Lemma 3.9(b) for $F_\lambda^{\bar{S}}$ (with ∞ from the source to the marks) a node u_Q per block Q , with capacity ∞ from u_Q to each site of Q and $(\lambda - \lambda_U)|Q| + \mu_B L_c^2 \mathbf{1}[Q \in \mathfrak{B}_F]$ from the source to u_Q . Putting u_Q on the source side means $Q \subseteq \mathbb{U}$. A finite cut has $\mathbb{M} \cup \mathbb{U} \subseteq \mathbb{X}$, and since the blocks tile Λ° (G2), its capacity is $\mathcal{E}^{\bar{S}}(\mathbb{X}, \mathbb{U}) + (\lambda - \lambda_U)|\Lambda^\circ|$ up to the constant of (3.3). The capacities are nonnegative ($\lambda_U < \lambda$). Hence the minimum cuts form a lattice, and the smallest source side is the set reachable from the source in the residual network of a maximum flow. (b) For fixed $\mathbb{U}^*$, $\mathcal{E}^{\bar{S}}(\cdot, \mathbb{U}^*) - F_\lambda^{\bar{S}}$ is constant on $\{\mathbb{X} \supseteq \mathbb{M} \cup \mathbb{U}^*\}$, and a smaller

minimizer there would give a smaller joint minimizer. (c) Let $\bar{S} \subseteq \bar{S}'$, and m, m' the smallest minimizers of $\mathcal{E} := \mathcal{E}^{\bar{S}}$ and $\mathcal{E}' := \mathcal{E}^{\bar{S}'} = \mathcal{E} + g$, $g(\mathbb{X}, \mathbb{U}) := \sum_{b \in \bar{S}' \setminus \bar{S}} w_b \mathbf{1}[\text{no end of } b \text{ in } \mathbb{X}]$, which does not increase when the pair grows. Minimality of m' for $\mathcal{E}'$ and $g(m \cup m') \leq g(m')$ give $\mathcal{E}(m') \leq \mathcal{E}(m \cup m')$. Combined with submodularity, this yields $\mathcal{E}(m \cap m') \leq \mathcal{E}(m)$. Thus $m \cap m'$ is a minimizer of $\mathcal{E}$, and $m \subseteq m'$.

For a state $(\mathbb{X}, \mathbb{U}, S)$ and a bond set A , we define the *two-sided insulation* by $\mathcal{A}_2(A) := \#\{b \in A : b \text{ has exactly one end in } \mathbb{X}\} + 2\#(S \cap A)$. Here a frozen bond counts 2 whatever its weight.

Definition 3.13 (Removal). Let σ be a hull-closed state with base B . For a removal cell $\Delta \not\subseteq B$ put $w(\Delta) := \mathcal{A}_2(E(\Delta))$, with $E(\Delta)$ the set of bonds with both ends in Δ and S the current frozen set. The *removal step* removes, all at once, every removal cell $\Delta \not\subseteq B$ with $w(\Delta) > w_* := (1 + c_s)|\partial\Delta|$, where $c_s \in (0, 10]$ is the removal slack: $\mathbb{X} := \mathbb{X} \cup \Delta$, $\mathbb{U} := \mathbb{U} \cup \Delta$, and the frozen bonds touching Δ are converted. A hull step follows every removal step.

Definition 3.14 (Cells, interfaces). Let σ be a state, $G' = G'(\sigma)$, and for $\hat{Q} \in \mathcal{Q}$ let $G' \cap \hat{Q}$ be the subgraph of G' induced on $\hat{Q} \setminus \mathbb{X}$, with components $\hat{\mathcal{K}}$. For a block $Q \not\subseteq \mathbb{X}$, the *cells* of Q are the sets $\Omega_{Q, \hat{\mathcal{K}}} := Q \cap \hat{\mathcal{K}}$ for the components $\hat{\mathcal{K}}$ of $G' \cap \hat{Q}(Q)$ that meet Q . For a block face F' between blocks $Q, Q' \not\subseteq \mathbb{X}$, an *interface* I is a nonempty set consisting of all G' -bonds across F' that join a given cell of Q to a given cell of Q' , and $F(I) := F'$; frozen bonds and hole faces belong to no interface. A bond $b \in I$ is *good* if its tube τ_b is a G' -path (its sites lie in $\Lambda^\circ \setminus \mathbb{X}$ and its bonds are G' -bonds), and $\text{good}(I)$ is the number of good bonds of I .

Definition 3.15 (Weak links). Let $\hat{Q} \in \mathcal{Q}$, $\hat{\mathcal{K}}$ a component of $G' \cap \hat{Q}$, and $f_{G'}(A, A')$ the number of G' -bonds between disjoint site sets A, A' . For $\chi' > 0$, $A \subseteq \hat{\mathcal{K}}$ is a χ' -*weak link* of $\hat{\mathcal{K}}$ if $0 < |A| \leq |\hat{\mathcal{K}}|/2$ and $f_{G'}(A, \hat{\mathcal{K}} \setminus A) < \chi'|A|/k_P$. The *weak-link constants* are $\chi := 2.0 \cdot 10^{-8}$ and $\chi_{N3} := \chi/3$.

Weak links are excluded by Theorem 4.16(c).

Definition 3.16 (Auxiliary material; the seal). Let $\sigma = (\mathbb{X}, \mathbb{U}, S)$ be a hull-closed state with ever-frozen set $\bar{S}$, and j a surviving block; put $V_j := j \setminus \mathbb{X}$. The *auxiliary functional* F_j^{aux} is $F_\lambda^{\bar{S} \cup \partial j}$, every bond of ∂j counted as frozen with the hull weight, on the sets Y with $\mathbb{X} \subseteq Y \subseteq \mathbb{X} \cup V_j$. Its minimizers are closed under union and intersection (Lemma 3.9(a)); the smallest one is M_j , and $A_j := M_j \setminus \mathbb{X}$ is the *auxiliary material* of j . The *seal step* freezes every G' -bond that crosses a face between two surviving blocks and has an end in $\bigcup_j A_j$; these bonds form S_1 , with the hull weight. A hull step follows every seal step.

The set M_j is obtained from one maximum flow inside j . A block with a thin piece, attached to the rest of the block by few bonds however large its contacts across the faces of j , has auxiliary material (Lemma 4.13). After the seal, the next hull step fills the auxiliary material (Lemma 4.14).

3.3 Phase 1 and Phase 2

Definition 3.17 (Phase 1; the terminal properties). *Phase 1* starts from the state $(\mathbb{M}, \emptyset, \emptyset)$ with $\bar{S} := \emptyset$ and repeats the following *outer step* until an outer step changes neither $\mathbb{U}$ nor $\bar{S}$, that is, until both loops are at rest: (1) $(\mathbb{X}, \mathbb{U}) :=$ the smallest minimizer of $\mathcal{E}^{\bar{S}}$ (Lemma 3.12);

(2) with $\mathbb{U}$ fixed, as long as some surviving block has auxiliary material, perform a seal step, add its bonds to $\bar{S}$, and perform the hull step $\mathbb{X} := H_{\lambda}^{\bar{S}}(\mathbb{M} \cup \mathbb{U})$. Its final state $(\mathbb{X}^1, \mathbb{U}^1, S^1)$, with $S^1 := \bar{S} \cap E(\Lambda^\circ \setminus \mathbb{X}^1)$, base $B^1 := \mathbb{M} \cup \mathbb{U}^1$ and ever-frozen set $\bar{S}^1 := \bar{S}$, is the *Phase-1 output*. The *terminal properties* of a hull-closed state with base B , frozen set S and ever-frozen set $\bar{S}$ are: (S-hull) $\mathbb{X} = H_{\lambda}^{\bar{S}}(B) = H_{\lambda}^{\bar{S}}(\mathbb{X})$; (S-rem) no removal cell $\Delta \not\subseteq B$ has $w(\Delta) > w_*$; (S-face) every interface I has $\text{good}(I) \geq c_g |F(I)|$; (S-plug) no component of any $G' \cap \hat{Q}$, $\hat{Q} \in \mathcal{Q}$, has a χ_{N3} -weak link.

We will show in Theorem 4.17 that Phase 1 terminates and that its output is the smallest minimizer of $\mathcal{E}^{\bar{S}^1}$, has no auxiliary material and no frozen bond, and has the four terminal properties. Since $\bar{S}$ does not change in the last outer step, the output is the pair returned by its step (1), and step (2) finds no auxiliary material. Smallest minimizers are canonical, and therefore no tie-break enters Phase 1.

Definition 3.18 (Scale cost, fragmented splits, the cut rule, Phase 2). Let σ be a state with slit graph G' and cells as in Definition 3.14; every site of $\Lambda^\circ \setminus \mathbb{X}$ lies in exactly one cell, and every cell in one component of G' . A *cell set* is a union U of cells; its cut $\partial_{G'} U$ is the set of G' -bonds with exactly one end in U , and $|\partial_{G'} U|$ is its number of (coarse) bonds. U is *c-connected* if it is connected in the graph on U whose edges are the G' -bonds inside U and the pairs of sites in a common cell. The distance of two interfaces is the least Euclidean distance (fine units) between the coarse faces dual to their bonds; for a finite set $\mathcal{B}$ of interfaces and $s > 0$, $\varphi_s(\mathcal{B})$ is the number of classes of the equivalence relation generated by “distance $\leq s$ ”. Since $\partial_{G'} U$ is a union of whole interfaces (Lemma 4.27(a)), the *scale cost* $J(U) := \sum_{j \geq 0: 2^j L \geq R_{\text{NF}}} (\varphi_{2^j L}(\partial_{G'} U) - 1)$ is defined, where $R_{\text{NF}} := L e^{c_3^{\text{NF}} F_{\text{unit}}}$ (fine), $F_{\text{unit}} := \sigma_F L^2$ with σ_F the wall tension (7.15), and $\theta_{\text{NF}}, c_3^{\text{NF}} > 0$ are the cut-rule constants (values in Section S1.1.1). A pair $(\mathfrak{K}, U)$ is *fragmented* if $\mathfrak{K}$ is a component of G' , $U \subsetneq \mathfrak{K}$ is a nonempty c-connected cell set, $\mathfrak{K} \setminus U$ is c-connected, and $J(U) > \theta_{\text{NF}} F_{\text{unit}} |\partial_{G'} U| / L_c^2$. The *cut rule* acts at a hull-closed state at which no removal cell is over threshold and no surviving block has auxiliary material: a firing freezes $\partial_{G'} U$ for the first fragmented pair $(\mathfrak{K}, U)$ in a fixed lexicographic order that depends only on C ; these bonds form S_{cut} , have the hull weight and count 2 in $\mathcal{A}_2$ and in $w(\Delta)$; nothing is filled. *Phase 2* starts from the Phase-1 output, with base B^1 and ever-frozen set $\bar{S}^1$, and repeats the following pass until a full pass changes nothing: (1) a hull step; (2) a removal step, then a hull step; (3) a seal step, then a hull step; (4) if (1) to (3) changed nothing, one firing of the cut rule, then a hull step. Its terminal state $(\mathbb{X}, \mathbb{U}, S)$ is the output of the construction.

The coarse trigger implies the lattice form. Since a coarse face carries at least r_0^2 fine plaquettes, we have $|\partial_{G'} U|_{\text{fine}} / L^2 \geq |\partial_{G'} U| / L_c^2$, and hence a split fragmented with the fine cut size is fragmented.

Proposition 3.19 (Termination, joint fixed point, (NF_*)). *Phase 2 terminates. Its terminal state has (S-hull), (S-rem), (S-face), (S-plug) on the global slit graph and (S-NF): for every component $\mathfrak{K}$ of the final G' and every nonempty c-connected cell set $U \subsetneq \mathfrak{K}$ with $\mathfrak{K} \setminus U$ c-connected,*

$$J(U) \leq \theta_{\text{NF}} F_{\text{unit}} |\partial_{G'} U| / L_c^2. \quad (\text{NF}_*)$$

If the Euclidean diameter of Λ° (fine units) is at most R_{NF} , then Phase 2 makes no firing and the output is the Phase-1 output.

3.4 The output and the regularization theorem

The output. The output $(\mathbb{X}, \mathbb{U}, S)$, with base $B = \mathbb{M} \cup \mathbb{U}$ and $S = S_1 \sqcup S_{\text{cut}} \subseteq E(\Lambda^\circ \setminus \mathbb{X})$, defines the slit domain $G' = G'(D) := G'(\mathbb{X}, S)$, the free set $\mathfrak{F}$ of (2.9) and the min-cut area $A_{G'}(C)$ of Definition 2.11. The fine slit graph is that of Section 2.5, and cells and interfaces are those of Definition 3.14. The *cell graph* of a component $\mathfrak{K}$ of G' has the cells in $\mathfrak{K}$ as vertices and the interfaces as edges. The weight of a cell Ω is $|\Omega|_{\text{fine}}$, its number of fine cubes. The $\mathcal{U}$ -cells are the parts of a partition of each cell graph given by Lemma 4.21 with $V = V_{\mathcal{U}} := \tilde{m}^{-3}$ (one fixed choice for each G'). A component of weight $< V_{\mathcal{U}}$ is a single $\mathcal{U}$ -cell.

Theorem 3.20 (Regularization theorem: terminal form). *For every $D \subseteq \Lambda_2^\circ \setminus \partial_2 \Lambda^\circ$:*

(R1) (Termination, determinism.) *Phase 1 and Phase 2 terminate; the output $(\mathbb{X}, S)$ is a deterministic function of D ; and $\mathbb{M} \subseteq \mathbb{X}$, so that $D \subseteq F(\mathbb{M}) \subseteq F(\mathbb{X}) \subseteq \mathfrak{F}$.*

The output satisfies the hull identity and the terminal properties, read on its slit graph: (N1) (S-hull), (N2) (S-face), (N3) (S-plug) and (N4) (S-rem) of Definition 3.17, and (N5) (S-NF) of Proposition 3.19. Consequently the following hold, with constants that depend only on the constants fixed before K and on none of β, τ, K, C, D , except c'_g and c_u , which see β only through the rounding of L_c and lie between two such constants (Section S1.1.1).

(R2) (Geometry.) *For every $W \subseteq \Lambda^\circ \setminus \mathbb{X}$, $|\partial_{\Lambda^\circ} W| \leq a_\eta |\partial_{G'} W| + b_\eta \lambda |W|$ with $(a_\eta, b_\eta) = (19/9, 10/9)$, where $\partial_{\Lambda^\circ} W$ is the set of bonds of E with exactly one end in W and $\partial_{G'} W$ the set of G' -bonds with exactly one end in W ; with $c_I := 2^{2/3}$, $|\partial_{G'} W| \geq (9c_I/38)|W|^{2/3}$ whenever $|W| \leq V_*$; and no component of G' has at most V_* sites.*

(R5) (Cell volume.) *Every cell satisfies $c_1 L_c^3 \leq |\Omega| < 8L_c^3$, with $c_1 := \min(c_g/2, ((1 - \eta_w)c_I/24)^3)$.*

(R6) (Poincaré inequality on cells.) *For every cell Ω of a block Q there is a connected set $\mathfrak{K}^b(\Omega)$ with $\Omega \subseteq \mathfrak{K}^b(\Omega) \subseteq \hat{Q}(Q)$ such that every $A \subseteq \mathfrak{K}^b(\Omega)$ with $0 < |A| \leq |\mathfrak{K}^b(\Omega)|/2$ has $f_{G'}(A, \mathfrak{K}^b(\Omega) \setminus A) \geq \chi |A|/(3k_P)$ (expansion form), and, for every $f : \Lambda^\circ \setminus \mathbb{X} \rightarrow \mathbb{R}$,*

$$\sum_{x \in \Omega} |f(x) - \bar{f}_\Omega|^2 \leq \frac{108}{\chi^2} k_P^2 \sum_{\{x,y\} \in E_{G'}(\mathfrak{K}^b(\Omega))} (f(x) - f(y))^2,$$

where $\bar{f}_\Omega$ is the average of f over Ω and $E_{G'}(\cdot)$ the set of G' -bonds with both ends in the set; each G' -bond lies in $E_{G'}(\mathfrak{K}^b(\Omega))$ for at most $N_{\text{ov}} \leq 27(k_P/L_c)^3$ cells Ω .

(R7) (Cells and $\mathcal{U}$ -cells.) *Every cell lies in one block and in one component of G' ; every interface lies in one block face, and that face lies in Ξ or contains no plaquette of Ξ . Every $\mathcal{U}$ -cell is a union of cells, connected in the cell graph, of weight in $[V_{\mathcal{U}}, (2+6/c_g)V_{\mathcal{U}})$, or a whole component of G' of weight in $(c_u V_{\mathcal{U}}, V_{\mathcal{U}})$, with $c_u := r_0^3 V_* \tilde{m}^3 = ((1 - \eta_w)c_I \tilde{m} L/24)^3$; and every $\mathcal{U}$ -cell has Euclidean diameter at most $C_{\mathcal{U}} \tilde{m}^{-1}$.*

(R8) (Interfaces.) *Every interface I has $\text{good}(I) \geq c_g |F(I)| \geq c_g \max(|I|, L_c^2)$; good bonds on one line are at distance $\geq L_c$; the tube of a good bond of I lies in $Q \cup Q' \subseteq \hat{Q}(Q) \cap \hat{Q}(Q')$, with one half in each cell of I ; a block face carries at most $1/c_g$ interfaces, and a cell has at most $6/c_g$ neighbouring cells. In fine units, $\text{good}_{\text{fine}}(I) \geq c'_g \max(|I|_{\text{fine}}, L^2)$ with $c'_g = c_g(1 + 1/L_c)^{-2}$.*

(R9) (Density bound.) *Every removal cell Δ satisfies $\mathcal{A}_2(E(\Delta)) \leq w_*$ and, counting every bond of E with an end in Δ , $\mathcal{A}_2(E(\Delta) \cup (\partial\Delta \cap E)) \leq w_* + 2|\partial\Delta| = (3 + c_s)|\partial\Delta|$.*

(R10) (Non-fragmentation.) (NF_{*}) *holds in the split form (S-NF) of Proposition 3.19.*

Item (R1) and the terminal properties (N1) to (N5) come from the construction: we prove them in Section 4 from Propositions 4.3(b), (d) and 3.19. Items (R2) and (R5) to (R10) then follow from the terminal state alone, through (N1) to (N5), (R10) being (N5). We prove them in Section 4. The construction gives more, namely one cell per surviving block, of volume above $(1 - 3\vartheta)L_c^3$, and one interface per block face (Theorem 4.16), but of the cell and interface properties later sections use only (R5) to (R8).

Below we write $P^{+R} := \{x : \text{dist}(x, P) \leq R\}$ and $r' := 2R_* + 2$, where R_* is the clearing-out radius of Theorem 4.23. We let $\mathcal{P}$ be the partition of $\mathbb{M}$ into the components of the graph joining two marks at distance at most r' . The constants C'_* , C_{free} are those of Proposition 4.24 (values in Section S1.1.1).

Theorem 3.21 (Regularization theorem: history form). (R3) (Block-local hull.) *At every hull-closed time of the full history (Definition 4.1), in particular at the output, for every block q , $|(\mathbb{X} \setminus B) \cap q| < 2L_c m_q$ unless it vanishes, where m_q is the number of marks in q . So hull material lies only in blocks containing a mark: frozen bonds, filled blocks, removal cells and box faces support no hull material, alone or jointly.*

(a) (Area per part.) *The Phase-1 output $(\mathbb{X}^1, \mathbb{U}^1, S^1)$ has $S^1 = \emptyset$. For $P \in \mathcal{P}$ put $Y_P := \mathbb{X}^1 \cap P^{+R_*}$. Then $\mathbb{X}^1 = \bigsqcup_{P \in \mathcal{P}} Y_P$, no bond joins Y_P to $Y_{P'}$ for $P \neq P'$, every block of $\mathbb{U}^1$ lies in one Y_P , and*

$$|\partial Y_P| + \lambda|Y_P \setminus \mathbb{U}^1| + \lambda_U|Y_P \cap \mathbb{U}^1| \leq C'_* N(P), \quad 2|\partial Y_P| + |Y_P|/\ell \leq C_{\text{free}} N(P).$$

(b) (Uniform reach.) *Every site of $\mathbb{X}^1$ lies within distance R_* of $\mathbb{M}$, uniformly in $|\mathbb{M}|$, Λ° , C and D ; so $Y_P \subseteq P^{+R_*}$.*

The constants C'_* , C_{free} and the ratio $R_*/(\ell \log \ell)$ are bounded independently of β , τ , K , C and D .

(R3) is Corollary 4.7, (a) is Proposition 4.24 and (b) is Theorem 4.23.

4 The regularization: proofs

In this section we prove Theorems 3.20 and 3.21 and Proposition 3.19. We first fix the class of histories to which the arguments apply.

Definition 4.1 (Admissible history). An *admissible history* is a finite sequence of states $\sigma_t = (\mathbb{X}_t, \mathbb{U}_t, S_t)$, $0 \leq t \leq T$, such that (H1) $\sigma_0 = (\mathbb{M}, \emptyset, \emptyset)$ and the first step is a hull step; (H2) every other step is a hull step or, at a hull-closed state, a *fill*, which adds a union $\mathbb{U}'$ of blocks to $\mathbb{U}$ and to $\mathbb{X}$ and converts the frozen bonds touching $\mathbb{U}'$, or a *freeze*, which adds to S a nonempty set of G' -bonds crossing block faces, each with its kind; (H3) every fill and every freeze is followed by a hull step. Phase 1 is admissible: its first outer step is a hull step, the fill of $\mathbb{U}^*$ and a hull step, every later outer step that changes $\mathbb{U}$ is a fill and a hull step (Lemmas 3.12(b), (c) and 4.2), and a seal step is a freeze. So is the *full history*, Phase 1 followed by Phase 2: removal steps are fills, seal steps and cut firings freezes.

4.1 The hull

We first show that the hull steps of an admissible history only enlarge the hole set (Lemma 4.2), and then derive the geometric consequences of the minimality of the hull.

Lemma 4.2 (One hull from the base). *Let (σ_t) satisfy (H1) to (H3) of Definition 4.1. At every hull-closed time t , $\mathbb{X}_t = H_{\lambda}^{\bar{S}_t}(B_t) = H_{\lambda}^{S_t}(\mathbb{X}_t)$ and $S_t \subseteq \bar{S}_t \subseteq E(\Lambda^\circ \setminus B_t)$, and every bond of $\bar{S}_t \setminus S_t$ touches $\mathbb{X}_t$. Every hull step $t \rightarrow t+1$ satisfies $\mathbb{X}_{t+1} = H_{\lambda}^{S_t}(\mathbb{X}_t) \supseteq \mathbb{X}_t$.*

Proof. We use that H is a closure operator, jointly monotone in the base and in the set of missing bonds (Lemma 3.9(d)). We argue by induction over the steps. By (H1) to (H3), each time is 0, or hull-closed, or follows a fill or a freeze at a hull-closed time t and is followed by a hull step. A hull step at a hull-closed time changes nothing, and the first hull step gives $H_{\lambda}^{\emptyset}(B_0) \supseteq B_0 = \mathbb{X}_0$. A freeze at t of a set A of G' -bonds, which do not touch $\mathbb{X}_t \supseteq B_t$, keeps B and $\mathbb{X}$ and gives $\bar{S}_{t+1} = \bar{S}_t \cup A$. Hence $\mathbb{X}_t = H_{\lambda}^{\bar{S}_t}(B_t) \subseteq H_{\lambda}^{\bar{S}_{t+1}}(B_t) = \mathbb{X}_{t+2}$. A fill of $\mathbb{U}'$ gives $B_{t+1} = B_t \cup \mathbb{U}'$ and drops from $\bar{S}$ only bonds touching B_{t+1} . Hence $\mathcal{R}(B_t, \bar{S}_t) \subseteq \mathcal{R}(B_{t+1}, \bar{S}_{t+1})$ and $\mathbb{X}_t \subseteq \mathbb{X}_{t+2}$. In this case, moreover, $\mathbb{U}' \subseteq B_{t+1} \subseteq \mathbb{X}_{t+2}$. In both cases $B_{t+1} \subseteq \mathbb{X}_{t+1} \subseteq \mathbb{X}_{t+2}$, and monotonicity and idempotence give $H_{\lambda}^{\bar{S}_{t+1}}(\mathbb{X}_{t+1}) = \mathbb{X}_{t+2}$. We now pass from $\bar{S}$ to S . Since a bond of $S_{t'}$ does not touch $\mathbb{X}_{t'} \supseteq B_{t'}$, we have $S_{t'} \subseteq \bar{S}_{t'}$. A bond of $\bar{S}_{t'} \setminus S_{t'}$ was converted, so it touches an earlier hole set and hence $\mathbb{X}_{t'}$, and by Lemma 3.9(b) it may be dropped: $H_{\lambda}^{\bar{S}_{t'}}(\mathbb{X}_{t'}) = H_{\lambda}^{S_{t'}}(\mathbb{X}_{t'})$. With $t' = t+1$ this is the claim on hull steps. At a hull-closed time, with $B_t \subseteq \mathbb{X}_t = H_{\lambda}^{\bar{S}_t}(B_t)$ and idempotence, the same identity gives $\mathbb{X}_t = H_{\lambda}^{S_t}(\mathbb{X}_t)$. Finally $\bar{S}_t \subseteq E(\Lambda^\circ \setminus B_t)$ by definition.

Proof of (R1) and (N1) to (N5). These properties follow from Propositions 4.3(b), (d) and 3.19.

Proposition 4.3 (Termination, determinism, terminal state). *(a) Order states by $\sigma \leq \sigma'$ iff $\mathbb{X} \subseteq \mathbb{X}'$, $\mathbb{U} \subseteq \mathbb{U}'$ and $\mathcal{R}(\sigma) \subseteq \mathcal{R}(\sigma')$. Under (H1) to (H3), every step that changes the state strictly increases it; in particular (H4) $\mathbb{X}_t \subseteq \mathbb{X}_{t+1}$ and $\mathcal{R}(\sigma_t) \subseteq \mathcal{R}(\sigma_{t+1})$ for every t , so that G' only shrinks, and a bond is frozen at most once along an admissible history. (b) Phase 1 and Phase 2 terminate, and their outputs are deterministic functions of D . (c) The terminal state of Phase 2 satisfies (S-hull), (S-rem) and (S-NF), and no surviving block has auxiliary material there. (d) $\mathbb{M} \subseteq B_t \subseteq \mathbb{X}_t$ at every time; hence, for the output, $D \subseteq F(\mathbb{M}) \subseteq F(\mathbb{X}) \subseteq \mathfrak{F}$.*

Sketch. Hull steps do not shrink $\mathbb{X}$ (Lemma 4.2), a fill adds blocks to $\mathbb{U}$, a freeze adds G' -bonds to S , and converted bonds stay missing. This gives (a). Phase 1 terminates (Theorem 4.17), and Phase 2 terminates because each effective step strictly increases the state in a finite box. Smallest minimizers are canonical and the cut rule uses a fixed order, which gives determinism. A terminal pass of Phase 2 changes nothing, which together with Lemmas 4.2 and 4.14(a) gives (c). Finally, (d) holds because the base stays in $\mathbb{X}$.

Below, we denote by $f(A, A')$ the number of bonds of E between disjoint site sets A, A' ($f_{G'}$ counts G' -bonds, Definition 3.15), by $s(A, A')$ and $s(E(A))$ the numbers of bonds of S between A and A' and with both ends in A , and we put $N := |\mathbb{M}|$. A state is *closed* if $\mathbb{X} = H_{\lambda}^S(\mathbb{X})$ for the hull weight. By Lemma 4.2, each hull-closed state of an admissible history is closed, and so is the output.

Proof of (R2). We apply Lemma 4.4 and Theorem 4.5(i), (ii) at the output, which is closed by (N1).

Lemma 4.4 (Two-sided minimality inequality). *Let $(\mathbb{X}, \mathbb{U}, S)$ be closed, $V := \Lambda^\circ \setminus \mathbb{X}$ and $W \subseteq V$. Then*

$$f(W, \mathbb{X}) + \sum_{b \in S, |b \cap W|=1} (w_b - 1) + \sum_{b \in S, b \subseteq W} w_b \leq |\partial_{G'} W| + \lambda |W|. \quad (4.1)$$

Consequently, with $(a_\eta, b_\eta) = (1 + \frac{1}{1-\eta_w}, \frac{1}{1-\eta_w}) = (\frac{19}{9}, \frac{10}{9})$, $|\partial_{\Lambda^\circ} W| \leq a_\eta |\partial_{G'} W| + b_\eta \lambda |W|$, and $N_2(W) := f(W, \mathbb{X}) + s(W, V \setminus W) + 2s(E(W)) \leq (|\partial_{G'} W| + \lambda |W|)/(1 - \eta_w)$.

Proof. Filling W does not lower the functional: $F_\lambda^S(\mathbb{X} \cup W) \geq F_\lambda^S(\mathbb{X})$. Since bonds of S do not touch $\mathbb{X}$, passing from $\mathbb{X}$ to $\mathbb{X} \cup W$ changes a bond between W and $\mathbb{X}$ from 1 to 0, a G' -bond between W and $V \setminus W$ from 0 to 1, a bond of S between W and $V \setminus W$ from w_b to 1 and a bond of S inside W from w_b to 0, and adds $\lambda |W|$. This is (4.1). The hull weight has $w_b - 1 = 1 - \eta_w$ and $w_b \geq 2(1 - \eta_w)$, which gives the bound on $N_2(W)$. Finally, $|\partial_{\Lambda^\circ} W| = f(W, \mathbb{X}) + s(W, V \setminus W) + |\partial_{G'} W| \leq |\partial_{G'} W| + N_2(W)$.

Theorem 4.5 (Geometry of the slit domain). *Let $(\mathbb{X}, \mathbb{U}, S)$ be closed and $c_I := 2^{2/3}$. For $W \subseteq \Lambda^\circ \setminus \mathbb{X}$: (i) $|\partial_{G'} W| \geq (9c_I/38)|W|^{2/3}$ whenever $|W| \leq V_* = (1 - \eta_w)^3 \rho_\lambda^3/2$; (ii) no component of G' has at most V_* sites; (iii) $|B_{G'}(x, r)| \geq \min\{(c_I r/(48a_\eta))^3, V_*\}$ for all $x \in \Lambda^\circ \setminus \mathbb{X}$ and integers $r \geq 0$; (iv) $\lambda_1^D(\Omega) \geq (c_I^2/(48a_\eta^2))|\Omega|^{-2/3}$ for $\Omega \subseteq \Lambda^\circ \setminus \mathbb{X}$ with $|\Omega| \leq V_*$.*

Proof. (i) In a box $\prod_{i=1}^3 \{1, \dots, n_i\}$, a line parallel to e_i that meets W carries a bond of the box with exactly one end in W unless its n_i sites lie in W , which happens for at most $|W|/n_i$ lines, and the lines meeting W number at least $3|W|^{2/3}$ by the discrete Loomis–Whitney inequality [LW49, Theorem 2] and the arithmetic-geometric mean inequality. Hence, in this box, such bonds number at least $3|W|^{2/3} - |W| \sum_i n_i^{-1} \geq c_I |W|^{2/3}$ if $|W|^{1/3} \leq 0.47 \min_i n_i$ (as $1 - 2^{2/3}/3 = 0.4708\dots$). Lemma 4.4 controls the boundary in the box up to $b_\eta \lambda |W|$, which is small below $V_* = (c_I \rho_\lambda/(2b_\eta))^3$ (as $(c_I/2)^3 = \frac{1}{2}$). Since $|W|^{1/3} \leq V_*^{1/3} < \rho_\lambda = L_c/12$ and each side of Λ° is at least $\ell \geq 308L_c$ (Lemma 3.3(c)), this and Lemma 4.4 give $c_I |W|^{2/3} \leq |\partial_{\Lambda^\circ} W| \leq a_\eta |\partial_{G'} W| + b_\eta \lambda |W|^{1/3} |W|^{2/3} \leq a_\eta |\partial_{G'} W| + \frac{c_I}{2} |W|^{2/3}$. Hence $|\partial_{G'} W| \geq (c_I/(2a_\eta))|W|^{2/3} = (9c_I/38)|W|^{2/3}$. (ii) A component $\mathfrak{K}$ of G' has $\partial_{G'} \mathfrak{K} = \emptyset$, so (i) excludes $0 < |\mathfrak{K}| \leq V_*$. (iii) (Sketch.) We put $v(r) := |B_{G'}(x, r)|$. Each G' -bond leaving $B_{G'}(x, r)$ ends at distance $r + 1$. Thus $|\partial_{G'} B_{G'}(x, r)| \leq 6(v(r + 1) - v(r))$, and (i) gives $v(r + 1) \geq v(r) + \alpha v(r)^{2/3}$ with $\alpha := c_I/(12a_\eta) \leq \frac{1}{4}$ while $v(r) \leq V_*$. Hence $v(r + 1)^{1/3} > v(r)^{1/3} + \alpha/4$, and $v(r)^{1/3} \geq 1 + \alpha r/4 \geq c_I r/(48a_\eta)$ until v exceeds V_* . (iv) By (i), each nonempty $A \subseteq \Omega$ has $|\partial_{G'} A| \geq h|A|$ with $h := (9c_I/38)|\Omega|^{-1/3}$, and the Cheeger argument (the coarea formula for f^2 and the Cauchy–Schwarz inequality, in a graph of maximal degree 6) gives $\lambda_1^D(\Omega) \geq h^2/12 \geq (c_I^2/(48a_\eta^2))|\Omega|^{-2/3}$.

Segment accounting. In Lemma 4.6 and Corollary 4.7, $B = \mathbb{M}' \cup U$ with $\mathbb{M}'$ a set of sites and U a union of blocks, $S \subseteq E(\Lambda^\circ \setminus B)$ carries weights $w_b \in [0, 2]$, $\mathbb{X} := H_\lambda^S(B)$ and $W \subseteq \mathbb{X} \setminus B$. At a hull-closed time of an admissible history, this setting holds with $\mathbb{M}' = \mathbb{M}$, $U = \mathbb{U}_t$ and $S = \tilde{S}_t$ with the hull weight (Lemma 4.2). A *segment* R of W is a maximal set of consecutive sites of W on an axis-parallel line, and $m(R) := |R|$. It has $m(R) - 1$ *interior bonds* and two *ends*, the positions just beyond its extremities. An end is an $\mathbb{X}$ -end, a *frozen*

end, an open end or a box end according as it is a bond to $\mathbb{X} \setminus W$, a bond of S to $\Lambda^\circ \setminus \mathbb{X}$, a bond of $E \setminus S$ to $\Lambda^\circ \setminus \mathbb{X}$, or on $\partial\Lambda^\circ$. With $x(R)$, $o(R)$ the numbers of $\mathbb{X}$ -ends and open ends, we put $c_W(R) := x(R) + \sum_{\text{frozen ends } b} (w_b - 1) + \sum_{\text{interior } b \in S} w_b - o(R) - \lambda m(R)/3$. A grid bond has its ends in different blocks. Such a bond lies on a block plane. The grid-type terms of R are its $\mathbb{X}$ -ends, frozen ends and interior frozen bonds that are grid bonds, and its box ends. The number $x_{\text{ng}}(R)$ counts the $\mathbb{X}$ -ends, frozen ends and interior frozen bonds that are not grid bonds. Similarly, S^n is the set of bonds of S that are not grid bonds, and $f_{\text{ng}}(A, A')$ counts the bonds between A and A' that are not grid bonds. Frozen bonds cross block faces and are therefore grid bonds. Hence $S^n = \emptyset$ at each hull-closed time of an admissible history.

Lemma 4.6 (Plane count). *Let $\lambda \geq 12/L_c$. Every segment R of W satisfies $c_W(R) \leq 2x_{\text{ng}}(R) - 2m(R)/L_c$.*

Proof. Insulation on block planes, which are at least L_c apart, contributes at most 2 per plane met, while the volume term is $\lambda m/3 \geq 4m/L_c$. Indeed, since $o = 2 - x - e_{\text{fr}} - e_b$, with e_{fr} , e_b the numbers of frozen and box ends, we have

$$c_W(R) = 2x + \sum_{\text{frozen ends}} w_b + \sum_{\text{interior frozen}} w_b + e_b - 2 - \lambda m/3. \quad (4.2)$$

The terms that are not grid bonds contribute at most $2x_{\text{ng}}(R)$, since $w_b \leq 2$. Each grid-type term has coefficient at most 2 and sits at its own bond position on a block plane perpendicular to the line. The $m+1$ bond positions of R lie in an interval of length m , which meets at most $m/L_c + 1$ block planes by (G1). Therefore $c_W(R) \leq 2x_{\text{ng}}(R) + 2(m/L_c + 1) - 2 - 4m/L_c$.

Corollary 4.7 (Block-local hull). *For every union N of blocks, $W := (\mathbb{X} \setminus B) \cap N$ is empty or satisfies $|W| < \frac{L_c}{3}(f_{\text{ng}}(W, B) + \sum_{v \in W} \deg_{S^n}(v))$, and every bond counted in $f_{\text{ng}}(W, B)$ ends in $\mathbb{M}' \setminus U$. Consequently, for every block q , $|(\mathbb{X} \setminus B) \cap q| < (L_c/3)(6m_q + s_q)$ unless it vanishes, with $m_q := |\mathbb{M}' \cap q|$ and s_q the number of endpoints in q of bonds of S^n , and $\mathbb{X} \subseteq B \cup \bigcup \{q : q \cap (\mathbb{M}' \cup V(S^n)) \neq \emptyset\}$: grid bonds of S of any weight in $[0, 2]$, filled blocks and box faces support no hull material, alone or jointly. At a hull-closed time of an admissible history m_q is the number of marks in q and $S^n = \emptyset$, which gives (R3).*

Proof. Let $W \subseteq \mathbb{X} \setminus B$ be nonempty. Then the set $\mathbb{X} \setminus W \supseteq B$ is strictly smaller than the smallest minimizer $\mathbb{X}$. Passing from $\mathbb{X}$ to $\mathbb{X} \setminus W$ changes the contribution of an $\mathbb{X}$ -end, a frozen end, an open end and an interior frozen bond by $+1$, $w_b - 1$, -1 and $+w_b$, and the volume term by $-\lambda|W|$ (bonds of S do not touch B , and each site of W lies in three segments), so that

$$F_\lambda^S(\mathbb{X} \setminus W) - F_\lambda^S(\mathbb{X}) = \sum_R c_W(R) > 0, \quad (4.3)$$

over the segments of W in the three directions. For $W = (\mathbb{X} \setminus B) \cap N$, an $\mathbb{X}$ -end that is not a grid bond has both ends in one block $q \subseteq N$, so it ends in $\mathbb{X} \cap N \setminus W \subseteq B$ but not in U (otherwise $q \subseteq U$), and it is not frozen: it is counted in $f_{\text{ng}}(W, B)$. A frozen end that is not a grid bond is counted once in the degree sum, an interior bond of S^n twice. Combining (4.3), Lemma 4.6 and $\sum_R m(R) = 3|W|$, we obtain $0 < \sum_R c_W(R) \leq 2\sum_R x_{\text{ng}}(R) - 6|W|/L_c \leq 2(f_{\text{ng}}(W, B) + \sum_{v \in W} \deg_{S^n}(v)) - 6|W|/L_c$. For $N = q$, $f_{\text{ng}}(W, B) \leq 6m_q$ and the degree sum is at most s_q . At a hull-closed time, such as the output, $S^n = \emptyset$.

For a hull-closed time t and a block face F' between surviving blocks Q, Q' , in the plane P' , we put $W_t := (\mathbb{X}_t \setminus B_t) \cap (Q \cup Q')$ and $\text{supp}_t(F') := f_{\text{ng}}(W_t, B_t)$. For a line $\ell \perp F'$, b_ℓ denotes its bond across F' . As $B_t \cap (Q \cup Q')$ consists of marks, $\text{supp}_t(F') \leq 6(m_Q + m_{Q'})$.

Lemma 4.8 (Local opacity). *At every hull-closed time t of an admissible history and for every such face F' : (a) the lines $\ell \perp F'$ such that $b_\ell \notin \bar{S}_t$ and $\ell \cap Z(F')$ contains a site of W_t and no mark, number fewer than $2\text{supp}_t(F')$ unless there is none; (b) without the condition on b_ℓ , these lines number fewer than $2\text{supp}_t(F')/\eta_w$ unless there is none. A frozen bond is a grid bond, so by (G5) it lies on the tube of a line $\ell \perp F'$ only if it is b_ℓ : frozen bonds block no tube of another line.*

Proof. In outline, a line on which hull material meets the tube zone carries a segment that is either long, and pays through the plane count, or short, and then has no grid-type term except b_ℓ and has deficit at least η_w . We put $d(R) := 2x_{\text{ng}}(R) - c_W(R)$ for the segments of W_t . By Lemma 4.6, $d(R) \geq 2m(R)/L_c \geq 0$, and (4.3) gives $\sum_R d(R) < 2\sum_R x_{\text{ng}}(R) \leq 2\text{supp}_t(F')$, as in the proof of Corollary 4.7. For a counted line ℓ let R_ℓ be the segment of W_t along ℓ through a site of $\ell \cap Z(F') \cap W_t$. Distinct lines give distinct segments. If $m(R_\ell) \geq L_c/2$, then $d(R_\ell) \geq 1$. Otherwise all bond positions of R_ℓ are at distance $< L_c$ from P' , so by (G1) the only possible grid-type term is b_ℓ , which is not an $\mathbb{X}$ -end, since its other end would lie in $\mathbb{X}_t \cap (Q \cup Q') \setminus W_t \subseteq B_t$, a mark of $\ell \cap Z(F')$. Thus, in this case, $d(R_\ell) = 2 + \lambda m/3 - \beta_\ell$, with $\beta_\ell = 2 - \eta_w$ if $b_\ell \in \bar{S}_t$ and $\beta_\ell = 0$ otherwise. Hence $d(R_\ell) \geq 1$ for each line counted in (a) and $d(R_\ell) \geq \eta_w$ for each line counted in (b). Summing these bounds, we obtain (a) and (b). For the last claim, a tube consists of b_ℓ and of bonds interior to Q or to Q' , which are not grid bonds.

4.2 Cells, giants and the seal

In this subsection we control the cells by sparseness. Since forced blocks are filled, a surviving block has fewer than ϑL_c^2 marks near it, its hole sites lie on few lines (Corollary 4.7), and one component of its free sites, the giant, contains all but a small volume. The seal excludes thin sets, at a cost of at most 726 frozen bonds per mark (Theorem 4.16).

Lemma 4.9 (Forced blocks are filled). *Let $\bar{S}$ be as in Definition 3.11. The smallest minimizer $(\mathbb{X}^*, \mathbb{U}^*)$ of $\mathcal{E}^{\bar{S}}$ has $\bigcup \mathfrak{B}_F \subseteq \mathbb{U}^*$. Hence at every hull-closed time of the full history other than the first every forced block lies in $\mathbb{U}$, and every surviving block q has $m_q \leq N(3q) < \vartheta L_c^2$.*

Proof. Suppose that $Q \in \mathfrak{B}_F$ and $Q \not\subseteq \mathbb{U}^*$. Then $Q \cap \mathbb{U}^* = \emptyset$, and we compare with $(\mathbb{X}^* \cup Q, \mathbb{U}^* \cup Q)$. Only the at most $|\partial Q|$ bonds leaving Q gain weight, at most 1 each, the volume terms change by $\lambda_U |Q| - \lambda |Q \cap \mathbb{X}^*| \leq \lambda_U |Q|$, and the penalty drops by $\mu_B L_c^2 > |\partial Q| + \lambda_U |Q|$ (Section S1.2), a contradiction. Along the full history $\mathbb{U}$ never decreases (Proposition 4.3(a)), and the second hull-closed time follows the first outer step of Phase 1 (if that step fills nothing, $\mathfrak{B}_F = \emptyset$).

Lemma 4.10 (Sparse blocks). *At every hull-closed time of an admissible history, every surviving block q satisfies: (a) $|\mathbb{X} \cap q| < 3L_c m_q$ unless $\mathbb{X} \cap q = \emptyset$; (b) unless there is none, fewer than $9m_q$ maximal runs of $\mathbb{X}$ in the three directions lie inside q and are joined by both end bonds to sites of $V_q := q \setminus \mathbb{X}$.*

Proof. (a) The set $\mathbb{X} \cap q$ consists of the m_q marks of q and fewer than $2L_c m_q$ sites of hull material (Corollary 4.7). (b) At most $3m_q$ runs contain a mark. A run without marks is a segment R of $W := (\mathbb{X} \setminus B) \cap q$ with two open ends inside q (in-block bonds are not frozen) and no grid-type term. For such a segment, (4.2) gives $d(R) := 2x_{\text{ng}}(R) - c_W(R) = 2 + \lambda m(R)/3$. As in Lemma 4.8, $d \geq 0$ for each segment of W and $\sum_R d(R) < 2f_{\text{ng}}(W, B) \leq 12m_q$. Hence fewer than $6m_q$ runs have no mark.

Lemma 4.11 (Two isoperimetric inequalities). (a) Let Q be a box with sides L_1, L_2, L_3 , $Z \subseteq Q$, $Q \setminus Z = A \sqcup B$, and n_μ the number of lines of Q in direction μ meeting both A and B . Then $\min(|A|, |B|) \leq 4|Z| + 6.4 \max_\mu (n_\mu L_\mu)$. (b) Let R be a rectangle of $\mathbb{Z}^2$ with sides $s_1, s_2 \geq s$, and $\sigma \subseteq R$ with $|\sigma| \leq s^2/4$. At least $2|\sigma|^{1/2}$ bonds of R have exactly one end in σ .

Proof. (a) Let a_μ be the number of lines in direction μ meeting A but not B . Then $|A| \leq L_\mu(a_\mu + n_\mu)$, and the lines meeting B are among the other $|Q|/L_\mu - a_\mu$. Hence, by the discrete Loomis–Whitney inequality [LW49, Theorem 2], $|B|^2 \leq \prod_\mu (|Q|/L_\mu - a_\mu) \leq \prod_\mu (|Q| - |A| + n_\mu L_\mu)/L_\mu$. With $\alpha := |A|/|Q|$, $\beta := 1 - \alpha$, $\zeta := |Z|/|Q|$ and $\nu := \max_\mu n_\mu L_\mu/|Q|$ this reads $(\beta - \zeta)^2 \leq (\beta + \nu)^3$. If $\nu > 0.1$, then $\min(|A|, |B|) \leq |Q|/2 < 6.4\nu|Q|$. If $\nu \leq 0.1$, then $(\beta + \nu)^{3/2} \leq \beta^{3/2} + 1.6\nu$, so $\beta(1 - \sqrt{\beta}) \leq \zeta + 1.6\nu$. Since $1 - \sqrt{\beta} \geq (1 - \beta)/2$, in this case $\alpha(1 - \alpha) \leq 2\zeta + 3.2\nu$, and $\min(|A|, |B|) \leq 2|Q|\alpha(1 - \alpha) \leq 4|Z| + 6.4\nu|Q|$. (b) Each row and each column that meets σ without lying in it carries such a bond, a different one for each. If no row or column lies in σ , the r rows and c columns meeting σ give $r + c \geq 2(rc)^{1/2} \geq 2|\sigma|^{1/2}$ bonds. If a row lies in σ (a column is symmetric), all columns meet σ , and if moreover a column lies in σ , at most $|\sigma|/s_2 + |\sigma|/s_1 \leq s/2$ rows and columns lie in σ ; either way there are at least $s \geq 2|\sigma|^{1/2}$ bonds.

Proposition 4.12 (The giant). At every hull-closed time of the full history other than the first, every surviving block q has $|V_q| > (1 - 3\vartheta)L_c^3$, and the graph G'_q of the G' -bonds with both ends in q has a unique component C_q , the giant, with $|V_q \setminus C_q| < 128L_c m_q$ unless $V_q = C_q$.

Proof. Since bonds inside a block are never frozen, on a line of q that meets two disjoint unions A, B of components of G'_q , two consecutive sites of V_q in A and in B are separated by a run of Lemma 4.10(b), a different run for each line. If $A \sqcup B = V_q$, Lemma 4.11(a) with $Z = \mathbb{X} \cap q$, $|Z| < 3L_c m_q$ and $L_\mu < 2L_c$ gives $\min(|A|, |B|) < 12L_c m_q + 6.4 \cdot 9m_q \cdot 2L_c < 128L_c m_q$. By Lemma 4.9, $m_q < \vartheta L_c^2$, and hence $|V_q| > (1 - 3\vartheta)L_c^3 > 3 \cdot 128L_c m_q$. If each component had fewer than $|V_q|/3$ sites, a union A of components with $|V_q|/3 \leq |A| < 2|V_q|/3$ would violate the bound. Hence the largest component C_q has at least $|V_q|/3$ sites, and the bound with $A = C_q$ gives $|V_q \setminus C_q| < 128L_c m_q$.

A thin set of a surviving block j is a nonempty $A \subseteq V_j$ with $|A| < \varepsilon_{\text{th}} L_c^3 := 0.008L_c^3$ and $\text{cut}_j(A) < |A|/L_c$, $\text{cut}_j(A)$ being the number of G' -bonds inside j between A and $V_j \setminus A$.

Lemma 4.13 (Thin sets). At every hull-closed time of the full history other than the first, every thin set A of a surviving block j satisfies $F_j^{\text{aux}}(\mathbb{X} \cup A) < F_j^{\text{aux}}(\mathbb{X})$. Hence a surviving block without auxiliary material has no thin set, and $V_j = C_j$ is connected in G'_j .

Proof. A thin set is enclosed by in-block insulation in each planar slice, and planar isoperimetry makes this insulation exceed the volume penalty. We fix a direction ν . A line of j in direction ν that meets A either meets $V_j \setminus A$, and then carries a bond counted in $\text{cut}_j(A)$ or a run of Lemma 4.10(b), or lies in $A \cup (\mathbb{X} \cap j)$, which happens for at most $(|A| + 3L_c m_j)/L_c$ lines. A

slice σ of A perpendicular to ν meets each line once. Hence, with $m_j < \vartheta L_c^2$ (Lemma 4.9), $|\sigma| \leq a := \text{cut}_j(A) + 9m_j + (|A| + 3L_cm_j)/L_c < (2\varepsilon_{\text{th}} + 12\vartheta)L_c^2 \leq 0.0164 L_c^2 \leq L_c^2/4$. The bonds of the cross-section of j (sides at least L_c) with exactly one end in σ end in $\mathbb{X}$ or in $V_j \setminus A$, and there are at least $2|\sigma|^{1/2} \geq 2|\sigma|/a^{1/2}$ of them (Lemma 4.11(b)). Summing over the slices, we obtain $f_j(A, \mathbb{X}) + \text{cut}_j(A) \geq 2|A|/a^{1/2} \geq 15.6|A|/L_c$, with $f_j(A, \mathbb{X})$ the number of bonds inside j from A to $\mathbb{X}$. Passing from $\mathbb{X}$ to $\mathbb{X} \cup A$ changes F_j^{aux} by $-f_j(A, \mathbb{X})$ on these hole faces, by at most 0 on the bonds of ∂j at A (from w_b to 1, or from 1 to 0), by $+\text{cut}_j(A)$ on the in-block bonds to $V_j \setminus A$, and by $+\lambda|A|$. In total the change is at most $-15.6|A|/L_c + 2\text{cut}_j(A) + 12|A|/L_c < 0$. If j has no auxiliary material, $\mathbb{X}$ minimizes F_j^{aux} , so j has no thin set. Moreover, a component of G'_j other than C_j has $\text{cut}_j = 0$ and fewer than $128L_cm_j < \varepsilon_{\text{th}}L_c^3$ sites (Proposition 4.12), so it would be thin.

Lemma 4.14 (The seal). *Let t be a hull-closed time of an admissible history and j a surviving block with $A_j \neq \emptyset$. (a) Some G' -bond that crosses a face between j and a surviving block has an end in A_j . (b) If a seal step at t is followed by a hull step with the same $\mathbb{U}$, then $A_j \subseteq \mathbb{X}_{t+2}$.*

Proof. On the sets between $\mathbb{X} := \mathbb{X}_t$ and $\mathbb{X} \cup A_j$, the functionals $F := F_\lambda^{\bar{S}_t}$ and F_j^{aux} differ by a constant unless a G' -bond leaves A_j across a face of j . Indeed, the other bonds of ∂j at A_j touch $\mathbb{X}$ or are frozen (a block adjacent to j that is not surviving lies in $\mathbb{X}$), and those at $V_j \setminus A_j$ keep their values. (a) Otherwise $F(\mathbb{X} \cup A_j) - F(\mathbb{X}) = F_j^{\text{aux}}(M_j) - F_j^{\text{aux}}(\mathbb{X}) < 0$, since $M_j \neq \mathbb{X}$ is the smallest minimizer, against $\mathbb{X} = H_\lambda^{\bar{S}_t}(B_t)$ and $\mathbb{X} \cup A_j \supseteq B_t$. (b) Let $X' := \mathbb{X}_{t+2} \supseteq \mathbb{X}$ (Lemma 4.2), $F' := F_\lambda^{\bar{S}_{t+2}}$, $A' := A_j \setminus X'$ and $C := \mathbb{X} \cup (A_j \cap X') \subseteq X'$. On the sets between C and $M_j = C \cup A'$ only the sites of A' vary. The bonds inside j at these sites count alike in F' and F_j^{aux} , and after the seal each bond of ∂j at them is frozen or touches $\mathbb{X}$, in both. Hence $F' - F_j^{\text{aux}}$ is constant on these sets. If $A' \neq \emptyset$, then $C \subsetneq M_j$, and submodularity of F' gives $F'(X' \cup A') - F'(X') \leq F'(C \cup A') - F'(C) = F_j^{\text{aux}}(M_j) - F_j^{\text{aux}}(C) < 0$, against the minimality of $X' = H_\lambda^{\bar{S}_{t+2}}(B_t)$.

Proposition 4.15 (Cost of the seal). *Along the full history, a block face F' between blocks Q , Q' carries at most $121(m_Q + m_{Q'})$ bonds frozen by seal steps, converted ones included; so seal steps freeze at most $C_{\text{seal}}N$ bonds ($C_{\text{seal}} := 6 \cdot 121 = 726$), all on faces F' with $m_Q + m_{Q'} > 0$.*

Proof. Each seal bond on F' has an end in auxiliary material, which the next hull step fills while Q and Q' still survive (Lemma 4.14(b), since in Phase 1 $\mathbb{U}$ is fixed during the inner loop, and in Phase 2 the hull step follows the seal before any removal). Consequently, at the last hull-closed time T at which Q and Q' both survive, each seal bond on F' has an end in W_T in the layer next to F' , hence in $Z(F')$, and distinct bonds lie on distinct lines $\ell \perp F'$. At most $m_Q + m_{Q'}$ of these lines carry a mark in $Z(F')$, and fewer than $2\text{supp}_T(F')/\eta_w \leq 120(m_Q + m_{Q'})$ of the others meet W_T in $Z(F')$ (Lemma 4.8(b)). A block has six faces.

Theorem 4.16 (Cells at rest). *Let t be a hull-closed time of the full history other than the first at which no surviving block has auxiliary material. Then: (a) every surviving block q carries exactly one cell, $\Omega_q = V_q$, connected in q , with $(1 - 3\vartheta)L_c^3 < |\Omega_q| < 8L_c^3$, and every other block lies in $\mathbb{X}$; (b) every block face F' between surviving blocks contains no G' -bond, or carries exactly one interface I ; then F' contains no cut bond and $\text{good}(I) \geq (1 - 134\vartheta)|F'|$; in particular (S -face) holds; (c) every component $\mathfrak{K}$ of $G' \cap \hat{Q}$, $\hat{Q} \in \mathcal{Q}$, satisfies $f_{G'}(A, \mathfrak{K} \setminus A) \geq h_\star|A|$ for $0 < |A| \leq |\mathfrak{K}|/2$, with $h_\star := 2.2 \cdot 10^{-11}/L_c$; since $3k_{ph_\star} \geq \chi$, (S -plug) holds.*

Proof. (a) By Lemma 4.13, $V_q = C_q$ is connected in q , and therefore lies in one component of $G' \cap \hat{Q}(q)$ and is the only cell of q . The volume bounds follow from Proposition 4.12 and (G1). A block that is not surviving lies in $\mathbb{U} \subseteq \mathbb{X}$. (b) By (a), the G' -bonds across F' , if any, join the cell of Q to the cell of Q' and form one interface. Cuts fire only at such times, where by (a) a cut, a union of whole interfaces (Lemma 4.27(a)), freezes all G' -bonds of each face its bonds cross. As G' only shrinks, a face with a G' -bond carries no cut bond. A line $\ell \perp F'$ is good unless $\ell \cap Z(F')$ contains a mark, or b_ℓ is a seal bond, or else $b_\ell \notin \bar{S}_t$ and $\ell \cap Z(F')$ contains a site of W_t . Indeed, on each remaining line the tube has no hole site, and its bonds are G' -bonds, since bonds inside a block are never frozen. The first case holds for at most $m_Q + m_{Q'}$ lines, the second for at most $121(m_Q + m_{Q'})$ lines (Proposition 4.15), and the third for fewer than $2 \text{supp}_t(F') \leq 12(m_Q + m_{Q'})$ lines (Lemma 4.8(a)). With $m_Q + m_{Q'} \leq N(3Q) < \vartheta L_c^2 \leq \vartheta |F'|$, we obtain $\text{good}(I) \geq (1 - 134\vartheta)|F'| \geq c_g|F'|$. (c) Expansion can fail only through a thin set, and there is none (Lemma 4.13). Let $A \subseteq \mathfrak{K}$, $0 < |A| \leq |\mathfrak{K}|/2$, and $\text{cut} := f_{G'}(A, \mathfrak{K} \setminus A)$. Since windows are unions of blocks, by (a) $\mathfrak{K}$ is a union of sets V_j of surviving blocks $j \subseteq \hat{Q}$, connected through faces that contain G' -bonds. We put $A_j := A \cap V_j$ and $B_j := V_j \setminus A_j$. A line of j meeting A_j and B_j carries a bond counted in cut or a run of Lemma 4.10(b). If $\min(|A_j|, |B_j|) \geq \varepsilon_{\text{th}} L_c^3$ for some j , Lemma 4.11(a) gives $\varepsilon_{\text{th}} L_c^3 \leq 12L_c m_j + 12.8L_c(\text{cut} + 9m_j)$, so $\text{cut} \geq (\varepsilon_{\text{th}} - 127.2\vartheta)L_c^2/12.8 \geq \varepsilon_{\text{th}} L_c^2/25.6$. Otherwise each block of $\mathfrak{K}$ is *in* ($|B_j| < \varepsilon_{\text{th}} L_c^3$) or *out* ($|A_j| < \varepsilon_{\text{th}} L_c^3$), not both, since $|V_j| > 2\varepsilon_{\text{th}} L_c^3$. If an in-block and an out-block share a face F' that contains G' -bonds, we follow each line $\perp F'$ from the face into either block until it reaches the majority side of that block through a bond counted in cut or a run, or stays in the minority side and $\mathbb{X}$ for the whole block. In this case the minority side of each block meets the face layer in at most $\text{cut} + 9\vartheta L_c^2 + (\varepsilon_{\text{th}} + 3\vartheta)L_c^2$ sites, all other bonds among the at least $(1 - 134\vartheta)L_c^2$ G' -bonds across F' (by (b)) join A to $\mathfrak{K} \setminus A$, and $3\text{cut} \geq (1 - 158\vartheta - 2\varepsilon_{\text{th}})L_c^2$. Otherwise all blocks are in, which is impossible since then $|A| \geq \sum_j (|V_j| - \varepsilon_{\text{th}} L_c^3) > |\mathfrak{K}|/2$, or all are out. In the latter case each nonempty A_j has fewer than $\varepsilon_{\text{th}} L_c^3$ sites and is not thin, so $\text{cut} \geq \sum_j \text{cut}_j(A_j) \geq |A|/L_c$. In all cases $\text{cut} \geq \min(3.1 \cdot 10^{-4} L_c^2, |A|/L_c) \geq h_*|A|$, as $|A| < k_P^3/2$ by (G4) (Section S1.2). Finally, $\chi_{N3}/k_P = \chi/(3k_P) \leq h_*$.

Theorem 4.17 (The Phase-1 output). *Phase 1 terminates. Its output $(\mathbb{X}^1, \mathbb{U}^1, S^1)$ is the smallest minimizer of $\mathcal{E}^{S^1}$, no surviving block has auxiliary material there, and: (i) every bond of $\bar{S}^1$ touches $\mathbb{X}^1$, so $S^1 = \emptyset$, and crosses a face between two blocks that together contain a mark; (ii) the four terminal properties (S-hull), (S-rem), (S-face), (S-plug) hold.*

Proof. Each outer step returns $\mathbb{U}^* \supseteq \mathbb{U}$ (Lemma 3.12(c), since $\bar{S}$ only grows), and each inner round freezes at least one bond (Lemma 4.14(a)). Each outer step except the last enlarges $\mathbb{U}$ or $\bar{S}$, which only grow in a finite box. Hence Phase 1 terminates. In the last outer step the inner loop performs no seal step, so $\bar{S} = \bar{S}^1$ throughout that step. Thus step (1) of the last outer step returns the smallest minimizer of $\mathcal{E}^{\bar{S}^1}$, with hole set $H_{\bar{S}^1}^{\bar{S}^1}(\mathbb{M} \cup \mathbb{U}^1)$ (Lemma 3.12(b)). This pair is the output, and step (2) finds no auxiliary material at it. (i) Each bond of $\bar{S}^1$ has an end in auxiliary material, filled by the next hull step (Lemma 4.14(b)). For the faces crossed by these bonds we use Proposition 4.15. (ii) Property (S-hull) follows from Lemma 4.2, and (S-face), (S-plug) follow from Theorem 4.16. For (S-rem), let $\Delta \not\subseteq B^1$ be a removal cell with $\Delta \not\subseteq \mathbb{X}^1$ (otherwise $w(\Delta) = 0$). We compare with $(\mathbb{X}^1 \cup \Delta, \mathbb{U}^1 \cup \Delta)$, admissible since Δ is a union of blocks. The bonds of $E(\Delta)$ with one end in $\mathbb{X}^1$ lose their weight 1 (no bond is frozen, by (i)), at most $|\partial\Delta|$ bonds gain at most 1, the volume terms grow by at most $\lambda_U|\Delta|$, and the

penalty does not grow. Hence $w(\Delta) \leq |\partial\Delta| + \lambda_U |\Delta| \leq (1 + c_s) |\partial\Delta| = w_*$, since $|\Delta|/|\partial\Delta| < \ell/2$ for a box with sides in $[\ell, 3\ell]$ (Section S1.2).

Lemma 4.18 (Expansion and Poincaré inequality on windows). *Let the state satisfy (S-plug), $\hat{Q} \in \mathcal{Q}$, and $\mathfrak{K}$ a component of $G' \cap \hat{Q}$. Then $f_{G'}(A, \mathfrak{K} \setminus A) \geq \chi |A|/(3k_P)$ for $A \subseteq \mathfrak{K}$ with $0 < |A| \leq |\mathfrak{K}|/2$, and $\sum_{x \in \mathfrak{K}} |f(x) - \bar{f}_{\mathfrak{K}}|^2 \leq (108/\chi^2) k_P^2 \sum_{\{x,y\} \in E_{G'}(\mathfrak{K})} (f(x) - f(y))^2$ for every f .*

Proof. The first claim is (S-plug), since $\chi_{N3}/k_P = \chi/(3k_P)$. The Poincaré inequality follows from the Cheeger argument and a median. Let μ be a median of f on $\mathfrak{K}$. Since each of $(f - \mu)_{\pm}$ is supported on at most $|\mathfrak{K}|/2$ sites, the argument of Theorem 4.5(iv) in the graph $(\mathfrak{K}, E_{G'}(\mathfrak{K}))$, with $h := \chi/(3k_P)$, bounds $\sum (f - \mu)_{\pm}^2$ by $12h^{-2} = 108k_P^2/\chi^2$ times the Dirichlet form of $(f - \mu)_{\pm}$. We add the two bounds, using $(a - b)^2 \geq (a_+ - b_+)^2 + (a_- - b_-)^2$, and replace μ by the mean.

Lemma 4.19 (Good bonds). *At every state satisfying (S-face), (R8) holds, a fine bond across $F(I)$ being good if the segment of L fine sites of its line centred at it is a path of the fine slit graph.*

Proof. (Sketch.) (S-face) and (G1) give the first claim, and bonds across block faces lie on block planes, at least L_c apart. A good tube is a G' -path in $Z(F') \subseteq Q \cup Q' \subseteq \hat{Q}(Q) \cap \hat{Q}(Q')$, by (G5) and (G4). Hence its half in Q lies in the component of $G' \cap \hat{Q}(Q)$ of its end, the cell of I in Q . The interfaces of one face are disjoint with at least $c_g |F'|$ bonds each, and a block has six faces. In fine units, a block face meets at most one exceptional layer on each side, so $|F'|_{\text{fine}} < r_0^2 |F'| (1 + 1/L_c)^2$, while each coarse good bond carries at least r_0^2 fine good bonds.

Theorem 4.20 (Cells). *At every closed state with (S-face) and (S-plug), such as the output, (R5), (R6) and the cell part of (R7) hold, with $\hat{\mathfrak{K}}^b(\Omega) := \hat{\mathfrak{K}}$ for the cell $\Omega = Q \cap \hat{\mathfrak{K}}$ of a block Q .*

Proof. For (R5), $|\Omega| \leq |Q| < 8L_c^3$ by (G1). An interface on a face F' of Q gives, by Lemma 4.19, at least $c_g L_c^2$ good half-tubes of $L_c/2$ sites in Ω , on distinct lines. Without an interface no G' -bond leaves Ω (inside Q a G' -bond joins two sites of one cell, and across a face it belongs to an interface), so Ω is a union of components of G' , each with more than $V_* = ((1 - \eta_w) c_I / 24)^3 L_c^3$ sites (Theorem 4.5(ii)). For (R6), we apply Lemma 4.18 in $\hat{\mathfrak{K}} \supseteq \Omega$ and use that the mean minimizes the sum of squares. A bond lies in at most $151^3 \leq 27(k_P/L_c)^3$ windows (G4), and the components $\hat{\mathfrak{K}}$ of distinct cells of one block are distinct. For (R7), we use that $\hat{\mathfrak{K}}$ is connected in G' , together with Lemma 3.5(A2).

Lemma 4.21 ($\mathcal{U}$ -cells). *Let $\mathcal{T}$ be a connected graph of maximal degree at most D , with vertex weights $w(v) \in (0, V)$ and total weight at least V . Then the vertex set of $\mathcal{T}$ splits into parts, each connected in $\mathcal{T}$, of weight in $[V, (2 + D)V)$. Applied to the cell graphs of the output with $D = 6/c_g$, cell weights $|\Omega|_{\text{fine}}$ and $V = V_{\mathcal{U}} = \tilde{m}^{-3}$, it gives the $\mathcal{U}$ -cell part of (R7), with $C_{\mathcal{U}}$ of Section S1.2, provided $\tilde{m}L \geq c_L/2$ (a lower bound on τ).*

Proof. We cut a rooted spanning tree from the leaves upward, cutting off a vertex with its uncut descendants once their weight reaches V . A cut part weighs less than $w(v) + DV < (1 + D)V$, and the light remainder at the root joins an adjacent cut part. In the application, the cell graph is connected (G' -bonds between cells are interface bonds) and has degree at most $6/c_g$ (Lemma 4.19). A cell weighs less than $8L^3 \leq \tilde{m}^{-3}$ by (3.1). A light component has more than V_* sites (Theorem 4.5(ii)), and hence weight more than $r_0^3 V_* = c_u V_{\mathcal{U}}$. Finally, a $\mathcal{U}$ -cell holds

fewer than $(2 + 6/c_g)\tilde{m}^{-3}/(c_1 L^3)$ connected cells by (R5), in blocks of Euclidean diameter less than $2\sqrt{3}L$.

4.3 Payment and reach of the Phase-1 output

In this subsection we obtain the payment of the Phase-1 output from three comparisons (Lemma 4.22). Filling everything near a point bounds the cost of a ball, deleting a region without marks bounds its cost by the bonds of a sphere around it, and a good sphere, chosen by averaging over the radii, makes the cost decay geometrically away from the marks. Consequently, no hole lies far from a mark (Theorem 4.23), and we pay for the pieces near the clusters of marks by comparison with the cheapest state that covers the marks and fills the forced blocks of the cluster (Proposition 4.24).

Throughout, $(\mathbb{X}^1, \mathbb{U}^1)$ is the Phase-1 output and $\mathcal{E} := \mathcal{E}^{\bar{S}^1}$. By Theorem 4.17, $(\mathbb{X}^1, \mathbb{U}^1)$ minimizes $\mathcal{E}$, and each bond of $\bar{S}^1$ touches $\mathbb{X}^1$ and has both ends within distance $4L_c$ of a mark. Hence, at $(\mathbb{X}^1, \mathbb{U}^1)$, the frozen bonds count as ordinary bonds and the penalty vanishes (Lemma 4.9). For a set A of sites we put $\mathcal{E}_A := \#\{b \in \partial\mathbb{X}^1 : b \subseteq A\} + \lambda|(\mathbb{X}^1 \setminus \mathbb{U}^1) \cap A| + \lambda_U|\mathbb{U}^1 \cap A|$, let $B_r(x) := Q_\infty(x, r) \cap \Lambda^\circ$, and for a fixed site x let $f(r) := \mathcal{E}_{B_r(x)}$, nondecreasing in r . Each competitor $(\mathbb{X}', \mathbb{U}')$ below has $\mathbb{M} \subseteq \mathbb{X}'$ and $\mathbb{U}' \subseteq \mathbb{X}'$ a union of blocks, and hence satisfies $\mathcal{E}(\mathbb{X}', \mathbb{U}') \geq \mathcal{E}(\mathbb{X}^1, \mathbb{U}^1)$.

Lemma 4.22 (Density, deletion, decay). *(a) (Upper density bound.) For every site x and every $r \geq 0$, $f(r) \leq g(r) := 6(2r + 4L_c)^2 + \lambda_U(2r + 4L_c)^3$. (b) (Deletion.) Let B be a set of sites with $\text{dist}(B, \mathbb{M}) \geq 6L_c$, and $\hat{B}$ the union of the blocks meeting B . Then $\mathcal{E}_B \leq \text{sl}(B) := f(B, \mathbb{X}^1 \setminus B) + \lambda|\mathbb{U}^1 \cap (\hat{B} \setminus B)|$. (c) (Decay.) Put $w := \lceil 120\ell/c_s \rceil$ and $w' := w + 2L_c$. If $\text{dist}(B_{r+w'}(x), \mathbb{M}) \geq 6L_c$, then $f(r) \leq f(r + w')/5$.*

Proof. (a) Let $\hat{B}$ be the union of the blocks meeting $B_r(x)$, a box of sides less than $2r + 4L_c$. We compare with $(\mathbb{X}^1 \cup \hat{B}, \mathbb{U}^1 \cup \hat{B})$: the bonds of $\partial\mathbb{X}^1$ with both ends in $\hat{B}$ disappear, at most $|\partial\hat{B}| \leq 6(2r + 4L_c)^2$ bonds appear, the frozen bonds still touch the hole set, each site of $\hat{B}$ ends with the weight λ_U , and the penalty does not grow. Hence $f(r) \leq \mathcal{E}_{\hat{B}} \leq |\partial\hat{B}| + \lambda_U|\hat{B}|$. (b) A forced block has a mark within distance $4L_c$ of each of its sites. Hence no block meeting B is forced, and no bond of $\bar{S}^1$ has an end in B . We compare with $(\mathbb{X}^1 \setminus B, \mathbb{U}')$, where $\mathbb{U}'$ is the union of the blocks of $\mathbb{U}^1$ that do not meet B . This competitor contains the marks and all forced blocks. In passing to it, the bonds of $\partial\mathbb{X}^1$ with both ends in B disappear, at most $f(B, \mathbb{X}^1 \setminus B)$ bonds appear, the frozen bonds still touch the hole set, the sites of $B \cap \mathbb{X}^1$ lose their weights, and those of $\mathbb{U}^1 \cap (\hat{B} \setminus B)$ change from λ_U to λ . (c) We put $A := B_{r+w'}(x) \setminus B_r(x)$ and apply (b) to $B_s := B_s(x)$ for $r \leq s < r + w$. Since a site of $\mathbb{X}^1$ at distance exactly $s + 1$ from x has at most six bonds into B_s , we have $\sum_s f(B_s, \mathbb{X}^1 \setminus B_s) \leq 6|\mathbb{X}^1 \cap A|$. A site of $\mathbb{U}^1$ outside B_s lies in a block meeting B_s for fewer than $2L_c$ values of s , and therefore $\sum_s \lambda|\mathbb{U}^1 \cap (\hat{B}_s \setminus B_s)| \leq 2\lambda L_c |\mathbb{U}^1 \cap A| = 24|\mathbb{U}^1 \cap A|$. Hence $\sum_s \text{sl}(B_s) \leq 30|\mathbb{X}^1 \cap A| \leq 30(\lambda^{-1} + \lambda_U^{-1})\mathcal{E}_A \leq (30\ell/c_s)\mathcal{E}_A$, using $\lambda^{-1} + \lambda_U^{-1} = L_c/12 + \ell/(2c_s) \leq \ell/c_s$. Thus some s has $\text{sl}(B_s) \leq \mathcal{E}_A/4$. The bonds and sites counted in $\mathcal{E}_A$ are not counted in $f(r)$. Therefore $\mathcal{E}_A \leq f(r + w') - f(r)$, and $f(r) \leq f(s) \leq \text{sl}(B_s) \leq (f(r + w') - f(r))/4$.

Theorem 4.23 (Clearing-out; uniform reach). *Let J be the least integer with $5^J > g((J+1)w')$, and $R_* := (J+1)w' + 6L_c$. Every site of $\mathbb{X}^1$ lies within distance R_* of a mark, and $R_* = O((\ell/c_s) \log(\ell/c_s))$, uniformly in $|\mathbb{M}|$, Λ° , C and D . This proves Theorem 3.21(b).*

Proof. Let $\text{dist}(x, \mathbb{M}) > R_*$. Then $\text{dist}(B_{(J+1)w'}(x), \mathbb{M}) > 6L_c$, and applying Lemma 4.22(c) J times from $r = w'$, together with Lemma 4.22(a), we obtain $f(w') \leq 5^{-J} f((J+1)w') \leq 5^{-J} g((J+1)w') < 1$. Hence no bond of $\partial\mathbb{X}^1$ has both ends in the box $B_{w'}(x)$, which therefore lies in $\mathbb{X}^1$ or misses it. In the first case $f(w') \geq \lambda_U |B_{w'}(x)| > 1$. Therefore $x \notin \mathbb{X}^1$. The arithmetic is in Section S1.3.

Proposition 4.24 (Cluster replacement; area per part). *Theorem 3.21(a) holds with $C'_\star := C_1 + 2C_{\text{seal}}$, $C_1 := 7 + 27 \cdot 24.6/\vartheta$, and $C_{\text{free}} := C'_\star(2 + 1/(2c_s) + 1/3696)$.*

Proof. By Theorem 4.17(i), $S^1 = \emptyset$. We compare with the competitor that replaces Y_P by the marks and the forced blocks of P . By Theorem 4.23 the sets Y_P cover $\mathbb{X}^1$. Marks of distinct parts are more than $2R_* + 2$ apart, and hence no bond joins P^{+R_*} to P'^{+R_*} . Moreover, a block of $\mathbb{U}^1$ is connected and lies in $\mathbb{X}^1$. The marks of a set $3Q$ are within distance $6L_c \leq r'$ of each other, and hence those of a forced block lie in one part. The union $U_F(P)$ of the forced blocks with marks in P lies in Y_P (by Lemma 4.9, and because a forced block lies within $4L_c$ of its marks) and has at most $27N(P)/(\vartheta L_c^2)$ blocks, since a mark lies in at most 27 sets $3Q$. The competitor $\mathbb{X}' := (\mathbb{X}^1 \setminus Y_P) \cup P \cup U_F(P)$, $\mathbb{U}' := (\mathbb{U}^1 \setminus Y_P) \cup U_F(P)$ contains the marks, $\mathbb{U}'$ is a union of blocks containing all forced blocks, and the change is confined to Y_P , which no bond joins to the rest of $\mathbb{X}^1$. A bond of $\bar{S}^1$ that loses contact with the hole set touches Y_P , and therefore the marks of its two blocks lie within $R_* + 4L_c < r'$ of P , hence in P . There are at most $C_{\text{seal}}N(P)$ such bonds (Proposition 4.15), each costing at most 2. Since $|\partial Q| + \lambda_U |Q| < 24.6L_c^2$ and $\lambda \leq 1$, minimality gives $|\partial Y_P| + \lambda |Y_P \setminus \mathbb{U}^1| + \lambda_U |Y_P \cap \mathbb{U}^1| \leq (6 + \lambda)N(P) + 24.6 \cdot 27N(P)/\vartheta + 2C_{\text{seal}}N(P) \leq C'_\star N(P)$. The second inequality follows from $|Y_P|/\ell \leq (\lambda\ell)^{-1}\lambda |Y_P \setminus \mathbb{U}^1| + (\lambda_U\ell)^{-1}\lambda_U |Y_P \cap \mathbb{U}^1|$, with $(\lambda\ell)^{-1} \leq 1/3696$ and $(\lambda_U\ell)^{-1} = 1/(2c_s)$.

4.4 Phase 2: release, density bound and cuts

In this subsection we bound the insulation in Phase 2 (Lemma 4.25) and prove the density bound (R9), the elementary properties of cuts and Proposition 3.19. Each hole site added in Phase 2 outside the removed cells is hull material, whose removal does not lower the hull functional, and the Phase-1 insulation $\text{ins}^1 := \partial\mathbb{X}^1$ is a fixed budget (Theorem 3.21(a)). Throughout, R_t is the union of the cells removed in Phase 2 before time t . A cell Δ removed in Phase 2 is removed at a hull-closed time t_Δ , at which $w(\Delta)$ is evaluated. We denote by s_Δ the number of bonds of $E(\Delta)$ frozen in Phase 2 before t_Δ , converted ones included, with no end in $\mathbb{X}^1$.

Lemma 4.25 (Local insulation; release). *(a) Let t be a hull-closed time of Phase 2, N' a box, $W' := (\mathbb{X}_t \setminus (\mathbb{X}^1 \cup R_t)) \cap N'$, $\text{Ins}_{N'}(Y) := \#\{b \in E(N') : \text{exactly one end in } Y\} + 2\#\{b \in \bar{S}_t \cap E(N') : \text{no end in } Y\}$, and $\text{cut}_{N'}$ the number of bonds across $\partial N'$ with inner end in W' whose outer end lies in $\mathbb{X}_t$ or which are frozen. Then*

$$\mathcal{A}_2(E(N'); t) \leq \text{Ins}_{N'}(\mathbb{X}^1 \cup R_t) + \text{cut}_{N'}. \quad (4.4)$$

(b) Every cell Δ removed in Phase 2 satisfies $w(\Delta) \leq |\text{ins}^1 \cap E(\Delta)| + 2s_\Delta + |\partial\Delta|$, hence $c_s|\partial\Delta| < |\text{ins}^1 \cap E(\Delta)| + 2s_\Delta$.

Proof. (a) We put $\mathbb{X} := \mathbb{X}_t$ and $V := \Lambda^\circ \setminus \mathbb{X}$. Since $B_t = \mathbb{M} \cup \mathbb{U}^1 \cup R_t \subseteq \mathbb{X}^1 \cup R_t \subseteq \mathbb{X}$ (Proposition 4.3(a)), W' misses B_t and $\text{Ins}_{N'}(\mathbb{X} \setminus W') = \text{Ins}_{N'}(\mathbb{X}^1 \cup R_t)$, while $\text{Ins}_{N'}(\mathbb{X}) = \mathcal{A}_2(E(N'); t)$, since the bonds of $\bar{S}_t$ with no end in $\mathbb{X}$ are those of S_t (Lemma 4.2). This proves

(a) if $W' = \emptyset$. Otherwise $F_{\lambda}^{\bar{S}_t}(\mathbb{X} \setminus W') > F_{\lambda}^{\bar{S}_t}(\mathbb{X})$, because $\mathbb{X} = H_{\lambda}^{\bar{S}_t}(B_t)$. Written bond by bond as in (4.3), this inequality reads $f_{E \setminus \bar{S}_t}(W', V) + \lambda|W'| < f(W', \mathbb{X} \setminus W') + \sum_{b \in \bar{S}_t \text{ between } W', V} (w_b - 1) + \sum_{b \in \bar{S}_t, b \subseteq W'} w_b$, with $f_{E \setminus \bar{S}_t}$ counting unfrozen bonds. When we pass from $\text{Ins}_{N'}(\mathbb{X} \setminus W')$ to $\text{Ins}_{N'}(\mathbb{X})$, a bond of $E(N')$ between W' and $\mathbb{X} \setminus W'$ gives -1 , a bond of $\bar{S}_t$ inside W' gives -2 , a bond between W' and V gives $+1$ if unfrozen and -1 if frozen, and the others give 0 . Inserting the inequality, we see that the bonds between W' and $\mathbb{X} \setminus W'$ inside N' cancel, frozen bonds inside N' with an end in W' get the coefficient $w_b - 2 \leq 0$, and the bonds counted in $\text{cut}_{N'}$ remain, each with coefficient at most 1 . Hence $\mathcal{A}_2(E(N'); t) < \text{Ins}_{N'}(\mathbb{X}^1 \cup R_t) + \text{cut}_{N'} - \lambda|W'|$.
(b) We take $N' := \Delta$ and $t := t_{\Delta}$. Since $\Delta \cap R_t = \emptyset$ and each bond of $\bar{S}^1$ touches $\mathbb{X}^1$ (Theorem 4.17), $\text{Ins}_{\Delta}(\mathbb{X}^1 \cup R_t) = |\partial \mathbb{X}^1 \cap E(\Delta)| + 2\#\{b \in \bar{S}_t \cap E(\Delta) : \text{no end in } \mathbb{X}^1\} \leq |\text{ins}^1 \cap E(\Delta)| + 2s_{\Delta}$, and $\text{cut}_{\Delta} \leq |\partial \Delta|$. The trigger gives the last claim.

Proposition 4.26 (Density bound; proof of (R9)). *At every state satisfying (S-rem), in particular at the output, (R9) holds, and $(3 + c_s)|\partial \Delta| < 54(3 + c_s)\ell^2$.*

Proof. Since a frozen bond is a missing pair of neighbours counted at both ends, the density count of Section 6 is two-sided, namely the hull's own count $\mathcal{A}_2$. If $\Delta \not\subseteq B$, (S-rem) gives $\mathcal{A}_2(E(\Delta)) = w(\Delta) \leq w_*$. If $\Delta \subseteq B \subseteq \mathbb{X}$, this count is 0 . A bond of $\partial \Delta \cap E$ weighs at most 2 , and $|\partial \Delta| < 54\ell^2$ by (G3). The constant c_0 of the conversion into fine plaquettes is in Section S1.2.

Phase 2 fills only by removal and freezes by the seal and the cut rule. By (H4), G' only shrinks and each bond is frozen at most once. The cut rule fires only at hull-closed states without auxiliary material, where Theorem 4.16 holds.

Lemma 4.27 (Elementary properties of cuts). *Let the firings $j = 1, 2, \dots$ of the cut rule have times t_j , components $\mathfrak{K}_j$ and sides U_j , $\mathfrak{K}_j \setminus U_j$. (a) Every bond of $\partial_{G'}U$ joins two different cells, so it crosses a block face; $\partial_{G'}U$ is a union of whole interfaces. (b) After the cut j no G' -bond joins U_j to $\mathfrak{K}_j \setminus U_j$; every later component of G' lies in U_j , in $\mathfrak{K}_j \setminus U_j$, or outside $\mathfrak{K}_j$. The sides of the cuts form a laminar family. (c) $|\partial_{G'}U_j| \geq c_g L_c^2$ for every cut. (d) Cut bonds, alone or with seal bonds, filled blocks and box faces, support no hull material, and block no tube of another interface. (e) If $J(U) > 0$, then $\partial_{G'}U$ contains two interfaces at distance $> R_{\text{NF}}$.*

Proof. (a) A G' -bond with both ends in a block Q lies in $G' \cap \hat{Q}(Q)$, and hence its ends lie in one cell. Therefore a bond of $\partial_{G'}U$ joins cells of different blocks, and all bonds across F' between a cell of U and a cell outside U lie in $\partial_{G'}U$. (b) The cut consists of all G' -bonds between U_j and $\mathfrak{K}_j \setminus U_j$, where $\mathfrak{K}_j$ is a component of G' at time t_j , and G' only shrinks (H4). Consequently, each later component lies in U_j , in $\mathfrak{K}_j \setminus U_j$, or outside $\mathfrak{K}_j$. If j_a fires before j_b , the sides of j_b lie in $\mathfrak{K}_{j_b}$, a component at time t_{j_b} , and hence each of them lies in a side of j_a or misses both. As the two sides of one cut are disjoint, any two sides are disjoint or nested. (c) By (a) the cut contains a whole interface I , and (S-face) holds at t_j (Theorem 4.16(b)). Thus $|\partial_{G'}U_j| \geq \text{good}(I) \geq c_g L_c^2$ (Lemma 4.19). (d) Cut bonds are grid bonds with the hull weight, and they therefore enter Corollary 4.7 and Lemma 4.8 as grid-type terms, as seal bonds do. A tube crosses no other block plane (G5). (e) If $J(U) > 0$, then $\varphi_s(\partial_{G'}U) \geq 2$ for some $s = 2^j L \geq R_{\text{NF}}$, and interfaces in different s -clusters are more than s apart.

Proof of Proposition 3.19. Termination is Proposition 4.3(b). At the terminal state a full pass changes nothing: the state is hull-closed, which with Lemma 4.2 is (S-hull); no removal cell is

over threshold, which is (S-rem); no surviving block has auxiliary material (Lemma 4.14(a)), and hence (S-face) and (S-plug) hold (Theorem 4.16); and the cut rule has no candidate, which is (S-NF) and contains the form used in Section 7.8. We now assume that the Euclidean diameter of Λ° is at most R_{NF} . Then no pair is fragmented, since the right side of (NF_*) is positive and $J(U) > 0$ would give two interfaces more than R_{NF} apart (Lemma 4.27(e)). Since, moreover, the Phase-1 output is hull-closed, satisfies (S-rem) and has no auxiliary material (Theorem 4.17), Phase 2 changes nothing in this case.

5 The min-cut lower tail

In this section we show that the frozen set of the construction of Section 3 lowers the min-cut area of Definition 2.11 by at most $\eta_T RT$, except with probability e^{-RT} . The proof has two ingredients. As Λ° is acyclic, a minimizer can be taken constant on each piece of the hole set (Section 5.2), and each part of the marks pays with its own marks for the frozen plaquettes it attaches (Sections 5.3 to 5.7). The units, Ξ , Λ_2° , $\partial_2 \Lambda^\circ$, $F(Y)$, F_S , $\mathfrak{F}$, $G'(D)$, $\mathbb{X}$ and $S = S_1 \sqcup S_{\text{cut}}$ are those of Sections 2.5, 3 and 3.4. A coarse face carries at most $\bar{r}_0^2$ fine plaquettes, $\bar{r}_0 = 2r_0 - 1$, and Ξ is a union of coarse faces (the plane of C is a special plane of $\mathfrak{G}(C)$). We write $\mathbb{P}_D := \text{Ber}(\delta)^{\otimes (\Lambda_2^\circ \setminus \partial_2 \Lambda^\circ)}$ for the law of D and $\mathbb{E}_D$ for its expectation.

5.1 Statement

Theorem 5.1 (Min-cut lower tail). *Fix $K > K_*$ and $\beta \geq \beta_0(K)$. For every rectangle C with $\min(R, T) \geq R_1(\beta)$ and every $\sigma \in (0, 1]$,*

$$\begin{aligned} \mathbb{P}_D(A_{G'(D)}(C) \leq (1 - \eta_T)RT) &\leq 2^{4|C|+5} e^{-RT}, \\ \mathbb{E}_D \exp(\sigma(RT - A_{G'(D)}(C))) &\leq \exp(\sigma \eta_T RT + 4|C| + 4), \end{aligned} \quad (5.1)$$

where

$$\eta_T := \frac{3c_T'' \bar{r}_0^2 C_3}{|\log \varepsilon''|}, \quad (5.2)$$

with ε'' of (5.8) and the constants c_T'' and C_3 of Section S1.4. Moreover $\eta_T = O(\bar{r}_0^2(C'_* + C_{\text{resp}})(1 + 1/c_s)/(\tau \log^2 K))$, so $\eta_T \rightarrow 0$ as $K \rightarrow \infty$, and as $\beta \rightarrow \infty$ at fixed $K > K_*$.

The conditions defining $\beta_0(K)$ (item (a) of Section S1.1.4) are lower bounds on τ , which persist for larger τ , while the constants c_T'' , C_3 , C_{free} , C'_* , C_{resp} are free of β , τ , K , C , D .

5.2 The deterministic side

In this subsection we let $(\mathbb{X}, S)$ be any state with $S \subseteq E(\Lambda^\circ \setminus \mathbb{X})$ and compute $A_{G'(D)}(C)$ with $\mathfrak{F} = F(\mathbb{X}) \cup F_S \cup \partial_2 \Lambda^\circ$. Then $F_S \cap F(\mathbb{X}) = \emptyset$.

Lemma 5.2 (Unfrozen area). *Every integer 2-chain Σ in Λ° with $\partial \Sigma = C$ satisfies $\#(\text{supp } \Sigma \setminus \partial_2 \Lambda^\circ) \geq RT$. Moreover $A_{G'(D)}(C) \leq RT$.*

Proof. There are RT disjoint closed dual paths that link C , namely nested closed paths of fine cubes in the planes perpendicular to Ξ through its plaquette rows. They lie inside Λ° off $\partial_2 \Lambda^\circ$ (Lemma 3.3(e)), and each of them crosses Ξ once (Section S2.3). As Λ° is acyclic, $\Sigma - \Xi$ is a boundary. Hence each path meets $\text{supp } \Sigma$ (intersection number ± 1) in a plaquette of its own. Finally $A_{G'(D)}(C) \leq |\text{supp } \Xi| = RT$.

Pieces. A *piece partition* is a partition $\mathcal{Y}$ of the sites of $\mathbb{X}$ (pieces need not be connected). For a site set Y , let ∂Y be the set of coarse bonds with exactly one end in Y , $\text{int } F(Y)$ the set of fine plaquettes whose two fine cubes lie in sites of Y , and $\partial^f Y$ the set of those with one fine cube in a site of Y and the other in a site of $\Lambda^\circ \setminus Y$. Then

$$F(Y) = \text{int } F(Y) \sqcup \partial^f Y \sqcup (F(Y) \cap \partial_2 \Lambda^\circ), \quad |\partial^f Y| \leq \bar{r}_0^2 |\partial Y|. \quad (5.3)$$

Lemma 5.3 (Constant height on pieces). *There is a minimizer $\tilde{\Sigma}$ of Definition 2.11 ($\partial \tilde{\Sigma} = C$, $\#(\text{supp } \tilde{\Sigma} \setminus \mathfrak{F}) = A_{G'(D)}(C)$), determined by $(\mathbb{X}, S)$, such that for every Y of every piece partition*

$$\text{supp } \tilde{\Sigma} \cap F(Y) \subseteq \partial^f Y \cup (\text{supp } \Xi \cap \text{int } F(Y)) \cup \partial_2 \Lambda^\circ. \quad (5.4)$$

Proof. By acyclicity a minimizer is the flat rectangle plus a boundary, and we may change the boundary freely on hole cubes, where all plaquettes are free. Let Σ be the first minimizer in a fixed order and $\Sigma - \Xi = \partial k$, k an integer function on the fine cubes of Λ° (as in Lemma 5.2). Put $\tilde{k} := k$ on the fine cubes of sites of $\Lambda^\circ \setminus \mathbb{X}$, $\tilde{k} := 0$ on those of sites of $\mathbb{X}$, and $\tilde{\Sigma} := \Xi + \partial \tilde{k}$. Then $\partial \tilde{\Sigma} = C$, and $\tilde{\Sigma}$, Σ differ only on faces of fine cubes of sites of $\mathbb{X}$, which lie in $F(\mathbb{X}) \subseteq \mathfrak{F}$. Hence $\#(\text{supp } \tilde{\Sigma} \setminus \mathfrak{F}) = A_{G'(D)}(C)$. For $p \in \text{int } F(Y)$ both cubes of p carry $\tilde{k} = 0$, and therefore $\tilde{\Sigma}_p = \Xi_p$. Together with (5.3), this is (5.4), and $\tilde{\Sigma}$ does not involve $\mathcal{Y}$.

Lemma 5.4 (C -rooted minimizer). *There is a minimizer Σ° , determined by $(\mathbb{X}, S)$ alone, such that, with $\text{supp } \Sigma^\circ$ its support in $\Lambda_2^\circ \setminus \partial_2 \Lambda^\circ$ and $a_\Sigma := |\text{supp } \Sigma^\circ|$: $\text{supp } \Sigma^\circ \subseteq \text{supp } \tilde{\Sigma}$; every component of $\text{supp } \Sigma^\circ$ (plaquettes sharing an edge are adjacent) contains a plaquette incident to an edge of C ; (5.4) holds for Σ° for every piece partition; and $a_\Sigma \geq RT$.*

Proof. In a fixed order, we replace each component of $\text{supp } \tilde{\Sigma} \setminus \partial_2 \Lambda^\circ$ with no plaquette at an edge of C by a free 2-chain on $\partial_2 \Lambda^\circ$ with the same boundary, which lies on the sphere $\partial \Lambda^\circ$ (the plaquettes of $\text{supp } \tilde{\Sigma}$ at an edge off $C \cup \partial \Lambda^\circ$ are pairwise adjacent). Then (5.4) passes to subsets, $a_\Sigma \geq RT$ is Lemma 5.2, and Σ° is a function of D , as $(\mathbb{X}, S)$ is (Theorem 3.20(R1)).

Proposition 5.5 (Free count with frozen sheets). *For a piece partition $\mathcal{Y}$, call $Y \in \mathcal{Y}$ touched if $\text{supp } \Sigma^\circ \cap F(Y) \neq \emptyset$, and $b \in S$ touched if $\text{supp } \Sigma^\circ$ meets the coarse face dual to b . Then*

$$a_\Sigma - A_{G'(D)}(C) = \#(\text{supp } \Sigma^\circ \cap \mathfrak{F}) \leq \bar{r}_0^2 \left[\sum_{Y \text{ touched}} (2|\partial Y| + |Y|/\ell) + \#\{b \in S \text{ touched}\} \right]. \quad (5.5)$$

So if $A_{G'(D)}(C) \leq (1-\eta)RT$, $\eta \in (0, 1]$, the right side of (5.5) is at least $a_\Sigma - (1-\eta)RT \geq \eta RT$.

Proof. The identity holds as $\partial_2 \Lambda^\circ \subseteq \mathfrak{F}$ and Σ° is a minimizer, and $a_\Sigma \geq RT$ (Lemma 5.4). Off $\partial_2 \Lambda^\circ$ the free set is $F(\mathbb{X}) \sqcup F_S$. A touched bond of S contributes at most $\bar{r}_0^2$. A face of Ξ carrying a plaquette of $\text{int } F(Y)$ has both its sites in Y . Its coarse column perpendicular to the plane of C (disjoint from those of the other faces) therefore either lies in Y , with $n_3 \geq \ell$ sites (Lemma 3.3(c)), or contains a bond of ∂Y . Hence $|\text{supp } \Xi \cap \text{int } F(Y)| \leq \bar{r}_0^2 (|\partial Y| + |Y|/\ell)$, and by (5.4), (5.3) a touched Y contributes at most $\bar{r}_0^2 (2|\partial Y| + |Y|/\ell)$.

5.3 The Phase-1 Peierls estimate

In this subsection we work with the Phase-1 output $(\mathbb{X}^1, \mathbb{U}^1)$ (no frozen bond), take $\mathcal{P}$, $r' = 2R_* + 2$ and the pieces Y_P , $P \in \mathcal{P}$, from Theorem 3.21, and compute $\mathfrak{F}^1$, $A^1 := A_{G'(D)}(C)$ and Σ° for $(\mathbb{X}^1, \emptyset)$. For a plaquette set Ξ' , let $Z(\Xi')$ be the set of sites with a fine cube having a face in Ξ' . Then $|Z(\Xi')| \leq 2|\Xi'|$, as a plaquette is a face of at most two fine cubes. We also write $Z(\Sigma^\circ) := Z(\text{supp } \Sigma^\circ)$. A part P is *touched* if Y_P is.

Lemma 5.6 (Charge and rooting). $\#(\text{supp } \Sigma^\circ \cap \mathfrak{F}^1) \leq \bar{r}_0^2 \sum_{P \text{ touched}} (2|\partial Y_P| + |Y_P|/\ell) \leq \bar{r}_0^2 C_{\text{free}} \sum_{P \text{ touched}} N(P)$, and every touched part satisfies $\text{dist}(P, Z(\Sigma^\circ)) \leq R_*$.

Proof. The idea is that each part pays for its own free plaquettes out of its own cost. We apply Proposition 5.5 to $\mathcal{Y}^1 := \{Y_P\}$ together with Theorem 3.21(a). A touched plaquette is a face of a fine cube of a site of Y_P . Thus $Z(\Sigma^\circ)$ meets $Y_P \subseteq P^{+R_*}$ (Theorem 3.21(b)).

Lemma 5.7 (Marks). Put $q_m := 3(\bar{r}_0 + 1)^3 \delta^{1/2} \leq 24r_0^3 \delta^{1/2}$. For every finite set W of sites, $\mathbb{P}_D(W \subseteq \mathbb{M}) \leq q_m^{|W|}$.

Proof. A plaquette marks at most two sites, and a site has at most $3(\bar{r}_0 + 1)\bar{r}_0^2 \leq 3(\bar{r}_0 + 1)^3$ plaquettes on its fine cubes. If $W \subseteq \mathbb{M}$, a choice for each $x \in W$ of such a plaquette in D takes at least $|W|/2$ distinct values, and there are at most $(3(\bar{r}_0 + 1)^3)^{|W|}$ choices.

Lemma 5.8 (Animals). In a graph of maximal degree $D_{\max}$, the number of connected vertex sets of size n containing a given vertex is at most $(eD_{\max})^{n-1}$. The number of plaquette sets $\Xi' \subseteq \Lambda_2^\circ$ with $|\Xi'| = a$ each of whose components (for edge adjacency) contains a plaquette incident to an edge of C is at most $2^{4|C|}(2\kappa_{\text{pl}})^a$, with $\kappa_{\text{pl}} := 12e$.

Proof. By [GK04, Lemma 2.1 and the sentence after it, p. 141] the first number is at most $\binom{Dn}{n-1}/n \leq (eD)^{n-1}$, $D := D_{\max}$. For the second, we choose the first plaquette at an edge of C of each component, the sizes and the components. There are $2^{4|C|}$ choices for the first plaquettes and 2^a for the sizes, and the components are counted by the first count with $D_{\max} = 12$. The details are in Section S2.3.

$$\text{Put } \varepsilon_c := 8e(2R_* + 1)^3(2r' + 1)^3 q_m.$$

Proposition 5.9 (Peierls bound for parts). Assume $\varepsilon_c \leq \frac{1}{2}$. For every finite set Z of sites and every $u_0 > 0$, $\mathbb{P}_D(\sum_{P \in \mathcal{P}: \text{dist}(P, Z) \leq R_*} N(P) \geq u_0) \leq 2^{|Z|+1} \varepsilon_c^{u_0}$.

Proof. Since the partition is random, we take the union bound over all candidate families of mark sets, not over the partition, and we use only that parts are pairwise disjoint subsets of $\mathbb{M}$, r' -connected by definition (the graph on P joining sites at distance at most r' is connected, see Theorem 3.21). Thus the event lies in the union, over $u \geq u_0$ and all families $(P_1, \dots, P_m)$ of pairwise disjoint r' -connected site sets with $\text{dist}(P_i, Z) \leq R_*$ and $\sum_i |P_i| = u$, of the deterministic events $\{P_1 \cup \dots \cup P_m \subseteq \mathbb{M}\}$, each of probability at most q_m^u (Lemma 5.7). A family is fixed by the multiset of roots $z_i \in Z$ within R_* of P_i , the sizes, marks $y_i \in P_i$ within R_* of z_i , and the sets $P_i \ni y_i$, connected in a graph of degree less than $(2r' + 1)^3$ (Lemma 5.8). There are $2^{|Z|+m}$ values for the multiset of roots, 2^u for the sizes and $(2R_* + 1)^3$ for each mark. Summing over $m \leq u$, we bound the probability by $2^{|Z|} \sum_{u \geq u_0} \varepsilon_c^u \leq 2^{|Z|+1} \varepsilon_c^{u_0}$.

Lemma 5.10 (Surface sum). *Let $k \geq 1$ and $j \in \{1, 2, 3\}$ be integers, $c_T(k) := 1 + \log(2^{k+1}\kappa_{\text{pl}})$ and $c_T := c_T(2) = 2 + \log 96$, $\eta \in (0, 1]$, $\varepsilon \in (0, \frac{1}{2}]$, $F_* \geq 1$, and E an event with $E \subseteq \bigcup_{\Xi'} E_{\Xi'}$, the union over the plaquette sets Ξ' of Lemma 5.8 with $|\Xi'| \geq RT$, where $\mathbb{P}_D(E_{\Xi'}) \leq 2^k |\Xi'| + j \varepsilon^{(|\Xi'| - (1-\eta)RT)/F_*}$. If $\eta |\log \varepsilon| / F_* \geq c_T(k)$, then $\mathbb{P}_D(E) \leq 2^{4|C|+j+2} e^{-RT}$, and every X with $0 \leq X \leq RT$ and $X < \eta RT$ off E satisfies $\mathbb{E}_D e^{\sigma X} \leq e^{\sigma \eta RT + 4|C|+4}$ for every $\sigma \in (0, 1]$.*

Proof. With $\lambda := \log(2^{k+1}\kappa_{\text{pl}})$ and $\omega := |\log \varepsilon| / F_*$, Lemma 5.8 bounds $\mathbb{P}_D(E)$ by $2^{4|C|+j}$ times a geometric series in $a \geq RT$ of ratio $e^{-(\omega-\lambda)} \leq e^{-1}$, and $\eta\omega \geq 1 + \lambda$. The moment bound follows from $0 \leq X \leq RT$ (Section S2.3).

Proposition 5.11 (Phase-1 output). *Assume $\varepsilon_c \leq \frac{1}{2}$ and $\eta_T^{(1)} := c_T \bar{r}_0^2 C_{\text{free}} / |\log \varepsilon_c| \leq 1$. Then for every rectangle C with $\min(R, T) \geq R_1(\beta)$ and every $\sigma \in (0, 1]$, the two bounds of (5.1) hold with A^1 , $\eta_T^{(1)}$ and $2^{4|C|+3}$ in place of $A_{G'(D)}(C)$, η_T and $2^{4|C|+5}$.*

Proof. We argue as in the proof of Theorem 5.1, without block families. On $E := \{A^1 \leq (1 - \eta_T^{(1)})RT\}$, Proposition 5.5 with $\mathcal{Y}^1$ and Lemma 5.6 give $\sum N(P) \geq u_0(|\Xi'|) := (|\Xi'| - (1 - \eta_T^{(1)})RT) / (\bar{r}_0^2 C_{\text{free}})$ over the parts within R_* of $Z(\Xi')$, where $\Xi' := \text{supp } \Sigma^\circ$ is a function of D (Lemma 5.4). Proposition 5.9 bounds the probability of this event for each deterministic Ξ' by $2^{2|\Xi'|+1} \varepsilon_c^{u_0(|\Xi'|)}$. Finally, Lemma 5.10 with $k = 2$, $j = 1$, $\varepsilon = \varepsilon_c$, $F_* = \bar{r}_0^2 C_{\text{free}}$ and $X := RT - A^1 \in [0, RT]$ (Lemma 5.2) gives both bounds.

Lemma 5.12 (Threshold K_*). *For $K > K_* = 377.38$ there is a β -independent $\gamma_T(K) > 0$ with*

$$|\log \varepsilon_c| \geq \tau \left(\frac{c_\delta(K)}{2K} - 14.9662 \right) - O(\log \tau + \log r_0) \geq \gamma_T(K) \tau - O(\log \tau + \log r_0), \quad (5.6)$$

and (5.6) holds with ε'' of (5.8) in place of ε_c .

Proof. One mark costs $\delta^{1/2} = e^{-\tau c_\delta(K)/(2K)}$, and the root and part entropies of Proposition 5.9 cost $e^{6 \log R_* + O(\log r_0)} \leq e^{14.9662 \tau + O(\log \tau + \log r_0)}$, since $6\pi^2 v_{\text{Cb}}(0) < 14.9662$. For the second inequality we use the lower bound on $c_\delta(K)$ of Lemma 2.6, not its asymptotic form, and take $\gamma_T(K) := (c_\delta^{\text{sp}}(K) - 1/(4a_*)) / (2K) - 14.9662$, which vanishes at $K = 377.3635$ (Section S1.4).

5.4 Phase 2: separators and leaves

In Sections 5.4 to 5.6 we bound the Phase-2 cut bonds by the insulation around them. As a bond can lie next to many cuts, the key is the Leaf Lemma: at one place and scale, n cuts have $n + 1$ disjoint sides, each crossing an annulus with its own insulation.

Setting. The cuts j , their times t_j , components $\mathfrak{K}_j$ and sides U_j are those of Section 4.4. We put $U'_j := \mathfrak{K}_j \setminus U_j$, $\text{cut}_j := \partial_{G'_{t_j}} U_j$ and $A_j := |\text{cut}_j|$. At a cut time the state is hull-closed without auxiliary material, so (S-face) and (S-plug) hold (Theorem 4.16) and each cell at time t_j lies in one block and has at least $c_1 L_c^3$ sites (Theorem 4.20). A c -path is a sequence of sites, consecutive ones joined by a G' -bond or lying in a common cell (a step moves less than $2L_c$ in each coordinate). The *box graph* is Λ° with all coarse nearest-neighbour bonds (holes and frozen bonds ignored), with box-components and box paths, and $\partial(Y, W)$ denotes the set of box bonds from Y to W . Bonds are *square-adjacent* if they are edges of a common unit square. The distance of two bonds is the least ℓ^∞ distance of their ends, and a family of bond sets is k_P -linked if the graph joining members with bonds at distance at most k_P is connected.

Definition 5.13 (Separators). Fix a cut j and write $U := U_j$, $U' := U'_j$. Let U^α be the box-components of U , and $\mathcal{W}(\alpha)$ the set of box-components W of $\Lambda^\circ \setminus U^\alpha$ with $W \cap U' \neq \emptyset$. Put $\mathfrak{S}_j^* := \bigcup_\alpha \bigcup_{W \in \mathcal{W}(\alpha)} \partial(U^\alpha, W)$, and call each $\partial(U^\alpha, W)$ a *piece* of $\mathfrak{S}_j^*$. Define $\mathfrak{S}_j^{**}$ and its pieces in the same way with U and U' exchanged, and put $\mathfrak{S}_j := \mathfrak{S}_j^* \cup \mathfrak{S}_j^{**}$.

Lemma 5.14 (Separators). *For every cut j : (a) every bond of $\mathfrak{S}_j^*$ joins a site of U to a site outside U and is a cut bond of j or insulating at time t_j (a bond of $\partial\mathbb{X}_{t_j}$ or of S_{t_j}); the same holds for $\mathfrak{S}_j^{**}$ with U' ; (b) every piece is square-connected; (c) every piece contains a cut bond of j ; distinct pieces of $\mathfrak{S}_j^*$ are disjoint, so $\mathfrak{S}_j^*$ has at most A_j pieces, and the same holds for $\mathfrak{S}_j^{**}$; (d) every cut bond of j lies in a piece of $\mathfrak{S}_j^*$ and in a piece of $\mathfrak{S}_j^{**}$; (e) every box path from U to U' contains a bond of $\mathfrak{S}_j^*$ and a bond of $\mathfrak{S}_j^{**}$; (f) the pieces of $\mathfrak{S}_j$ are k_P -linked.*

Proof. The separators are box-graph objects, and we use Timár's lemma only in the box graph (in the slit graph its conclusion fails), whereas the slit graph enters only through (a) and the paths of (f). Items (a), (c) to (e) are elementary (Section S2.3).

(b) The set $\Pi := \partial(U^\alpha, W)$ is a minimal set of box bonds separating a site $y \in U^\alpha$ from a site $x \in W$. It separates, because each box neighbour of W outside W lies in U^α . It is minimal, because for $e = (u, v) \in \Pi$, $u \in U^\alpha$, $v \in W$, a box path inside U^α from y to u , then e , then a box path inside W from v to x , meets Π only in e . The unit squares generate the cycle space of the box graph over $\mathbb{Z}_2$, since the cubical complex of a box is contractible. By [Tim13, Lemma 1], for a minimal separating edge set Π and a generating set $\mathcal{C}$ of the cycle space, each partition (Π_1, Π_2) of Π admits a cycle of $\mathcal{C}$ meeting both Π_1 and Π_2 . Taking for $\mathcal{C}$ the unit squares, we obtain the square-connectedness of Π .

(f) A cell is $\Omega = Q \cap \hat{\mathfrak{K}}$ with $\hat{\mathfrak{K}}$ a component of $G'_{t_j} \cap \hat{Q}(Q)$. In particular, two sites of one cell are joined by a G' -path inside $\hat{Q}(Q)$, a box of sides less than k_P (G4), lying in $\mathfrak{K}_j$ if the cell does. Bonds with all ends in $\hat{Q}(Q)$ are at distance at most $k_P - 1$. A G'_{t_j} -path in $\mathfrak{K}_j$ enters or leaves U^α only through cut bonds, and such a bond with outer end v lies in the piece $\partial(U^\alpha, W)$, where $W \ni v$ is a box-component of $\Lambda^\circ \setminus U^\alpha$. Here $v \in \mathfrak{K}_j \setminus U = U'$, as a site of U box-adjacent to U^α lies in U^α .

Same α , different W . A G' -bond inside U' joins box-adjacent sites outside U^α and therefore stays in one box-component of $\Lambda^\circ \setminus U^\alpha$. For $W_1 \neq W_2$ in $\mathcal{W}(\alpha)$ we take a c-path in U' from $W_1 \cap U'$ to $W_2 \cap U'$ (U' is c-connected). Along it the box-component of $\Lambda^\circ \setminus U^\alpha$ changes only at same-cell steps, between two sites of a cell $\Omega' \subseteq U'$ lying in components $W \neq \tilde{W}$. The G' -path inside $\hat{Q}$ joining them is a box path from W to $\tilde{W}$ and hence meets U^α . Its first entry crosses a cut bond of $\partial(U^\alpha, W)$ and its last exit one of $\partial(U^\alpha, \tilde{W})$, since before the entry (after the exit) the path stays in W (in $\tilde{W}$). As both bonds lie in $\hat{Q}$, these two pieces contain bonds at distance at most k_P . Following the c-path, we conclude that all pieces $\partial(U^\alpha, W)$, $W \in \mathcal{W}(\alpha)$, are linked.

Different α . A G' -bond inside U stays in one U^α . Thus, along a c-path in U , the box-component changes only at same-cell steps between sites of a cell $\Omega \subseteq U$ lying in $U^{\alpha_1} \neq U^{\alpha_2}$. The G' -path inside $\hat{Q}$ joining them leaves U^{α_1} and enters U^{α_2} through cut bonds of pieces of α_1 and of α_2 , both in $\hat{Q}$. Each α has a piece, since the box-component of $\Lambda^\circ \setminus U^\alpha$ containing a site of U' lies in $\mathcal{W}(\alpha)$. Combining this with the first case, we find that all pieces of $\mathfrak{S}_j^*$ are linked. The same holds for the pieces of $\mathfrak{S}_j^{**}$ (U and U' exchanged), and by (d) a cut bond lies in a piece of each. Hence all pieces of $\mathfrak{S}_j$ are linked.

For a site x and $r > 0$, $B_\infty(x, r)$ is the set of bonds with both ends within ℓ^∞ distance r

of x . Put $c_{\text{sh}} := c_1/216$, and let $\text{side}_{\min}$ be the least coarse side of Λ° .

Lemma 5.15 (Admissible cuts). *For a site x and $27L_c \leq \rho \leq \text{side}_{\min}/24$, a cut j is admissible at (x, ρ) if (i) a cut bond of j has both ends within distance $\rho/2 + 2$ of x ; (ii) U_j and U'_j (at time t_j) each contain a site farther than 9ρ from x ; (iii) $|\text{cut}_j \cap B_\infty(x, 9\rho)| \leq c_{\text{sh}}\rho L_c/2$. If j is admissible at (x, ρ) , then $B_\infty(x, 9\rho)$ contains at least $c_{\text{sh}}\rho L_c$ bonds of each of $\mathfrak{S}_j^*$ and $\mathfrak{S}_j^{**}$.*

Proof. Both sides of the cut cross about $\rho/(9L_c)$ disjoint shells around x , and in each shell the separator, which contains all box bonds between the sides it separates (Lemma 5.14(e)), is large by a box Cheeger inequality. In a box $B = [n_1] \times [n_2] \times [n_3] \subset \mathbb{Z}^3$ (shell boxes need not be cubes), with $n := \max_i n_i$, each $A \subseteq B$ with $|A| \leq |B|/2$ has at least $|A|/(2n)$ bonds to $B \setminus A$. To see this, we route all ordered pairs along coordinate-ordered paths. At most $n|B|$ of them use a given bond, and at least $|A||B|/2$ join A to $B \setminus A$. The *shell boxes* form a tensor-product partition of Λ° with sides in $[6L_c, 12L_c)$, and the shell Sh_i is the union of the boxes at index distance $3i$, $3i+1$ or $3i+2$ from the box of x . A c -path moves less than $2L_c$ per step and cells lie in one block. It follows that each of the at least $\rho/(9L_c)$ disjoint, face-connected shells Sh_i with $\rho/(36L_c) + 1 \leq i \leq \rho/(4L_c) - 1$ lies within 9ρ of x and contains a whole cell of U_j and one of U'_j , by (i) and (ii). In each shell, the Cheeger inequality in one shell box or in the union of two adjacent ones gives $(c_1/24)L_c^2$ bonds of $\mathfrak{S}_j^*$ and as many of $\mathfrak{S}_j^{**}$. Summing over the shells, we obtain $(\rho/(9L_c))(c_1/24)L_c^2 = c_{\text{sh}}\rho L_c$ bonds of each. The full argument is in Section S2.3.

Lemma 5.16 (Leaf Lemma). *Let $j_1, \dots, j_n$ be distinct cuts admissible at (x, ρ) . Let I be the set of bonds $b \in B_\infty(x, 9\rho)$ such that for some i , $b \in \mathfrak{S}_{j_i}$, b is not a cut bond of j_i , and b is insulating at time t_{j_i} . Then $|I| \geq (n+1)c_{\text{sh}}\rho L_c/4$.*

Proof. The $2n$ sides U_{j_i} , U'_{j_i} (at the respective times) are pairwise disjoint or nested (Lemma 4.27(b)) and distinct, since the component of a later cut lies in one side of an earlier cut or outside its component. Hence the minimal sides, the *leaves*, are pairwise disjoint. We show that there are at least $n+1$ of them by adding the cuts in firing order. When a cut j is added, its two sides are leaves, since an earlier side contains $\mathfrak{K}_j$ or is disjoint from it. An earlier leaf stops being one only if it contains $\mathfrak{K}_j$, while the earlier sides containing $\mathfrak{K}_j$ form a chain, with at most one minimal element. Thus each cut adds two leaves and removes at most one. For a leaf $L = U_j$ (resp. U'_j), Lemma 5.15 gives at least $c_{\text{sh}}\rho L_c$ bonds of $\mathfrak{S}_j^*$ (resp. $\mathfrak{S}_j^{**}$) in $B_\infty(x, 9\rho)$, at most $c_{\text{sh}}\rho L_c/2$ of them are cut bonds of j (Lemma 5.15(iii)), and the others are insulating at t_j and have an end in L (Lemma 5.14(a)), so they lie in I . Since a bond has two ends and the leaves are disjoint, a bond of I is counted for at most two leaves.

5.5 Phase 2: charges and domination

A cut fires only on a fragmented split, whose scale cost exceeds its area. We spread the area over thin clusters and charge each to its own insulation (Leaf Lemma), which gives the doubly exponentially small rate ε_D .

Scales. For the scales $s = 2^j L \geq R_{\text{NF}}$ of Definition 3.18 (lattice lengths) put $\rho(s) := s/(128\bar{r}_0)$ (coarse units). Points of two sites at coarse distance d are at lattice Euclidean distance at most $\sqrt{3}\bar{r}_0(d+1)$. Consequently, (U1) two cut bonds whose ends are pairwise within coarse distance d lie in one s -cluster if $\sqrt{3}\bar{r}_0(d+1) \leq s$, and (U2) cut bonds of different s -clusters have all their ends at coarse distance greater than $s/(\sqrt{3}\bar{r}_0) - 1 > 73\rho(s) - 1$. The *charge boxes* at scale s partition the coarse lattice into cubes Q of side $\lceil \rho(s) \rceil$ with centres x_Q .

Thin clusters, good scales. Fix a cut j and a scale s . An s -cluster c of cut_j is *thin* if it contains a bond with an end x_c such that $|c \cap B_\infty(x_c, 10\rho(s))| \leq c_{\text{sh}}\rho(s)L_c/2$, and *fat* otherwise. For a thin cluster we fix such a bond and end x_c . The scale s is *good* for j if $\varphi_s \geq 2$ and at least half of the s -clusters are thin. Let φ_s^{thin} be the number of thin s -clusters.

Lemma 5.17 (Charges). *Assume $R_{\text{NF}} \geq \max(3456\bar{r}_0L_c, 2048\bar{r}_0L_c/(c_{\text{sh}}\theta_{\text{NF}}F_{\text{unit}}))$. For every cut j : (a) $J'_j := \sum_{s \text{ good}} (\varphi_s - 1) \geq J(U_j)/2$; (b) with $q_j(s, c) := A_j(\varphi_s - 1)/(J'_j\varphi_s^{\text{thin}})$ (s good, c a thin s -cluster), $\sum_{s \text{ good}} \sum_{c \text{ thin}} q_j(s, c) = A_j$ and $q_j(s, c) < 4L_c^2/(\theta_{\text{NF}}F_{\text{unit}})$; (c) for good s and thin c , with $Q \ni x_c$ the charge box, j is admissible at $(x_Q, \rho(s))$ and $x_{c'} \notin Q$ for every other thin s -cluster c' .*

Proof. (a) A fat cluster has more than $c_{\text{sh}}\rho L_c/2$ bonds, $\rho = \rho(s)$, and clusters are disjoint subsets of cut_j . Consequently, at a scale with $\varphi_s \geq 2$ that is not good, $\varphi_s - 1 < 2\varphi_s^{\text{fat}} < 4A_j/(c_{\text{sh}}\rho L_c) = 512\bar{r}_0A_j/(c_{\text{sh}}sL_c)$. Summing over dyadic $s \geq R_{\text{NF}}$ gives less than $1024\bar{r}_0A_j/(c_{\text{sh}}R_{\text{NF}}L_c) \leq J(U_j)/2$, by the assumption on R_{NF} and $J(U_j) > \theta_{\text{NF}}F_{\text{unit}}A_j/L_c^2$ (the split is fragmented).

(b) $\sum_c q_j(s, c) = A_j(\varphi_s - 1)/J'_j$, and these sum to A_j over good s . At a good scale $\varphi_s - 1 < \varphi_s \leq 2\varphi_s^{\text{thin}}$, so $q_j(s, c) < 2A_j/J'_j \leq 4A_j/J(U_j) < 4L_c^2/(\theta_{\text{NF}}F_{\text{unit}})$.

(c) Take for x_Q a central site of Q , so that $|x_c - x_Q|_\infty \leq \rho/2 + 1$, $\rho = \rho(s)$. (i) The chosen bond of c has both ends within $\rho/2 + 2$ of x_Q . (ii) Since $\varphi_s \geq 2$, there is a cut bond in another s -cluster, with one end in U_j and one in U'_j . By (U2) its ends are farther than 9ρ from x_Q . (iii) $B_\infty(x_Q, 9\rho) \subseteq B_\infty(x_c, 10\rho)$, and by (U1) all cut bonds of j in $B_\infty(x_c, 10\rho)$ lie in c , which is thin. If $x_c, x_{c'} \in Q$, the two chosen bonds lie in one cluster by (U1). The numerical checks and the range $27L_c \leq \rho \leq \text{side}_{\min}/24$ of Lemma 5.15 are in Section S1.4.

Classes. For a bond set Y , $\mathfrak{B}(Y)$ is the set of blocks containing an end of a bond of Y . The *classes* $\mathcal{J}$ are the connected components of the graph on cuts with $j \sim j'$ if $\mathfrak{B}(\mathfrak{S}_j) \cap \mathfrak{B}(\mathfrak{S}_{j'}) \neq \emptyset$. We put $\mathfrak{X}(\mathcal{J}) := \bigcup_{j \in \mathcal{J}} \mathfrak{S}_j$, $\mathfrak{B}(\mathcal{J}) := \mathfrak{B}(\mathfrak{X}(\mathcal{J}))$ and $A(\mathcal{J}) := \sum_{j \in \mathcal{J}} A_j$. Let S_{cut} be the set of all bonds frozen by the cut rule and $S_1^{(2)}$ the set of bonds frozen by seal steps in Phase 2, and put $\text{ins}^1 := \partial\mathbb{X}^1$. The *payable bonds* form $B_{\text{pay}} := \text{ins}^1 \cup S_{\text{cut}} \cup S_1^{(2)} \cup \partial\mathbb{X}_{\text{final}} \cup \bigcup_\Delta \partial\Delta$, the last union over the cells removed in Phase 2. We denote by m_q the number of marks of a block q , and by $h(\mathfrak{B})$ the number of sites in the blocks of $\mathfrak{B}$ that are hull material at some time of Phase 2. As a bond is frozen at most once (Proposition 4.3), each cut bond belongs to exactly one cut.

Lemma 5.18 (Classes). (a) *Distinct classes have disjoint $\mathfrak{X}$'s and disjoint $\mathfrak{B}$'s.* (b) *The cut bonds of every $j \in \mathcal{J}$ lie in $\mathfrak{X}(\mathcal{J})$, and every cut bond lying in $\mathfrak{X}(\mathcal{J})$ belongs to a cut of $\mathcal{J}$; so $|\mathfrak{X}(\mathcal{J}) \cap S_{\text{cut}}| = A(\mathcal{J})$.* (c) *$\mathfrak{B}(\mathcal{J})$ is a union of at most $2A(\mathcal{J})$ sets, each $*$ -connected in the block lattice (26-adjacency), which are linked at block distance at most $k_b := \lceil k_P/L_c \rceil + 2$.* (d) *Every bond of $\mathfrak{X}(\mathcal{J}) \setminus B_{\text{pay}}$ is, at some time of Phase 2, a hole face of a site $y \notin \mathbb{X}^1$ that is hull material at that time and lies in a block of $\mathfrak{B}(\mathcal{J})$. So $|\mathfrak{X}(\mathcal{J}) \setminus B_{\text{pay}}| \leq 6h(\mathfrak{B}(\mathcal{J})) \leq 12L_c \sum_{q \in \mathfrak{B}(\mathcal{J})} m_q$.*

Proof. (a) A common bond or block relates two cuts. (b) The first claim is Lemma 5.14(d). For the second, a cut bond of j'' in $\mathfrak{S}_j$ lies in $\mathfrak{S}_{j''}$, so $j'' \sim j$. (c) The set $\mathfrak{S}_j$ has at most $2A_j$ pieces (Lemma 5.14(c)), which are square-connected, hence with $*$ -connected blocks, and k_P -linked (Lemma 5.14(b), (f)). Bonds at distance $\leq k_P$ have blocks at block distance $\leq k_P/L_c + 1 \leq k_b$, and related cuts share a block.

(d) Let $b \in \mathfrak{S}_j \setminus B_{\text{pay}}$, $j \in \mathcal{J}$. Cut bonds and bonds frozen at t_j lie in $S_{\text{cut}} \cup S_1^{(2)} \subseteq B_{\text{pay}}$ (the bonds frozen in Phase 1 are converted, Theorem 4.17). It follows by Lemma 5.14(a) that $b = (y, u)$ is a hole face at t_j , with $y \in \mathbb{X}_{t_j}$ and $u \in \mathfrak{R}_j \subseteq G'_{t_j}$. If $y \in \mathbb{X}^1$, then $b \in \partial \mathbb{X}^1 = \text{ins}^1$, as $u \in G'_{t_j} \subseteq G'_{\mathbb{X}^1}$ (G' only shrinks). If y lies in a cell Δ removed in Phase 2 before t_j , then $b \in \partial \Delta$, as $u \notin \Delta \subseteq \mathbb{X}_{t_j}$. Otherwise $y \notin \mathbb{X}^1 \cup R_{t_j} \supseteq \mathbb{M} \cup \mathbb{U}^1 \cup R_{t_j} = B_{t_j}$ (Phase 2 fills only by removal, and R_t is as in Section 4.4), so y is hull material at t_j , also if it lies in a cell removed after t_j . Its block contains an end of b and hence lies in $\mathfrak{B}(\mathcal{J})$. A site carries six bonds. For the last bound, let $q \not\subseteq \mathbb{U}^1$ be a block. If its removal cell is never removed, the hull material $(\mathbb{X}_t \setminus B_t) \cap q = (\mathbb{X}_t \cap q) \setminus \mathbb{M}$ only grows in t . If the cell is removed at time t_Δ , all hull material of q before t_Δ is hull material just before t_Δ (removal cells are unions of blocks, Lemma 3.3(c)). Corollary 4.7 at the final time, or just before the removal, gives fewer than $2L_c m_q$ sites per block.

The hole faces of (d) whose other end is absorbed later (*transient* faces) can number L_c times the local sources. We absorb them into ε_D and attach them to the contour only through blocks.

Proposition 5.19 (Domination). *For every set $\mathcal{J}$ of cuts, $A(\mathcal{J}) \leq \varepsilon_D |\mathfrak{X}(\mathcal{J})|$ with $\varepsilon_D := 3.2768 \cdot 10^7 \bar{r}_0 L_c / (c_{\text{sh}} \theta_{\text{NF}} F_{\text{unit}} R_{\text{NF}})$, and if $\mathcal{J}$ is a class and $\varepsilon_D \leq \frac{1}{2}$,*

$$A(\mathcal{J}) \leq 2\varepsilon_D \left(|\mathfrak{X}(\mathcal{J}) \cap B_{\text{pay}} \setminus S_{\text{cut}}| + 12L_c \sum_{q \in \mathfrak{B}(\mathcal{J})} m_q \right). \quad (5.7)$$

Proof. We assign the charge $q_j(s, c)$ of Lemma 5.17 to the pair (Q, s) with $Q \ni x_c$. Fix (Q, s) and let $j_1, \dots, j_n$ be the cuts of $\mathcal{J}$ with a charge at (Q, s) . Each has exactly one, of size less than $4L_c^2 / (\theta_{\text{NF}} F_{\text{unit}})$, and is admissible at $(x_Q, \rho(s))$ (Lemma 5.17(b), (c)). The Leaf Lemma gives $I_Q \subseteq B_\infty(x_Q, 9\rho(s)) \cap \mathfrak{X}(\mathcal{J})$ with $|I_Q| \geq (n+1)c_{\text{sh}}\rho(s)L_c/4$. Hence the charge at (Q, s) is at most $4nL_c^2 / (\theta_{\text{NF}} F_{\text{unit}}) \leq 16L_c |\mathfrak{X}(\mathcal{J}) \cap B_\infty(x_Q, 9\rho(s))| / (c_{\text{sh}} \theta_{\text{NF}} F_{\text{unit}} \rho(s))$. A bond lies in $B_\infty(x_Q, 9\rho(s))$ for at most 20^3 charge boxes Q of one scale, and $\sum_s 1/\rho(s) \leq 256\bar{r}_0/R_{\text{NF}}$ over dyadic $s \geq R_{\text{NF}}$. Summing over (Q, s) , we obtain $A(\mathcal{J}) \leq 20^3 \cdot 16 \cdot 256\bar{r}_0 L_c |\mathfrak{X}(\mathcal{J})| / (c_{\text{sh}} \theta_{\text{NF}} F_{\text{unit}} R_{\text{NF}}) = \varepsilon_D |\mathfrak{X}(\mathcal{J})|$. For (5.7), $\mathfrak{X}(\mathcal{J})$ is the disjoint union of $\mathfrak{X} \cap S_{\text{cut}}$, $\mathfrak{X} \cap B_{\text{pay}} \setminus S_{\text{cut}}$ and $\mathfrak{X} \setminus B_{\text{pay}}$. The first has $A(\mathcal{J})$ elements and the last is bounded by Lemma 5.18(b), (d). Thus $A \leq \varepsilon_D (A + R)$ with R the bracket of (5.7), and $A \leq \varepsilon_D R / (1 - \varepsilon_D) \leq 2\varepsilon_D R$.

5.6 Phase 2: responses

A cut changes G' only on its own bonds, which cross block faces and carry the hull weight $2 - \eta_w$. The hull, the removal and the seal therefore respond to it as to marks, and we pay for the responses locally with the marks, the cut bonds and the bonds of $\text{ins}^1 = \partial \mathbb{X}^1$. Each frozen bond is a grid bond and supports no hull material (Corollary 4.7). Hence a response creates no new source, so paying for the responses involves no recursion. Let $m(Z)$ be the number of marks in Z , and let $n_{\text{cut}}(Z)$ and $n_{\text{seal}}(Z)$ be the numbers of cut bonds and of bonds frozen by seal steps in Phase 2 (converted ones included) with both ends in Z . For a removal cell Δ_0 , $\mathcal{N}(\Delta_0)$ is the set of the at most 27 removal cells within index distance 1 of Δ_0 , with union $U(\Delta_0)$, and $\mathcal{N}_2(\Delta_0)$ is the set of the at most 125 removal cells within index distance 2 of Δ_0 , with union $U_2(\Delta_0)$.

Response units. These are (1) the bonds of $S_1^{(2)}$, (2) the bonds of $\partial\mathbb{X}_{\text{final}} \setminus \text{ins}^1$ (hole faces created in Phase 2 and present at the end), and (3) the pairs (Δ, b) with Δ a cell removed in Phase 2 and $b \in \partial\Delta$. A unit is located at its bond, and $\text{units}(Z)$ denotes the number of units with an end in Z . We assign each unit to the removal cell of the first of its ends in a fixed order. Transient hole faces are not units. Each bond of $B_{\text{pay}} \setminus S_{\text{cut}}$ lies in ins^1 or is the bond of a unit.

Lemma 5.20 (Units). *For every removal cell Δ_0 , $\text{units}(\Delta_0) \leq C_a m(U_2(\Delta_0)) + C_b n_{\text{cut}}(U_2(\Delta_0)) + C_c |\text{ins}^1 \cap E(U_2(\Delta_0))|$, with $C_a := 4362 + 5808/c_s$, $C_b := 2 + 4/c_s$ and $C_c := 1 + 2/c_s$.*

Proof. We bound the units near Δ_0 by the seal count or by comparison with the Phase-1 output (Lemma 4.25). Write $U := U(\Delta_0)$, $m := m(U_2(\Delta_0))$, $n := n_{\text{cut}}(U)$ and $i := |\text{ins}^1 \cap E(U)|$. A seal bond with an end in a block q lies on one of the six faces of q , and the face shared with q' carries at most $121(m_q + m_{q'})$ of them (Proposition 4.15). Summing over the blocks of Δ_0 , resp. of U , we find that at most $1452m$ units of type (1) have an end in Δ_0 and that $n_{\text{seal}}(U) \leq 1452m$. A unit (Δ, b) with an end in Δ_0 has $\Delta \in \mathcal{N}(\Delta_0)$, and by Lemma 4.25(b) the units of type (3) of the cells of $\mathcal{N}(\Delta_0)$ number less than $u_3 := c_s^{-1}(i + 2n_{\text{seal}}(U) + 2n)$. We turn to the units of type (2). For $1 \leq r \leq L_c$ the boxes $N_r := \{x : \text{dist}(x, \Delta_0) \leq r\}$ lie in U and have disjoint outer layers. A final hole face with an end in Δ_0 lies in $E(N_r)$ and has exactly one end in $\mathbb{X}_\infty$, so it is counted in $\mathcal{A}_2(E(N_r); \infty)$, which (4.4) bounds by $\text{Ins}_{N_r}(\mathbb{X}^1 \cup R_\infty) + \text{cut}_{N_r}$. A bond of $E(N_r)$ with exactly one end in $\mathbb{X}^1 \cup R_\infty$ is a bond of ins^1 or the bond of a unit of type (3) of a cell of $\mathcal{N}(\Delta_0)$, and a bond of $\bar{S}_\infty \cap E(N_r)$ with no end in $\mathbb{X}^1$ was frozen in Phase 2. Hence $\text{Ins}_{N_r}(\mathbb{X}^1 \cup R_\infty) \leq i + u_3 + 2n_{\text{seal}}(U) + 2n$. The sites counted in cut_{N_r} are hull material in the outer layer of N_r , each with at most three bonds leaving N_r , and the blocks of U hold fewer than $2L_c m$ sites of hull material (Corollary 4.7). Consequently, $\text{cut}_{N_r} \leq 6m$ for some r . Adding these bounds, we obtain $\text{units}(\Delta_0) \leq 1452m + 2u_3 + i + 2n_{\text{seal}}(U) + 2n + 6m$, which gives the claim (Section S1.4).

Lemma 5.21 (Charging map). *There are weights $\varpi(u, z) \geq 0$, for u a response unit and z a source (a mark, a cut bond or a bond of ins^1), with $\varpi(u, z) = 0$ if $\text{dist}(u, z) > R_{\text{resp}} := 10\ell$ and $\sum_z \varpi(u, z) \geq 1$ for every u , such that every mark receives $\sum_u \varpi(u, z) \leq 125C_a$, every cut bond at most $125C_b$, and every bond of ins^1 at most $125C_c$.*

Proof. Since Lemma 5.20 bounds the units of a cell by the sources around it, we spread each unit over those sources. For a unit u assigned to Δ_0 put $\varpi(u, z) := c_z/D(\Delta_0)$ for the sources z of $U_2(\Delta_0)$, that is, the marks in $U_2(\Delta_0)$, the cut bonds and the bonds of ins^1 in $E(U_2(\Delta_0))$. Here $c_z = C_a, C_b$ or C_c according to the kind of z , and $D(\Delta_0)$ is the right side of Lemma 5.20. Then $\sum_z \varpi(u, z) = 1$, and $D(\Delta_0) \geq \text{units}(\Delta_0) > 0$ if a unit is assigned to Δ_0 . A source lies in $U_2(\Delta)$ for at most 125 cells Δ and therefore receives at most $125c_z$. A unit assigned to Δ_0 has an end in Δ_0 , a source of $U_2(\Delta_0)$ lies in a cell at index distance at most 2, and cells have sides less than 3ℓ . Hence $\text{dist}(u, z) \leq 9\ell + 1 \leq R_{\text{resp}}$.

Put $C_{\text{resp}} := 375C_a + 125C_c C'_\star$ (fixed before K , see Section S1.4). A part P owns the bonds of ∂Y_P , at most $C'_\star N(P)$ bonds of ins^1 (Theorem 3.21(a)). The marks and the ins^1 bonds of P therefore receive at most $(125C_a + 125C_c C'_\star)N(P) \leq C_{\text{resp}}N(P)$, and a cut bond receives at most $125C_b \leq C_{\text{resp}}$.

5.7 Counting and the proof of Theorem 5.1

In this subsection $(\mathbb{X}, S)$ is the Phase-2 output, with $\mathfrak{F}$ and the minimizer Σ° of Lemma 5.4 computed for it, and we write $\Xi' := \text{supp } \Sigma^\circ$ and $a := a_\Sigma = |\Xi'| \geq RT$. Put $R' := R_* + 2R_{\text{resp}}$ and

$$\varepsilon'' := 8e(2R' + 4L_c + 1)^3(2r' + 1)^3 q_m. \quad (5.8)$$

The hole set only grows (Proposition 4.3). Hence $\mathbb{X}^1 \subseteq \mathbb{X}$.

Pieces. The sets Y_P , $P \in \mathcal{P}$ (Theorem 3.21(a)), the sets $Y_\Delta := \Delta \setminus \mathbb{X}^1$ for the cells Δ removed in Phase 2 (a removed cell lies in $\mathbb{X}$), and the sets $Y_{\Delta'} := (\mathbb{X} \setminus \mathbb{X}^1) \cap \Delta'$ for the removal cells Δ' not removed in Phase 2 partition $\mathbb{X}$. A piece is touched if Ξ' meets its $F(\cdot)$, and a part P is touched if Y_P is. The cell of a touched Y_Δ or $Y_{\Delta'}$ is a *touched cell*. Let u_{touch} be the number of response units that have an end in a touched cell or are of type (1) with a coarse face containing a plaquette of Ξ' (each unit counted once), c_{touch} the number of bonds of $S \cap S_{\text{cut}}$ whose coarse face contains a plaquette of Ξ' , and i_{touch} the number of bonds of ins^1 with an end in a touched cell.

Proposition 5.22 (Phase-2 free count). $\#(\Xi' \cap \mathfrak{F}) \leq \bar{r}_0^2 [C_{\text{free}} \sum_{P \text{ touched}} N(P) + 20(u_{\text{touch}} + c_{\text{touch}} + i_{\text{touch}}) + (26L_c/\ell) \sum_{\Delta' \text{ touched}} m(\Delta')]$, the last sum over the touched cells Δ' .

Proof. Roughly speaking, the removal cells cut Phase-2 hole material into pieces of diameter less than 9ℓ , bounded by Phase-1 insulation or response units, whose volume enters divided by ℓ . We apply Proposition 5.5 to these pieces. A touched part costs at most $C_{\text{free}}N(P)$ (Lemma 5.6). A touched Y_Δ costs $2|\partial Y_\Delta| + |Y_\Delta|/\ell$, with $|Y_\Delta|/\ell < 27\ell^2 \leq 4.5|\partial \Delta|$. A bond of ∂Y_Δ either lies in $\partial \Delta$, in which case it is the bond of a unit (Δ, b) of type (3), or joins $\mathbb{X}^1 \cap \Delta$ to $\Delta \setminus \mathbb{X}^1$, in which case it is a bond of ins^1 with both ends in Δ .

A touched $Y := Y_{\Delta'}$ costs $2|\partial Y| + |Y|/\ell$. A bond (y, v) of ∂Y , $y \in Y$, is of one of four types: (a) $v \in G'$: a final hole face with no end in $\mathbb{X}^1$, the bond of a unit of type (2); (b) $v \in \mathbb{X}^1$, so the bond lies in ins^1 ; (c) v lies in a cell Δ removed in Phase 2, so $(\Delta, (y, v))$ is a unit of type (3); (d) $v \in (\mathbb{X} \setminus \mathbb{X}^1) \cap \Delta''$ for another cell Δ'' not removed in Phase 2. For a bond of type (d), we walk from y along its coordinate line into Δ' . If the walk leaves Y inside Δ' , it leaves through a bond from Y to $\Delta' \setminus Y \subseteq G' \cup \mathbb{X}^1$, of type (a) or (b) with both ends in Δ' , and distinct walks end at distinct bonds. Otherwise the walk crosses Δ' inside Y , through at least ℓ sites, and we charge $1/\ell$ to each of them. Since a site lies on three lines, each with at most two type-(d) bonds at its ends, it receives at most $6/\ell$. Hence $|\partial Y|$ is at most the number of bonds of types (a) to (c) with an end in Y , plus those of types (a), (b) inside Δ' , plus $6|Y|/\ell$. Moreover, $|Y| < 2L_c m(\Delta')$ by Corollary 4.7 at the final time, as Y is hull material (Phase 2 fills only by removal).

Since the Phase-1 output has no frozen bond, a touched bond of S lies in $S_1^{(2)}$ or in $S \cap S_{\text{cut}}$. In the first case it is a unit of type (1), counted in u_{touch} , and in the second it is counted in c_{touch} . Each unit or bond of ins^1 charged above has an end in the touched cell charging it, so it is counted in u_{touch} or i_{touch} , with coefficients adding up to at most 20 (Section S1.4).

The attached family. Let $\mathcal{T}$ be the set of classes with a cut bond counted in c_{touch} , and $\mathcal{F}$ the smallest set of classes containing $\mathcal{T}$ with the closure property: if $\varpi(u, z) > 0$ for a cut bond z of a class $\mathcal{J}'$, where (i) u is a unit counted in u_{touch} , or (ii) the bond of the unit u lies in $\mathfrak{X}(\mathcal{J})$ for some $\mathcal{J} \in \mathcal{F}$, then $\mathcal{J}' \in \mathcal{F}$. Put $\mathfrak{B}_{\mathcal{F}} := \bigsqcup_{\mathcal{J} \in \mathcal{F}} \mathfrak{B}(\mathcal{J})$ and $A_{\mathcal{F}} := \sum_{\mathcal{J} \in \mathcal{F}} A(\mathcal{J})$. Let $Z_{\mathcal{F}}$ be the set of sites of the blocks of $\mathfrak{B}_{\mathcal{F}}^{+1}$ (the blocks at block distance at most 1 from $\mathfrak{B}_{\mathcal{F}}$)

and $\mathcal{Q}_{\text{att}}$ the set of parts P with $\text{dist}(P, Z(\Xi') \cup Z_{\mathcal{F}}) \leq R'$, and put $N_{\mathcal{Q}} := \sum_{P \in \mathcal{Q}_{\text{att}}} N(P)$. Consider the condition

$$8\varepsilon_D(1 + 8L_c)(C'_* + 3C_{\text{resp}} + 6)(48C_{\text{resp}} + 2) \leq 1, \quad (5.9)$$

which holds for $\beta \geq \beta_0(K)$, as ε_D is doubly exponentially small. We assume it in Lemma 5.23 and Proposition 5.24.

Lemma 5.23 (Attached family). *(a) the units counted in u_{touch} and the units whose bond lies in $\bigcup_{\mathcal{F}} \mathfrak{X}(\mathcal{J})$ number together at most $2C_{\text{resp}}(N_{\mathcal{Q}} + A_{\mathcal{F}})$; (b) $A_{\mathcal{F}} \leq 8\varepsilon_D(1 + 8L_c)(C'_* + 3C_{\text{resp}} + 6)N_{\mathcal{Q}} \leq N_{\mathcal{Q}}/(48C_{\text{resp}} + 2)$; (c) $c_{\text{touch}} \leq A_{\mathcal{F}}$, $i_{\text{touch}} \leq C'_*N_{\mathcal{Q}}$ and $|\mathfrak{B}_{\mathcal{F}}| \leq C_{\mathfrak{B}}N_{\mathcal{Q}}$ ($C_{\mathfrak{B}}$ of Section S1.4); (d) $\mathfrak{B}_{\mathcal{F}}$ has $m \leq 2A_{\mathcal{F}}$ *-components.*

Proof. (a) A unit of the first family has an end within 3ℓ of $Z(\Xi')$, since a touched cell meets $Z(\Xi')$ and has sides less than 3ℓ , and since a unit of type (1) whose face meets Ξ' has both ends in $Z(\Xi')$. A unit of the second family has an end in $Z_{\mathcal{F}}$. Thus a source z with $\varpi(u, z) > 0$ lies within $R_{\text{resp}} + 3\ell + 1$ of $Z(\Xi') \cup Z_{\mathcal{F}}$ (Lemma 5.21). If such a source is a mark, it is a mark of a part of $\mathcal{Q}_{\text{att}}$. If it is a cut bond, it belongs to a class of $\mathcal{F}$ (closure property), and there are at most $A_{\mathcal{F}}$ of these (Lemma 5.18(b)). If it is a bond of $\text{ins}^1 = \bigsqcup_P \partial Y_P$, its owner P has a site of $Y_P \subseteq P^{+R_*}$ within $R_{\text{resp}} + 3\ell + 2$ of $Z(\Xi') \cup Z_{\mathcal{F}}$ and so lies in $\mathcal{Q}_{\text{att}}$. As each unit has total weight at least 1, the loads after Lemma 5.21 imply that each family has at most $C_{\text{resp}}N_{\mathcal{Q}} + 125C_bA_{\mathcal{F}} \leq C_{\text{resp}}(N_{\mathcal{Q}} + A_{\mathcal{F}})$ elements.

(b) We sum (5.7) over $\mathcal{J} \in \mathcal{F}$. The bonds of $\mathfrak{X}(\mathcal{J}) \cap B_{\text{pay}} \setminus S_{\text{cut}}$ are bonds of ins^1 or of units. A bond of ins^1 lies in ∂Y_P for one part P , which has an end of it in $Y_P \subseteq P^{+R_*}$, so P lies within $R_* + 1$ of $Z_{\mathcal{F}}$ and belongs to $\mathcal{Q}_{\text{att}}$. A part owns at most $C'_*N(P)$ such bonds (Theorem 3.21(a)). The units are bounded by (a), and the marks in blocks of $\mathfrak{B}_{\mathcal{F}}$ belong to parts of $\mathcal{Q}_{\text{att}}$. Therefore

$$A_{\mathcal{F}} \leq 2\varepsilon_D[C'_*N_{\mathcal{Q}} + 2C_{\text{resp}}(N_{\mathcal{Q}} + A_{\mathcal{F}}) + 12L_cN_{\mathcal{Q}}].$$

The coefficient of $A_{\mathcal{F}}$ on the right is $4\varepsilon_DC_{\text{resp}} \leq \frac{1}{2}$ by (5.9). Hence $A_{\mathcal{F}} \leq 4\varepsilon_D(C'_* + 2C_{\text{resp}} + 12L_c)N_{\mathcal{Q}}$, which gives the first inequality. The second inequality is (5.9).

(c) A touched cut bond belongs to a cut of a class in $\mathcal{T}$ (Lemma 5.18(b)). Thus $c_{\text{touch}} \leq A_{\mathcal{F}}$. A bond counted in i_{touch} has an end within 3ℓ of $Z(\Xi')$, so its owner lies within $R_* + 3\ell + 1 \leq R'$ of $Z(\Xi')$, in $\mathcal{Q}_{\text{att}}$, and $i_{\text{touch}} \leq C'_*N_{\mathcal{Q}}$ (Theorem 3.21(a)). For $|\mathfrak{B}_{\mathcal{F}}|$, we charge each block of $\mathfrak{B}(\mathcal{J})$ to a bond of $\mathfrak{X}(\mathcal{J})$ with an end in it if that bond lies in B_{pay} (at most two blocks per bond). Otherwise (Lemma 5.18(d)) the block is that of a site y of hull material or one of its six face neighbours, and it is charged to the first mark of the block of y (Corollary 4.7), with at most seven blocks per mark. Counting as in (a) and (b) gives $|\mathfrak{B}_{\mathcal{F}}| \leq C_{\mathfrak{B}}N_{\mathcal{Q}}$ (Section S1.4).

(d) By Lemma 5.18(c), $\mathfrak{B}(\mathcal{J})$ is a union of at most $2A(\mathcal{J})$ *-connected sets.

Proposition 5.24 (Contour inequality). $\#(\Xi' \cap \mathfrak{F}) \leq \bar{r}_0^2 C_3 N_{\mathcal{Q}}$, $|\mathfrak{B}_{\mathcal{F}}| \leq C_{\mathfrak{B}}N_{\mathcal{Q}}$ and $m \leq M_m N_{\mathcal{Q}}$, with C_3 , $C_{\mathfrak{B}}$ free of β , τ , K , C , D and $M_m = O(\varepsilon_D L_c C_{\text{resp}})$ (Section S1.4).

Proof. We insert Lemma 5.23 into Proposition 5.22. Touched parts lie in $\mathcal{Q}_{\text{att}}$ (Lemma 5.6). By Lemma 5.23(c) and the proof of (a), $u_{\text{touch}} + c_{\text{touch}} + i_{\text{touch}} \leq (C_{\text{resp}} + C'_*)N_{\mathcal{Q}} + (C_{\text{resp}} + 1)A_{\mathcal{F}}$. The marks of the touched cells belong to parts of $\mathcal{Q}_{\text{att}}$, so $\sum_{\Delta'} m(\Delta') \leq N_{\mathcal{Q}}$, and $26L_c/\ell \leq 26/308$. Finally, $A_{\mathcal{F}} \leq N_{\mathcal{Q}}/(48C_{\text{resp}} + 2)$ (Lemma 5.23(b)). This gives the first bound (Section S1.4), and the others are Lemma 5.23(c), (b), (d).

Lemma 5.25 (Counting parts). *Assume $\varepsilon'' \leq \frac{1}{2}$. For a fixed set Z of sites, a fixed set $\mathfrak{B}'$ of blocks and $u_0 > 0$, $\mathbb{P}_D(\sum_{P \in \mathcal{P}: \text{dist}(P, Z \cup \mathfrak{B}') \leq R'} N(P) \geq u_0) \leq 2^{|Z|+|\mathfrak{B}'|+1}(\varepsilon'')^{u_0}$.*

Proof. The argument is that of Proposition 5.9, with roots in $Z \cup \mathfrak{B}'$ (at most $2^{|Z|+|\mathfrak{B}'|+m}$ multisets) and the mark y_i within R' of its root ($(2R' + 4L_c + 1)^3$ choices, as block sides are less than $2L_c$).

Anchored block sets. Put $d_0 := \lceil R'/L_c \rceil + 2$. The block distance is the ℓ^∞ distance of block indices, and $q(x)$ is the block of a site x . Since block sides are at least L_c , sites at distance r lie in blocks at block distance at most $r/L_c + 1$. A block set $\mathfrak{B}'$ is *anchored to* a plaquette set Ξ' if the graph on $\{o\} \cup \{*\text{-components of } \mathfrak{B}'\}$, in which o is joined to a component C when C has a block within block distance d_0 of $q(x)$ for some $x \in Z(\Xi')$, and two components are joined when they have blocks within block distance d_0 , is connected.

Lemma 5.26 (Counting block families). *(a) $\mathfrak{B}_{\mathcal{F}}$ is anchored to $\Xi' = \text{supp } \Sigma^\circ$. (b) For every plaquette set Ξ' with $|\Xi'| = a$, the number of block sets anchored to Ξ' with b blocks and m $*$ -components is at most $2^{2a+2b+m}(2d_0 + 1)^{3m}(26e)^b$.*

Proof. (a) We build $\mathcal{F}$ by the closure. Put $\mathcal{F}_0 := \mathcal{T}$, and let $\mathcal{F}_{t+1}$ be $\mathcal{F}_t$ with the classes forced by $\mathcal{F}_t$. A class of $\mathcal{T}$ has a touched cut bond, whose ends lie in $Z(\Xi')$ and whose blocks lie in its $\mathfrak{B}$. A class $\mathcal{J}' \in \mathcal{F}_{t+1} \setminus \mathcal{F}_t$ has a cut bond z with $\varpi(u, z) > 0$, hence $\text{dist}(u, z) \leq R_{\text{resp}}$, for a unit u of type (i) or (ii) relative to $\mathcal{F}_t$. In case (i), u has an end within 3ℓ of $Z(\Xi')$, so the block of an end of z is within block distance $(R_{\text{resp}} + 3\ell + 1)/L_c + 1 \leq d_0$ of $q(x)$ for some $x \in Z(\Xi')$. In case (ii), the block of an end of z is within block distance $(R_{\text{resp}} + 1)/L_c + 2 \leq d_0$ of $\mathfrak{B}(\mathcal{J})$, $\mathcal{J} \in \mathcal{F}_t$, as u has an end in a block of $\mathfrak{B}(\mathcal{J})$. In both cases this block lies in $\mathfrak{B}(\mathcal{J}')$, as $z \in \mathfrak{X}(\mathcal{J}')$, and $\mathfrak{B}(\mathcal{J}')$ is a union of $*$ -connected sets linked at block distance $k_b \leq d_0$ (Lemma 5.18(c)), each lying in one component of $\mathfrak{B}_{\mathcal{F}}$. By induction on t , each component meeting $\mathfrak{B}(\mathcal{J})$, $\mathcal{J} \in \mathcal{F}_t$, is connected to o , and each component of $\mathfrak{B}_{\mathcal{F}}$ meets some $\mathfrak{B}(\mathcal{J})$.

(b) Along a breadth-first spanning tree from o (fixed orders), the block set is fixed by the following data:

- the component sizes, with 2^b choices;
- the anchors of the children of o in $Z(\Xi')$, where $|Z(\Xi')| \leq 2a$, and of each component among its blocks, with 2^{2a+b+m} choices in all;
- the root positions relative to the anchors, with $(2d_0 + 1)^{3m}$ choices;
- the components, for which Lemma 5.8 with $D_{\text{max}} = 26$ gives $(26e)^b$ choices (see Section S2.3).

Proof of Theorem 5.1. The conditions of item (a) of Section S1.1.4 include (5.9) and

$$2^{29} \cdot 26e \cdot (\varepsilon'')^{1/(3C_{\mathfrak{B}})} \leq \frac{1}{2}, \quad 2(2d_0 + 1)^3(\varepsilon'')^{1/(3M_m)} \leq \frac{1}{2}. \quad (5.10)$$

On $E := \{A_{G'(D)}(C) \leq (1 - \eta_T)RT\}$, as $a = A_{G'(D)}(C) + \#(\Xi' \cap \mathfrak{F})$, Proposition 5.24 gives $N_{\mathcal{Q}} \geq \frac{1}{3}(u_1(a) + b/C_{\mathfrak{B}} + m/M_m)$ with $b := |\mathfrak{B}_{\mathcal{F}}|$ and $u_1(a) := (a - (1 - \eta_T)RT)/(\bar{r}_0^2 C_3)$.

Reduction to deterministic pairs. The minimizer Σ° and the block family $\mathfrak{B}_{\mathcal{F}}$ are functions of D , defined through fixed orders: the output $(\mathbb{X}, S)$ (Theorem 3.20(R1)), the minimizer (Lemma 5.4), the classes, the weights ϖ (Lemma 5.21) and the smallest closed family $\mathcal{F}$. Thus

on E the pair $(\text{supp } \Sigma^\circ, \mathfrak{B}_{\mathcal{F}})$ takes one deterministic value $(\Xi', \mathfrak{B}')$, with Ξ' a plaquette set as in Lemma 5.8 and $\mathfrak{B}'$ anchored to Ξ' (Lemma 5.26(a)). Hence $E \subseteq \bigcup_{(\Xi', \mathfrak{B}')} E(\Xi', \mathfrak{B}')$, the union over all such deterministic pairs, where $E(\Xi', \mathfrak{B}')$ is the event that the parts P within R' of $Z(\Xi') \cup \bigcup \mathfrak{B}'^{+1}$ have $\sum N(P) \geq \frac{1}{3}(u_1(|\Xi'|) + |\mathfrak{B}'|/C_{\mathfrak{B}} + m(\mathfrak{B}')/M_m)$. In $E(\Xi', \mathfrak{B}')$ the random partition enters only as a family of disjoint r' -connected mark sets (Theorem 3.21), over all of which Lemma 5.25 sums.

The sum. By Lemma 5.25 with $Z = Z(\Xi')$ ($|Z| \leq 2a$) and the block set $\mathfrak{B}'^{+1}$ (at most $27b$ blocks), $\mathbb{P}_D(E(\Xi', \mathfrak{B}')) \leq 2^{2a+27b+1}(\varepsilon'')^{(u_1+b/C_{\mathfrak{B}}+m/M_m)/3}$. Summing over the anchored $\mathfrak{B}'$ with Lemma 5.26(b), we obtain the factors $(2^{29} \cdot 26e(\varepsilon'')^{1/(3C_{\mathfrak{B}})})^b$ and $(2(2d_0 + 1)^3(\varepsilon'')^{1/(3M_m)})^m$, each at most 2^{-b} , resp. 2^{-m} , by (5.10). Hence the union over $\mathfrak{B}'$ has probability at most $2^{4a+3}(\varepsilon'')^{u_1(a)/3}$. Finally, Lemma 5.10 with $k = 4$, $j = 3$, $\varepsilon = \varepsilon''$, $F_* = 3\bar{r}_0^2 C_3$ and $X := RT - A_{G'(D)}(C)$, for which $\eta_T |\log \varepsilon''|/F_* = c_T'' \geq c_T(4)$ and $0 \leq X \leq RT$ (Lemma 5.2), gives both bounds of (5.1).

The order of η_T follows from $C_3 = O((C'_* + C_{\text{resp}})(1 + 1/c_s))$ and Lemma 5.12 (Section S1.4).

6 Stage 1: the Mayer expansion to the Debye scale on slit domains

In this section we prove [GM82, Prop. 3], with the iterated Mayer expansion of [GM81a], for the gas whose kernel is the Neumann Green function of a slit graph (Theorem 6.5), and we derive heat-kernel, covariance and depletion bounds on outputs. Since the Neumann field of a distant source is enhanced near a free edge of a frozen sheet, in Sections 6 and 7 we use no pointwise bound on a Neumann kernel or its gradient near such an edge other than the diagonal lower bounds and heat-kernel bounds listed next. We use only diagonal lower bounds (Lemma 6.1), exact L^1 sums (Lemma 6.2), heat-kernel decay from the isoperimetry (R2) (Theorem 6.8, Lemma 6.14) and, in Lemma 7.32, the free-space solid angle.

Conventions. The lattice is the fine lattice (spacing 1), $\beta_{\text{GM}} = 4\pi^2\tau$, and distances are Euclidean unless $d_{G'}$ is written. Slit graphs $G' = (V, E)$, $V = \Lambda' \setminus \mathbb{X}$, $E = E(V) \setminus S$, their components $\mathfrak{K}$, the bound $d_{G'}(x, y) \geq \|x - y\|_1$ and the outputs $G' = G'(D)$ are those of §2.5. We define $(-\Delta_{G'} f)(y) := \sum_{y' \sim y} (f(y) - f(y'))$ over the G' -neighbours of y . For $m > 0$, $v_{G'}^{(m)} := (-\Delta_{G'} + m^2)^{-1}$, $v_{G'} := v_{G'}^{(M_Y)}$, $v := v_{\mathbb{Z}^3}^{(M_Y)}$, $v_{\Lambda'}$ is $v^{(M_Y)}$ of the box Λ' with all its bonds, and v_{Cb} is the Coulomb kernel of $\mathbb{Z}^3$. On $\mathbb{R}_0^V := \{\varphi : \sum_{x \in \mathfrak{K}} \varphi(x) = 0 \text{ for all } \mathfrak{K}\}$, $G_N := (-\Delta_{G'})^+$ is the inverse of $-\Delta_{G'}$. The *image energies* are $\Phi(x) = \Phi_{G'}(x) := \frac{1}{2}\beta_{\text{GM}}(v_{G'}(x, x) - v(0))$ and $I_{G'}(x) := v_{G'}(x, x) - v_{\Lambda'}(x, x)$, and the *depleted set* is $\mathfrak{D} := \{x \in V : \Phi_{G'}(x) > \theta_0\}$, $\theta_0 = 0.01$. For a matrix K on V and $r \geq 0$, $\|K\|_r$ is the larger of $\sup_x \sum_y |K(x, y)|e^{r d_{G'}(x, y)}$ and the same expression with x and y exchanged. This norm is submultiplicative, and $\|K\|_{2 \rightarrow 2} \leq \|K\|_0 \leq \|K\|_r$ for all such K and r . The activity of a unit monopole is $z := e^{-\beta_{\text{GM}} v_{\text{Cb}}(0)/2}$, and $m_D^2 := 2\beta_{\text{GM}} z$ [GM82, (1.8a)]. Hence $m_D = (8\pi^2\tau)^{1/2} e^{-\pi^2 v_{\text{Cb}}(0)\tau}$ (numerical values are in §S1.5). The mass $\tilde{m} \in [m_D/2, 2m_D]$ and the Pauli-Villars mass $M_Y = \tilde{m}/\lambda_{\text{PV}}$ are fixed in (6.11). Until then M_Y is any mass in $I_Y := [m_D/(2\lambda_{\text{PV}}), 2m_D/\lambda_{\text{PV}}]$, and all statements hold uniformly in that range. Besides the loop, the letter C denotes the parameter $C \in (0, 1)$ of Theorem 6.5 (the notation of [GM82]), a generic constant, such as those of (S5), (6.7) and Proposition 6.6, and, in Theorem 6.11 and Appendix C, the covariance $(C_0^{-1} + \nu)^{-1}$. The context fixes the meaning.

6.1 Kernel facts on every slit graph

Lemma 6.1 (Thomson monotonicity). *Let $H \subseteq H'$ be graphs with $V(H) \subseteq V(H')$, $E(H) \subseteq E(H')$ and $V(H)$ finite; $H' = \mathbb{Z}^3$ is allowed. Let f be supported on $V(H)$. (i) For every $m > 0$, $(f, v_H^{(m)} f) \geq (f, v_{H'}^{(m)} f)$. (ii) If f has zero sum on every component of H , then it has zero sum on every component of H' , and $(f, (-\Delta_H)^+ f) \geq (f, (-\Delta_{H'})^+ f)$; for $H' = \mathbb{Z}^3$ the right side is $(f, v_{\mathbb{Z}^3} f)$. (iii) For every slit graph $G' \subseteq \Lambda'$ and $x \in V$: $v_{G'}^{(m)}(x, x) \geq v_{\mathbb{Z}^3}^{(m)}(0)$ and $v_{G'}(x, x) \geq v_{\Lambda'}(x, x)$. Hence $\Phi_{G'} \geq 0$ and $I_{G'} \geq 0$.*

Proof. In the variational formula $(f, v_H^{(m)} f) = \sup_{\varphi} [2(f, \varphi) - \sum_{\{y, y'\} \in E(H)} (\varphi(y) - \varphi(y'))^2 - m^2 \sum_{y \in V(H)} \varphi(y)^2]$, restricting φ from $V(H')$ to $V(H)$ keeps (f, φ) and does not raise the energy, which is (i). In (ii), f is orthogonal to the kernels of both Laplacians, and the formula holds at $m = 0$. We obtain (iii) by taking $f = \delta_x$ in (i).

Lemma 6.2 (Neumann resolvent). *Let G' be a slit graph and $0 < m < m'$. Then: (i) $v_{G'}^{(m)}$ is symmetric and entrywise nonnegative, and $v_{G'}^{(m)}(x, y) = 0$ when x and y lie in different components; (ii) $\sum_{y \in V} v_{G'}^{(m)}(x, y) = m^{-2}$ for every $x \in V$; (iii) $v_{G'}^{(m)} - v_{G'}^{(m')} = (m'^2 - m^2) v_{G'}^{(m)} v_{G'}^{(m')}$, and this matrix is entrywise nonnegative and positive semidefinite, with row sums $m^{-2} - m'^{-2}$; (iv) $\text{spec}(-\Delta_{G'}) \subseteq [0, 12]$.*

Proof. Since $\deg_{G'} \leq 6$, the matrix $P := I - (-\Delta_{G'})/6$ is symmetric, nonnegative and stochastic, with no entry across components, and the series $v_{G'}^{(m)} = (6 + m^2)^{-1} \sum_{n \geq 0} (6/(6 + m^2))^n P^n$ gives (i) and (ii). Part (iii) is the resolvent identity, and (iv) is Gershgorin's theorem.

Proposition 6.3 (Representation; one zero mode per component). *Let G' be a slit graph, $\beta_{\text{GM}} > 0$, $k \in \mathbb{R}$ and $f : V \rightarrow \mathbb{R}$. Put $\tilde{Z}(k, f) := \sum_{[n]} \exp(-\frac{1}{2\beta_{\text{GM}}} \sum_{b \in E} (\nabla_b(n - kf))^2)$, where n runs over $(2\pi\mathbb{Z})^V$ modulo the addition of a constant in $2\pi\mathbb{Z}$ on each component. Let $u := G_N - v_{G'}$ on $\mathbb{R}_0^V$ and let $d\mu_u$ be the centred Gaussian measure on $\mathbb{R}_0^V$ with covariance u . Then: (a) $u = M_Y^2(-\Delta_{G'})^+(-\Delta_{G'} + M_Y^2)^{-1}$ on $\mathbb{R}_0^V$, and u is positive definite there. (b) $\tilde{Z}(k, f) = \mathcal{N}_0 \sum_q e^{ik(q, f)} e^{-\beta_{\text{GM}}(q, G_N q)/2}$, the sum over $q \in \mathbb{Z}^V$ with zero sum on every component, where $\mathcal{N}_0 = \prod_{\mathfrak{R}} \mathcal{N}_{\mathfrak{R}} > 0$ depends only on G' and β_{GM} . An isolated vertex carries only $q = 0$ and has $\mathcal{N}_{\mathfrak{R}} = 1$. (c) With $\theta := \sum_{\mathfrak{R}} \theta_{\mathfrak{R}} 1_{\mathfrak{R}}$,*

$$\sum_{q \text{ neutral}} e^{ik(q, f)} e^{-\beta_{\text{GM}}(q, G_N q)/2} = \int_{[0, 2\pi)^{\mathcal{K}}} \prod_{\mathfrak{R}} \frac{d\theta_{\mathfrak{R}}}{2\pi} \int d\mu_u(\varphi) Z^Y(\varphi + k\beta_{\text{GM}}^{-1/2} f + \beta_{\text{GM}}^{-1/2} \theta), \quad (6.1)$$

where $Z^Y(\psi) := \sum_{q \in \mathbb{Z}^V} e^{i\beta_{\text{GM}}^{1/2}(q, \psi)} e^{-\beta_{\text{GM}}(q, v_{G'} q)/2}$ is the Yukawa gas on G' , with no neutrality constraint. (d) $Z^Y(\psi) > 0$ for real ψ , and $Z^Y = \prod_{\mathfrak{R}} Z_{\mathfrak{R}}^Y(\psi|_{\mathfrak{R}})$. Consequently the coefficients ρ_s of the expansion of $\ln Z^Y$ in Theorem 6.5 vanish unless all their points lie in one component.

Proof. We carry out the argument of [GM82, (2.4) to (2.12)], written there for a measure on $\mathbb{Z}^3$ with charges confined to Λ and no zero mode (p. 20), on each component $\mathfrak{R}$. On $\mathfrak{R}$, rooting at $x_0 \in \mathfrak{R}$ and Poisson summation over $(2\pi\mathbb{Z})^{|\mathfrak{R}|-1}$ give the neutral charges $\hat{q} = (q, -\sum q)$, with $\langle q, A^{-1} q \rangle = \langle \hat{q}, G_N \hat{q} \rangle$ for the restriction A of $-\Delta_{\mathfrak{R}}$ to $\mathfrak{R} \setminus \{x_0\}$. One $\theta_{\mathfrak{R}}$ per component imposes neutrality, whatever their number and sizes. Finally, $G_N = v_{G'} + u$, with (a) from $(-\Delta_{G'} + M_Y^2)^{-1} 1_{\mathfrak{R}} = M_Y^{-2} 1_{\mathfrak{R}}$ (Lemma 6.2(ii)). To prove $Z^Y > 0$ (cited in [GM82, p. 21]), we use Poisson summation over $\mathbb{Z}^V$, which writes $Z^Y(\psi)$ as a sum of positive Gaussians in

$n - \beta_{\text{GM}}^{1/2}\psi$, $n \in (2\pi\mathbb{Z})^V$, with covariance $\beta_{\text{GM}}v_{G'}$. The factorization and (d) hold because $v_{G'}$ and the potentials $v_{G'}^\ell$ ([GM81a, (2.5), p. 6, and (4.1), p. 17]), from which [GM82, §3] builds ρ_s through the tree formula [GM81a, (3.7)], vanish across components (Lemma 6.2(i)).

Proposition 6.4 (Stability on G'). *Let $M_1 \geq M_Y$ be the mass of [GM82, (3.20)], and on a slit graph G' let $v_{G'}^1 := v_{G'}^{(M_1)}$, with the coincident-site rule [GM81a, (5.1a)], and $v_{G'}^2 := v_{G'} - v_{G'}^1$. Then the stability bounds [GM81a, (4.2)] hold on G' with $\gamma_1 = \frac{1}{2}v_{\mathbb{Z}^3}^1(0)$, with the same $\varepsilon_1 > 0$ as on $\mathbb{Z}^3$ [GM81a, Prop. 8], and with $\varepsilon_2 = \delta_2 = 0$. Moreover $v_{G'}^\ell(x, y) \geq 0$, and*

$$\tilde{v}^1 := \sup_y \sum_x v_{G'}^1(x, y) = M_1^{-2}, \quad \tilde{v}^2 := \sup_y \sum_x v_{G'}^2(x, y) = M_Y^{-2} - M_1^{-2} \quad (6.2)$$

exactly, as on $\mathbb{Z}^3$ [GM81a, (5.3)].

Proof. By Lemma 6.1 each Yukawa energy on G' is at least its $\mathbb{Z}^3$ value for the same configuration. Hence the margins of [GM81a, Prop. 8] transfer. At level one, with the rule (5.1a) and $m(x) := \sum_i m_i \delta_{x_i x}$, $E_{G'} = \frac{1}{2}(m, v_{G'}^1, m) + \frac{1}{2} \sum_{x_i=x_j, \text{sign } m_i = -\text{sign } m_j} |m_i m_j| v_{G'}^1(x_i, x_j) \geq E_{\mathbb{Z}^3}$ by Lemma 6.1(i), (iii), and $E_{\mathbb{Z}^3} \geq \gamma_1 + \varepsilon_1(-1 + \sum_i m_i^2)$, coincident positions included. At level two, $\frac{1}{2}(m, v_{G'}^2, m) \geq 0$ by Lemma 6.2(iii), that is (4.2) with $\varepsilon_2 = \delta_2 = 0$. Positivity and (6.2) are Lemma 6.2(i) to (iii).

6.2 The Stage 1 theorem

Theorem 6.5 (Stage 1 on slit domains). *Fix $\kappa_S, \mu < \infty$ and $C, \delta_{\text{GM}} \in (0, 1)$. There are $\lambda_0 = \lambda_0(\kappa_S, \mu, C, \delta_{\text{GM}}) > 0$ and, for $\lambda_{\text{PV}} \leq \lambda_0$, $\beta_{\text{GM}}^0 = \beta_{\text{GM}}^0(\lambda_{\text{PV}}, \kappa_S, \mu, C, \delta_{\text{GM}})$ such that for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ and every $M_Y \in I_Y$ the following hold on every slit graph G' , with constants independent of G' and M_Y . Write $\xi = (m, x)$ with $m \in \mathbb{Z} \setminus \{0\}$ and $x \in V$, $-\xi := (-m, x)$, and $\int d\xi := \sum_m \sum_{x \in V}$.*

(S1) Expansion. *For real ψ on V , $Z^Y(\psi) = \exp(-V_{\text{eff}}(\psi))$ with*

$$-V_{\text{eff}}(\psi) = \rho_0 + \sum_{s \geq 1} \frac{1}{s!} \int d\xi_1 \cdots d\xi_s \rho_s(\xi_1, \dots, \xi_s) \varepsilon(\xi_1) \cdots \varepsilon(\xi_s), \quad (6.3)$$

where $\varepsilon(\xi) := e^{im\beta_{\text{GM}}^{1/2}\psi(x)} - 1$, the coefficients $\rho_s = \rho_s^{G'}$ are real and symmetric, ρ_0 is a constant, and the series converges absolutely, uniformly in real ψ . With Proposition 6.3, for every k and f ,

$$\tilde{Z}(k, f) = \mathcal{N}_0 \int \prod_{\mathfrak{K}} \frac{d\theta_{\mathfrak{K}}}{2\pi} \int d\mu_u(\varphi) \exp[-V_{\text{eff}}(\varphi + k\beta_{\text{GM}}^{-1/2}f + \beta_{\text{GM}}^{-1/2}\theta)]. \quad (6.4)$$

(S2) Coefficient bounds. *For $s \geq 1$ and every $a_1 \subseteq V$,*

$$\begin{aligned} \int_{x_1 \in a_1} d\xi_1 \int d\xi_2 \cdots \int d\xi_s |\rho_s(\xi_1, \dots, \xi_s)| e^{2\kappa_S(-1 + \sum_i |m_i|)} e^{(1-\delta_{\text{GM}})A L_{G'}(x_1, \dots, x_s)} \\ \leq s! |a_1| \frac{m_D^2}{\beta_{\text{GM}}} e^{-\mu(s-1)} (1-C)^{-s-10} \end{aligned} \quad (6.5)$$

with $A = 0$, where $L_{G'}(a_1, \dots, a_n)$ is the tree length of [GM82, (1.11a), (1.11b)] computed with the distance $3^{-1/2}d_{G'}$ and auxiliary points in V , $L_{G'}(x_1, \dots, x_s) := L_{G'}(\{x_1\}, \dots, \{x_s\})$,

and $0 \cdot \infty := 0$ (by Proposition 6.3(d), $\rho_s = 0$ unless $x_1, \dots, x_s$ lie in one component). Since $L_{G'}(a_1, \dots, a_s) \leq L_{G'}(x_1, \dots, x_s)$ whenever $x_i \in a_i$, for all $a_1, \dots, a_s \subseteq V$

$$\begin{aligned} \int_{x_1 \in a_1} d\xi_1 \cdots \int_{x_s \in a_s} d\xi_s |\rho_s(\xi_1, \dots, \xi_s)| e^{2\kappa_S(-1 + \sum_i |m_i|)} \\ \leq s! |a_1| \frac{m_D^2}{\beta_{\text{GM}}} e^{-(1 - \delta_{\text{GM}})A L_{G'}(a_1, \dots, a_s) - \mu(s-1)} (1 - C)^{-s-10}. \end{aligned} \quad (6.6)$$

(S3) Symmetries. $\rho_s(-\xi_1, \dots, -\xi_s) = \rho_s(\xi_1, \dots, \xi_s)$; in particular $\sum_m m \rho_1(m, x) = 0$ at every $x \in V$. Moreover $V_{\text{eff}}(\psi + 2\pi\beta_{\text{GM}}^{-1/2}I) = V_{\text{eff}}(\psi)$ for every $I \in \mathbb{Z}^V$.

(S4) One-body term. The bounds of Proposition 6.6 hold.

(S5) Dipole kernel. $\nu(x_1, x_2) := \beta_{\text{GM}} \sum_{m_1, m_2} m_1 m_2 \rho_2((m_1, x_1), (m_2, x_2))$ is real and symmetric, and $\|\nu\|_{2 \rightarrow 2} \leq \|\nu\|_0 \leq C e^{-\mu} m_D^2$, which is at most $\frac{1}{2} \tilde{m}^2$ once $C e^{-\mu} \leq \frac{1}{8}$.

With M_Y fixed in (6.11), on every output (S2) holds also with $A = \sqrt{3} \kappa_Y M_Y$ (Proposition 6.10), and, if $\delta_{\text{GM}} \leq \frac{1}{2}$, (S5) in the form $\|\nu\|_{r_{\text{cov}}} \leq C e^{-\mu} m_D^2$ (r_{cov} of Theorem 6.11).

Proof of Theorem 6.5. Removing bonds raises all quadratic energies (Lemma 6.1) and conserves the mass of the walk (Lemma 6.2), so the stability margins of [GM81a] and the pinned L^1 norms of the links are those of $\mathbb{Z}^3$, and no image factor enters. The proof of [GM82, Prop. 3], with the results of [GM81a] that it uses, therefore runs on G' with the changes (a) to (h) of Table 6.1. For the same reason we use all other steps verbatim. They are the following.

- In [GM81a]: §2, (2.5) to (2.25), Prop. 2, (2.26) (its finite pieces “remain arbitrary until we come to doing estimates”, p. 6), Prop. 3, (2.30), App. A (only (2.27) is used), Prop. 4, (3.7), App. B, (B.4) (stated for any real symmetric pair array $w(ij)$, p. 32, and used here for the $v_{G'}^\ell$, $\ell \geq 1$), (3.8) to (3.12), (4.3) ((4.2) for all subconfigurations: Proposition 6.4), Lemma 5, (4.8), App. C (combinatorial), Prop. 6, Cor. 7, Lemma 9 (through the pinned norms of (a) only).
- In [GM82]: (3.1) to (3.18) (for real ψ , (3.14) uses only $|1 + \varepsilon(\xi)| = 1$, p. 25), (4.1) to (4.11), Lemma 6 (any metric), (4.14) to (4.22), Prop. 7, Cor. 8 (through $\|v^\ell\|_A$ only), (4.25) (γ_1 is the $\mathbb{Z}^3$ constant), Prop. 9, (4.26) to (4.30), (4.32) (through the above, with $L_{G'}$ and the rate $(1 - \delta_{\text{GM}})A$).

Numbers refer to the DESY preprints (§1.4), read with the evident corrections $\varepsilon(\xi) = -1 + \exp[im\beta_{\text{GM}}^{1/2}\varphi(x)]$ in the statement of [GM82, Prop. 3] (as in its (3.4)) and $2\beta_{\text{GM}}m_D^{-2}\rho_1(\pm 1, x) \rightarrow 1$ at its end. The constants depend only on $(\lambda_{\text{PV}}, \kappa_S, \mu, C, \delta_{\text{GM}})$ and, on outputs, on those of Theorem 6.8.

(a) *Pinned norms.* [GM81a, p. 18] uses translation invariance and a preparatory majorization by functions of differences of the arguments, and [GM82, p. 32] uses translation invariance. Instead, we take the norms in the max form [GM81a, (4.5b)], $\|\sigma_\alpha\|(m_\alpha) := \max_{i \in \alpha} \sup_{x \in V} \sum_{x_\alpha \in V^n: x_i = x} |\sigma_\alpha(\xi_\alpha)|$, with the weights of [GM82, (4.12)] inserted in the same way, and $\tilde{v}^\ell := \sup_y \sum_x |v_{G'}^\ell(x, y)|$. We sum the positions (“trimming the tree”, [GM81a, p. 18]) after re-rooting the tree at the vertex containing the pinned constituent, in order of decreasing tree distance from it. Each summed vertex is then a leaf attached by one link (j, k) , and $\sum_{x_j} |v_{G'}^\ell(x_j, x_k)| \leq \tilde{v}^\ell$ uniformly in x_k . The s -integrals and tree sums are bounded by [GM81a, Lemma 5] independently of the positions. Hence [GM81a, (4.7), (4.10), (4.13), (4.16), (4.17)] hold as printed, and $\int d\xi_\alpha |\sigma_\alpha| \leq |V| \|\sigma_\alpha\|$.

Table 6.1: Changes to [GM82, Prop. 3] and the results of [GM81a] it uses, on a slit graph G' .

change	steps	use of $\mathbb{Z}^3 \rightarrow$ on G'
(a) pinning	[GM81a, (4.5a), (4.6), pp. 18 to 20]; [GM82, (4.12), (4.13), p. 32]	translation invariance $\rightarrow$ pinned norms in max form, re-rooted sums
(b) stability	[GM81a, Prop. 8, (5.1a) to (5.13), App. D]; [GM82, Lemma 5, (3.25)]	Fourier analysis $\rightarrow$ Proposition 6.4: same $\gamma_1, \varepsilon_1; \varepsilon_2 = \delta_2 = 0$
(c) Gaussian	[GM82, (2.4) to (2.12), pp. 20, 21]	measure on $\mathbb{Z}^3$, no zero mode, $Z_\Lambda(\phi) > 0$ cited $\rightarrow$ Proposition 6.3: one zero mode per component, $Z^Y > 0$ proved
(d) resolvents	[GM82, (3.19) to (3.22)]	resolvents of $\mathbb{Z}^3 \rightarrow$ of G' ; rule (3.19) ($=$ [GM81a, (5.1a)]), same M_1 (3.20)
(e) metric	[GM82, (3.23), (3.24)]	$3^{-1/2}\ x-y\ _1 \rightarrow \ x-y\ _{G'} := 3^{-1/2}d_{G'}(x, y)$, in $\ \cdot\ _A$ and $L_{G'}$
(f) link norms	[GM82, Lemma 5, (3.26); (4.23), (4.24), p. 34]	momentum space $\rightarrow$ (6.2), Proposition 6.10; Debye condition $\lambda_{\text{PV}}^2 \leq C^3(1-C)^8 e^{-2\kappa_S - \mu}/24$
(g) σ^R	[GM82, (4.33), (4.34), p. 36]	$v(0) \rightarrow v_{G'}(x, x)$ in $\sigma^R = \exp[-\beta_{\text{GM}} m^2 v(0)/2]$
(h) tree length	[GM82, (4.30) to (4.32), pp. 35, 36; (4.12), p. 32]	$L(a_1, \dots, a_s)$ on the right of (4.31) $\rightarrow L_{G'}(x_1, \dots, x_s)$ inside the sum over all positions

(f) *Link norms.* For $A \geq 0$ put $\|v_{G'}^\ell\|_A := \sup_y \sum_x |v_{G'}^\ell(x, y)| e^{A\|x-y\|_{G'}}$. At $A = 0$ the values are (6.2), on all slit graphs. On an output, Proposition 6.10 gives $\|v_{G'}\|_A, \|v_{G'}^2\|_A \leq 3M_Y^{-2}$ and $\|v_{G'}^1\|_A \leq 3M_1^{-2}$ for $A \leq \sqrt{3}\kappa_Y M_Y$, since $\|x-y\|_{G'} = 3^{-1/2}d_{G'}(x, y)$, the rate of (6.13) is nondecreasing in the mass, and $M_Y \leq M_1 \leq m_W$. These bounds replace [GM82, (3.26)] in [GM82, (4.23)] and in [GM82, (4.24)]. In [GM82, (4.23)], $3M_1^{-2}$ is a power of β_{GM} , absorbed by $e^{\beta_{\text{GM}}\varepsilon_1}$. In [GM82, (4.24)], $3M_Y^{-2}$ replaces $2(\delta M)^{-2}$ at $A = (1-\delta)M$, so the factor $\frac{1}{4}\delta^2$ of the Debye condition [GM82, p. 34] becomes $\frac{1}{6}$. The condition then holds on I_Y once $\lambda_{\text{PV}}^2 \leq C^3(1-C)^8 e^{-2\kappa_S - \mu}/24$ (§S1.5), which is the only changed threshold and does not depend on G' .

(g) *The leading one-body term.* For a single constituent the recursion [GM82, (3.12)] gives $\sigma^{\ell+1} = \sigma^\ell \exp[-\beta_{\text{GM}} m^2 v_{G'}^\ell(x, x)/2]$, hence $\sigma^R(m, x) = \exp[-\beta_{\text{GM}} m^2 v_{G'}(x, x)/2]$ in place of [GM82, (4.33)]. The remainder in [GM82, (4.34)] is bounded as on [GM82, p. 36] by $e^{-\mu}(1 - e^{-\mu})^{-2} m_D^2 \beta_{\text{GM}}^{-1} (1-C)^{-9}$, uniformly in x , through the pinned norm of [GM82, Prop. 9]. It is bounded in the same way, up to a constant factor, with the weights $e^{2\kappa_S(|m|-1)}$, since we use the pinned norm with weight $e^{(2\kappa_S + \mu)\sum_j |m_j|}$ and $\sum_{j \geq 2} |m_j| \geq t-1$. Thus, with $A = 0$ and uniformly in x and G' ,

$$\rho_1(m, x) = e^{-\beta_{\text{GM}} m^2 v_{G'}(x, x)/2} + R(m, x), \quad \sum_m |R(m, x)| e^{2\kappa_S(|m|-1)} \leq C e^{-\mu} m_D^2 / \beta_{\text{GM}}. \quad (6.7)$$

It replaces the final assertion of [GM82, Prop. 3], whose error at fixed μ is of order $e^{-\mu}$.

(h) *The tree length inside the sum.* We state the bound (S2) with the tree length of the points inside the integral, which is what the chain [GM82, (4.30) to (4.32)] gives. By [GM82, (4.30)], $|\rho_s(\xi_1, \dots, \xi_s)| \leq s! \sum_{t \geq s} \binom{t}{s} \sum_{[\alpha]: C(\alpha) = \{1, \dots, t\}} \int d\xi_{s+1} \dots d\xi_t |\sigma_\alpha^R(\xi_\alpha)|$. By [GM82, (4.9)] for the sets $\{x_i\}$, $L_{G'}(x_1, \dots, x_s) \leq \tilde{L}(x_1, \dots, x_t)$ for all $t \geq s$ and $x_{s+1}, \dots, x_t \in V$, where $\tilde{L}$ is the least length of a tree on exactly the vertices $x_1, \dots, x_t$ [GM82, (4.8)], computed with $3^{-1/2}d_{G'}$. In [GM82, (4.31)] the restriction $x_i \in a_i$ ($i \geq 2$) enters only through $L(a_1, \dots, a_s) \leq \tilde{L}(x_1, \dots, x_t)$, since the pinned norm [GM82, (4.12)], taken in the max form of change (a), sums all positions with the weight $e^{A\tilde{L}(x_\alpha)}$. Hence (4.31) holds with $x_2, \dots, x_s$ summed over V and the factor $e^{(1-\delta_{\text{GM}})AL_{G'}(x_1, \dots, x_s)}$ on the left in place of $e^{-(1-\delta_{\text{GM}})AL(a_1, \dots, a_s)}$ on the right.

Combining this with (4.32), [GM82, Prop. 9] and the sum over $x_1 \in a_1$, we obtain (6.5). The bound (6.6), the form of [GM82, (1.18)], follows from it as stated.

With these changes the proof of [GM82, Prop. 3] gives (S1) and (S2), uniformly in $M_Y \in I_Y$, since the Debye condition is the only threshold involving M_Y . Together with Proposition 6.3, (S1) gives (6.4). For (S3), the activities are equal, the potentials depend on the charges through $m_i m_j$ and the rule (3.19), both invariant under $m \mapsto -m$, and $\varepsilon(\xi)$ is $2\pi\beta_{\text{GM}}^{-1/2}$ -periodic in $\psi(x)$. (S4) is Proposition 6.6. For (S5), ν is real and symmetric because ρ_2 is, and it vanishes across components (Proposition 6.3(d)). By $|m_1 m_2| \leq C e^{2\kappa_S(|m_1|+|m_2|-1)}$, (6.6) at $s = 2$ with $a_1 = \{x_1\}$ and any $a_2 \subseteq V$ gives $\sum_{x_2 \in a_2} |\nu(x_1, x_2)| \leq C e^{-\mu} m_D^2 e^{-(1-\delta_{\text{GM}})A L_{G'}(\{x_1\}, a_2)}$. With $A = 0$ and $a_2 = V$ this is $\|\nu\|_0 \leq C e^{-\mu} m_D^2$, at most $\frac{1}{8} m_D^2 \leq \frac{1}{2} \tilde{m}^2$ once $C e^{-\mu} \leq \frac{1}{8}$. On outputs we take $A = \sqrt{3} \kappa_Y M_Y$ and, for an integer $k \geq 0$, $a_2 = \{x_2 : d_{G'}(x_1, x_2) \geq k\}$, whose tree length with $\{x_1\}$ is at least $3^{-1/2} k$. With these choices and $\gamma_\nu := (1 - \delta_{\text{GM}}) \kappa_Y M_Y$, the tail sums $T_k := \sum_{x_2: d_{G'}(x_1, x_2) \geq k} |\nu(x_1, x_2)|$ satisfy $T_k \leq C e^{-\mu} m_D^2 e^{-\gamma_\nu k}$. Summation by parts, $e^{rd} = 1 + \sum_{1 \leq k \leq d} (e^{rk} - e^{r(k-1)})$, gives for $0 \leq r < \gamma_\nu$

$$\begin{aligned} \sum_{x_2} |\nu(x_1, x_2)| e^{r d_{G'}(x_1, x_2)} &= T_0 + \sum_{k \geq 1} (e^{rk} - e^{r(k-1)}) T_k \\ &\leq \left(1 + \frac{1 - e^{-r}}{e^{\gamma_\nu r} - 1}\right) C e^{-\mu} m_D^2 \leq \left(1 + \frac{r}{\gamma_\nu - r}\right) C e^{-\mu} m_D^2, \end{aligned}$$

which is at most $2C e^{-\mu} m_D^2$ for $r \leq \gamma_\nu/2$. For $\delta_{\text{GM}} \leq \frac{1}{2}$, $\gamma_\nu \geq \frac{1}{2} \kappa_Y M_Y \geq 2\kappa_0 \tilde{m} \geq 2r_{\text{cov}}$ (Proposition 6.10), so $\|\nu\|_{r_{\text{cov}}} \leq 2C e^{-\mu} m_D^2$: this is (S5) on outputs, with the constant C doubled.

Proposition 6.6 (Additive one-body bound). *Let $\rho^{\text{ref}}(m) := \rho_1^{\mathbb{Z}^3}(m)$, the one-point coefficient of the Yukawa gas on $\mathbb{Z}^3$, and $\Phi = \Phi_{G'}$. Uniformly in $x \in V$ and G' ,*

$$|\rho_1(\pm 1, x) - \rho^{\text{ref}}(1) e^{-\Phi(x)}| \leq C e^{-\mu} \rho^{\text{ref}}(1), \quad (6.8)$$

$$\rho_1(\pm 1, x) \leq \rho^{\text{ref}}(1) (1 + C e^{-\mu}), \quad (6.9)$$

$$\sum_{|m| \geq 2} |\rho_1(m, x)| m^2 e^{2\kappa_S(|m|-1)} \leq C m_D^2 / \beta_{\text{GM}}. \quad (6.10)$$

Consequently $|\rho_1(\pm 1, x) - \rho^{\text{ref}}(1)| \leq \rho^{\text{ref}}(1) (1 - e^{-\Phi(x)} + C e^{-\mu})$.

Proof. The coefficient $\rho^{\text{ref}}(1)$ carries the same self-energy factor as ρ_1 up to the image term $e^{-\Phi}$, so no error of order $\beta_{\text{GM}} M_Y$ arises. Subtracting from (6.7) its form $\rho^{\text{ref}}(1) = e^{-\beta_{\text{GM}} v(0)/2} + R^\infty$, $|R^\infty| \leq C e^{-\mu} m_D^2 / \beta_{\text{GM}}$, on $\mathbb{Z}^3$ gives (6.8) and (6.9), since $e^{-\beta_{\text{GM}} v_{G'}(x, x)/2} = e^{-\beta_{\text{GM}} v(0)/2} e^{-\Phi(x)}$, $\Phi \geq 0$ (Lemma 6.1), and $m_D^2 / \beta_{\text{GM}} \leq 2\rho^{\text{ref}}(1) (1 - 2C e^{-\mu})^{-1}$ (as $e^{-\beta_{\text{GM}} v(0)/2} \geq z$). In (6.10) the leading terms are at most $e^{-2\beta_{\text{GM}} v(0)} \leq e^{-\mu} m_D^2 / \beta_{\text{GM}}$, and in the remainders m^2 is absorbed by the μ -part of the weight $e^{(2\kappa_S + \mu) \sum_j |m_j|}$ of change (g), since $m^2 e^{-\mu|m|} \leq 4e^{-2\mu}$ for $|m| \geq 2$ and $\mu \geq 1$. Hence the Debye condition of change (f) suffices at the same κ_S .

Remark 6.7. The multiplicative form $\rho_1(\pm 1, x) = \rho^{\text{ref}}(1) e^{-\Phi(x)} (1 + \mathcal{O}(e^{-\mu}))$ fails at a first-layer corner site of a box: a tight neutral pair, which feels only its dipole image, makes the one-site density exponentially larger in β_{GM} (§S3.1.5). So Proposition 6.6 is additive only.

The masses $\tilde{m}$ and M_Y . For $M \in I_Y$ let ρ_M^{ref} be the one-point coefficient of the Yukawa gas on $\mathbb{Z}^3$ with kernel $(-\Delta_{\mathbb{Z}^3} + M^2)^{-1}$, and $\tilde{m}(M)^2 := \beta_{\text{GM}} \sum_m m^2 \rho_M^{\text{ref}}(m)$. By (6.7) and (6.10) on $\mathbb{Z}^3$ (whose terms $|m| \geq 2$ carry the weight $e^{2\kappa_S(|m|-1)} \geq e^{2\kappa_S}$), $\tilde{m}(M)^2 = 2\beta_{\text{GM}} e^{-\beta_{\text{GM}} v_{\mathbb{Z}^3}^{(M)}(0)/2} + \mathcal{O}(e^{-\mu} + e^{-2\kappa_S}) m_D^2 = m_D^2(1 + \mathcal{O}(\beta_{\text{GM}} M)) + \mathcal{O}(e^{-\mu} + e^{-2\kappa_S}) m_D^2$, as $v_{\mathbb{Z}^3}^{(M)}(0) = v_{\text{Cb}}(0) - \mathcal{O}(M)$. Since $\beta_{\text{GM}} m_D \rightarrow 0$, $\tilde{m}(M) \in [m_D/2, 2m_D]$ once μ and κ_S exceed a threshold depending on C (the remainder of (6.7) carries $(1 - C)^{-9}$ [GM82, p. 36]) and $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$. Since M_1 and the rule (3.19) do not depend on M , the mass M enters only through $v_M^2 = v^{(M)} - v^{(M_1)}$, so each term of the expansion of ρ_M^{ref} is continuous in M . The t -th term is $\mathcal{O}(te^{-\mu(t-1)})$ uniformly in $M \in I_Y$ [GM82, p. 36]. With $M_- := m_D/(2\lambda_{\text{PV}})$, we have $0 \leq v_M^2 \leq v_{M_-}^2$ entrywise by the resolvent identity, so at fixed t the terms are dominated by those with $v_{M_-}^2$ in place of v_M^2 , whose bound in [GM82] needs exactly the Debye condition of change (f) at $M = M_-$. By dominated convergence, $M \mapsto \tilde{m}(M)/\lambda_{\text{PV}}$ is a continuous map of I_Y into itself. Its fixed points form a nonempty closed set. We let M_Y be the smallest fixed point and put

$$\tilde{m} := \tilde{m}(M_Y) = \lambda_{\text{PV}} M_Y \in [m_D/2, 2m_D], \quad \rho^{\text{ref}} := \rho_{M_Y}^{\text{ref}}, \quad \tilde{m}^2 = \beta_{\text{GM}} \sum_m m^2 \rho^{\text{ref}}(m). \quad (6.11)$$

The inverse mass $\tilde{m}^{-1}$, which like ρ^{ref} is determined by the gas on $\mathbb{Z}^3$ alone, before any construction, is the unit of length of Sections 3 to 5, and ρ^{ref} is the reference density of Proposition 6.6 and of Section 7.

6.3 Heat kernel and covariance on outputs of the construction

We prove the statements of this subsection in Appendix D (with full proofs of Proposition 6.10 and Theorem 6.11 in Section S2.4), from (R2) and, for Proposition 6.9, from (R5) and (R6), without any Poincaré inequality or volume-growth hypothesis. The *coarse slit graph* of an output is $\Gamma = (V_c, E_c)$, $V_c := \Lambda^\circ \setminus \mathbb{X}$, $E_c := E(V_c) \setminus S$, the graph G' of Sections 3 to 5 (§2.5). For $W \subseteq V_c$, $\partial_\Gamma W$ is the set of bonds of E_c with exactly one end in W . A *coarse cell* is a site of the grid (Definition 3.4), a box of fine sides in $[r_0, \bar{r}_0]$, not a cell of Definition 3.14. In fine units put

$$V_*^f := r_0^3 V_*/2, \quad R_{\text{HK}} := (1 - \eta_w) c_I c_\rho L/3, \quad K_0 := (1 - \eta_w) c_I c_\rho c_L/6 \leq \tilde{m} R_{\text{HK}} \leq 2K_0, \quad (6.12)$$

with $c_I = 2^{2/3}$, $c_\rho = 1/12$, $V_* = (1 - \eta_w)^3 \rho_\lambda^3/2$ (coarse units). The bounds on $\tilde{m} R_{\text{HK}}$ follow from Definition 3.1 (§S1.5). Let $p_t(x, y)$ be the kernel of $e^{t\Delta_{G'}}$, the transition probability of the walk X_t that jumps across each bond of G' at rate 1, and let $\tau(x, \rho) := \inf\{t : d_{G'}(x, X_t) > \rho\}$.

Theorem 6.8 (Heat kernel at scale R_{HK}). *Let G' be an output, and let C_H (a function of c_F of Proposition 6.9) and $\varepsilon_1^{\text{HK}}$ be the absolute constants of §S1.5. There are $C_{\text{mo}}, c_{\text{ex}}, C_{\text{gs}}, c_{\text{gs}}, C_M > 0$, depending only on C_H , such that for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, $x, y \in V$ and $d := d_{G'}(x, y)$: (HK0) $p_t(x, y) \leq \exp(-\alpha d + 6(\cosh \alpha - 1)t)$ for all $\alpha, t \geq 0$; in particular $p_t(x, y) \leq e^{-d/17}$ for $t \leq d$; (HK1) $p_t(x, y) \leq \max(C_H t^{-3/2}, 4/V_*^f) \leq C_H \max(t^{-3/2}, R_{\text{HK}}^{-3})$ for all $t > 0$; (HK1') $\mathbb{E}_x d_{G'}(x, X_t) \leq C_{\text{mo}} t^{1/2}$ for $0 \leq t \leq R_{\text{HK}}^2$; (HK2) $\mathbb{P}_x(\tau(x, \rho) < t) \leq \sqrt{2} \exp(-c_{\text{ex}} \min(\rho^2/t, \rho))$ for $\rho \geq 1$ and $0 < t \leq \varepsilon_1^{\text{HK}} R_{\text{HK}}^2$; (HK3) $p_t(x, y) \leq C_{\text{gs}} t^{-3/2} \exp(-c_{\text{gs}} \min(d^2/t, d))$ for $1 \leq t \leq 2\varepsilon_1^{\text{HK}} R_{\text{HK}}^2$; (HK4) $\mathbb{E}_x e^{\theta d_{G'}(x, X_s)} \leq 1 + C_M \theta s^{1/2}$ for $0 < s \leq \varepsilon_1^{\text{HK}} R_{\text{HK}}^2$ and $0 \leq \theta \leq \min(s^{-1/2}, c_{\text{ex}}/2)$.*

Proposition 6.9 (Coarse-to-fine transfer). *Let G' be an output, read on the fine lattice. (i) Isoperimetry. $|\partial_{G'} A| \geq c_F |A|^{2/3}$ for every $A \subseteq V$ with $|A| \leq V_*^f$, where the absolute constant c_F (§S1.5) does not depend on r_0 . (ii) Faber–Krahn, components. For $\Omega \subseteq V$ with $|\Omega| \leq V_*^f$, $\lambda_1^D(\Omega) \geq (c_F^2/12)|\Omega|^{-2/3}$, where $\lambda_1^D(\Omega)$ is the smallest eigenvalue of $-\Delta_{G'}$ on functions vanishing off Ω ; and every component of G' has more than $2V_*^f$ vertices. (iii) Cheeger transfer. If a set K of coarse cells satisfies $|\partial_K W| \geq h_c \min(|W|, |K \setminus W|)$ for every $W \subseteq K$, then its blow-up K^f satisfies $|\partial_{K^f} A| \geq r_0^{-1} \min\{\frac{1}{2}, h_c/C_{\text{bl}}\} \min(|A|, |K^f \setminus A|)$ for every $A \subseteq K^f$, with the absolute constant C_{bl} of §S1.5. (iv) Cell Poincaré inequality. For every fine cell Ω of a block Q there is a connected set $\mathfrak{R}^b$ with $\Omega \subseteq \mathfrak{R}^b \subseteq V \cap \hat{Q}(Q)$ such that $\sum_{x \in \Omega} |\varphi(x) - \bar{\varphi}_\Omega|^2 \leq C_W^f k_\square^2 \sum_{b \in E(\mathfrak{R}^b)} (\nabla_b \varphi)^2$ for every $\varphi : V \rightarrow \mathbb{R}$, with $C_W^f := 108C_{\text{bl}}^2/\chi^2$ and $k_\square := r_0(k_P + 3)$; every fine site and every fine bond lies in at most $N_{\text{ov}} \leq 27(k_P/L_c)^3$ of the sets $\mathfrak{R}^b$, resp. $E(\mathfrak{R}^b)$; and $|\Omega| \geq c_1 L^3$. (v) Diameter. Under (L8-2) of §7.6, every $\mathfrak{R}^b$ has $d_{G'}$ -diameter less than L' .*

Proposition 6.10 (Resolvent decay at R_{HK}). *Let G' be an output, $\kappa_W := \min(1, 1/(2C_M))$ and $m_W := \min(1, c_{\text{ex}}/2)$. For every mass $0 < M \leq m_W$,*

$$\sup_x \sum_y v_{G'}^{(M)}(x, y) e^{\theta_M d_{G'}(x, y)} \leq 3M^{-2}, \quad \theta_M := \kappa_W M \min(1, (\varepsilon_1^{\text{HK}})^{1/2} M R_{\text{HK}}). \quad (6.13)$$

In particular, if $M_Y \leq m_W$,

$$\sup_x \sum_y v_{G'}(x, y) e^{\kappa_Y M_Y d_{G'}(x, y)} \leq 3M_Y^{-2}, \quad \kappa_Y := \kappa_W \min(1, (\varepsilon_1^{\text{HK}})^{1/2} K_0/\lambda_{\text{PV}}), \quad (6.14)$$

and the same bound holds for $v_{G'}^2$ (since $0 \leq v_{G'}^2 \leq v_{G'}$ entrywise) and for $v_{G'}^1$ at mass $M_1 \leq m_W$. Moreover $\kappa_Y M_Y \geq 4\kappa_0 \tilde{m}$, where $\kappa_0 := \kappa_W (\varepsilon_1^{\text{HK}})^{1/2} K_0$.

Since C_M depends only on the absolute constant C_H (Theorem 6.8), the constant κ_W does not depend on τ , K , C or D , and this is the only property of κ_W that we use.

Theorem 6.11 (Covariance bounds). *Let G' be an output and $C_0 := (M_Y^{-2} \Delta_{G'}^2 - \Delta_{G'} + \tilde{m}^2)^{-1}$ on $\mathbb{R}^V$. Assume $\lambda_{\text{PV}} \leq \frac{1}{4}$, $M_Y \leq m_W$ and $\tilde{m} \leq \min(1/68, m_W/2)$ (all true for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$). Put $\kappa'_0 := \min(\kappa_0/2, (2c_{\text{gs}})^{1/2})$ and $r_{\text{cov}} := \min(\kappa_0, \kappa'_0) \tilde{m}$, and let K_1, K'_1, K''_1 and $K_3 \leq 3.5$ be the constants of §S1.5, which depend only on $(c_F, c_\rho, c_L, \lambda_{\text{PV}})$. Then for all $x, y \in V$: (H1) $C_0(x, x) \leq K_1 M_Y$ (the zero modes included); (H1') $0 \leq C_0(x, y) \leq K'_1 M_Y e^{-\kappa'_0 \tilde{m} d_{G'}(x, y)}$; (H2) $\|C_0\|_{\kappa_0 \tilde{m}} \leq K_3 \tilde{m}^{-2}$; (H_C) if ν is symmetric with $\|\nu\|_{r_{\text{cov}}} \leq \varepsilon_\nu \tilde{m}^2$ and $2K_3 \varepsilon_\nu \leq 1$, then $C := (C_0^{-1} + \nu)^{-1}$ exists and is positive definite, $C(x, x) \leq 2K_1 M_Y$, $|C(x, y)| \leq 2K''_1 M_Y e^{-r_{\text{cov}} d_{G'}(x, y)}$ and $\|C\|_{r_{\text{cov}}} \leq 2K_3 \tilde{m}^{-2}$; the same bounds hold for every matrix C_s with $|C_s(x, y)| \leq |C(x, y)|$ entrywise and $C_s(x, x) \leq C(x, x)$, such as the interpolated covariances of Appendix C.*

6.4 Depletion

In this subsection we bound the number of depleted sites per cell. The face kernel of Theorem 6.15 contains all collective effects of the walls. Chebyshev's inequality turns (R9) into a site count, and sparse insulation costs one logarithm, which Section 7 absorbs. Throughout, $\Lambda' = \Lambda^\circ$, $M := M_Y$, and $u_x := v_{\Lambda'}(\cdot, x)$ is the kernel of the box without holes. The *missing pairs* are the (y, y') with $y \in V$, $|y - y'| = 1$, and $y' \in \mathbb{X}$ (the set $\mathfrak{M}_{\mathbb{X}}$) or $\{y, y'\} \in S$ (the set $\mathfrak{M}_S$). Their union is $\mathfrak{M} := \mathfrak{M}_{\mathbb{X}} \cup \mathfrak{M}_S$. A frozen bond gives one pair at each end (frozen bonds are *two-sided*), and $N_2(U) := \#\{(y, y') \in \mathfrak{M} : y \in U\}$. For $x, y \in V$ put $g_x(y) := \sum_{y' : (y, y') \in \mathfrak{M}} (u_x(y) - u_x(y'))$.

Lemma 6.12 (Boundary identities, frozen bonds two-sided). *For every slit graph $G' \subseteq \Lambda'$ and $x \in V$: (a) for $y \in V$,*

$$((-\Delta_{G'} + M^2)u_x|_V)(y) = \delta_x(y) - g_x(y);$$

$$(b) I_{G'}(x) = \sum_{(y,y') \in \mathfrak{M}} v_{G'}(x,y) [u_x(y) - u_x(y')].$$

Proof. For (a), at $y \in V$ the operator $-\Delta_{\Lambda'} + M^2$ is $-\Delta_{G'} + M^2$ plus the terms of the pairs of $\mathfrak{M}$ at y , and $(-\Delta_{\Lambda'} + M^2)u_x = \delta_x$. For (b) we apply $v_{G'}$ to (a) at x .

Lemma 6.13 (Insulation density). *There is an absolute constant A_0^{AD} such that on every output, for every z and every $r \geq 1$ (fine units), $N_2(V \cap B(z,r)) \leq A_0^{\text{AD}} r^2 + A_0^{\text{AD}} r^3 / (r_0 \rho_\lambda)$.*

Proof. Informally, the two-sided count is forced, and it reduces the density bound to the minimality of the hull on a ball. At the final state Lemma 4.4 applies to all sets $W \subseteq V_c$ of coarse cells, with $\partial_\Gamma W$ as the coarse cut $\partial_{G'} W$. In the two-sided minimality inequality (4.1) a frozen bond with one end in W carries $w_b - 1 = 1 - \eta_w$ and one inside W carries $w_b \geq 2(1 - \eta_w)$ ($w_b = 2 - \eta_w$, the hull weight). Hence the coarse count $N_2^c(W)$ of missing pairs with first entry in W is at most $(|\partial_\Gamma W| + \lambda|W|)/(1 - \eta_w)$. We first consider $r \geq r_0$, and let W be the set of coarse cells of V_c that meet $B(z,r)$. Then $|W| \leq C(r/r_0)^3$, and each bond of $\partial_\Gamma W$ joins a coarse cell meeting $B(z,r)$ to a coarse cell of V_c that does not, so $|\partial_\Gamma W| \leq C(r/r_0)^2$. Since a coarse missing pair carries fewer than $4r_0^2$ fine ones, $N_2(V \cap B(z,r)) \leq 4r_0^2 \cdot \frac{10}{9} (C(r/r_0)^2 + C(r/r_0)^3 / \rho_\lambda)$. For $1 \leq r < r_0$, the fine missing pairs with first entry in $B(z,r)$ cross at most six coarse grid planes, each in at most Cr^2 bonds, counted with both orientations.

Lemma 6.14 (Green bound). *On every output, for all $x, y \in V$, $v_{G'}(x,y) \leq C_{\text{Gr}}[(1 + |x - y|)^{-1} + \eta_G]$ with $\eta_G := \tilde{m}(K_0^{-1} + \lambda_{\text{PV}}^2 K_0^{-3})$ and C_{Gr} depending only on the constants of Theorem 6.8; and $v_{G'}(x,y) = 0$ when x and y lie in different components.*

Proof. We split the time integral in $v_{G'} = \int_0^\infty e^{-M^2 t} p_t dt$, with $R := R_{\text{HK}}$. On $[0, 1]$ we use $p_t \leq 1$ and (HK0). On $[1, 2\varepsilon_1^{\text{HK}} R^2]$, (HK3), split at $t = d_{G'}(x,y)$, gives $C/(1 + |x - y|)$. Beyond this interval, (HK1) gives $C(R^{-1} + M^{-2}R^{-3}) \leq C\eta_G$, since $R \geq K_0/\tilde{m}$ and $M = \tilde{m}/\lambda_{\text{PV}}$. Across components $v_{G'} = 0$ by Lemma 6.2(i). The zero mode of a component $\mathfrak{K}$ contributes $M^{-2}/|\mathfrak{K}| \leq 1/(2M^2 V_*^f)$ (Proposition 6.9(ii)), which lies inside $M^{-2}R^{-3}$.

Theorem 6.15 (Face-kernel bound). *On every output, for every $x \in V$, with $r(y) := |x - y|$, $I_{G'}(x) \leq C_F S_3(x) + C_F \eta_G S_2(x)$, where $S_3(x) := \sum_{(y,y') \in \mathfrak{M}} (1 + r(y))^{-3} e^{-M_Y r(y)/2}$, $S_2(x) := \sum_{(y,y') \in \mathfrak{M}} (1 + r(y))^{-2} e^{-M_Y r(y)/2}$, $C_F := C_{\text{Gr}} C'_L$ with C'_L the constant of Lemma D.2, and $S_2(x) \leq C_S \log(2/M_Y)$ with $C_S := C A_0^{\text{AD}} (1 + \lambda_{\text{PV}} / (c_\rho c_L))$ (§S1.5).*

Proof. We take absolute values in Lemma 6.12(b), so no cancellation across a sheet is used. For each pair, $0 \leq v_{G'}(x,y) \leq C_{\text{Gr}}[(1 + r(y))^{-1} + \eta_G]$ by Lemma 6.14, and $|u_x(y) - u_x(y')| \leq C'_L (1 + r(y))^{-2} e^{-M_Y r(y)/2}$ by (L2) of Lemma D.2, which applies with $M = M_Y$ since $|y - y'| = 1$, the sides of Λ° are at least $\ell_D \geq \frac{1}{2}\tilde{m}^{-1} \geq M_Y^{-1}$ (Lemma 3.3(c)), and $M_Y \leq \frac{1}{5}$ for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$. A frozen bond appears twice, each time with its own first entry, and the Neumann kernel appears only through its values. For the bound on S_2 we sum Lemma 6.13 over dyadic shells $2^k \leq r < 2^{k+1}$. Each shell inside $r_0 \rho_\lambda$ gives at most $8A_0^{\text{AD}}$, which up to $r \leq M_Y^{-1}$ yields the logarithm, for the shells beyond we use $e^{-M_Y r/2}$, and the growth term gives at most $C A_0^{\text{AD}} / (M_Y r_0 \rho_\lambda) \leq 2C A_0^{\text{AD}} \lambda_{\text{PV}} / (c_\rho c_L)$ (§S1.5(f)).

We put $\varepsilon_0 := \theta_0/(2\pi^2\tau)$, so that $\Phi(x) > \theta_0$ if and only if $v_{G'}(x, x) - v(0) > \varepsilon_0$, and $\varepsilon_F := \varepsilon_0/(4C_F)$. With the constants L_g, C_R of §S1.5, let $R_{\text{dep}} := \lceil 32A_0^{\text{AD}}L_g/\varepsilon_F \rceil = \lceil C_R\tau/\theta_0 \rceil$, and let (F) be the condition $C_F\eta_G C_S \log(2/M_Y) \leq \varepsilon_0/4$. The constant $c_0 := z\tau \bar{r}_0^2(w_* + 2|\partial\Delta|)$, maximized over removal cells Δ , comes from the two-sided bound (R9) (Proposition 4.26). It satisfies $c_0 < 25(3 + c_s)$ (§S1.2).

Theorem 6.16 (Depletion per removal cell). *Let G' be an output and assume $R_{\text{dep}} \leq r_0\rho_\lambda$, $R_{\text{dep}} \leq \ell_D$ and (F). Then for every removal cell Q , $\#(\mathfrak{D} \cap Q) \leq C(z\theta_0)^{-1}[c_0 + C_{\text{box}} + c_0^{\text{sp}} \log(C'\tau/\theta_0)]$, where $c_0^{\text{sp}} \leq c_0$ is the part of the insulation lying in sparse blocks (defined in the proof), C_{box} is the box-face term, and C, C' are absolute up to C_R and C_F .*

Proof. (i) *Reduction to S_3 .* Since $v_{G'}(x, x) - v(0) = I_{G'}(x) + [v_{\Lambda'}(x, x) - v(0)]$, we have $\mathfrak{D} \subseteq \{I_{G'} > \varepsilon_0/2\} \cup \{\Phi_{\Lambda'} > \theta_0/2\}$, where $\Phi_{\Lambda'}(x) := \frac{1}{2}\beta_{\text{GM}}(v_{\Lambda'}(x, x) - v(0))$. On the first set Theorem 6.15 and (F) give $S_3(x) > \varepsilon_F$. (ii) *Far insulation.* The pairs at distance more than $R := R_{\text{dep}}$ from x contribute to $S_3(x)$ at most $\sum_{k \geq 0} (2^k R)^{-3} N_2(V \cap B(x, 2^{k+1}R)) e^{-M_Y 2^k R/2}$. By Lemma 6.13, with $R \leq r_0\rho_\lambda$ and $M_Y r_0\rho_\lambda \geq c_\rho c_L/(2\lambda_{\text{PV}})$, this is at most $16A_0^{\text{AD}}L_g/R \leq \varepsilon_F/2$ (§S1.5(g)). There the shells with $2^{k+1}R \leq r_0\rho_\lambda$ give $16A_0^{\text{AD}}/R$, and the others a logarithm. Hence each $x \in \mathfrak{D}$ outside the box set has $S_3^{\text{near}}(x) > \varepsilon_F/2$, where S_3^{near} counts the pairs with first entry within R of x . (iii) *Blocks.* We cover Q by grid cubes q of side R . Let $3q$ be the block of $3 \times 3 \times 3$ such cubes centred at q , put $N_{3q} := N_2(V \cap 3q)$, and call q dense if $N_{3q} \geq R^2$, sparse otherwise. A dense q has $\#(\mathfrak{D} \cap q) \leq R^3 \leq RN_{3q}$. For a sparse q , Chebyshev's inequality gives $\#(\mathfrak{D} \cap q) \leq (2/\varepsilon_F) \sum_{x \in q} S_3^{\text{near}}(x) \leq (2/\varepsilon_F) N_{3q} (4\pi \log(4R) + C_3)$ plus the box set. Each pair lies in at most 27 blocks $3q$, and the pairs counted for $q \cap Q \neq \emptyset$ lie within $2R \leq 2r_0\rho_\lambda < \ell_D$ of Q , hence in the 27 removal cells Q' around Q , as the removal cells form a product grid with sides at least ℓ_D . Each of these cells has $N_2(Q') \leq 4r_0^2(w_* + 2|\partial\Delta|) \leq 4c_0/(z\tau)$ by (R9), since a coarse pair carries fewer than $4r_0^2$ fine ones and removed cells contain no site of V . Hence $A_{\text{dense}} + A_{\text{sparse}} \leq 4 \cdot 27^2 c_0/(z\tau)$, where $A_\bullet := \sum_q \bullet N_{3q}$. We put $c_0^{\text{sp}} := z\tau A_{\text{sparse}}/(4 \cdot 27^2) \leq c_0$ and substitute $R \leq 2C_R\tau/\theta_0$ and $2/\varepsilon_F = 16\pi^2 C_F\tau/\theta_0$. (iv) *Box faces.* By the image representation of the box kernel (proof of Lemma D.2) and (L1), $v_{\Lambda'}(x, x) - v(0) \leq C \sum_F d_F(x)^{-1} + CM_Y$, the sum over the faces F of Λ' , $d_F(x)$ the layer distance of x from F . Hence the set $\{\Phi_{\Lambda'} > \theta_0/2\}$ lies within distance $C\tau/\theta_0$ of $\partial\Lambda'$ and meets a removal cell, of side at most $3\ell_D$, in at most $C\ell_D^2\tau/\theta_0 \leq C_{\text{box}}/(z\theta_0)$ sites, since $\ell_D^2\tau \leq C\tau/\tilde{m}^2 \leq C/z$.

Proposition 6.17 (Depletion per $\mathcal{U}$ -cell). *Under the hypotheses of Theorem 6.16, for every $\mathcal{U}$ -cell $\Delta_{\mathcal{U}}$ of the output, $(\tilde{m}^2/\beta_{\text{GM}}) \#(\mathfrak{D} \cap \Delta_{\mathcal{U}}) \leq c_{\text{dep}} := A_0^D + A_1^D \log(C'\tau/\theta_0)$, with $A_0^D := CN_{\text{cov}}(c_0 + C_{\text{box}})/\theta_0$, $A_1^D := CN_{\text{cov}}c_0^{\text{sp}}/\theta_0$ and $N_{\text{cov}} \leq (C_{\mathcal{U}}\tilde{m}^{-1}/\ell_D + 6)^3$.*

Proof. Steps (i) to (iv) of the proof of Theorem 6.16 apply to any region U in place of Q , the pairs being counted in the union of the blocks $3q$ over the cubes q meeting U . By (R7) the cell $\Delta_{\mathcal{U}}$ has diameter at most $C_{\mathcal{U}}\tilde{m}^{-1}$. Hence that union lies in a cube of side $C_{\mathcal{U}}\tilde{m}^{-1} + 4R$, which meets at most N_{cov} removal cells (sides $\geq \ell_D \geq R$), each with $N_2 \leq 4c_0/(z\tau)$. Finally $\tilde{m}^2/\beta_{\text{GM}} \leq 4m_D^2/\beta_{\text{GM}} = 8z$. Since a $\mathcal{U}$ -cell lies in one component and depletion is counted site by site, the number of components plays no role.

Lemma 6.18 (The coupling condition for depletion). *Let $\theta_0 = 0.01$ and*

$$C_*^{\text{dep}} := \max\{17.8 C_R/(c_\rho c_L), 3.6 \cdot 10^3 C_F C_S (K_0^{-1} + \lambda_{\text{PV}}^2 K_0^{-3})\}.$$

If $\tau \geq 2$, $2.494\tau \geq \ln(C_*^{\text{dep}}/\theta_0) + 2.5\ln\tau$ and $m_D/2 \leq \tilde{m} \leq 2m_D$, then $R_{\text{dep}} \leq r_0\rho_\lambda$, $R_{\text{dep}} \leq \ell_D$ and (F) hold.

Proof. By hypothesis, $C_*^{\text{dep}}\tau^{5/2}e^{-2.494\tau} \leq \theta_0$. With $\tilde{m} \leq 2m_D$, $r_0\rho_\lambda = c_\rho L \geq c_\rho c_L/(2\tilde{m})$ and $\log(2/M_Y) \leq \ln(1/m_D) \leq 2.4944\tau$, the three conclusions are arithmetic (§S1.5).

The value $\theta_0 = 0.01$ is dictated by Section 7, which needs $\varepsilon_\nu + \theta' \leq 1/7$ with $\theta' \geq 1 - e^{-\theta_0}$. The condition of Lemma 6.18 is a lower bound on τ that persists.

7 Stage 2: the single-scale expansion and the deterministic area law

In this section we prove Theorem 7.57: for each output $G' = G'(D)$ of the construction and each rectangle C with $\min(R, T) \geq R_1(\beta)$, $W_{G'}(C) \leq \exp(-\sigma_V[A_{G'}(C) - \eta_S RT] + c_N(R + T))$. We carry out the single-scale analysis of [GM82, Section 6, Appendices C to E], with the lemmas of [BF80, Section 9] that it uses, on G' . The structural changes are the following. First, the outer partition functions are graded by the class of their walls (their periods around the handles of the insulating set), and the Kirkwood–Salsburg system of [BF80, Appendix 4] is solved for all classes at once in a weighted ℓ^1 algebra. Second, the area weight of a class comes from one bound at fixed class (Proposition 7.53), the only place where the handles enter the bounds. In Subsections 7.1 to 7.5 we derive the class-graded polymer expansion (Corollary 7.28), and in Subsections 7.6 to 7.9 we bound it.

7.0 Standing conventions

We use the following conventions in Section 7 and Appendices A to C: *cells* (Definition 3.14) may be disconnected; interfaces, walls and tubes are defined on cells, and polymers, seeds and the Kirkwood–Salsburg system on $\mathcal{U}$ -cells (Theorem 3.20 (R7)); $|X|$ is the number of $\mathcal{U}$ -cells of X , never a Debye volume, and the constants of Appendix B are per $\mathcal{U}$ -cell. We work in fine lattice units, $a = 1$. Sums over sites and bonds are plain sums, each bond counted once. The collar scale L' is fixed by $\tilde{m}L' = K_{\text{col}}\beta_{\text{GM}}$, $K_{\text{col}} = 2$, where $\beta_{\text{GM}} = 4\pi^2\tau$. The min-cut area $A_{G'}(C)$ of Definition 2.11 (a support count) and the forced wall content $A(p)$ of Lemma 7.43 (an ℓ^1 quantity) are related by Theorem 7.14 and Lemma 7.55. Constants C , c are absolute or depend only on the constants fixed before K (Section 8), and change from line to line. We write $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for the ℓ^2 inner product and norm on V or E , and $\|\varphi\|_Y^2 := \sum_{x \in Y} \varphi(x)^2$.

7.1 Representation

Throughout, $G' = (V, E)$ is the fine slit graph of Subsection 2.5 for the set (2.9), with set of components $\text{Comp}(G') \ni \mathfrak{K}$. From Subsection 7.2 on, G' will be an output of the construction.

Definition 7.1 (Operators and the loop on G'). (a) For an oriented bond $b = (x, y) \in E$ and $\varphi : V \rightarrow \mathbb{R}$, $\nabla\varphi(b) := \varphi(y) - \varphi(x)$, and $\Delta := -\nabla^*\nabla$ is the Neumann Laplacian of G' . (b) On $\mathbb{R}^V$, $u^{-1} := M_Y^{-2}(-\Delta)(-\Delta + M_Y^2)$ and $C_0 := (u^{-1} + \tilde{m}^2)^{-1} = F(-\Delta)$, $F(t) := (M_Y^{-2}t^2 + t + \tilde{m}^2)^{-1}$. (c) For an integer 2-chain Ξ in Λ° with $\partial\Xi = C$ and $b = (x, y) \in E$, ξ_b is the coefficient in Ξ of the plaquette p_b dual to b , oriented as ∂c_y (c_y the cube of y), and $j(b) := 2\pi\xi_b$ (the opposite orientation of Subsection 2.5 amounts to $\Xi \mapsto -\Xi$ and leaves (2.8) unchanged, $m \mapsto -m$).

With $n = 2\pi m$, (2.8) reads $Z(k, \Xi) := Z_{G'}(k, \Xi) = \sum_{[n]} \exp(-\frac{1}{2\beta_{\text{GM}}} \sum_{b \in E} (\nabla n(b) - kj(b))^2)$, $[n]$ running over $(2\pi\mathbb{Z})^V$ modulo functions constant on each component; $W_{G'}(C) = Z(1, \Xi)/Z(0, \Xi)$ does not depend on Ξ (Proposition 2.10). (d) The Neumann solid angle $f = f_{G'}$ solves $-\Delta f = \nabla^* j$; $R_f := \nabla f - j$ and $E_{G'} := \|R_f\|^2$.

The operator C_0 is positive definite, since $u^{-1} \geq 0$. As C_0 and u^{-1} are both functions of $-\Delta$ and $\Delta \mathbf{1}_{\mathfrak{K}} = 0$, we have

$$C_0 \mathbf{1}_{\mathfrak{K}} = \tilde{m}^{-2} \mathbf{1}_{\mathfrak{K}} \quad (\mathfrak{K} \in \text{Comp}(G')), \quad \tilde{m}^2 C_0 - 1 = -C_0 u^{-1} = -u^{-1} C_0. \quad (7.1)$$

Since both ends of each bond lie in one component, $\nabla^* j \perp \ker \Delta$. Hence f exists, unique up to one constant per component, and R_f , $E_{G'}$ do not depend on the choice.

Lemma 7.2 (Solid angle; completing the square). *Let Π be the orthogonal projection of $\ell^2(E)$ onto $\text{ran } \nabla$. Then $\nabla f = \Pi j$ and $R_f = -(1 - \Pi)j$; hence $\nabla^* R_f = 0$, $\langle \nabla \varphi, R_f \rangle = 0$ for every $\varphi : V \rightarrow \mathbb{R}$, $\langle j, R_f \rangle = -E_{G'}$, and $E_{G'} = \min_{\varphi} \|\nabla \varphi - j\|^2$. For every $n : V \rightarrow \mathbb{R}$ and $k \in \mathbb{Z}$, $\sum_E (\nabla n - kj)^2 = \sum_E (\nabla(n - kf))^2 + k^2 E_{G'}$, so $Z(k, \Xi) = e^{-k^2 E_{G'}/2\beta_{\text{GM}}} \tilde{Z}(k, \Xi)$ with $\tilde{Z}(k, \Xi) := \sum_{[n]} \exp(-\frac{1}{2\beta_{\text{GM}}} \sum_E (\nabla(n - kf))^2)$.*

Proof. The solid angle is the gradient part of the loop current, since f solves the Euler–Lagrange equation $\nabla^*(\nabla \varphi - j) = 0$ of $\|\nabla \varphi - j\|^2$. Hence $\nabla f = \Pi j$ and $R_f \in \text{ran}(1 - \Pi) = \ker \nabla^*$, and the identities follow. For the last one we write $\nabla n - kj = \nabla(n - kf) + kR_f$ and use $R_f \perp \text{ran } \nabla$.

The input from Stage 1. We use (S1) to (S5) of Theorem 6.5, with $\xi = (m, x)$ and $\int d\xi$ as there, u and $d\mu_u$ of Proposition 6.3 (on $\mathbb{R}_0^V$, where u inverts u^{-1}), and $P := 2\pi\beta_{\text{GM}}^{-1/2}$. By (6.4) with $\theta_{\mathfrak{K}} = \beta_{\text{GM}}^{1/2} c_{\mathfrak{K}}$ there is $\mathcal{N}_0 > 0$, independent of k and Ξ , such that

$$\tilde{Z}(k, \Xi) = \mathcal{N}_0 \int_{[0, P) \text{Comp}(G')} \prod_{\mathfrak{K}} \frac{dc_{\mathfrak{K}}}{P} \int d\mu_u(\varphi) e^{-V_{\text{eff}}(\varphi + c|k\Xi)}, \quad c := \sum_{\mathfrak{K}} c_{\mathfrak{K}} \mathbf{1}_{\mathfrak{K}}, \quad (7.2)$$

$$-V_{\text{eff}}(\varphi | k\Xi) = \rho_0 + \sum_{s \geq 1} \frac{1}{s!} \int d\xi_1 \cdots d\xi_s \rho_s(\xi_1, \dots, \xi_s) \varepsilon_k(\xi_1) \cdots \varepsilon_k(\xi_s), \quad (7.3)$$

with $\varepsilon_k(\xi) := e^{im(\beta_{\text{GM}}^{1/2}\varphi(x) + kf(x))} - 1$ and the real coefficients $\rho_s = \rho_s^{G'}$ of (S1), and the series converges absolutely, uniformly in real φ . By (S3), $V_{\text{eff}}(\varphi + PI | k\Xi) = V_{\text{eff}}(\varphi | k\Xi)$ for $I \in \mathbb{Z}^V$. On outputs the dipole kernel ν of (S5) has $\|\nu\|_{r_{\text{cov}}} \leq \varepsilon_{\nu} \tilde{m}^2$, with $2K_3 \varepsilon_{\nu} \leq 1$. The space $\mathbb{R}_0^V$, the norms $\|\cdot\|_r$, the image energy $\Phi = \Phi_{G'} \geq 0$ and the depleted set $\mathfrak{D}$ are those of the conventions of Section 6. The quantities ρ^{ref} , $\tilde{m}$ and $M_Y = \tilde{m}/\lambda_{\text{PV}}$ are as in (6.11), and (S4') denotes Proposition 6.6 together with (6.10) for ρ^{ref} in place of $\rho_1(\cdot, x)$.

Proposition 7.3 ([GM82, (6.1) to (6.4)] on G'). *Assume (S1) to (S3). (a) $Z(k, \Xi) = e^{-k^2 E_{G'}/2\beta_{\text{GM}}} \tilde{Z}(k, \Xi)$, with $\tilde{Z}$ given by (7.2): one zero-mode variable $c_{\mathfrak{K}} \in [0, P)$ per component of G' . (b) The integrand of (7.2) is P -periodic in each $c_{\mathfrak{K}}$; hence (a) does not change when f is changed by a constant on each component. (c) ρ_0 cancels in $W_{G'}(C)$. From now on $\rho_0 = 0$.*

This is [GM82, (6.1) to (6.4)], by Lemma 7.2 and (7.2), except that there is one zero mode per component (GM's Gaussian measure is defined on $(a\mathbb{Z})^3$ and has none), the loop entering only through the phases ε_k , and that $e^{-k^2 E_{G'}/2\beta_{\text{GM}}}$ replaces $e^{-k^2 (J, v_{\text{Cb}} J)/2\beta}$. Part (b) is (S3) with $I = \mathbf{1}_{\mathfrak{K}}$, and part (c) is [GM82, (6.5)]: ρ_0 gives the same factor in $Z(1, \Xi)$ and $Z(0, \Xi)$.

Cells, cell averages and block spins. We write $-V_{\text{eff}}(\varphi \mid k\Xi) = \int d\xi \rho_1(\xi) \varepsilon_k(\xi) + E'(\varphi)$, where E' collects the terms $s \geq 2$ of (7.3). The cells $\Omega = \Omega_{Q, \hat{\mathfrak{R}}}$ of Definition 3.14 partition V , each in one component. For a cell Ω and $x \in \Omega$ let $A(x) = A_\Omega$ and $\bar{f}(x) = \bar{f}_\Omega$ be the averages of φ and f over Ω , $\delta := \varphi - A$, $\delta f := f - \bar{f}$ and $\tilde{\delta} := \delta + k\beta_{\text{GM}}^{-1/2} \delta f$. Then $\sum_{x \in \Omega} \tilde{\delta}(x) = 0$ for each cell, and $\beta_{\text{GM}}^{1/2} \varphi + kf = (\beta_{\text{GM}}^{1/2} A + k\bar{f}) + \beta_{\text{GM}}^{1/2} \tilde{\delta}$. The block spins are the elements of $\mathcal{H} := \{h : V \rightarrow P\mathbb{Z} \text{ constant on every cell}\}$.

Proposition 7.4 (Peierls identity per cell). *Assume (S3) and $\sum_m m^2 |\rho^{\text{ref}}(m)| < \infty$. (a) For every real φ on V , $e^{\int d\xi \rho_1(\xi) \varepsilon_k(\xi)} = \Theta(\varphi) e^{G(\varphi)}$ with $\Theta(\varphi) := \sum_{h \in \mathcal{H}} e^{-\frac{1}{2} \tilde{m}^2 \|\varphi + k\beta_{\text{GM}}^{-1/2} f - h\|^2}$, $G = G_1 + G_2$, $e^{G_1} = \prod_\Omega r_\Omega(A_\Omega)$, $G_2 = G_{2,1} + G_{2,2} + G_{2,3}$,*

$$r_\Omega(A) := \frac{\exp\{|\Omega| \sum_m \rho^{\text{ref}}(m) [e^{im(\beta_{\text{GM}}^{1/2} A + k\bar{f}_\Omega)} - 1]\}}{\sum_{n \in \mathbb{Z}} \exp\{-\frac{1}{2} \tilde{m}^2 |\Omega| (A + k\beta_{\text{GM}}^{-1/2} \bar{f}_\Omega - nP)^2\}}, \quad (7.4)$$

$$\begin{aligned} G_{2,1} &:= \int d\xi \rho^{\text{ref}}(m) [e^{im\beta_{\text{GM}}^{1/2} \tilde{\delta}(x)} - 1 - im\beta_{\text{GM}}^{1/2} \tilde{\delta}(x) + \frac{1}{2} m^2 \beta_{\text{GM}} \tilde{\delta}(x)^2], \\ G_{2,2} &:= \int d\xi \rho^{\text{ref}}(m) [e^{im(\beta_{\text{GM}}^{1/2} A(x) + k\bar{f}(x))} - 1] [e^{im\beta_{\text{GM}}^{1/2} \tilde{\delta}(x)} - 1 - im\beta_{\text{GM}}^{1/2} \tilde{\delta}(x)], \\ G_{2,3} &:= \int d\xi (\rho_1(m, x) - \rho^{\text{ref}}(m)) [e^{im(\beta_{\text{GM}}^{1/2} \varphi(x) + kf(x))} - 1]. \end{aligned} \quad (7.5)$$

Θ , r_Ω , G_1 and the $G_{2,i}$ are real, $\Theta > 0$, $r_\Omega > 0$, and e^G does not depend on h . The identity holds for every partition of V in place of the cells. (b) Under (S1) to (S3),

$$\tilde{Z}(k, \Xi) = \mathcal{N}_0 \sum_{h \in \mathcal{H}} \int \prod_{\hat{\mathfrak{R}}} \frac{dc_{\hat{\mathfrak{R}}}}{P} \int d\mu_u(\varphi) e^{-\frac{1}{2} \tilde{m}^2 \|\varphi + c + k\beta_{\text{GM}}^{-1/2} f - h\|^2} e^{G(\varphi+c)} e^{E'(\varphi+c)}. \quad (7.6)$$

Part (a) is [GM82, (6.6), (6.8) to (6.11)] and (b) is [GM82, (6.12)], with cells in place of blocks, $|\Omega|$ in place of L^3 and $\rho^{\text{ref}}(m)$ in place of $\rho_1(m, 0)$ in [GM82, (6.7), (6.9), (6.10)], since on G' the one-body coefficient depends on the site (the difference is $G_{2,3}$). Also, (6.12) needs the evident sign correction. In (b) we move the sum over h out of the integrals by Tonelli's theorem, as the moduli of the h -terms sum to $e^{\int \rho_1 \varepsilon_k} |e^{E'}|$.

The deficit term. Since $e^{im\beta_{\text{GM}}^{1/2} h} = 1$ for $h \in \mathcal{H}$, (S3) for ρ_1 and ρ^{ref} gives, for each $h \in \mathcal{H}$ and with $u_h := \beta_{\text{GM}}^{1/2}(\varphi + k\beta_{\text{GM}}^{-1/2} f - h)$, $G_{2,3} = \sum_m \sum_{x \in V} (\rho_1(m, x) - \rho^{\text{ref}}(m)) (\cos(mu_h(x)) - 1)$, and $G_{2,3}(Y)$ is the same sum over $x \in Y$. For a union Y of $\mathcal{U}$ -cells, $N_{\mathfrak{D}}(Y)$ denotes the number of its $\mathcal{U}$ -cells that meet the depleted set $\mathfrak{D} = \{\Phi > \theta_0\}$, $\theta_0 = 0.01$.

Proposition 7.5 (Deficit on the depleted set). *Assume (S3) and (S4'), and put*

$$\theta' := 1 - e^{-\theta_0} + C(e^{-\mu} + e^{-2\kappa s}). \quad (7.7)$$

Uniformly in real φ , in $h \in \mathcal{H}$ and in G' : (a) $|G_{2,3}(V \setminus \mathfrak{D})| \leq \theta' \cdot \frac{1}{2} \tilde{m}^2 \|\varphi + k\beta_{\text{GM}}^{-1/2} f - h\|_{V \setminus \mathfrak{D}}^2$; (b) for every $Y \subseteq V$, $-C(e^{-\mu} + e^{-2\kappa s}) \frac{\tilde{m}^2}{\beta_{\text{GM}}} \#(\mathfrak{D} \cap Y) \leq G_{2,3}(\mathfrak{D} \cap Y) \leq C \frac{\tilde{m}^2}{\beta_{\text{GM}}} \#(\mathfrak{D} \cap Y)$; (c) if G' is an output of the construction, then $e^{-C c_{\text{dep}} N_{\mathfrak{D}}(Y)} \leq e^{G_{2,3}(\mathfrak{D} \cap Y)} \leq e^{C c_{\text{dep}} N_{\mathfrak{D}}(Y)}$ for every union Y of $\mathcal{U}$ -cells, with c_{dep} the constant of Proposition 6.17.

Sketch. Off $\mathfrak{D}$, $\Phi \leq \theta_0$ and (S4') give $\sum_m |\rho_1(m, x) - \rho^{\text{ref}}(m)| m^2 \leq \theta' \tilde{m}^2 / \beta_{\text{GM}}$ (Section S1.6), and $|\cos(mu) - 1| \leq \frac{1}{2} m^2 u^2$ gives (a). On $\mathfrak{D}$ the deviation is at most the density, and its negative part is at most $C e^{-\mu} \rho^{\text{ref}}(1)$ plus the tails, as $e^{-\Phi} \leq 1$. This proves (b). Part (c) follows from (b) and Proposition 6.17, which gives $(\tilde{m}^2 / \beta_{\text{GM}}) \#(\mathfrak{D} \cap \Delta_{\mathcal{U}}) \leq c_{\text{dep}}$ for each $\mathcal{U}$ -cell $\Delta_{\mathcal{U}}$.

In Lemma 7.29 we will move the quadratic part of $G_{2,3}(V \setminus \mathfrak{D})$ into the covariance. Throughout Section 7 we assume the *standing hypothesis*

$$\varepsilon_\nu + \theta' \leq \frac{1}{7}, \quad (7.8)$$

which holds for $\theta_0 = 0.01$, μ and κ_S large, and $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ (Lemma 6.18 and Section 8).

Walls. For $h \in \mathcal{H}$ we put $n := (\beta_{\text{GM}}^{1/2} / 2\pi)h \in \mathbb{Z}^V$, define the *wall field* $w := \nabla h - k\beta_{\text{GM}}^{-1/2}j$ on E , and put $N := (\beta_{\text{GM}}^{1/2} / 2\pi)w$, that is, $N(b) = n(y) - n(x) - k\xi_b \in \mathbb{Z}$ for $b = (x, y)$. The wall bonds are $\Sigma_E := \{b \in E : N(b) \neq 0\}$, the wall sites Σ_V their endpoints, and $N^\vee := \sum_{b \in E} N(b)p_b$ (p_b oriented as in Definition 7.1 (c)).

Lemma 7.6 (Walls). *Let $h \in \mathcal{H}$ and $k \in \mathbb{Z}$. (a) $N(b) \neq 0$ only on interface bonds, and N is constant on each interface I ; we write N_I . (b) Frozen bonds, bonds into $\mathbb{X}$ and bonds leaving Λ° are not in E ; they carry no wall. (c) $\partial N^\vee = -kC + \partial Y$ for an integer 2-chain Y supported in $\mathfrak{F} = F(\mathbb{X}) \cup F_S \cup \partial_2 \Lambda^\circ$: the wall spans C relative to the insulating set, and may end on faces of hole cubes, on frozen faces and on $\partial \Lambda^\circ$. (d) For every cycle γ of G' , $\sum_{b \in \gamma} N(b) = -k \text{lk}(\gamma, C)$. Moreover $N(h) = N(h')$ if and only if $h - h'$ is constant on every component of G' ; so N determines the class $[h]$ of h modulo $\mathcal{C}_P := \{\sum_{\mathfrak{K}} P n_{\mathfrak{K}} \mathbf{1}_{\mathfrak{K}} : n_{\mathfrak{K}} \in \mathbb{Z}\}$. (e) (Relatively closed walls.) For an integer field $M : E \rightarrow \mathbb{Z}$ with $M^\vee := \sum_{b \in E} M(b)p_b$, the following are equivalent: (i) M is relatively closed: $\partial M^\vee = \partial Y$ for an integer 2-chain Y supported in $\mathfrak{F}$; (ii) M has zero periods: $\sum_{b \in \gamma} M(b) = 0$ for every cycle γ of G' ; (iii) $M = \nabla n'$ for some $n' : V \rightarrow \mathbb{Z}$.*

Proof. In outline, a wall is where the cell-constant field fails to absorb the jump of the loop current, which is constant on interfaces because they are aligned with Ξ . (a) $\nabla h = 0$ on bonds inside a cell, which do not cross Ξ (Lemma 3.5 (A1)). Other bonds join two blocks, and an interface lies in one block face, which lies in Ξ or misses it (Theorem 3.20 (R7)). (b) is the definition of E . (c) Extend n by 0 to $\mathbb{X}$ and let $U := \sum_{x \in V} n(x)c_x$. A plaquette of a cube of V not dual to a bond of E is a face of a hole cube, a frozen face or in $\partial_2 \Lambda^\circ$, so $\partial U = \sum_{b \in E} \nabla n(b)p_b + Y_1$ with $\text{supp } Y_1 \subseteq \mathfrak{F}$. Likewise, $\sum_{b \in E} \xi_b p_b = \Xi - Y_2$, with Y_2 the part of Ξ in $\mathfrak{F}$ (Ξ is interior to Λ° by Lemma 3.5, and $F(\mathbb{X})$ contains all faces of hole cubes). So $N^\vee = \partial U - Y_1 - k\Xi + kY_2$ and $\partial N^\vee = -kC + \partial(kY_2 - Y_1)$. (d) $\sum_{b \in \gamma} \nabla n(b) = 0$, and $\sum_{b \in \gamma} \xi_b$ counts the crossings of γ with Ξ , that is, $\text{lk}(\gamma, C)$. If $N(h) = N(h')$, then $\nabla(n - n') = 0$. (e) Since the box is acyclic, the relative cycle $M^\vee - Y$ is a boundary ∂W , which a cycle of G' crosses as often inwards as outwards (plaquettes dual to bonds of E are not in $\mathfrak{F}$). This proves (i) $\Rightarrow$ (ii). Integrating along paths, we obtain (ii) $\Leftrightarrow$ (iii), and the computation of (c) with $k = 0$ gives (iii) $\Rightarrow$ (i).

Handles. In $\mathbb{Z}^3$ the period condition of Lemma 7.6 (d) is local, and walls with zero periods are removed by $h \mapsto h - Pn'$ (Lemma 7.6 (e)). In G' a cycle γ around a handle of the insulating

set with $\text{lk}(\gamma, C) = 0$ may be crossed once each, with opposite signs, by two separated wall pieces. We therefore write the shift in source form (Proposition 7.8) and grade the outer partition functions by the class of their walls (Definition 7.25).

Proposition 7.7 (Unfolding and translation). *Assume (S1) to (S3) and (S5). For each $h \in \mathcal{H}$ let $g = g_h : V \rightarrow \mathbb{R}$ be equivariant: $g_{h+c} = g_h + c$ for $c \in \mathcal{C}_P$. Put $H := h - k\beta_{\text{GM}}^{-1/2}f$, $C^{-1} := C_0^{-1} + \nu$, $E(\psi) := E'(\psi + g) + \frac{1}{2}\langle \psi, \nu\psi \rangle$, $g_c := \tilde{m}^2 C_0 H$, $F_1 := \frac{1}{2}\tilde{m}^2 \|g - H\|^2 + \frac{1}{2}\langle g, u^{-1}g \rangle$ and $F_2 := \langle \psi, C_0^{-1}(g - g_c) \rangle$. Then $C^{-1} \geq \frac{1}{2}C_0^{-1} > 0$, and with $d\mu_C$ the centred Gaussian measure on $\mathbb{R}^V$ with covariance C ,*

$$\tilde{Z}(k, \Xi) = \sum_{[h] \in \mathcal{H}/\mathcal{C}_P} \mathcal{N} \int d\mu_C(\psi) e^{E(\psi)} e^{G(\psi+g)} e^{-F_1} e^{-F_2}, \quad (7.9)$$

where $\mathcal{N} > 0$ depends on G' , M_Y , $\tilde{m}$ and ν , and not on k , Ξ or h . Each term of (7.9) does not depend on the representative of $[h]$.

This is [GM82, (6.13) to (6.20)] with $\chi_\Lambda \equiv 1$ on V and E the E of [GM82, (6.14c)], except that the sum over h is unfolded over $\mathcal{H}/\mathcal{C}_P$ and the mass acts at every site. The integrand of (7.6) is invariant under $(\varphi + c, h) \mapsto (\varphi + c + c', h + c')$, $c' \in \mathcal{C}_P$, since each cell lies in one component, r_Ω is P -periodic, and the phases of E' shift by multiples of 2π . Hence the sum over c' turns the zero-mode integrals into Lebesgue measure on $\mathbb{R}^{\text{Comp}(G')}$. As $u^{-1}\mathbf{1}_{\mathcal{R}} = 0$, the measure becomes $e^{-\frac{1}{2}\langle \hat{\varphi}, u^{-1}\hat{\varphi} \rangle} d\hat{\varphi}$ on $\mathbb{R}^V$, and $C_0 = (u^{-1} + \tilde{m}^2)^{-1}$ is one operator on $\mathbb{R}^V$ (in [GM82, (6.15)] the mass term is $\tilde{m}^2 \chi_\Lambda$). By (S5), $C^{-1} \geq \frac{1}{2}C_0^{-1}$, and equivariance with (7.1) gives the independence of the representative.

7.2 The shift and the localized F_1

From now on G' is an output of the construction and $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, so that $L' \geq 16$ and $L/2 < L'$. We write $d = d_{G'}$ for the graph distance of G' ($d(x, y) = \infty$ for sites in different components) and $N_r(A) := \{x \in V : d(x, A) \leq r\}$. Each cell lies in one $\mathcal{U}$ -cell and each $\mathcal{U}$ -cell in one component (Theorem 3.20 (R7)). A good tube of an interface bond b (Subsection 7.3) is a straight path of G' of L sites with middle bond b .

The wall neighbourhood. The set $\hat{\Sigma}$ is the union of the $\mathcal{U}$ -cells Δ with $d(\Delta, \Sigma_V) < L'$, as on [GM82, p. 47] with $\mathcal{U}$ -cells for unit cubes and d for GM's distance $3^{-1/2}\|x - y\|_1$ of [GM82, (1.11a), (3.23)]. The *wall components* S_β are the unions of the $\mathcal{U}$ -cells of the connected components of the graph on the $\mathcal{U}$ -cells in $\hat{\Sigma}$, two being adjacent when a bond of G' joins them. Put $\Sigma_\beta := \Sigma_V \cap S_\beta$, let $E(S_\beta)$ be the set of bonds of E with both ends in S_β , and $w_\beta := w\mathbf{1}_{E(S_\beta)}$. The *wall sources* and the *standard cutoff* are $s_\beta := M_Y^{-2}(-\Delta + M_Y^2)\nabla^* w_\beta$ and $\chi_\beta := \zeta(d(\cdot, \Sigma_\beta))$, with $\zeta = 1$ on $[0, L'/2]$, $\zeta(t) = 3 - 4t/L'$ on $[L'/2, 3L'/4]$ and $\zeta = 0$ on $[3L'/4, \infty)$. The transition set A_β is the set of endpoints of bonds $\{x, y\} \in E$ with $\chi_\beta(x) \neq \chi_\beta(y)$. The shift is

$$g := H - \sum_{\beta} \chi_\beta C_0 s_\beta, \quad D := g - H = \sum_{\beta} D_\beta, \quad D_\beta := -\chi_\beta C_0 s_\beta. \quad (7.10)$$

Proposition 7.8 (Source-form shift). *For every $h \in \mathcal{H}$ and $k \in \{0, 1\}$: (a) every wall bond lies in $E(S_\beta)$ for exactly one β , so $w = \sum_{\beta} w_\beta$ with disjoint supports; $\{x : d(x, \Sigma_\beta) < L'\} \subseteq S_\beta$;*

every interface carrying a wall lies in one $E(S_\beta)$, and the good tubes of its bonds lie in S_β ; (b) $\chi_\beta = 0$ off S_β , $\chi_\beta = 1$ on Σ_β , $d(A_\beta, \Sigma_\beta) \geq L'/2 - 1 \geq 7$, $N_1(A_\beta) \subseteq S_\beta$, and χ_β is $(4/L')$ -Lipschitz for d ; (c) $u^{-1}H = \sum_\beta s_\beta$ and $\text{supp } s_\beta \subseteq N_1(\Sigma_\beta)$; (d) $g = H$ off $\bigcup_\beta \text{supp } \chi_\beta$, in particular $g + k\beta_{\text{GM}}^{-1/2}f = h$ on $V \setminus \hat{\Sigma}$; on S_β , $g = H - \chi_\beta C_0 s_\beta$; and g is equivariant; (e) $F'_2 := C_0^{-1}(g - g_c)$ satisfies

$$F'_2 = \sum_\beta F'_{2,\beta}, \quad F'_{2,\beta} := -[C_0^{-1}, \chi_\beta] C_0 s_\beta, \quad \text{supp } F'_{2,\beta} \subseteq N_1(A_\beta) \subseteq S_\beta, \quad (7.11)$$

and on $N_1(A_\beta)$ one has $s_\beta = 0$ and $F'_2 = -C_0^{-1}\chi_\beta C_0 s_\beta$; (f) given S_β , the functions s_β , $D|_{S_\beta}$ and $F'_{2,\beta}$ depend on h only through $N|_{E(S_\beta)}$. No extension of $h|_{S_\beta}$ to V enters the construction.

Proof. Roughly speaking, since $-\Delta f = \nabla^* j$, the jumps of h across Ξ cancel j on each bond with $N = 0$, and hence $u^{-1}H$ is a pure wall source. Near each wall component, the shift subtracts the response $C_0 s_\beta$ to that source. Parts (a), (b), (d) and (f) are support checks. A geodesic of length less than L' to Σ_β stays in $\hat{\Sigma}$, hence in one wall component, and good tubes lie within $L/2 < L'$ of their middle bond. The function ζ is $(4/L')$ -Lipschitz. Since $A_\beta \subseteq \{L'/2 - 1 < d(\cdot, \Sigma_\beta) \leq 3L'/4 + 1\}$ and $L' \geq 16$, we have $N_1(A_\beta) \subseteq \{d(\cdot, \Sigma_\beta) < L'\} \subseteq S_\beta$. Finally, $\nabla c = 0$ for $c \in \mathcal{C}_P$, and $w_\beta = 2\pi\beta_{\text{GM}}^{-1/2}N$ on $E(S_\beta)$, whose wall bonds have endpoint set Σ_β . (c) $-\Delta H = \nabla^* \nabla h - k\beta_{\text{GM}}^{-1/2} \nabla^* j = \nabla^* w$, so $u^{-1}H = M_Y^{-2}(-\Delta + M_Y^2) \nabla^* w = \sum_\beta s_\beta$ by (a). Moreover, $\nabla^* w_\beta$ is supported in Σ_β , and $-\Delta + M_Y^2$ enlarges supports by at most one step. (e) By (7.1), $C_0^{-1}g_c = \tilde{m}^2 H$ and $C_0^{-1}H - \tilde{m}^2 H = u^{-1}H$, and by (c) we obtain $F'_2 = \sum_\beta (s_\beta - C_0^{-1}\chi_\beta C_0 s_\beta)$. A site $x \in N_1(\Sigma_\beta) \setminus \Sigma_\beta$ is adjacent to some $z \in \Sigma_\beta$, and $\chi_\beta(x) = 1$, since otherwise $z \in A_\beta$, against (b). Thus $\chi_\beta = 1$ on $N_1(\Sigma_\beta) \supseteq \text{supp } s_\beta$, and hence $s_\beta - C_0^{-1}\chi_\beta C_0 s_\beta = -[C_0^{-1}, \chi_\beta] C_0 s_\beta$. The kernel $C_0^{-1}(x, y) = (M_Y^{-2}\Delta^2 - \Delta + \tilde{m}^2)(x, y)$ vanishes unless $d(x, y) \leq 2$. Hence $([C_0^{-1}, \chi_\beta]v)(x) = \sum_y C_0^{-1}(x, y)(\chi_\beta(y) - \chi_\beta(x))v(y)$ vanishes unless χ_β changes along a path of at most two bonds from x , which puts x in $N_1(A_\beta) \subseteq S_\beta$. Since $d(A_\beta, \Sigma_\beta) \geq 7 \geq 3$, the sets $N_1(A_\beta)$ and $N_1(\Sigma_\beta)$ are disjoint, so $s_\beta = 0$ on $N_1(A_\beta)$, and there $F'_2 = -C_0^{-1}\chi_\beta C_0 s_\beta$.

A construction through an extension h_β^e of $h|_{S_\beta}$ that is constant on the components of $V \setminus S_\beta$, as in [GM82, (6.24)], is not available on G' : such an extension exists only if w_β has zero periods on all cycles of G' , and a wall spanning the central hole of a solid torus of $\mathbb{X}$ has period ± 1 on a meridian of the torus. When an extension exists, (7.10) agrees with [GM82, (6.24), (6.25)] up to the choice of cutoff and C_0 for C_{00} .

Proposition 7.9 (Exact identity). *For every $h \in \mathcal{H}$, $k \in \mathbb{Z}$ and $D : V \rightarrow \mathbb{R}$, with $g := H + D$,*

$$2F_1 = \tilde{m}^2 \|D\|^2 + \|w + \nabla D\|^2 + M_Y^{-2} \|\Delta D - \nabla^* w\|^2 - k^2 \beta_{\text{GM}}^{-1} E_{G'}. \quad (7.12)$$

Proof. The idea is that the gradient of g differs from the wall field plus ∇D by the Biot–Savart field, which is orthogonal to gradients and has inner product $-E_{G'}$ with j . Hence the perimeter energy separates exactly. As $u^{-1} = -\Delta + M_Y^{-2}\Delta^2$ and $-\Delta = \nabla^* \nabla$, we have $\langle g, u^{-1}g \rangle = \|\nabla g\|^2 + M_Y^{-2} \|\Delta g\|^2$. Now $\nabla g = \nabla h - k\beta_{\text{GM}}^{-1/2}(j + R_f) + \nabla D = (w + \nabla D) - k\beta_{\text{GM}}^{-1/2} R_f$ and, by Lemma 7.2, $\langle w + \nabla D, R_f \rangle = -k\beta_{\text{GM}}^{-1/2} \langle j, R_f \rangle = k\beta_{\text{GM}}^{-1/2} E_{G'}$, so $\|\nabla g\|^2 = \|w + \nabla D\|^2 - k^2 \beta_{\text{GM}}^{-1} E_{G'}$. Next $\Delta g = \Delta h + k\beta_{\text{GM}}^{-1/2} \nabla^* j + \Delta D = \Delta D - \nabla^* w$. Adding $\tilde{m}^2 \|g - H\|^2 = \tilde{m}^2 \|D\|^2$ gives (7.12).

Corollary 7.10 (Localization). *Let D be the shift (7.10). For each wall component put*

$$2F_1(S_\beta) := \tilde{m}^2 \|D_\beta\|^2 + \|w_\beta + \nabla D_\beta\|^2 + M_Y^{-2} \|\Delta D_\beta - \nabla^* w_\beta\|^2, \quad (7.13)$$

and for every union Y of wall components and of $\mathcal{U}$ -cells outside $\hat{\Sigma}$, $F_1^{\text{loc}}(Y) := \sum_{\beta: S_\beta \subseteq Y} F_1(S_\beta)$. Then: (a) $F_1 = F_1^{\text{loc}}(V) - \frac{1}{2} k^2 \beta_{\text{GM}}^{-1} E_{G'}$; (b) $F_1(S_\beta) \geq 0$, and given S_β it depends on h only through $N|_{E(S_\beta)}$; if $E(S_\beta)$ contains no Ξ -bond, it does not depend on (k, Ξ) ; (c) for every such Y ,

$$F_1^{\text{loc}}(Y) \geq \frac{1}{2} \tilde{m}^2 \sum_{x \in Y} (g - H)^2 + \frac{1}{2} \sum_{b \in E(Y)} |\nabla g + k \beta_{\text{GM}}^{-1/2} R_f|^2; \quad (7.14)$$

(d) $Z(k, \Xi)$ is the right side of (7.9) with e^{-F_1} replaced by $e^{-F_1^{\text{loc}}(V)}$: the prefactor $e^{-k^2 E_{G'}/2\beta_{\text{GM}}}$ of Lemma 7.2 cancels, so $E_{G'}$ never enters the perimeter constant.

Proof. Each term of (7.12) but the last is a sum of squares of functions supported in the wall components: ∇D_β is supported on $E(S_\beta)$ and ΔD_β on S_β , since a bond leaving S_β from $\text{supp } \chi_\beta$ would end in $A_\beta \subseteq S_\beta$. This gives (a). Part (b) is Proposition 7.8 (f), and off Ξ -bonds $N = \nabla n$ does not involve k . For (c) we drop the terms with M_Y^{-2} , as $\nabla g + k \beta_{\text{GM}}^{-1/2} R_f = w + \nabla D$. Part (d) follows from Lemma 7.2, Proposition 7.7 and (a).

7.3 The wall tension and the spanning area

Tubes on the fine lattice. Let $b = (x, y)$ be a bond of an interface $I = I(\Omega, \Omega')$, oriented from Ω to Ω' , with $y = x + \sigma e_\mu$, $\sigma = \pm 1$. Put $\tau_b^- := \{x - i\sigma e_\mu : 0 \leq i < L/2\}$, $\tau_b^+ := \{y + i\sigma e_\mu : 0 \leq i < L/2\}$ and $\tau_b := \tau_b^- \cup \tau_b^+$, a segment of L sites (L is even). The bond b is *good* if $\tau_b \subseteq V$ and the $L-1$ consecutive bonds of τ_b lie in E . We write $\text{good}(I)$ for the number of good bonds of I . On the fine lattice the face rule gives $\text{good}(I) \geq c'_g \max(|I|, L^2)$ for each interface, with $c'_g := c_g(1 + 1/L_c)^{-2} \geq \frac{9}{16} c_g$ (Theorem 3.20 (R8), Lemma 4.19).

Lemma 7.11 (Tube inequality). *Let $z_1, \dots, z_L$ be a path, let $\tilde{H} = c_-$ on $\{z_1, \dots, z_{L/2}\}$ and $\tilde{H} = c_+$ on $\{z_{L/2+1}, \dots, z_L\}$, $J := c_+ - c_-$, and let $m(L/2) \leq 1/\sqrt{2}$. Then for every g , $\frac{1}{2} m^2 \sum_{i \leq L} (g(z_i) - \tilde{H}(z_i))^2 + \frac{1}{2} \sum_{i < L} (g(z_{i+1}) - g(z_i))^2 \geq \frac{1}{6} m^2 (L/2) J^2$.*

Sketch. With $\bar{g}_\pm$ the averages of g on the two halves, we write $J = (c_+ - \bar{g}_+) + (\bar{g}_+ - \bar{g}_-) + (\bar{g}_- - c_-)$ and apply Cauchy–Schwarz to each term. Intuitively, the mass term supplies the lower bound if g does not follow the jump, and the gradient term supplies it if g does.

Lemma 7.12 (Tube geometry). *(a) If $b \in I(\Omega, \Omega')$ is good, then $\tau_b^- \subseteq \Omega$ and $\tau_b^+ \subseteq \Omega'$. (b) Good tubes on a common lattice line are site-disjoint. Hence every site lies in at most 3 good tubes and every bond in at most one. (c) No bond of τ_b other than b crosses Ξ , and $\xi, \nabla h, N$ are constant on every interface.*

Sketch. A good tube is a straight path of G' crossing one block face at its middle bond, and by (G1) of Definition 3.4 blocks are at least L long. The chain Ξ lies in grid planes (Lemma 3.5 (A1)), and for (c) we use Lemma 7.6 (a).

Theorem 7.13 (Wall tension). *Let $h \in \mathcal{H}$, $k \in \{0, 1\}$, and let Y be a union of wall components and of $\mathcal{U}$ -cells outside $\hat{\Sigma}$. With $J_I := 2\pi\beta_{\text{GM}}^{-1/2}N_I$,*

$$F_1^{\text{loc}}(Y) \geq \frac{1}{18} \tilde{m}^2 \frac{L}{2} \sum_{I \subseteq Y} \text{good}(I) J_I^2 \geq 2\sigma_F \sum_{b \in E(Y)} N(b)^2, \quad \sigma_F := \frac{\pi^2 c'_g}{18} \frac{\tilde{m}^2 L}{\beta_{\text{GM}}}, \quad (7.15)$$

where $I \subseteq Y$ means that both cells of I lie in Y . We put $F_{\text{unit}} := \sigma_F L^2$. There is no perimeter term.

Proof. The constant $\frac{1}{18}$ is the $\frac{1}{6}$ of the tube inequality divided by the overlap 3 of good tubes, and the remaining factor 2 in front of σ_F is used in Subsection 7.8. By comparison, [GM82, Lemma 12, (C.9)] gives $\frac{1}{60}$ with a perimeter term instead. For a good bond b of an interface $I \subseteq Y$, $\tau_b \subseteq Y$, and with $\vartheta_b := D + J_b \mathbf{1}_{\tau_b^+}$ we have $w + \nabla D = \nabla \vartheta_b$ on the bonds of τ_b and $g - H = \vartheta_b - J_b \mathbf{1}_{\tau_b^+}$ on τ_b (Lemma 7.12 (a), (c)). Hence Lemma 7.11 ($m = \tilde{m}$, $c_- = 0$, $c_+ = J_b$, $\tilde{m}L/2 \leq c_L/2 \leq 1/\sqrt{2}$) bounds the part of the right side of (7.14) on τ_b below by $\frac{1}{6} \tilde{m}^2 \frac{L}{2} J_b^2$. Summing over good bonds, we count each site at most three times and each bond at most once (Lemma 7.12 (b)). The second inequality follows from $\sum_{E(Y)} N^2 = \sum_{I \subseteq Y} |I| N_I^2$ and $\text{good}(I) \geq c'_g |I|$.

Theorem 7.14 (Walls span the loop). *Let $k = 1$, let $n : V \rightarrow \mathbb{Z}$ be arbitrary, and $N(b) := \nabla n(b) - \xi_b$ on E . Then $\sum_{b \in E} N(b)^2 \geq \#\{b \in E : N(b) \neq 0\} \geq A_{G'}(C)$.*

Proof. Correcting Ξ by the boundary of the integer field gives a 2-chain spanning C , whose support outside the free set $\mathfrak{F}$ is exactly the set of wall bonds. With U as in the proof of Lemma 7.6 (c), $\Sigma := \Xi - \partial U$ is an integer 2-chain in Λ° with $\partial \Sigma = C$. The coefficient of p_b in Σ is $\xi_b - \nabla n(b) = -N(b)$ for $b \in E$, and all other plaquettes of Λ° are faces of hole cubes, frozen faces or faces in $\partial_2 \Lambda^\circ$, hence lie in $\mathfrak{F}$. So $\text{supp } \Sigma \setminus \mathfrak{F} = \{p_b : b \in E, N(b) \neq 0\}$, and Definition 2.11 gives $A_{G'}(C) \leq \#\{N \neq 0\} \leq \sum_E N^2$, since N is integer valued.

7.4 The fluctuation term F_2 is small

Throughout this subsection we assume $\lambda_{\text{PV}} \leq \frac{1}{4}$ and $\tilde{m} \leq M_Y \leq 1$ (true for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$), and put $R_s := (-\Delta + s)^{-1}$ for $s > 0$ and

$$r_\pm := \frac{1}{2} M_Y^2 (1 \pm (1 - 4\lambda_{\text{PV}}^2)^{1/2}), \quad \mathcal{G} := C_0 M_Y^{-2} (-\Delta + M_Y^2). \quad (7.16)$$

As $r_+ + r_- = M_Y^2$ and $r_+ r_- = \tilde{m}^2 M_Y^2$, partial fractions in $t = -\Delta$ give $C_0 = \varpi(R_{r_-} - R_{r_+})$ and $\mathcal{G} = c_+ R_{r_-} - c_- R_{r_+}$, with $\varpi := M_Y^2/(r_+ - r_-)$ and $c_\pm := r_\pm/(r_+ - r_-)$ (bounds in Section S1.6). For X a union of wall components and of $\mathcal{U}$ -cells outside $\hat{\Sigma}$, $F_2(X) := \langle \mathbf{1}_X \psi, F_2' \rangle = \sum_{\beta: S_\beta \subseteq X} \langle \psi, F_{2,\beta}' \rangle$ (Proposition 7.8 (e)).

Lemma 7.15 (Resolvent bounds). (a) (Combes–Thomas.) *Let $\rho : V \rightarrow \mathbb{R}$ satisfy $|\rho(x) - \rho(y)| \leq \alpha$ on every bond of G' , and let $\kappa_{\text{CT}} := s - 6(\cosh \alpha - 1) > 0$. Then $\|e^\rho R_s e^{-\rho}\| \leq \kappa_{\text{CT}}^{-1}$.* (b) *Let $B \subseteq E$ have endpoint set V_B , and let $A \subseteq V$ with $d(A, V_B) \geq r \geq 0$. Then $\|\mathbf{1}_A \mathcal{G} \nabla^* \mathbf{1}_B\| \leq 8.71 \tilde{m}^{-2} e^{-\tilde{m}r/3}$.* (c) *If $\chi : V \rightarrow [0, 1]$ is Λ_χ -Lipschitz for d , then $\|[C_0^{-1}, \chi]\| \leq \Lambda_\chi(144M_Y^{-2} + 6) \leq 150 \Lambda_\chi M_Y^{-2}$.*

Proof. All three are operator norms from the Schur test and the degree bound 6, so missing bonds enter only through the degree. (a) is the method of [CT73, Theorem 1, p. 257] in a lattice form not stated there: with $e^\rho(-\Delta + s)e^{-\rho} = -\Delta + s + \mathfrak{B}$, pairing the two orientations of each bond gives $\text{Re}\langle f, \mathfrak{B}f \rangle = -\sum_{\text{bonds}} 2(\cosh t - 1)\text{Re}(\bar{f}(x)f(y))$, $t := \rho(x) - \rho(y)$, which is at least $-6(\cosh \alpha - 1)\|f\|^2$. As $-\Delta \geq 0$, it follows that $\text{Re}\langle f, (-\Delta + s + \mathfrak{B})f \rangle \geq \kappa_{\text{CT}}\|f\|^2$. The antisymmetric part of $\mathfrak{B}$ drops out, so ρ may have kinks (as d has at slit edges and hole corners), and the second-order form matters: in (b), at $s = r_- \leq 1.08\tilde{m}^2$, $\alpha = \tilde{m}/3$, the first-order quantity $s - 6(e^\alpha - 1)$ is negative. (b) We apply (a) with $\rho := \frac{\tilde{m}}{3} \min\{d(\cdot, V_B), r\}$ to the two resolvents of $\mathcal{G}$, and the Schur test to $e^\rho \nabla^* e^{-\rho}$. (c) We use $[C_0^{-1}, \chi] = M_Y^{-2}(\Delta[\Delta, \chi] + [\Delta, \chi]\Delta) - [\Delta, \chi]$. The numerical constants are derived in Section S1.6.

Theorem 7.16 (F_2 is controlled by F_1 ; [GM82, Lemma 14] on G'). *For every $h \in \mathcal{H}$, $k \in \{0, 1\}$ and every union X of wall components and of $\mathcal{U}$ -cells outside $\hat{\Sigma}$, $\sum_{x \in X} F_2'(x)^2 = \sum_{\beta: S_\beta \subseteq X} \|F_{2,\beta}'\|^2 \leq \tilde{m}^2 \varepsilon(L') F_1^{\text{loc}}(X)$, where*

$$\varepsilon(L') := C_\varepsilon c_g^{-1} \lambda_{\text{PV}}^4 \tilde{m}^{-9} (\tilde{m}L/2)^{-1} (\tilde{m}L')^{-2} e^{-\tilde{m}L'/3}, \quad (7.17)$$

with the absolute constant C_ε of Section S1.6.

Proof. The idea is that the commutator is localized where χ_β changes, at distance at least $L'/2 - 1$ from the walls, and that there the Combes–Thomas decay outweighs the polynomial loss, because $\tilde{m}L' \propto \beta_{\text{GM}}$. By Proposition 7.8 (e), $F_{2,\beta}' = -[C_0^{-1}, \chi_\beta] \mathbf{1}_{A'} \mathcal{G} \nabla^* \mathbf{1}_B w_\beta$ with $A' := N_1(A_\beta)$, $B := \text{supp } w_\beta$, $d(A', V_B) \geq L'/2 - 2$ and $\Lambda_\chi = 4/L'$. We apply Lemma 7.15 (b), (c), and bound $\sum_\beta \|w_\beta\|^2$ by Theorem 7.13 with $\text{good}(I) \geq c_g'|I|$ (Section S1.6).

This is [GM82, Appendix D, (D.3) to (D.16)], modified as follows:

- (1) operator norms (Lemma 7.15) in place of GM's pointwise bounds [GM82, (D.10)], from [GM82, Lemma B.3], which fail within distance 2 of a missing bond, at the price $M_Y^{-4} \tilde{m}^{-4}$;
- (2) the source form of Proposition 7.8 (e) in place of [GM82, (D.3) to (D.5)], since h_β^e need not exist on G' ;
- (3) Theorem 7.13 in place of [GM82, Lemma 12, (C.21)], so there is no perimeter term P_F , and the bound is uniform in Λ° , C , Ξ , the hole set $\mathbb{X}$, S , h , k and X .

Corollary 7.17 (Gaussian F_2 factors). *Let C' be positive semidefinite on $\mathbb{R}^V$ with $\|C'\| \leq t_C \tilde{m}^{-2}$, and let $d\mu_{C'}$ be the centred Gaussian measure with covariance C' . For every $p > 0$ and every X as in Theorem 7.16,*

$$\int d\mu_{C'}(\psi) e^{-pF_2(X)} = \exp(\tfrac{1}{2}p^2 \langle \mathbf{1}_X F_2', C' \mathbf{1}_X F_2' \rangle) \leq \exp(\tfrac{1}{2}p^2 t_C \varepsilon(L') F_1^{\text{loc}}(X)).$$

Proof. Since $F_2(X) = \langle \psi, \mathbf{1}_X F_2' \rangle$ is linear in ψ , its Laplace transform is Gaussian. The bound then follows from $\langle F, C' F \rangle \leq \|C'\| \|F\|^2$ and Theorem 7.16. We will apply the corollary with C' of Lemma 7.29 ($C' \leq \alpha_D C_0 \leq \frac{7}{6} \tilde{m}^{-2}$, so $t_C = \frac{7}{6} < 2$) and with block-diagonal restrictions $\sum_j \mathbf{1}_{Y_j} C_0 \mathbf{1}_{Y_j}$ ($t_C = 1$).

Corollary 7.18 (Collar). *Let p_2, p_3, p', q' and c_λ be the Hölder exponents and the constant of Subsection 7.6, and $\varepsilon_0 := \min\{(1 - 1/p_3 - c_\lambda)/(2p_2), 2(p'/q')^2\}$ (values in Section S1.6). There is β_{GM}^0 such that $\varepsilon(L') \leq \varepsilon_0$ for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$.*

Proof. We have $\tilde{m}L' = 2\beta_{\text{GM}}$, while for $\varepsilon(L') \leq \varepsilon_0$ we need only $\tilde{m}L' \geq 27 \log(1/\tilde{m}) + O(1)$, with $\log(1/\tilde{m}) \leq 2.4944\tau$ and $K_{\text{col}} = 2 > 27 \cdot 2.4944/(4\pi^2) = 1.706$ (Section S1.6). A collar with $\tilde{m}L'$ fixed, as in [GM82, p. 47], does not suffice on G' .

Proposition 7.19 (Collar consistency). *Let $D_E := C_U \tilde{m}^{-1}$ bound the Euclidean diameter of $\mathcal{U}$ -cells (Theorem 3.20 (R7)) and $\eta_{\text{col}} := (2\sqrt{2}L + D_E + 3)/L'$. For every wall component S_β , with $n_I(\beta)$ the number of interfaces of S_β carrying a wall: (a) $F_1(S_\beta) \geq \frac{\pi^2 c'_g}{9} \frac{\tilde{m}^2 L^3}{\beta_{\text{GM}}} n_I(\beta) = 2F_{\text{unit}} n_I(\beta)$; (b) S_β has at most $n_I(\beta) \frac{4\pi}{3c_u} (\tilde{m}L')^3 (1 + \eta_{\text{col}})^3$ $\mathcal{U}$ -cells; (c) for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, every $\mathcal{U}$ -cell of $\hat{\Sigma}$ receives a share at least Φ_U of F_1^{loc} , that is, $F_1(S_\beta) \geq \Phi_U \cdot \#\{\mathcal{U}\text{-cells of } S_\beta\}$, where the constant Φ_U (Section S1.6) satisfies $\Phi_U \rightarrow \infty$ as $\beta_{\text{GM}} \rightarrow \infty$.*

Sketch. (a) is Theorem 7.13 with $Y = S_\beta$, $J_I^2 \geq 4\pi^2/\beta_{\text{GM}}$ and $\text{good}(I) \geq c'_g L^2$. (b) Each $\mathcal{U}$ -cell of S_β lies within d -distance L' of a wall interface, which lies in one block face, and $\mathcal{U}$ -cells are disjoint with at least $c_u \tilde{m}^{-3}$ sites (R7). (c) We divide (a) by (b). Then $\Phi_U \rightarrow \infty$, because the energy is exponential in β_{GM} , whereas the collar is polynomial in $\tilde{m}L' \propto \beta_{\text{GM}}$.

7.5 The class-graded expansion

By Corollary 7.10 (d), $Z(k, \Xi) = \sum_{[h]} \mathcal{N} \int d\mu_C e^E Q_h$ with $Q_h := e^{G(\psi+g)} e^{-F_1^{\text{loc}}(V)} e^{-F_2}$. We expand the Gaussian integral at fixed h and then resum the walls, graded by their class.

Definition 7.20 (Closures, wall polymers, seeds). Fix $h \in \mathcal{H}$. (a) The closure $\text{cl}(S)$ of a union S of $\mathcal{U}$ -cells is S together with the $\mathcal{U}$ -cells joined to S by a bond of G' . Closures of wall components are *linked* if they share a $\mathcal{U}$ -cell; a *wall polymer* $\mathcal{S}_\gamma(h)$ is the union of a maximal linked family of closures. (b) The wall polymers and the $\mathcal{U}$ -cells contained in no wall polymer form a partition $\tilde{\mathcal{Y}}(h)$ of V , whose members are the *elements*. A set is *admissible* (for h) if it is a union of elements; $\text{sat}_h(S)$ is the union of the elements that meet S . (c) X_1 is the union of the $\mathcal{U}$ -cells that contain an endpoint of a bond of G' crossing Ξ , and Δ_1 is a fixed $\mathcal{U}$ -cell. The *seeds* are $Y_1(h) := \text{sat}_h(X_1)$ in the expansion of $Z(1, \Xi)$ and $Y_1(h) := \text{sat}_h(\Delta_1)$ in that of $Z(0, \Xi)$. (d) $\mathfrak{M}(h) := \{X \supseteq Y_1(h) : X \text{ admissible}\}$. (e) For admissible Y , $G(Y) := \sum_{\Omega \subseteq Y} \ln r_\Omega(A_\Omega) + \sum_i G_{2,i}(Y)$ (the site sums of (7.5) restricted to Y) and $Q_h(Y) := e^{G(Y)} e^{-F_1^{\text{loc}}(Y)} e^{-F_2(Y)}$ ($F_2(Y)$ as in Subsection 7.4); $E(Y)$ and $E(Y, s)$ are defined in Appendix C.

An admissible set is a union of wall components and of $\mathcal{U}$ -cells outside $\hat{\Sigma}$ (a $\mathcal{U}$ -cell of $\hat{\Sigma}$ joined by a bond to S_β belongs to S_β). Hence $F_1^{\text{loc}}(Y)$ and $F_2(Y)$ are defined. An element that meets X_1 lies in each admissible $X \supseteq X_1$, since the elements partition V and admissible sets are unions of elements. Therefore $\mathfrak{M}(h) = \{X \supseteq X_1 : X \text{ admissible}\}$ (with Δ_1 for $k = 0$, as on [GM82, p. 51]), which is the rule of [GM82, p. 50].

Lemma 7.21 (Factorization). *Let Y, Y' be disjoint admissible sets. Then $Q_h(Y \cup Y') = Q_h(Y)Q_h(Y')$; in particular $Q_h = Q_h(X)Q_h(V \setminus X)$ for admissible X . Moreover $Q_h(Y)$, and every term of $E(Y, s)$, depend on (ψ, h) only through $\psi|_Y$ and $N|_{E(Y)}$. The same holds with $G^\natural$ of Lemma 7.29 in place of G .*

Sketch. This is [GM82, (E.4)] with admissible sets for unions of unit cubes. The action is a sum of cell terms ($\ln r_\Omega$), site terms ($G_{2,i}$, the terms of $G^\natural$ and of E) and wall-component terms ($F_1(S_\beta)$, $F'_{2,\beta}$), and an admissible set contains each cell and wall component entirely

or not at all. As $g + k\beta_{\text{GM}}^{-1/2}f = h + D$ with h cell-constant and $P\mathbb{Z}$ -valued, each phase is $e^{im\beta_{\text{GM}}^{1/2}(\psi+D)}$ (r_Ω is P -periodic), and $D|_Y$, $F_1^{\text{loc}}(Y)$, $F_2(Y)$ depend on h only through $N|_{E(Y)}$ (Proposition 7.8 (f), Corollary 7.10 (b)).

The factorization holds for admissible sets only (F_1^{loc} does not split inside a wall polymer). For this reason we saturate the seed, as X_1 itself need not be admissible.

Proposition 7.22 (Cluster expansion at fixed h ; [GM82, Lemma E.1, Proposition E.2] on G'). *Assume (S1) to (S5) and the hypotheses of Lemma C.2, and let $\mathbf{C} \in \{C, C'\}$ (C' of Lemma 7.29). Let $U = V$, or $U = V \setminus X_0$ for a union X_0 of $\mathcal{U}$ -cells; let h be a cell-constant configuration on U with values in $P\mathbb{Z}$ whose wall polymers in U have their closures in U (for $U = V$, any $h \in \mathcal{H}$); and let $Y_1 \subseteq U$ be nonempty and admissible for h on U (it may depend on h and need not be connected). With $\mathbf{C}_U := \mathbf{1}_U \mathbf{C} \mathbf{1}_U$ on $\mathbb{R}^U$,*

$$\int d\mu_{\mathbf{C}_U}(\psi) e^{E(U)} Q_h(U) = \sum_X \mathcal{K}_h(X) \int d\mu_{\mathbf{C}_U}(\psi) e^{E(U \setminus X)} Q_h(U \setminus X), \quad (7.18)$$

where X runs over the admissible subsets of U containing Y_1 , $\mathcal{K}_h(X) := \sum_{\bar{y}: X(\bar{y})=X} \mathcal{K}_{\bar{y}}$ over the sequences $\bar{y} = (Y_1, \dots, Y_n)$ of pairwise disjoint nonempty admissible sets with union $X(\bar{y}) = X$, and

$$\mathcal{K}_{\bar{y}} := \int_{[0,1]^{n-1}} ds \int d\mu_s^{\bar{y}}(\psi) e^{E_{\bar{y}}(X(\bar{y}), s)} \kappa(\bar{y}, s) Q_h(X(\bar{y})), \quad (7.19)$$

with $d\mu_s^{\bar{y}}$ (covariance (C.1) at $s_n = 0$), $E_{\bar{y}}$ and the operator $\kappa(\bar{y}, s)$ of Appendix C, built from $\mathbf{C}_U$. Every sequence has length at most $e_h + 1$, e_h the number of elements of U not contained in Y_1 ; all sums are finite, all integrals converge absolutely, and $\int d\mu_{\mathbf{C}_U} e^{E(Y)} Q_h(Y) \in (0, \infty)$ for every admissible $Y \subseteq U$. $\mathcal{K}_{\bar{y}}$ depends only on $\bar{y}$, the $X(\bar{y}) \times X(\bar{y})$ block of $\mathbf{C}$ and $N|_{E(X(\bar{y}))}$, not on U .

We give the proof in Appendix C.

Admissibility in terms of walls. For a set W of wall bonds with endpoint set $V(W)$, $\hat{\Sigma}(W)$ is the union of the $\mathcal{U}$ -cells Δ with $d(\Delta, V(W)) < L'$. For a union X of $\mathcal{U}$ -cells, ∂X is the set of bonds of G' joining X to $X^c := V \setminus X$, $W_X := \Sigma_E \cap E(X)$ and $W_{X^c} := \Sigma_E \cap E(X^c)$. We consider the conditions (in) $\text{cl}(\hat{\Sigma}(W_X)) \subseteq X$, (out) $\text{cl}(\hat{\Sigma}(W_{X^c})) \subseteq X^c$, and (∂) $N = 0$ on ∂X .

Lemma 7.23 (Admissibility). *Let $k \in \{0, 1\}$, $h \in \mathcal{H}$, and let X be a union of $\mathcal{U}$ -cells with $X \supseteq X_1$ if $k = 1$ and $X \supseteq \Delta_1$ if $k = 0$. Then $X \in \mathfrak{M}(h)$ if and only if (in), (out) and (∂) hold. In that case the elements of $\tilde{\mathcal{Y}}(h)$ inside X are determined by X and $N|_{E(X)}$, and those inside X^c by X and $N|_{E(X^c)}$.*

Sketch. Each wall component, with its closure, belongs to the side that contains its walls (Proposition 7.8 (a)), and no element straddles ∂X when ∂X carries no wall: under (∂) , $\hat{\Sigma} = \hat{\Sigma}(W_X) \cup \hat{\Sigma}(W_{X^c})$, no $\mathcal{U}$ -cell of one being adjacent to one of the other by (in), (out), and the wall components inside X are the components of $\hat{\Sigma}(W_X)$, with $W_X = \text{supp } N|_{E(X)}$.

At $k = 1$, since both ends of each Ξ -bond lie in $X_1 \subseteq X$, the set ∂X contains no Ξ -bond, and (∂) is equivalent to $h(x) = h(y)$ for all $\{x, y\} \in \partial X$, as at $k = 0$. By Lemmas 7.21 and 7.23, for admissible $X \supseteq X_1$ the activity $\mathcal{K}_h(X)$ with first set $\text{sat}_h(X_1)$ depends on h only through $N|_{E(X)}$.

Wall fields. A *wall field* on a union U of $\mathcal{U}$ -cells is a function $N : E(U) \rightarrow \mathbb{Z}$, extended by 0 to E , that vanishes on the bonds inside cells and is constant on each interface. For $h \in \mathcal{H}$ and $k \in \{0, 1\}$, the restriction $N(h)|_{E(U)}$ is a wall field (Lemma 7.6 (a)). The objects of Subsection 7.2 ($S_\beta, \chi_\beta, s_\beta, D, F_1(S_\beta), F_1^{\text{loc}}, F_{2,\beta}', F_2(Y)$) are functions of $w := 2\pi\beta_{\text{GM}}^{-1/2}N$ alone (Proposition 7.8 (f), Corollary 7.10 (b)). For a wall field N we define $G(Y)$, $G^\sharp(Y)$ (Lemma 7.29), $E(Y)$ and $E(Y, s)$ by their formulas with $\psi + g + k\beta_{\text{GM}}^{-1/2}f$ replaced by $\psi + D$ in all phases and all cell averages (so $A_\Omega + k\beta_{\text{GM}}^{-1/2}\bar{f}_\Omega$ becomes $\text{avg}_\Omega(\psi + D)$) and $\tilde{\delta}$ becomes $(\psi + D) - \text{avg}_\Omega(\psi + D)$, and put $Q_N(Y) := e^{G(Y)}e^{-F_1^{\text{loc}}(Y)}e^{-F_2(Y)}$.

Lemma 7.24 (Wall form). (a) If $N = N(h)|_{E(U)}$ with $h \in \mathcal{H}$ and $k \in \{0, 1\}$, then $Q_N(Y) = Q_h(Y)$, and $E(Y, s)$ is the same in both definitions, for every admissible $Y \subseteq U$. (b) Definition 7.20 applies, and Lemma 7.21, Proposition 7.22 and Lemma 7.23 hold, for every wall field N on $E(U)$ whose wall polymers have their closures in U , with V replaced by U and without the condition on the seed in Lemma 7.23; Theorem 7.13 holds for every wall field; and Proposition 7.5 holds with $\|\varphi + k\beta_{\text{GM}}^{-1/2}f - h\|$ replaced by $\|\psi + D\|$.

Sketch. (a) As in the proof of Lemma 7.21, h drops out. (b) These proofs use h only through (a) and the walls (the action consists of cell, site and wall-component terms). In Theorem 7.13, (7.14) becomes (7.13) without its last term, and $w = 0$ inside cells. The proof of Proposition 7.5 is pointwise in $\beta_{\text{GM}}^{1/2}(\psi + D)(x)$, and Appendix C uses Lemma C.1 and the Poincaré inequality on cells, pointwise in (ψ, D) (Section S2.5).

Definition 7.25 (Classes and outer partition functions). (a) The *m3-graph* has the sites of V as vertices, joined by the bonds of E and by the pairs of sites in a common cell; a closed walk γ in it has the length $|\gamma|$, the sum of the $d_{G'}$ -distances of its steps. The *class* of a function N on E (extended by 0 to the same-cell pairs) is $[N] : \gamma \mapsto \sum_{e \in \gamma} N(e)$ (oriented sum) on closed m3-walks. (b) A wall field N is *long* if $[N](\gamma) = 0$ for every closed m3-walk γ with $|\gamma| < 2L'$. $\mathcal{N}^{\text{long}}(U)$ is the set of long wall fields on $E(U)$, and $\Gamma_{\text{long}} := \{[N] : N \in \mathcal{N}^{\text{long}}(V)\}$; it is a group. (c) For a union X of $\mathcal{U}$ -cells and $p \in \Gamma_{\text{long}}$,

$$\mathcal{Z}_p(X) := \sum_N \mathcal{N} \int d\mu_C(\psi) e^{E(X^c)} Q_N(X^c), \quad (7.20)$$

the sum over the $N \in \mathcal{N}^{\text{long}}(X^c)$ that satisfy (out) and have $[N] = p$, with the integrand of Lemma 7.24; $\mathcal{Z}(X) := (\mathcal{Z}_p(X))_{p \in \Gamma_{\text{long}}}$, $\mathcal{Z}_p(X) := 0$ for $p \notin \Gamma_{\text{long}}$, $\mathcal{Z}_p := \mathcal{Z}_p(\emptyset)$ and $\mathcal{Z} := \mathcal{Z}_0$. (d) For $k \in \{0, 1\}$ and a union X of $\mathcal{U}$ -cells, with $X \supseteq X_1$ if $k = 1$, an *inside field* at coupling k is a wall field N_X on $E(X)$ that satisfies (in) and for which $[(N_X + k\xi)\mathbf{1}_{E(X)}]$ vanishes on the closed m3-walks in X ; its *outer class* is $p_k(N_X) := -[(N_X + k\xi)\mathbf{1}_{E(X)}]$.

Integrating along m3-paths, we see that $[N] = 0$ if and only if $N = \nabla n$ with $n : V \rightarrow \mathbb{Z}$ constant on each cell (on the cycles of G' alone n need not be constant on cells, which may be disconnected). With Lemmas 7.6 (d) and 7.24 (a) this gives $\mathcal{Z} = \mathcal{Z}(0, \Xi)$: the configurations of $\mathcal{Z}_0$ are the walls of the $[h] \in \mathcal{H}/\mathcal{C}_P$ at $k = 0$. A long wall field has period 0 around each plaquette of G' (a closed walk of length 4). For $X \supseteq X_1$, ξ vanishes off $E(X)$, so by Lemma 7.6 (d) an inside field at $k = 1$ has the periods $-\text{lk}(\cdot, C)$ on the cycles of G' in X (it is not long in general), and $p_1(N_X)(\gamma) = -\text{lk}(\gamma, C) - \sum_{e \in \gamma \cap E(X)} N_X(e)$ on the cycles of G' .

Lemma 7.26 (Long fields). (a) A closed $m\beta$ -walk γ with $|\gamma| < 2L'$ that contains a wall bond of a wall component S_β has all its sites in S_β . Hence, if $N \in \mathcal{N}^{\text{long}}(U)$ has its wall polymers' closures in U and $Y \subseteq U$ is a union of elements of N , then $N\mathbf{1}_{E(Y)}$ and $N\mathbf{1}_{E(U \setminus Y)}$ are long, and their classes add up to $[N]$. (b) Let $h \in \mathcal{H}$, $k = 1$ and $X \in \mathfrak{M}(h)$. Then $N(h)|_{E(X)}$ is an inside field at $k = 1$, $N(h)|_{E(X^c)}$ is long and satisfies (out), and $[N(h)|_{E(X^c)}] = p_1(N(h)|_{E(X)}) \in \Gamma_{\text{long}}$.

Proof. A closed walk of length less than $2L'$ stays at $d_{G'}$ -distance less than L' from each of its sites. (a) So all sites of γ lie in S_β (Proposition 7.8 (a)), all wall bonds of γ lie in $E(S_\beta)$, and S_β lies in one element. Thus the period of $N\mathbf{1}_{E(Y)}$ on γ is 0 or that of N , and N vanishes on the bonds between Y and $U \setminus Y$ (Lemma 7.23). (b) Put $N := N(h) = \nabla n - \xi$, $N_X := N|_{E(X)}$ and $N' := N|_{E(X^c)}$. By Lemma 7.23, N_X satisfies (in), N' satisfies (out), and $N = 0$ on ∂X . Since ξ vanishes off $E(X)$, we have $N_X + \xi\mathbf{1}_{E(X)} = \nabla n$ on $E(X)$ and $[N'] = -[N_X + \xi\mathbf{1}_{E(X)}] = p_1(N_X)$, as $[N + \xi] = 0$. A short closed walk through a site of X carries no wall of N' , by (out), and on one inside X^c the period of $N' = \nabla n$ is 0. Hence N' is long.

For unions $Z \subseteq Y$ of $\mathcal{U}$ -cells and a wall field N_Y on $E(Y)$ of which Z is a union of elements, we write $\mathcal{K}_{N_Y}^{(Z)}(Y)$ for the activity of Proposition 7.22 at N_Y with first set Z , and $\tilde{\mathcal{K}}(\Delta)$ for that of a wall-free $\mathcal{U}$ -cell Δ ($n = 1$). The finiteness of $\sum_{N_Y} |\mathcal{K}_{N_Y}^{(Z)}(Y)|$ and of $\sup_p \mathcal{Z}_p(X)$ used below is Proposition 7.47, proved without Theorem 7.27.

Theorem 7.27 (Resummation). Assume the hypotheses of Proposition 7.22. (a) For $X \supseteq X_1$ and an inside field N_X at $k = 1$, let $\mathcal{K}_{N_X}(X) := \mathcal{K}_{N_X}^{(\text{sat}(X_1))}(X)$. Then

$$Z(1, \Xi) = \sum_{X \supseteq X_1} \sum_{N_X} \mathcal{K}_{N_X}(X) \mathcal{Z}_{p_1(N_X)}(X),$$

the inner sum over the inside fields at $k = 1$, with absolutely convergent sums. (b) (Kirkwood-Salsburg identity.) Let $X_0 \subsetneq V$ be a union of $\mathcal{U}$ -cells and $\Delta \subseteq X_0^c$ a $\mathcal{U}$ -cell. For every $p \in \Gamma_{\text{long}}$,

$$\mathcal{Z}_p(X_0) = \sum_{Y \ni \Delta} \sum_{N_Y} \mathcal{K}_{N_Y}^{(\text{sat}(\Delta))}(Y) \mathcal{Z}_{p-[N_Y]}(X_0 \cup Y), \quad (7.21)$$

the sums over the unions $Y \subseteq X_0^c$ of $\mathcal{U}$ -cells containing Δ and the $N_Y \in \mathcal{N}^{\text{long}}(Y)$ satisfying (in), with first set $\text{sat}_{N_Y}(\Delta)$; the sums converge absolutely. The term $Y = \Delta$, $N_Y = 0$ is $\tilde{\mathcal{K}}(\Delta) \mathcal{Z}_p(X_0 \cup \Delta)$. (c) (Prescribed walls.) Let R be a family of disjoint unions $\mathcal{S}$ of $\mathcal{U}$ -cells, each with a long wall field $N_{\mathcal{S}}$ on $E(\mathcal{S})$ of which it is a wall polymer, $\sigma := (\bigcup R, \sum_{\mathcal{S}} N_{\mathcal{S}})$, and for $r \in \Gamma_{\text{long}}$ let $Z_r(R)$ be the part of Z_r from the configurations N in which every $\mathcal{S} \in R$, with $N|_{E(\mathcal{S})} = N_{\mathcal{S}}$, is an element. Then $Z_r(R) = \sum_{Y \supseteq \bigcup R} \sum_{N_Y} \mathcal{K}_{N_Y}^{(\bigcup R)}(Y) \mathcal{Z}_{r-[N_Y]}(Y)$, the sum over the $N_Y \in \mathcal{N}^{\text{long}}(Y)$ satisfying (in) with $N_Y|_{E(\bigcup R)} = \sum_{\mathcal{S}} N_{\mathcal{S}}$ of which $\bigcup R$ is a union of elements.

Proof. At fixed walls the expansion factorizes over the elements, admissibility splits into conditions on the two sides (Lemma 7.23), and the class of the walls is additive, so this class is the only datum of the outer side that the inside constrains. (a) By Corollary 7.10 (d) and Proposition 7.22 with $U = V$ and first set $\text{sat}_h(X_1)$, $Z(1, \Xi) = \sum_{[h]} \mathcal{N} \sum_{X \in \mathfrak{M}(h)} \mathcal{K}_h(X) \int d\mu_C e^{E(X^c)} Q_h(X^c)$. Fix $X \supseteq X_1$. The map $[h] \mapsto (N(h)|_{E(X)}, N(h)|_{E(X^c)})$ is a bijection from $\{[h] : X \in \mathfrak{M}(h)\}$ onto the pairs

(N_X, N') of an inside field at $k = 1$ and a configuration N' of $\mathcal{Z}_{p_1(N_X)}(X)$. It maps into this set by Lemma 7.26 (b), and it is injective by Lemma 7.6 (d). It is onto, since $N := N_X + N'$ has $[N + \xi] = [(N_X + \xi)\mathbf{1}_{E(X)}] + [N'] = 0$, hence $N = N(Pn)$ with n cell-constant, and (in), (out), (∂) give $X \in \mathfrak{M}(Pn)$ (Lemma 7.23). The activity depends only on N_X (Lemmas 7.21, 7.23). Since both ends of each Ξ -bond lie in $X_1 \subseteq X$, the outer integral depends only on N' and is the summand of (7.20) (Lemma 7.24 (a)). The outer integrals are ≥ 0 (Lemma 7.42), and Proposition 7.47 allows the sums to be exchanged. Only inside fields with $p_1(N_X) \in \Gamma_{\text{long}}$ occur, by Lemma 7.26 (b), and for them Proposition 7.47 gives $\sum_{N_X} |\mathcal{K}_{N_X}(X)| < \infty$. (b), (c) In (b) we fix a configuration N of $\mathcal{Z}_p(X_0)$ and expand on $U = X_0^c$ with first set $\text{sat}_N(\Delta)$ by Proposition 7.22 in wall form (Lemma 7.24 (b)). In (c) we fix a configuration N of $\mathcal{Z}_r(R)$ and expand in the same way on $U = V$ with first set $\bigcup R$, possibly disconnected. By Lemma 7.23 in wall form and Lemma 7.26 (a), the map $N \mapsto (Y, N\mathbf{1}_{E(Y)}, N\mathbf{1}_{E(U \setminus Y)})$ is a bijection onto the triples of Y , N_Y as stated and a configuration of $\mathcal{Z}_{p-[N_Y]}(X_0 \cup Y)$ in (b), and of $\mathcal{Z}_{r-[N_Y]}(Y)$ in (c). The summand factorizes (Lemma 7.21), and Proposition 7.47 allows the sums to be exchanged. For $Y = \Delta$ and $N_Y = 0$, the $\mathcal{U}$ -cell Δ is a wall-free element.

In (b) the removal of Y moves the class by $[N_Y]$, and each pair (X, p) , $X \neq \emptyset$, enters the identity for $(X \setminus \Delta, p)$ through the wall-free term $Y = \Delta$, the form of [BF80, (A4.3), (A4.5)]. The class cannot be dropped: removing the polymer of one of the two walls in the paragraph headed *Handles* (Section 7.1) leaves outer walls of nonzero class, which are not the walls of a height on the complement.

Corollary 7.28 (Loop expansion). *Under the hypotheses of Theorem 7.27, in the finite box,*

$$W_{G'}(C) = \sum_{X \supseteq X_1} \sum_{N_X} \mathcal{K}_{N_X}(X) \frac{\mathcal{Z}_{p_1(N_X)}(X)}{Z}, \quad (7.22)$$

the inner sum over the inside fields at $k = 1$, with absolutely convergent sums and $\mathcal{Z}_{p_1(N_X)}(X) \geq 0$.

Proof. Divide Theorem 7.27 (a) by $Z = Z(0, \Xi) > 0$, and use Lemma 7.42. Here $\mathcal{Z}_p(X)/Z$ is a ratio of two absolutely convergent sums in the finite box, which Theorem 7.49 only bounds.

7.6 Brydges–Federbush Section 9 on slit domains

In this subsection we bound the activities of Section 7.5 on G' by [BF80, Section 9] and [GM82, Section 7]. On G' the argument differs from these references in the following points. There is a one-body deficit, whose quadratic part is moved into the covariance (Lemma 7.29). There is no extension of the wall field across Ξ and no Poincaré inequality across a slit (Lemmas 7.32, 7.33). Finally, the covariance rate is small and independent of τ (Proposition 7.36).

Throughout, X is a polymer (a union of elements of $\tilde{Y}(h)$, Definition 7.20), $|X|$ its number of $\mathcal{U}$ -cells, $\vartheta := \psi + D$ with $D := g - H$ the field in the phases, and $\tilde{\delta} := P_X \vartheta$, where $P_X \vartheta := \vartheta - \text{avg}_\Omega \vartheta$ on each cell $\Omega \subseteq X$ and $P_X \vartheta := 0$ off X . We put $C_u := 2 + 6/c_g$, so that $\#\Delta < C_u \tilde{m}^{-3}$ for each $\mathcal{U}$ -cell Δ by (R7), and $\|A\|_r := \sup_x \sum_y |A(x, y)| e^{r d_{G'}(x, y)}$ for a matrix A on V and $r \geq 0$. Constants C, C_n depend only on those of Theorem 6.5 and Proposition 6.6.

The deficit decomposition. By Proposition 7.4 the inhomogeneity enters only through $G_{2,3}$ of (7.5), which by (S3) is $\sum_{m \neq 0} \sum_{x \in V} (\rho_1(m, x) - \rho^{\text{ref}}(m)) (\cos(m\beta_{\text{GM}}^{1/2}\vartheta(x)) - 1)$. Let $G_{2,3}^{\mathfrak{D}}$ be its part with $x \in \mathfrak{D}$, and put

$$\begin{aligned} \mu_{\mathfrak{D}}(x) &:= \beta_{\text{GM}} \sum_{m \neq 0} m^2 (\rho_1(m, x) - \rho^{\text{ref}}(m)) 1_{V \setminus \mathfrak{D}}(x), \\ G_{2,3}^{\natural} &:= \sum_{m \neq 0} \sum_{x \in V \setminus \mathfrak{D}} (\rho_1(m, x) - \rho^{\text{ref}}(m)) (\cos(m\beta_{\text{GM}}^{1/2}\vartheta(x)) - 1 + \tfrac{1}{2}m^2\beta_{\text{GM}}\psi(x)^2). \end{aligned} \quad (7.23)$$

Then, identically in ψ ,

$$G_{2,3} = G_{2,3}^{\mathfrak{D}} - \tfrac{1}{2} \langle \psi, \mu_{\mathfrak{D}} \psi \rangle + G_{2,3}^{\natural}. \quad (7.24)$$

We write $G_{2,3}^{\mathfrak{D}}(Y)$, $G_{2,3}^{\natural}(Y)$ for the sums over $x \in Y$. Put $G^{\natural} := G_1 + G_{2,1} + G_{2,2} + G_{2,3}^{\mathfrak{D}} + G_{2,3}^{\natural}$ and $C' := (C_0^{-1} + \nu + \mu_{\mathfrak{D}})^{-1}$, where the operator $\mu_{\mathfrak{D}}$ depends only on G' (through ρ_1).

Lemma 7.29 (Covariance step). *Assume the standing hypothesis (7.8), $\varepsilon_{\nu} + \theta' \leq 1/7$, with ε_{ν} of Section 7.1 (the output form of (S5)) and θ' from (7.7). Then, uniformly in ψ , h , k , Ξ and G' , with $C_{\mathfrak{D}}$ the constant C of Proposition 7.5(b): (a) $|\mu_{\mathfrak{D}}(x)| \leq \theta' \tilde{m}^2$ for every x ; for every union Y of $\mathcal{U}$ -cells, $-C(e^{-\mu} + e^{-2\kappa s}) c_{\text{dep}} N_{\mathfrak{D}}(Y) \leq G_{2,3}^{\mathfrak{D}}(Y) \leq C_{\mathfrak{D}} c_{\text{dep}} N_{\mathfrak{D}}(Y)$ and*

$$|G_{2,3}^{\natural}(Y)| \leq \tfrac{1}{2} \theta' \tilde{m}^2 \sum_{x \in Y \setminus \mathfrak{D}} (\vartheta(x)^2 + \psi(x)^2).$$

(b) *At a site $x \notin \mathfrak{D}$ with $D(x) = 0$, the n -th derivative in $\psi(x)$ of the x -term of $G_{2,3}^{\natural}$ is at most $C_n \theta' \tilde{m}^2 \beta_{\text{GM}} |\psi(x)|^{4-n}$ for $0 \leq n \leq 4$ and at most $C_n \theta' \tilde{m}^2 \beta_{\text{GM}}^{n/2-1}$ for $n \geq 4$. At $x \in \mathfrak{D}$, the n -th derivative ($n \geq 1$) of the x -term of $G_{2,3}^{\mathfrak{D}}$ is at most $C_n (\tilde{m}^2 / \beta_{\text{GM}}) \beta_{\text{GM}}^{n/2}$. (c) C' exists, is positive definite, and $C' \leq \alpha_D C_0$ as forms, with $\alpha_D := (1 - \varepsilon_{\nu} - \theta')^{-1} \leq 7/6$. (d) The bounds of Theorem 6.11 hold for C' and for every interpolated covariance C'_s of (C.2): $C'_s(x, x) = C'(x, x) \leq \alpha_D K_1 M_Y$, $|C'_s(x, y)| \leq 2K_1'' M_Y e^{-r_{\text{cov}} d_{G'}(x, y)}$, $\|C'_s\|_{r_{\text{cov}}} \leq 2K_3 \tilde{m}^{-2}$ and $\|C'_s\|_{2 \rightarrow 2} \leq \alpha_D \tilde{m}^{-2}$. (e) As measures on $\mathbb{R}^V$,*

$$e^{G(\psi+g)} d\mu_C(\psi) = \mathfrak{c} e^{G^{\natural}} d\mu_{C'}(\psi), \quad \mathfrak{c} := \left(\frac{\det C'}{\det C} \right)^{1/2}, \quad (7.25)$$

and $\mathfrak{c}$ does not depend on h , k or Ξ . (f) Propositions 7.3, 7.7 and 7.22, Theorem 7.27 and Corollary 7.28 hold with $(C, G, \mathcal{N})$ replaced by $(C', G^{\natural}, \mathfrak{c}\mathcal{N})$, and $W_{C'}(C)$ and the ratios $\mathcal{Z}_p(X)/Z$ are unchanged.

Proof. At a site with $D(x) = 0$, $\partial_{\psi(x)} G_{2,3} = -\mu_{\mathfrak{D}}(x)\psi(x) + O(\theta' \tilde{m}^2 \beta_{\text{GM}} |\psi(x)|^3)$ with $\theta' \geq 1 - e^{-\theta_0}$ fixed: no power of β_{GM} , whereas [BF80, (9.813) to (9.819)] need $\beta_{\text{GM}}^{1/2}$ per derivative of a one-body coefficient of order $\tilde{m}^2 / \beta_{\text{GM}}$. For this reason the quadratic part is moved into the covariance, as ν is, and $G_{2,3}^{\natural}$ is of fourth order (type vi of [BF80, (9.811)]). (a) Arguing as in the proof of Proposition 7.5(a), with m^2 replaced by $|m|^n$, we obtain, for $x \notin \mathfrak{D}$ and $n \geq 2$,

$$\beta_{\text{GM}} \sum_{m \neq 0} |m|^n |\rho_1(m, x) - \rho^{\text{ref}}(m)| \leq C_n \theta' \tilde{m}^2, \quad (7.26)$$

with $C_2 = 1$ once C in (7.7) is large, and at each x the left side is at most $C_n \tilde{m}^2$. For $n = 2$ this bounds $\mu_{\mathfrak{D}}$, and together with $|\cos y - 1 + \frac{1}{2}z^2| \leq \frac{1}{2}y^2 + \frac{1}{2}z^2$ ($y = m\beta_{\text{GM}}^{1/2}\vartheta(x)$, $z = m\beta_{\text{GM}}^{1/2}\psi(x)$)

it bounds $G_{2,3}^{\natural}$. The bound on $G_{2,3}^{\mathfrak{D}}(Y)$ is Proposition 7.5(b) with Proposition 6.17. (b) If $D(x) = 0$, then $\vartheta(x) = \psi(x)$ and $f(\psi) := \cos(m\beta_{\text{GM}}^{1/2}\psi) - 1 + \frac{1}{2}m^2\beta_{\text{GM}}\psi^2$ vanishes to fourth order with $|f^{(n)}| \leq |m|^n\beta_{\text{GM}}^{n/2}$. At $x \in \mathfrak{D}$, $|\partial_{\psi}^n \cos(m\beta_{\text{GM}}^{1/2}\vartheta)| \leq |m|^n\beta_{\text{GM}}^{n/2}$. We then sum with (7.26). (c) By the Schur test $\|\nu\|_{2 \rightarrow 2} \leq \|\nu\|_{r_{\text{cov}}} \leq \varepsilon_{\nu}\tilde{m}^2$, by (a) $\|\mu_{\mathfrak{D}}\|_{2 \rightarrow 2} \leq \theta'\tilde{m}^2$, and $C_0^{-1} = u^{-1} + \tilde{m}^2 \geq \tilde{m}^2$ (Definition 7.1). Hence $C_0^{-1} + \nu + \mu_{\mathfrak{D}} \geq (1 - \varepsilon_{\nu} - \theta')C_0^{-1} \geq \frac{6}{7}C_0^{-1} > 0$, and inversion reverses the order of positive definite forms. (d) Theorem 6.11 applies to $C' = (C_0^{-1} + \nu')^{-1}$, $\nu' := \nu + \mu_{\mathfrak{D}}$, as $2K_3\|\nu'\|_{r_{\text{cov}}} \leq 2K_3(\varepsilon_{\nu} + \theta')\tilde{m}^2 \leq \tilde{m}^2$ ($K_3 \leq 3.5$). The diagonal bound follows from (c). Block compressions keep the diagonal and positivity and increase no other bound. (e) By (7.24), $e^{G(\psi+g)} = e^{G^{\natural}}e^{-\frac{1}{2}\langle\psi, \mu_{\mathfrak{D}}\psi\rangle}$. Since $C^{-1} = C_0^{-1} + \nu$ (Proposition 7.7), we have $e^{-\frac{1}{2}\langle\psi, \mu_{\mathfrak{D}}\psi\rangle}d\mu_C(\psi) = (\det 2\pi C)^{-1/2}e^{-\frac{1}{2}\langle\psi, C'^{-1}\psi\rangle}d\psi = \mathfrak{c}d\mu_{C'}(\psi)$. The operators C_0 , ν , $\mu_{\mathfrak{D}}$ depend only on G' , M_Y and $\tilde{m}$, so the factor $\mathfrak{c}$ does not depend on h , k or Ξ . As an identity of measures, (7.25) preserves absolute integrability. (f) We insert (7.25) in Proposition 7.7. Those proofs use only factorization over cells and sites, the margin of Lemma C.2 ($q_M \leq 1/6$ for C') and the quadratic bound on $|E(X, s)|$, and $\mathfrak{c}$ cancels in $W_{G'}(C)$ and in each ratio $\mathcal{Z}_p(X)/Z$.

From now on we build all activities and all $\mathcal{Z}_p(X)$ with $(C', G^{\natural}, \mathfrak{c}\mathcal{N})$. For a polymer X , $\mathcal{N}(X)$ is the set of wall configurations $N_X := N|_{E(X)}$ of cell-constant labels h_X with values in $P\mathbb{Z}$ ($P = 2\pi\beta_{\text{GM}}^{-1/2}$) whose walls lie in $E(X)$. At $k = 0$ the statements of this subsection and Lemma 7.48 hold for the long wall fields of Definition 7.25 (Proposition 7.46). We write $n_I(X)$ for the number of interfaces with both cells in X , and put $A_w := 2F_{\text{unit}}$.

Lemma 7.30 (Wall entropy). (a) $F_1^{\text{loc}}(Y) \geq A_w \sum_{I \subseteq Y} N_I^2$ for every union Y of elements of $\tilde{Y}(h)$, where N_I is the constant value of N on the interface I (Lemma 7.6(a)). (b) For every polymer X and every $c_4 > 0$ with $e^{-c_4 A_w} \leq \frac{1}{3}$, $\sum_{N_X \in \mathcal{N}(X)} e^{-c_4 F_1^{\text{loc}}(X)} \leq (1 + 3e^{-c_4 A_w})^{n_I(X)} \leq e^{c'_4 |X|}$ with $c'_4 := 9C_u e^{-c_4 A_w} / (c_g c_1 (\tilde{m}L)^3)$.

Proof. We adapt the proof of [BF80, Lemmas 9.2, 9.3] to G' as follows. (1) The face sum of [BF80, (9.117)] becomes the interface sum. Indeed, walls are supported on interfaces (Lemma 7.6(a)), and by Theorem 7.13 with $\text{good}(I) \geq c'_g L^2$, which holds by (R8) in fine units and Lemma 4.19, an interface $I \subseteq Y$ gives at least $\frac{1}{18}\tilde{m}^2 \frac{L}{2} c'_g L^2 \cdot 4\pi^2 N_I^2 / \beta_{\text{GM}} = 2F_{\text{unit}} N_I^2$. (2) The sum over h becomes the injection $N_X \mapsto (N_I)_I \in \mathbb{Z}^{n_I(X)}$, each interface giving $\sum_n e^{-c_4 A_w n^2} \leq 1 + 3e^{-c_4 A_w}$. (3) The cube count becomes $n_I(X) \leq 3C_u |X| / (c_g c_1 (\tilde{m}L)^3)$, since a cell has at least $c_1 L^3$ sites by (R5) and at most $6/c_g$ neighbours by (R8), and a $\mathcal{U}$ -cell has fewer than $C_u \tilde{m}^{-3}$ sites. We also use $\log(1 + 3t) \leq 3t$. By Corollary 7.10(d), no perimeter term arises. With $c_4 = c'/4$ (Table S1.4), c'_4 is doubly exponentially small in τ , as $A_w \geq (\pi^2 c'_g / 72) c_L^3 (\tilde{m}\beta_{\text{GM}})^{-1}$ ($\tilde{m}L \geq c_L/2$).

In Debye units, for the Peierls factor r_{Ω} of (7.4), we put $v_{\Omega} := \tilde{m}^3 |\Omega|$, $\hat{\rho}(m) := \tilde{m}^{-3} \rho^{\text{ref}}(m)$, $b_D := \beta_{\text{GM}} \tilde{m}$, $\hat{P} := 2\pi b_D^{-1/2}$, and

$$\hat{R}_v(x) := \frac{\exp\{v \sum_m \hat{\rho}(m) [e^{imb_D^{1/2}x} - 1]\}}{\sum_{n \in \mathbb{Z}} \exp\{-\frac{1}{2}v(x - n\hat{P})^2\}}, \quad (7.27)$$

so that $r_{\Omega}(A) = \hat{R}_{v_{\Omega}}(\tilde{m}^{-1/2}(A + k\beta_{\text{GM}}^{-1/2}\bar{f}_{\Omega}))$. By (R5) in fine units and $c_L/2 \leq \tilde{m}L \leq c_L$, we have $v_{\Omega} \in [v_-, v_+] := [c_1 c_L^3 / 8, 8c_L^3]$. Let $\eta_* := \sup_{0 < s \leq \pi} [1 - 2b_D \sum_m \hat{\rho}(m)(1 - \cos ms)/s^2] \leq 1 - 4/\pi^2 + C_S e^{-2\kappa_S}$.

Lemma 7.31 (Derivative bounds). *Fix $\eta_H \in (\eta_*, 1)$. There are $b_0 > 0$ and c_7, c'_7, c''_7 , depending only on $v_\pm, \eta_H, C_S$ and κ_S , such that for $b_D \leq b_0$, $v \in [v_-, v_+]$, all real x and all integers $N \geq 0$,*

$$|\partial_x^N \hat{R}_v(x)| \leq c_7 (c'_7 b_D^{1/6})^N e^{c''_7 N \log N} e^{\frac{1}{2} \eta_H v x^2}.$$

Consequently, uniformly in h , with $\partial A_\Omega / \partial \psi(x) = 1/|\Omega|$ ($x \in \Omega$),

$$|\partial_A^N r_\Omega(A_\Omega)| \leq c_7 (c'_7 b_D^{1/6} \tilde{m}^{-1/2})^N e^{c''_7 N \log N} \exp\left(\frac{1}{2} \eta_H \tilde{m}^2 \sum_{x \in \Omega} \vartheta(x)^2\right).$$

The first bound is proved as [BF80, Lemma 9.7, (9.51)], with $\tilde{l}_D = 1$, $L^3 \rightarrow v$, $\beta \rightarrow b_D$, $\rho_i \rightarrow \hat{\rho}(m)$, $\tau \rightarrow \hat{P}$, except for the following points. (1) There are infinitely many charges. The Taylor expansions of the factors I, II of [BF80, (9.53)] converge for $|\operatorname{Im} x| \leq b_D^{-1/6}$ by the tails of (S4'), as $e^{|m|b_D^{1/3}} e^{-2\kappa_S|m|}$ is summable. (2) The constants are uniform in $v \in [v_-, v_+] \subset (0, \infty)$, not for a fixed volume L^3 . (3) We need an explicit exponent for $\alpha_D p' \eta_H < 1$ in (L8-1). The inequalities $1 - \cos s \geq 2s^2/\pi^2$ on $[0, \pi]$ and $2\beta_{\text{GM}} \rho^{\text{ref}}(1) \geq (1 - Ce^{-2\kappa_S}) \tilde{m}^2$ bound η_* , and each $\eta_H \in (\eta_*, 1)$ is admissible in [BF80, (9.59)]. The second bound follows by $\hat{P}$ -periodicity, $A_\Omega + k\beta_{\text{GM}}^{-1/2} \tilde{f}_\Omega = h_\Omega + \text{avg}_\Omega \vartheta$ with $h_\Omega \in P\mathbb{Z}$, and $|\Omega|(\text{avg}_\Omega \vartheta)^2 \leq \sum_{x \in \Omega} \vartheta(x)^2$. With $\gamma := \eta_H \tilde{m}^2$ it is the analogue on G' of [GM82, (7.9), (7.10)]: each derivative of G_1 costs $b_D^{1/6}$, times $e^{\frac{1}{2}\gamma \sum_X \vartheta^2}$.

Hölder exponents and the hypotheses (L8-1) to (L8-4). We fix $\eta_H \in (\eta_*, 1)$, Hölder exponents $p_1, \dots, p_4 > 1$ with $\sum_i 1/p_i = 1$ and p_1 even, and $p' > p_3$, q' with $1/p' + 1/q' = 1/p_3$ (values in Section S1.7.1). Let C_W^f , $k_\square = r_\square L$, N_{ov} and $\mathfrak{R}_\Omega^b \supseteq \Omega$, with sites in the box $\hat{Q}(Q)$ of fine side at most $k_\square$, be those of the cell Poincaré inequality, Proposition 6.9(iv), and $C'_P := C_W^f r_\square^2 N_{\text{ov}}$, so that $C'_P L^2 = C_W^f k_\square^2 N_{\text{ov}}$. The hypotheses are: (L8-1) the standing hypothesis (7.8), and $\hat{\eta} := \alpha_D p' \eta_H < 1$ (hence $\alpha_D p_3 \eta_H < 1$); (L8-2) $C'_P (\tilde{m}L)^2 \leq \eta_H/8$, so that $\theta_P := (4\hat{\eta}/\eta_H) C'_P (\tilde{m}L)^2 \leq \frac{1}{2}$ and $\nu_0 := \max(\hat{\eta}, \theta_P) < 1$; (L8-3) $\varepsilon(L') \leq 2(p'/q')^2$, with $\varepsilon(L')$ of Theorem 7.16; (L8-4) each $\mathfrak{R}_\Omega^b$ has $d_{G'}$ -diameter less than L' . Here (L8-1) holds at the values of Section S1.7.1, (L8-2) by the choice of c_L ($\tilde{m}L \leq c_L$), (L8-3) by Corollary 7.18, and (L8-4) follows from (L8-2) by Proposition 6.9(v). Here $\hat{\eta}$ is the constant denoted $\hat{\beta} < 1$ in [BF80, (9.638)]. A cell $\Omega \subseteq Q$ is *near* if the ℓ^∞ -distance from the sites of $\hat{Q}(Q)$ to C is at most $\frac{1}{2}$, and *far* otherwise. For a union Y of elements, $E(\mathcal{N}(Y))$ is the union of the $E(\mathfrak{R}_\Omega^b)$ over the near cells $\Omega \subseteq Y$.

The free-space solid angle. Let $\Omega_\Xi(y) \in (-2\pi, 2\pi)$ be the signed solid angle of the flat rectangle Ξ seen from a point y off its plane, increasing by 4π when y crosses Ξ in the direction in which $\xi_b = +1$. For a site x of G' (a dual site of $\mathbb{Z}^3$, never in the plane of Ξ) we put $f^C(x) := \frac{1}{2} \Omega_\Xi(x)$ and, for each dual bond $b = (x, x')$ of $\mathbb{Z}^3$, $R^C(b) := f^C(x') - f^C(x) - j(b)$, where $j(b) = 2\pi \xi_b$, which on the bonds of G' is the jump field of Ξ (Section 7.1). The field f^C depends only on C , not on G' , unlike the Neumann solid angle $f_{G'}$ of Lemma 7.2, and R^C differs from R_f . Put $S(D) := \sum_{b: d(b, C) \leq D} |R^C(b)|^2$ (sum over dual bonds of $\mathbb{Z}^3$) and $D_{\mathcal{N}} := \sqrt{3} k_\square + 1$.

Lemma 7.32 (Solid-angle primitive). *(a) Assume (L8-4). Let Y be a union of elements of $\tilde{Y}(h)$ and $\Omega \subseteq Y$ a cell. For every $\vartheta \in \mathbb{R}^V$, every function a on the sites of $\mathfrak{K}_\Omega^b$ and every $\varepsilon \in (0, 1]$,*

$$\begin{aligned} \|P_\Omega \vartheta\|^2 &\leq (1 + \varepsilon) C_W^f k_\square^2 \|\nabla \vartheta + w_Y\|_{E(\mathfrak{K}_\Omega^b)}^2 + (1 + 1/\varepsilon) k^2 \beta_{\text{GM}}^{-1} \delta_\Omega(a)^2, \\ \delta_\Omega(a) &:= (C_W^f)^{1/2} k_\square \|\nabla a - j\|_{E(\mathfrak{K}_\Omega^b)} + \|P_\Omega a\|_\Omega, \end{aligned}$$

where $w_Y := \sum_{S_\beta \subseteq Y} w_\beta$. If Ω is far, there is an a with $\delta_\Omega(a) = 0$. (b) For every dual bond b of $\mathbb{Z}^3$, $d(b, C) \geq \frac{1}{2}$ and $|R^C(b)| \leq 2\pi/d(b, C)$; at every dual site, $|f^C| < \pi$. (c) On every output G' , $\sum_{\Omega \text{ near}} \|R^C\|_{E(\mathfrak{K}_\Omega^b)}^2 \leq N_{\text{ov}} \|R^C\|_{E(\mathcal{N}(V))}^2 \leq N_{\text{ov}} S(D_{\mathcal{N}})$, and if $|C| \geq 8k_\square$, then

$$S(D_{\mathcal{N}}) \leq C_R |C| (\log k_\square + C_{\text{sh}}), \quad C_R := 24\pi^3, \quad C_{\text{sh}} := 92.$$

Proof. Heuristically, on a box crossed by C the trivial primitive pays for a flat sheet of jumps (energy $\asymp k_\square^2$), while the solid angle pays only for the Biot–Savart residual (energy $\asymp k_\square \log k_\square$ per unit length of C). Since freezing only deletes terms of free-space ℓ^2 sums, the Neumann field and its slit-edge enhancement never enter.

(a) We first show that $w = w_Y$ on $E(\mathfrak{K}_\Omega^b)$. Let $b' \in E(\mathfrak{K}_\Omega^b)$ be a wall bond. It belongs to one component $w_{\beta'}$ and has a wall site z as an endpoint. Fix $x \in \Omega$. Then $d_{G'}(x, z) < L'$ by (L8-4), so each $\mathcal{U}$ -cell met by a G' -path of length $d_{G'}(x, z)$ from x to z lies in $\hat{\Sigma}$, consecutive ones joined by a bond, and the $\mathcal{U}$ -cell of x lies in $S_{\beta'}$ (Proposition 7.8). This $\mathcal{U}$ -cell also lies in the element of Y containing Ω , and each S_β lies in one element (Definition 7.20). Hence $S_{\beta'} \subseteq Y$. Other bonds carry $w = 0 = w_Y$. Next put $a_\Omega := h - k\beta_{\text{GM}}^{-1/2}a$ on the sites of $\mathfrak{K}_\Omega^b$. Here, at $k = 0$, for long wall fields, the label is replaced by the primitive of Lemma 7.45 (Proposition 7.46), and at $k = 1$, for the inside fields of outer class in Γ_{long} , by the primitive constructed in the proof of Theorem 7.57. As h is constant on cells, $P_\Omega h = 0$ and $P_\Omega \vartheta = P_\Omega(\vartheta + a_\Omega) + k\beta_{\text{GM}}^{-1/2} P_\Omega a$. On $E(\mathfrak{K}_\Omega^b)$, since $w = \nabla h - k\beta_{\text{GM}}^{-1/2}j$, we have $\nabla(\vartheta + a_\Omega) = \nabla \vartheta + w_Y + k\beta_{\text{GM}}^{-1/2}(j - \nabla a)$. The Poincaré inequality of Proposition 6.9(iv) for $\vartheta + a_\Omega$ on $\mathfrak{K}_\Omega^b$ and the triangle inequality give $\|P_\Omega \vartheta\| \leq (C_W^f)^{1/2} k_\square \|\nabla \vartheta + w_Y\|_{E(\mathfrak{K}_\Omega^b)} + k\beta_{\text{GM}}^{-1/2} \delta_\Omega(a)$. We then square, using $(x + y)^2 \leq (1 + \varepsilon)x^2 + (1 + 1/\varepsilon)y^2$. Now let $\Omega \subseteq Q$ be far, and let Π be the plane of Ξ . A dual bond crossing Π crosses it at the centre of a plaquette, and $j(b) \neq 0$ iff that plaquette lies in Ξ . The plaquettes crossed by bonds of $\hat{Q}(Q)$ form a rectangle. If this rectangle contained plaquettes in and outside Ξ , it would contain two sharing an edge $e \in \partial \Xi = C$ at ℓ^∞ -distance $\frac{1}{2}$ from a site of $\hat{Q}(Q)$, which is excluded for a far cell. Hence $j = 2\pi\sigma \nabla 1_{\Pi^+}$ on $E(\hat{Q}(Q))$ for some $\sigma \in \{-1, 0, 1\}$, where Π^+ is the set of sites on one side of Π . With this σ , the function $a := 2\pi\sigma 1_{\Pi^+}$ has $\nabla a = j$ on $E(\mathfrak{K}_\Omega^b)$ and is constant on Ω , as Q lies on one side of Π (Lemma 3.5, (A3)). Thus $\delta_\Omega(a) = 0$.

(b) Primal edges are at distance at least $\frac{1}{2}$ from dual bonds. Off C , $\nabla \Omega_\Xi$ extends continuously across Ξ as the Biot–Savart field of C . Hence $|R^C(b)| \leq \frac{1}{2} \max_{y \in b} J(y)$, where $J(y) := \oint_C |dl| |y - l|^{-2}$. Let $y = y' + he$ with $y' \in \Pi$ and $h \geq 0$. The length of $\{l \in C : |y' - l| \leq t\}$ vanishes for $t < d(y', C)$ and is at most $2\pi t$ (it lies in the boundary of a convex subset of a disc of radius t). Integration by parts and the substitution $t = h \tan \varphi$ give $J(y) \leq 4\pi/d(y, C)$. Finally, $|\Omega_\Xi| < 2\pi$, since the radial projection of the rectangle from a point off Π lies in an open hemisphere.

(c) For a near cell $\Omega \subseteq Q$, the bonds of $\mathfrak{R}_\Omega^b$ are dual bonds of $\mathbb{Z}^3$ joining sites of $\hat{Q}(Q)$, a box of diameter at most $\sqrt{3}k_\square$ within Euclidean distance $\frac{\sqrt{3}}{2}$ of C , so $d(b, C) \leq D_{\mathcal{N}}$. Each $|R^C(b)|^2$ is a fixed nonnegative number attached to a dual bond of $\mathbb{Z}^3$, independent of G' . As noted at the start of the proof, the passage from $\mathbb{Z}^3$ to G' only deletes terms: freezing a plaquette deletes a bond, and a hole deletes sites and bonds. With the overlap N_{ov} this gives the first two inequalities. For the last one, we use the number $\mathcal{N}(r)$ of dual bonds with $d(b, C) \leq r$, which vanishes for $r < \frac{1}{2}$. The midpoints of these bonds lie within $r + \frac{1}{2}$ of C and form three translates of $\mathbb{Z}^3$. Hence, with $r_* := \frac{1}{2} + \frac{\sqrt{3}}{2}$, $\mathcal{N}(r) \leq 3|\{y : d(y, C) \leq r + r_*\}| \leq 3\pi|C|(r + r_*)^2 + 16\pi(r + r_*)^3$. By (b) and summation by parts, $\sum_{d(b, C) \leq D} d(b, C)^{-2} = \mathcal{N}(D)D^{-2} + 2 \int_{1/2}^D \mathcal{N}(r)r^{-3}dr$. The term $3\pi|C|r^2$ gives $6\pi|C|\log D + \mathcal{O}(|C|)$, and the other terms give $\mathcal{O}(|C| + D)$. The factor $(2\pi)^2$ of (b) gives $C_R = (2\pi)^2 \cdot 6\pi = 24\pi^3$. At $D = D_{\mathcal{N}} \leq 2k_\square$ and $|C| \geq 8k_\square$ the remaining constants add up to $C_{\text{sh}} = 92$ (Section S1.7.2).

For a polymer X put

$$B_X(\vartheta) := \frac{1}{2}\tilde{m}^2 \sum_{x \in X} \vartheta(x)^2 + \frac{2\tilde{m}^2}{\eta_H} \sum_{x \in X} (P_X \vartheta)(x)^2, \quad T_3 := \left[\int d\mu_{C'}(\psi) e^{p_3 \eta_H B_X(\psi + D)} \right]^{1/p_3}.$$

With $\gamma = \eta_H \tilde{m}^2$ we have $\frac{1}{2}p_3\gamma \sum_X \vartheta^2 + 2p_3\tilde{m}^2 \sum_X \tilde{\delta}^2 = p_3\eta_H B_X(\vartheta)$. Hence T_3 is the third factor of the Hölder split (Proposition 7.35), and it is the analogue on G' of [BF80, (9.64)]. The near-cell factor is

$$\mathbf{n}'(Y) := 2k^2\beta_{\text{GM}}^{-1} \left[\theta_P \|R^C\|_{E(\mathcal{N}(Y))}^2 + \frac{4\hat{\eta}\tilde{m}^2}{\eta_H} \sum_{\Omega \subseteq Y \text{ near}} \|P_\Omega f^C\|_\Omega^2 \right].$$

Lemma 7.33 (Untranslation). *Assume (L8-1) to (L8-4). For every h , every polymer X and every s ,*

$$T_3 \leq \exp \left\{ \frac{c_8 |X| + \mathbf{n}'(X)}{p'} + \frac{F_1^{\text{loc}}(X)}{p_3} \right\},$$

with the constant c_8 of Section S1.7.3, and $\mathbf{n}' = 0$ for $k = 0$. If $\min(R, T) \geq R_1(\beta) = (2r_\square + 5)L$, then $\mathbf{n}'(V) \leq c_{\text{near}}|C|$, with the constant c_{near} of Section S1.7.3, and uniformly for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$,

$$c_{\text{near}} \leq 0.1264 \theta_P C_R + \frac{2\theta_P C_R C_{\text{sh}} + 1024\pi^2 \hat{\eta} (2r_\square + 5)^2 c_L^2 / (\eta_H c_1)}{\beta_{\text{GM}}}.$$

Proof. We absorb the shift D by a translation, which turns $\frac{1}{2}\langle \psi, C_0^{-1} \psi \rangle + F_2 + F_1^{\text{loc}}$ into one quadratic form of $\psi + D$ (“applying (7.8) to (7.10) backwards”, [BF80, p. 226], in local form). Lemma 7.32 bounds the supremum of the exponent, and only near cells leave a remainder.

Step 1 (local untranslation identity). Let Y be a union of elements, D_Y, w_Y, s_Y the sums of $D_\beta, w_\beta, s_\beta$ over $S_\beta \subseteq Y$, and $\mathcal{Q}_Y(\vartheta) := \tilde{m}^2 \|\vartheta\|^2 + \|\nabla \vartheta + w_Y\|^2 + M_Y^{-2} \|\Delta \vartheta - \nabla^* w_Y\|^2$. Then for all $\psi \in \mathbb{R}^V$

$$\frac{1}{2}\langle \psi, C_0^{-1} \psi \rangle + F_2(Y) + F_1^{\text{loc}}(Y) = \frac{1}{2} \mathcal{Q}_Y(\psi + D_Y). \quad (7.28)$$

Indeed, since $C_0^{-1} = \tilde{m}^2 - \Delta + M_Y^{-2} \Delta^2$ and $-\Delta = \nabla^* \nabla$, the right side is $\frac{1}{2}\langle \psi, C_0^{-1} \psi \rangle + \langle \psi, C_0^{-1} D_Y + s_Y \rangle + \frac{1}{2} \mathcal{Q}_Y(D_Y)$ with $s_Y = M_Y^{-2} (-\Delta + M_Y^2) \nabla^* w_Y$. By Proposition 7.8, $D_\beta = -\chi_\beta C_0 s_\beta$ and $(1 - \chi_\beta) s_\beta = 0$, so $C_0^{-1} D_\beta + s_\beta = -[C_0^{-1}, \chi_\beta] C_0 s_\beta = F'_{2,\beta}$, supported in S_β . Therefore the cross term is $F_2(Y)$. Since the functions $D_\beta, \nabla D_\beta, w_\beta, \Delta D_\beta, \nabla^* w_\beta$ are supported

in S_β or $E(S_\beta)$, terms of different β have disjoint supports, and $\frac{1}{2}\mathcal{Q}_Y(D_Y) = \sum_{S_\beta \subseteq Y} F_1(S_\beta) = F_1^{\text{loc}}(Y)$ (Proposition 7.9, Corollary 7.10).

Step 2 (the Gaussian integral at fixed Y). We show

$$J_Y := \int d\mu_{C_0}(\psi) \exp\{\hat{\eta}B_Y(\psi + D) - F_1^{\text{loc}}(Y) - F_2(Y)\} \leq \exp\{c_8|Y| + \mathbf{n}'(Y)\}. \quad (7.29)$$

Each S_β lies in Y or is disjoint from it, so $D = D_Y$ on Y . Put $N := \hat{\eta}\tilde{m}^2 1_Y + (4\hat{\eta}\tilde{m}^2/\eta_H)P_Y$, so that $\hat{\eta}B_Y(\vartheta) = \frac{1}{2}\langle\vartheta, N\vartheta\rangle$ ($P_Y = 1_Y P_Y 1_Y$ is an orthogonal projection). By (7.28) and $\vartheta = \psi + D_Y$, J_Y is the ratio of the integrals over $\mathbb{R}^V$ of $\exp\{-\frac{1}{2}\mathcal{Q}_Y(\vartheta) + \frac{1}{2}\langle\vartheta, N\vartheta\rangle\}$ and of $e^{-\frac{1}{2}\langle\psi, C_0^{-1}\psi\rangle}$, with quadratic part $-\frac{1}{2}\langle\vartheta, (C_0^{-1} - N)\vartheta\rangle$. For positivity, we use [BF80, (9.637), (9.640) to (9.642)], with the cell Poincaré inequality, summed with overlap N_{ov} ($\|P_Y\vartheta\|^2 \leq C'_P L^2 \|\nabla\vartheta\|^2$), for the Neumann Laplacian of the cubes in (9.642). We obtain $\langle\vartheta, N\vartheta\rangle \leq \hat{\eta}\tilde{m}^2 \|\vartheta\|^2 + \theta_P \|\nabla\vartheta\|^2 \leq \nu_0 \langle\vartheta, C_0^{-1}\vartheta\rangle$ ($C_0^{-1} \geq \tilde{m}^2 - \Delta$), so $K := C_0^{1/2} N C_0^{1/2}$ satisfies $0 \leq K \leq \nu_0 < 1$. Hence $J_Y = \det(1 - K)^{-1/2} \exp\{\sup_\vartheta [-\frac{1}{2}\mathcal{Q}_Y(\vartheta) + \frac{1}{2}\langle\vartheta, N\vartheta\rangle]\}$.

We first bound the supremum. For a far cell we apply Lemma 7.32(a) with $\delta_\Omega(a) = 0$ and $\varepsilon \rightarrow 0$, and for a near cell with $\varepsilon = 1$, $a = f^C$ and $\delta_\Omega(f^C)^2 \leq 2C_W^f k_\square^2 \|R^C\|_{E(\mathbb{R}_\Omega^b)}^2 + 2\|P_\Omega f^C\|_\Omega^2$. Summing over the cells of Y with overlap N_{ov} and multiplying by $2\hat{\eta}\tilde{m}^2/\eta_H$ gives, as $(2\hat{\eta}\tilde{m}^2/\eta_H) \cdot 2C_W^f k_\square^2 N_{\text{ov}} = \theta_P$, $\frac{1}{2}\langle\vartheta, N\vartheta\rangle \leq \frac{1}{2}\hat{\eta}\tilde{m}^2 \|\vartheta\|_Y^2 + \theta_P \|\nabla\vartheta + w_Y\|^2 + \mathbf{n}'(Y)$. Since $\mathcal{Q}_Y(\vartheta) \geq \tilde{m}^2 \|\vartheta\|^2 + \|\nabla\vartheta + w_Y\|^2$, the exponent is at most $-\frac{1}{2}(1 - \hat{\eta})\tilde{m}^2 \|\vartheta\|^2 - (\frac{1}{2} - \theta_P) \|\nabla\vartheta + w_Y\|^2 + \mathbf{n}'(Y) \leq \mathbf{n}'(Y)$.

For the determinant, we have $-\frac{1}{2}\log \det(1 - K) \leq \text{tr } K/(2(1 - \nu_0))$ [BF80, Appendix 3], and $\text{tr } K = \text{tr}(N C_0) \leq \hat{\eta}(1 + 4/\eta_H)\tilde{m}^2 K_1 M_Y \#Y \leq 2(1 - \nu_0)c_8|Y|$ by $P_Y \leq 1_Y$, Theorem 6.11, $M_Y = \tilde{m}/\lambda_{\text{PV}}$ and $\#Y < C_u \tilde{m}^{-3}|Y|$.

Step 3 (assembly). We follow [BF80, (9.65) to (9.626)], adapted to G' as follows. (1) The convexity step [BF80, (9.67) to (9.611)] (Table S1.3, S3) is applied to the block compressions $C'_\epsilon = \sum_J 1_{Y_J} C'_J 1_{Y_J}$ of (C.2), $C'_s = \sum_\epsilon \lambda_\epsilon(s) C'_\epsilon$. All factors are finite. Indeed, for $\frac{1}{2}\langle\vartheta, N_1\vartheta\rangle = p_3\eta_H B_X(\vartheta)$ we have $\alpha_D N_1 \leq N$ (as $p_3 < p'$), N commutes with each 1_{Y_J} , and $C'_\epsilon \leq \alpha_D C_{0\epsilon}$, so $\|N_1^{1/2} C'_\epsilon N_1^{1/2}\| \leq \nu_0 < 1$ by the positivity step. (2) Lemma 7.29(c), $C'_\epsilon \leq \alpha_D C_{0\epsilon}$ with $C_{0\epsilon} := \sum_J 1_{Y_J} C_0 1_{Y_J}$, replaces (9.618) to (9.622) in the monotonicity step (9.612) to (9.617). (3) After Hölder's inequality (9.625) with exponents p'/p_3 , q'/p_3 , both integrals factor over the sets Y_J , unions of elements (9.626), and the first factor is at most $\exp\{(c_8|X| + \mathbf{n}'(X))/p'\}$ by (7.29). (4) Corollary 7.17 with $t_C = 1$ and (L8-3) replaces (9.632): $(\int d\mu_{C_0} e^{(q'/p')F_2(Y_j)})^{1/q'} e^{F_1^{\text{loc}}(Y_j)/p'} \leq e^{F_1^{\text{loc}}(Y_j)/p_3}$, as $1/p_3 - 1/p' = 1/q'$. For $k = 0$, $\mathbf{n}' = 0$, since $\mathbf{n}'$ carries the factor k^2 .

Step 4 (the perimeter bound). By Lemma 7.32(c) and $k \leq 1$, the first part of $\mathbf{n}'(V)$ is at most $2\beta_{\text{GM}}^{-1}\theta_P C_R |C|(\log k_\square + C_{\text{sh}})$, as $|C| = 2(R + T) \geq 4R_1(\beta) \geq 8k_\square$. By Lemma 7.32(b), $\|P_\Omega f^C\|_\Omega^2 \leq \pi^2 |\Omega| \leq 8\pi^2 L^3$. It remains to count the near cells. A near cell $\Omega \subseteq Q$ lies in a block $Q \subseteq \hat{Q}(Q)$ all of whose sites are within ℓ^∞ -distance $k_\square + \frac{1}{2}$ of C . The ℓ^∞ -neighbourhood of width s of C has volume at most $(2s)^2(|C| + 8s)$, and for $s = k_\square + 2$ and $L \geq 4$, $2s \leq (2r_\square + 5)L$ and $8s \leq (8r_\square + 20)L$. Blocks are disjoint with at least L^3 sites, and by (R5) a block contains fewer than $8/c_1$ cells. Hence, as $|C| \geq 4R_1(\beta) = (8r_\square + 20)L$, $N_{\text{near}} \leq \frac{8}{c_1}(2r_\square + 5)^2(|C| + (8r_\square + 20)L)/L \leq \frac{16}{c_1}(2r_\square + 5)^2|C|/L$, and the second part of $\mathbf{n}'(V)$ is at most $2\beta_{\text{GM}}^{-1} \frac{4\hat{\eta}\tilde{m}^2}{\eta_H} 8\pi^2 L^3 N_{\text{near}} \leq \frac{1024\pi^2 \hat{\eta}}{\eta_H c_1} (2r_\square + 5)^2 c_L^2 \frac{|C|}{\beta_{\text{GM}}}$, using $\tilde{m}L \leq c_L$. Adding the two parts gives $\mathbf{n}'(V) \leq c_{\text{near}}|C|$. The bound on c_{near} uniform in $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ is arithmetic with $\log k_\square \leq \log(r_\square c_L) + \log(1/\tilde{m})$ and the value of $\tilde{m}$ (Section S1.7.3).

By Proposition 7.7, $E(\psi) = E'(\psi + g) + \frac{1}{2}\langle\psi, \nu\psi\rangle$, where E' collects the Mayer terms of order $s \geq 2$ of Theorem 6.5. We write $E(X)$ for the part of E localized in X (Section 7.5). By (S3), $G_{2,1}$ and $G_{2,2}$ are real. Put $c_\lambda := \frac{1}{4}(1 - 1/p_3)$, $c'_\lambda := C(e^{-\mu} + \varepsilon_\nu)$, $c_E := c'_\lambda + \theta'$, $G_2^\natural := G_{2,1} + G_{2,2} + G_{2,3}^\natural + G_{2,3}^\natural$, and assume

$$2\alpha_D p_4 c_E \leq \frac{1}{2}, \quad \frac{p_4 \alpha_D c_E^2}{1 - 2\alpha_D p_4 c_E} + c_E \leq c_\lambda. \quad (7.30)$$

At the values of Section S1.7.4, (7.30) holds whenever $c_E \leq 1/70$, which holds for $\theta_0 = 0.01$ and $\mu, \kappa_S, \beta_{\text{GM}}$ large.

Lemma 7.34 (Fluctuation factor). *Assume $\varepsilon_\nu + \theta' \leq 1/7$ and (7.30). For every h , every polymer X and every s ,*

$$T_4 := \left[\int d\mu_{C'_s} |e^{E(X)} e^{G_2^\natural(X)} e^{-2\tilde{m}^2 \sum_X \tilde{\delta}^2} |^{p_4} \right]^{1/p_4} \leq \exp\{c_9 |X| + C_{\mathfrak{D}} c_{\text{dep}} N_{\mathfrak{D}}(X) + c_\lambda F_1^{\text{loc}}(X)\},$$

with $c_9 := 2\alpha_D c_E K_1 C_u / \lambda_{\text{PV}}$.

Proof. The proof of [BF80, Lemma 9.9, (9.72) to (9.718)] carries over to G' with the following changes. (1) In (9.72), the bracket of $G_{2,1}$ lies in $[0, \frac{1}{2}y^2]$ ($y = m\beta_{\text{GM}}^{1/2}\tilde{\delta}$) and that of $G_{2,2}$ is at most y^2 , so by (S4') $G_{2,1}(X) + G_{2,2}(X) \leq \frac{3}{2}(1 + Ce^{-2\kappa_S})\tilde{m}^2 \sum_X \tilde{\delta}^2 \leq 2\tilde{m}^2 \sum_X \tilde{\delta}^2$. (2) By Lemma 7.29(a), $e^{G_{2,3}^\natural(X)} \leq e^{C_{\mathfrak{D}} c_{\text{dep}} N_{\mathfrak{D}}(X)}$ and $|G_{2,3}^\natural(X)| \leq \frac{1}{2}\theta'\tilde{m}^2 \sum_X (\vartheta^2 + \psi^2)$. Thus c'_λ of (9.712) becomes c_E . (3) The bounds (9.73) to (9.712), $|E(X)| \leq \frac{1}{2}c'_\lambda \tilde{m}^2 \sum_X (\vartheta^2 + \psi^2)$, hold on G' . Indeed, $|\varepsilon_k| \leq |m|\beta_{\text{GM}}^{1/2}|\vartheta|$ as in (9.78), since the phase at $\psi + g$ is $m\beta_{\text{GM}}^{1/2}(\vartheta + h)$ with $h \in P\mathbb{Z}$, and (S2) of Theorem 6.5, uniform in G' , gives (9.711) by the Schur test [BF80, Lemma 9.6] and $\|\nu\| \leq \varepsilon_\nu \tilde{m}^2$ (S5). (4) In (9.716) to (9.718), $C'_s \leq \alpha_D \tilde{m}^{-2}$ (Lemma 7.29(d)) replaces the lower bound on C_s^{-1} : $K_4 := 2p_4 c_E \tilde{m}^2 C_s^{1/2} 1_X C_s^{1/2}$ has $\|K_4\| \leq 2\alpha_D p_4 c_E \leq \frac{1}{2}$ and $-\frac{1}{2} \log \det(1 - K_4) \leq \text{tr } K_4 \leq p_4 c_9 |X|$ (Theorem 6.11). Condition (7.30) is (9.718), and $\frac{1}{2}\tilde{m}^2 \|1_X D\|^2 \leq F_1^{\text{loc}}(X)$ (Corollary 7.10) replaces $\frac{1}{2} \int f^2 \leq F_1$.

Let κ' be a term of the expansion of an activity, with the parameters of [GM82, (7.3) to (7.5)] referring to $\mathcal{U}$ -cells (the cubes $(a_1, \dots, a_t)$ of a Mayer factor replaced by its number t of points and its set of $\mathcal{U}$ -cells, Lemma A.6), and put $\mathcal{P}'_G := e^{n'(V)/p'}$ and

$$T_1 := \left[\int d\mu_{C'_s} \prod_i |\psi(x_i)|^{p_1} \right]^{1/p_1}, \quad T_2 := \left[\int d\mu_{C'_s} e^{-p_2 F_2(X)} \right]^{1/p_2}.$$

Proposition 7.35 (Hölder split). *Assume (L8-1) to (L8-4), (7.30) and*

$$p_2 \varepsilon(L') \leq \frac{1}{2}(1 - 1/p_3 - c_\lambda). \quad (7.31)$$

Fix h, X, s and a term κ' . (a) There is a kernel $k((x_i)) \geq 0$ such that

$$\begin{aligned} & \int d\mu_{C'_s} |e^{E(X,s)} \kappa' e^{G_2^\natural(X)} e^{-F_1^{\text{loc}}(X)} e^{-F_2(X)}|_0 \\ & \leq \int_{x_i \in X} k((x_i)) \int d\mu_{C'_s} e^E e^{G_2^\natural(X)} e^{-F_2(X)} \prod_i |\psi(x_i)| e^{\frac{1}{2}\gamma \sum_X \vartheta^2} e^{-F_1^{\text{loc}}(X)}. \end{aligned}$$

(b) The right side of (a) is at most $\int k((x_i)) T_1 T_2 e^{-F_1^{\text{loc}}(X)} T_3 T_4$, with T_3 as in Lemma 7.33 and T_4 as in Lemma 7.34. (c) $T_1 \leq c_T^{\sum_j n_j} \prod_j n_j!$, where n_j is the number of points x_i in the j -th $\mathcal{U}$ -cell and c_T depends only on $p_1, r_C, \alpha_D K_1/\lambda_{\text{PV}}, K_1''/\lambda_{\text{PV}}, c_u$ and $C_{\mathcal{U}}$; and $T_2 \leq e^{p_2 \varepsilon(L') F_1^{\text{loc}}(X)}$. (d) ([GM82, Lemma 13] on G' .)

$$T_2 e^{-F_1^{\text{loc}}(X)} T_3 T_4 \leq \mathcal{P}'_G e^{c_{13}|X| + C_{\mathfrak{D}} c_{\text{dep}} N_{\mathfrak{D}}(X)} e^{-c' F_1^{\text{loc}}(X)},$$

with $c_{13} := c_8/p' + c_9$ and $c' := 1 - 1/p_3 - c_\lambda - p_2 \varepsilon(L') \geq c'_0 := \frac{1}{2}(1 - 1/p_3 - c_\lambda)$; at the values of Section S1.7.4, $c'_0 = 1/16$. For $k = 0$, $\mathcal{P}'_G = 1$.

Proof. (a) is the analogue on G' of [GM82, (7.9)], that is, of [BF80, (9.22), (9.23), (9.81) to (9.823)], with the differentiated factors of Appendix B, the rates of Proposition 7.36 and Lemma 7.31 for the derivatives of G_1 . By (S3), E and $G_2^{\natural}$ are real for real ψ . (b) is [GM82, (7.11)], that is [BF80, (9.25)], since $(e^{\frac{1}{2}\gamma \sum_X \vartheta^2} e^{2\tilde{m}^2 \sum_X \tilde{\delta}^2})^{p_3} = e^{p_3 \eta_H B_X(\vartheta)}$. (c) For T_1 we use [BF80, (9.31) to (9.33)], with Lemma 7.29(d) in place of the Neumann series of (9.32) and (R7) for the number of $\mathcal{U}$ -cells within a given distance. The bound on T_2 is Corollary 7.17 with $t_C = \alpha_D \leq 2$. (d) By (c) and Lemmas 7.33, 7.34, the exponents of $F_1^{\text{loc}}(X)$ add up to $-1 + p_2 \varepsilon(L') + 1/p_3 + c_\lambda = -c'$, and (7.31) gives $c' \geq c'_0$. Finally, $e^{n'(X)/p'} \leq \mathcal{P}'_G$, as n' is a sum of nonnegative cell terms.

Remark. The perimeter enters only through $\mathcal{P}'_G$: far cells cost nothing, and $E_{G'}$ (Corollary 7.10(d)) never enters, since $e^{-k^2 E_{G'}/2\beta_{\text{GM}}}$ of Proposition 7.3 cancels against $F_1^{\text{loc}}(V) = F_1 + \frac{1}{2}k^2 \beta_{\text{GM}}^{-1} E_{G'}$ and later uses of F_1^{loc} are lower bounds. So no $\log(1 + \min(R, T))$ appears in Theorem 7.57, and $\mathcal{P}'_G \leq e^{2c_{\text{near}}(R+T)/p'}$.

We measure distances between $\mathcal{U}$ -cells in Debye units, $\text{dist}(A, B) := \tilde{m} d_{G'}(A, B)$. The covariance rate in these units, $r_C := r_{\text{cov}}/\tilde{m}$, does not depend on τ (Theorem 6.11).

Proposition 7.36 (Rates). *Let $0 < \delta_2 < \bar{\delta} < r_C$ and put $c'_{\text{rate}} := \frac{1}{2}(r_C - \bar{\delta})$. Every step of [BF80, Section 9] that uses the rate conditions [BF80, (9.123), (9.124)], the covariance bounds [BF80, (9.32), (9.34)] or [BF80, (9.95)] holds on G' with the bounds of Lemma 7.29(d) for C'_s and with (S2) of Theorem 6.5. No lower bound on any rate is used; any τ -independent margin, in particular c'_{rate} , suffices, and the pinning factor of [BF80, Lemmas 9.10, 9.11] is used with the margin $c_p := (r_C - \bar{\delta})/18$.*

Proof. The rate conditions [BF80, (9.123), (9.124)] require a rate of at least $(1 - \delta_1)/\tilde{l}_D \geq \frac{1}{2}$. On G' the rate r_C is small, but these conditions enter only through margins. The changes are listed below. (i) *Tree factor* $e^{\tilde{\delta}d}$, $d = \sum_i \text{dist}(\Delta'_i, \Delta''_i)$: link i decays at rate r_C (Lemma 7.29(d)), pinning costs $e^{9c_p d}$, and the rest $r_C - \bar{\delta} - 9c_p = c'_{\text{rate}}$ pays the positions of Δ''_i , as $d_{G'}$ dominates the Euclidean distance and by (R7) at most $\frac{4\pi}{3}(r + 2C_{\mathcal{U}})^3/c_u$ $\mathcal{U}$ -cells lie within Debye distance r of a given one. (ii) *Mayer factors*. The factors $e^{\tilde{\delta}L_0}$ and e^{rO_0} of [GM82, (7.3), (7.4)] are paid by the tree decay of (S2) at the Stage 1 rate $\sqrt{3}\kappa_Y M_Y \geq 4\sqrt{3}r_{\text{cov}} > \tilde{\delta}\tilde{m}$ (Proposition 6.10) and by $e^{-\mu(s-1)}$ of (S2) (see the *Tree weights* step of the proof of Proposition B.7). (iii) *Wick's theorem*. The rate $\frac{1}{2}$ is used only to obtain (9.32) by a Neumann series, whereas Lemma 7.29(d) gives the pointwise bound directly. The contraction count holds for all rates, with constants depending on $r_C, K_1''/\lambda_{\text{PV}}, c_u, C_{\mathcal{U}}$. (iv) *Clustering*. The factor $e^{-(1-2\delta_1)\text{dist}(X_1, W)}$ of [BF80, (9.122)] is not used, since each application has $W \subseteq Z$ or $W \supseteq Z$. (v) [BF80, (A4.25), (A4.26)] are used only for the infinite-volume limit of [BF80, Lemma A4.1], and we work in the finite box. In (9.95) the exponent is read with the minus sign that (9.124) has.

Seed configurations and the seeded activity. A pair (U, k) is *admissible* if $U = V$ and $k \in \{0, 1\}$, or $U = X_0^c := V \setminus X_0$ with $X_0 \subsetneq V$ a union of $\mathcal{U}$ -cells and $k = 0$. A *seed configuration* in (U, k) is a pair $\sigma = (Z, N_Z)$ with $Z \subseteq U$ a nonempty, possibly disconnected union of $\mathcal{U}$ -cells, $Z \supseteq X_1$ if $k = 1$, and $N_Z \in \mathcal{N}(Z)$. For $X \supseteq Z$, $\mathcal{N}_\sigma(X)$ is the set of $N_X \in \mathcal{N}(X)$ with $N_X|_{E(Z)} = N_Z$ for which Z is a union of elements of N_X . The *seeded activity* $\mathcal{K}_{N_X}^{(Z)}(X)$ is the activity of Proposition 7.22 at N_X with first set $Y_1 = Z$, built with $(C', G^\natural)$: $\mathcal{K}_{N_X}^{(Z)}(X) = \sum_{\bar{y}} \int_{[0,1]^{n-1}} ds \int d\mu_{C'_s}(\psi) e^{E(X,s)} \kappa(\bar{y}, s) Q_{N_X}^\natural(X)$, the sum over $\bar{y} = (Z, Y_2, \dots, Y_n)$ with the Y_j ($j \geq 2$) disjoint unions of elements and $\bigcup_j Y_j = X$, and $Q_{N_X}^\natural(X) := e^{G^\natural(X)} e^{-F_1^{\text{loc}}(X)} e^{-F_2(X)}$. We write $X = Z \sqcup X_\Sigma \sqcup X_o$, where X_Σ consists of the non-seed $\mathcal{U}$ -cells in wall polymers and X_o of the other non-seed $\mathcal{U}$ -cells (the *lone* cells). A cell is of *class B* if it is one of the $\mathcal{U}$ -cells of $\bigcup_\beta S_\beta$, and of *class A* otherwise. Since $\text{supp } D \cup \text{supp } F'_2 \subseteq \bigcup_\beta S_\beta$ (Proposition 7.8 (b), (e)), $D = F'_2 = 0$ on every class-A cell. As in [BF80, p. 214], $|\cdot|_0$ is the absolute value taken inside the sums and integrals that result when all differentiations in κ' are performed.

Lemma 7.37 (Seeded polymer sum). *Let (U, k) be admissible, $\sigma = (Z, N_Z)$ a seed configuration in (U, k) , $c'_1 > 0$, c'_4 as in Lemma 7.30 with $c_4 = c'_1$, $e^{a''} \geq 4C_\varphi$ ($C_\varphi < \infty$ the link sum of Lemma A.6), and $c_2^L := \log(2eD_\mathcal{U}) + 4 + c'_4 + a'' + \alpha$. Then*

$$\sum_{\substack{X \supseteq Z \\ X \subseteq U}} \sum_{N_X \in \mathcal{N}_\sigma(X)} |\mathcal{K}_{N_X}^{(Z)}(X)| e^{\alpha|X|} \leq \sup_{(\cdot)} e^{c'_1 F_1^{\text{loc}}(X, N_X) + c_2^L |X| + \delta_2 d} \int d\mu_{C'_s} |e^{E(X,s)} \kappa' Q_{N_X}^\natural(X)|_0,$$

where the supremum runs over $n, T, \bar{y}$, the elementary terms, $s \in [0, 1]^{n-1}$ and $N_X \in \mathcal{N}_\sigma(X)$; κ' is the elementary term without its tree weight $q(T, s)$ (Lemma A.1) and with every Mayer factor $\mathcal{E}$ replaced by $\sum_{\eta^A} e^{c_{10} L_{\eta^A}} e^{c'_{10} t} \mathcal{E}_{\eta^A}$ ([BF80, (9.18), (9.18a), (9.19)]); and $d := \sum_i \text{dist}(\Delta'_i, \Delta''_i)$. The constant c_2^L depends only on $D_\mathcal{U}, c'_4, C_\varphi, \alpha$, not on L', β_{GM} or τ . The bound also holds for the sum restricted by $X \cap W \neq \emptyset$ if $W \subseteq Z$ or $W \supseteq Z$ (the restriction is then void).

This is [BF80, Lemmas 9.1, 9.4] on G' in the distance-zero form, for seeds that may depend on the configuration, be disconnected, omit X_1 when $k = 0$, and lie in X_0^c (Proposition A.7). In the supremum we take elementary terms with the same link data (Lemma A.6) together, as in the definition of κ' above. That is, κ' contains, inside $|\cdot|_0$, the sums over the assignments of the points of each $\mathcal{E}$ -factor to the cells of its set of cells, over their positions and over their charges (and, for $t = 2$, over the forms), and Proposition A.7 uses this reading.

Lemma 7.38 (Class split). *Insert Proposition 7.35 and the Wick bound into the right side of Lemma 7.37. In the resulting bound, the factor $e^{c'_2|X|}$ of [BF80, (9.823)], with $e^{c_2^L|X|}$ and $e^{c_{13}|X|}$, is at most $\prod_{\Delta \subseteq X} e^{c_o} \cdot \prod_{\Delta \text{ class B}} e^{c_B(b_D)} \cdot \prod_{\Delta \cap \mathcal{D} \neq \emptyset} e^{C_{\mathcal{D}} c_{\text{dep}}}$ (products over $\mathcal{U}$ -cells), and the near-cell factor is pulled out globally as $\mathcal{P}'_G$. Here c_o does not depend on L', b_D or τ ; every class-B cell lies in X_Σ or in the seed; every factor on a class-A cell carries in addition a nonnegative power of b_D ; and, with the leftover pinning exponent $q_1 = 2$ of Lemma B.9, $c_B(b_D) \leq 2(C_\mathcal{E}^{1/2} + C_G^{1/2})b_D^{-1/2}$, with $C_\mathcal{E}, C_G$ the constants of Proposition B.10, independent of L' . The scale L' enters only through the set of cells in X_Σ , through $\varepsilon(L')$, which is decreasing, and through the monotone hypotheses (L8-3), (L8-4) and (7.31).*

Proposition 7.39 (Per-cell smallness). *Insert Proposition 7.35 and the Wick bound into Lemma 7.37 with $c'_1 = c'/4$, and distribute F_1^{loc} by the fractions of Table S1.4. Then for*

$\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, in the resulting bound, every non-seed $\mathcal{U}$ -cell Δ of X carries a factor at most ε_Δ , where

$$\begin{aligned}\varepsilon_\Delta &:= K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}}} (b_D / \lambda_{\text{PV}})^{1/6} \quad (\Delta \subseteq X_o), \\ \varepsilon_\Delta &:= K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}} + c_B(b_D) - (c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})} \quad (\Delta \subseteq X_\Sigma),\end{aligned}$$

and every seed cell carries at most $K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}}}$, times $e^{c_B(b_D)}$ if it is of class B . Here $K_{\text{cell}} := K_B e^{(c_L^2 + c_{13})C_u} \max(1, \lambda_{\text{PV}}^{1/6})$ with K_B independent of b_D , L' and τ , $\Phi_{\mathcal{U}}$ is the share of Proposition 7.19, and the exponent $p_0 = 1/6$ is attained.

Lemma 7.38 and Proposition 7.39 are [BF80, Section 9.8, (9.81) to (9.824)] on G' , made quantitative, and are proved in Appendix B. With $p_0 := 1/6$ we put

$$c_{16}^D := 2K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}}} \max\left\{(b_D / \lambda_{\text{PV}})^{p_0}, e^{c_B(b_D) - (c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}\right\}. \quad (7.32)$$

Theorem 7.40 (Removal of the logarithm). *Fix α . There is τ_0 such that $c_{16}^D \leq 1$ for $\tau \geq \tau_0$, and $c_{16}^D \rightarrow 0$ as $\tau \rightarrow \infty$. Consequently the depletion factor $e^{C_{\mathfrak{D}} c_{\text{dep}} N_{\mathfrak{D}}(X)}$, and with it the $\log \tau$ in c_{dep} , is absorbed by the per-cell smallness of Proposition 7.39.*

Proof. The per-cell gains are exponentially small in τ , while the depletion cost is $O(\log \tau)$. The first branch of (7.32) is at most 1 if $\alpha \leq p_0 \log(\lambda_{\text{PV}}/b_D) - C_{\mathfrak{D}} c_{\text{dep}} - \log(2K_{\text{cell}})$, which holds for $\tau \geq \tau_0$, as $\log(1/b_D) \geq \pi^2 v_{\text{Cb}}(0)\tau - \frac{3}{2} \log \tau - C$ ($\tilde{m} \leq 2m_D$), $c_{\text{dep}} = O(\log \tau)$ (Proposition 6.17) and $\log K_{\text{cell}}$ is affine in α . In the second branch, $(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}}) \geq c b_D^{-1} \beta_{\text{GM}}^{-3}$ (Proposition 7.19 with $\tilde{m}L \geq c_L/2$, $\tilde{m}L' = 2\beta_{\text{GM}}$) outweighs $c_B(b_D) \leq C b_D^{-1/2}$ (Lemma 7.38). A depleted cell that is lone or in X_Σ has its factor $e^{C_{\mathfrak{D}} c_{\text{dep}}}$ inside $\varepsilon_\Delta \leq \frac{1}{2}$ (Section S1.7.5).

Let $\tilde{\mathcal{K}}(\Delta)$ be the activity of the single element Δ at $k = 0$ (first set Δ , $n = 1$), with no wall on Δ .

Lemma 7.41 (Single cell). *For $\tau \geq \tau_0$ and every wall-free $\mathcal{U}$ -cell Δ , $\tilde{\mathcal{K}}(\Delta) \geq \frac{1}{2}$.*

Proof. Positivity at $k = 0$ allows us to use Jensen's inequality. A supremum bound does not suffice, since $\sup_\psi (-G_{2,3}^{\mathfrak{D}}(\Delta))$ is only bounded by $C(e^{-\mu} + e^{-2\kappa_S})c_{\text{dep}}$, which grows like $\log \tau$.

A single $\mathcal{U}$ -cell that is an element lies in no S_β , so $D = 0$ on Δ and $F_1^{\text{loc}}(\Delta) = F_2(\Delta) = 0$. The label is constant on Δ and, at $k = 0$, enters each phase only through a multiple of 2π . Hence $\vartheta = \psi$ on Δ , $\tilde{\mathcal{K}}(\Delta) = \int d\mu_{C'_\Delta} e^{E(\Delta) + G^{\natural}(\Delta)}$ with C'_Δ the $\Delta \times \Delta$ block of C' , the exponent is real by (S3), and Jensen's inequality gives

$$\tilde{\mathcal{K}}(\Delta) \geq \exp\left(-\mathbb{E}|E(\Delta)| - \mathbb{E}|G_1(\Delta)| - \mathbb{E}|G_{2,1}(\Delta) + G_{2,2}(\Delta)| - \mathbb{E}(G_{2,3}^{\mathfrak{D}}(\Delta))_- - \mathbb{E}|G_{2,3}^{\natural}(\Delta)|\right),$$

expectations under $d\mu_{C'_\Delta}$. Put $x_b := \alpha_D K_1 b_D / \lambda_{\text{PV}}$. Then $\beta_{\text{GM}} C'(x, x) \leq x_b$ by Lemma 7.29(d) and $M_Y = \tilde{m} / \lambda_{\text{PV}}$. Moreover, $\#\Delta < C_u \tilde{m}^{-3}$, and Δ contains at most C_u / v_- cells. As we will show below, each expectation is at most $C(b_D^{1/2} + c_{\text{dep}} b_D)$, with C depending only on the constants fixed before τ_0 . Since $c_{\text{dep}} b_D = O(\log \tau) e^{-2.494\tau} \rightarrow 0$, the sum is then at most $\ln 2$ for $\tau \geq \tau_0$.

$G_{2,3}^{\mathfrak{D}}$. At $x \in \mathfrak{D}$ the x -term is negative only through the excess $\rho_1(m, x) > \rho^{\text{ref}}(m)$, at most $C e^{-\mu} \rho^{\text{ref}}(1)$ for $m = \pm 1$ (Proposition 6.6, $e^{-\Phi} \leq 1$) and bounded by the $|m| \geq 2$

moment bound otherwise. With $1 - \cos u \leq \frac{1}{2}u^2$, $\mathbb{E}\psi(x)^2 = C'(x, x)$ and Proposition 6.17, $\mathbb{E}(G_{2,3}^{\mathfrak{D}}(\Delta))_- \leq C(e^{-\mu} + e^{-2\kappa_S})\tilde{m}^2 \alpha_D K_1 M_Y \#(\mathfrak{D} \cap \Delta) \leq C(e^{-\mu} + e^{-2\kappa_S})\alpha_D K_1 c_{\text{dep}} b_D / \lambda_{\text{PV}}$.

The other four terms are moment bounds with $\beta_{\text{GM}} C'(x, x) \leq x_b$. First, $\mathbb{E}|E(\Delta)| \leq C\#\Delta(m_D^2/\beta_{\text{GM}})x_b^{3/2} \leq Cb_D^{1/2}$, as each term of $E(\Delta)$ has at least three factors ε or $\tilde{\varepsilon} := \varepsilon - im\beta_{\text{GM}}^{1/2}\psi$ and the weight $e^{2\kappa_S(\sum |m_i|-1)}$ of (S2) absorbs the charges. Second, $\mathbb{E}|G_1(\Delta)| \leq Cb_D$, as $\log \hat{R}_v(x) = O(vb_D x^4 + e^{-\pi^2 v/b_D})$ for $|x| \leq b_D^{-1/2}$ (Taylor's formula and the periodization), with a Gaussian tail beyond. Third, $\mathbb{E}|G_{2,1} + G_{2,2}| \leq C(b_D + b_D^{1/2})$, as the brackets are of fourth and third order and $\text{Var } \tilde{\delta}(x) \leq 2\alpha_D(K_1/\lambda_{\text{PV}} + 1/v_-)\tilde{m}$. Finally, $\mathbb{E}|G_{2,3}^{\mathfrak{d}}(\Delta)| \leq 3C_4\theta' C_u(\alpha_D K_1)^2 b_D / \lambda_{\text{PV}}^2$ by Lemma 7.29(b) with $n = 0$ and $\mathbb{E}\psi^4 = 3C'(x, x)^2$.

The bound $\tilde{\mathcal{K}}(\Delta) \geq \frac{1}{2}$ replaces [BF80, (A4.16) to (A4.21)], which use translation invariance, whereas here positivity at $k = 0$ (Lemma 7.42) and the diagonal bound of Theorem 6.11 suffice.

7.7 The Kirkwood–Salsburg system in the class algebra

In this subsection we solve the Kirkwood–Salsburg system of Theorem 7.27 (b) by the argument of [BF80, Appendix 4] in a weighted ℓ^1 algebra over Γ_{long} , with the area weight of the classes (Theorem 7.49). The area decay of $\mathcal{Z}_p(X)/Z$ will then follow from that of Z_p/Z (Subsection 7.8). We build activities and outer partition functions with $(C', G^{\mathfrak{d}}, \mathfrak{c}\mathcal{N})$ (Lemma 7.29 (f)).

Lemma 7.42 (Positivity). *At $k = 0$ and for real ψ , the functions E , G and $G^{\mathfrak{d}}$ of Lemma 7.24 are real, and $e^{E(Y)}Q_N^{\mathfrak{d}}(Y) > 0$ for every wall field N and every union Y of its elements. Hence $\mathcal{Z}_p(X) \geq 0$ for every X and p , $Z > 0$, and every activity with $n = 1$ is positive.*

Proof. By (S3) the terms of m and $-m$ in the charge sums of E , G_1 , $G_{2,1}$, $G_{2,2}$ are complex conjugate, and $G_{2,3}^{\mathfrak{D}}$, $G_{2,3}^{\mathfrak{d}}$, F_1^{loc} , F_2 are real. Hence $e^{E(Y)}Q_N^{\mathfrak{d}}(Y)$ is the exponential of a real number. The outer partition functions $\mathcal{Z}_p(X)$ and the activities with $n = 1$ are integrals of such factors, with $\mathfrak{c}\mathcal{N} > 0$, and Z contains $N = 0$. Positivity is used in the union bound, step (1) of the proof of Proposition 7.53.

Lemma 7.43 (Forced area). (Definition and Lemma.) *For an additive function p on the cycle space of G' let $A(p)$ be the minimum of $\sum_{e \in E} |N(e)|$ over the $N \in \mathbb{Z}^E$ with $[N](\gamma) = p(\gamma)$ for every cycle γ of G' , and $A(p) := \infty$ if no such N exists; for $p \in \Gamma_{\text{long}}$, $A(p)$ is that of the restriction of p to the cycles of G' . Then $A(0) = 0$, $A(-p) = A(p)$, $A(p+q) \leq A(p) + A(q)$, and $A([N]) \leq \sum_E |N|$ for every $N \in \mathbb{Z}^E$; in particular $A < \infty$ on Γ_{long} . Fix $\varepsilon_E \in (0, c'_0/8]$ and put $\sigma' := \varepsilon_E \sigma_F$ and $\omega(p) := e^{\sigma' A(p)}$. Then $\omega(0) = 1 \leq \omega$ and $\omega(p+q) \leq \omega(p)\omega(q)$ on Γ_{long} , so $\mathcal{A}_\omega := \{a : \Gamma_{\text{long}} \rightarrow \mathbb{C} : \|a\|_\omega := \sum_p |a(p)|\omega(p) < \infty\}$, with $(a * b)(p) := \sum_q a(q)b(p-q)$, is a commutative Banach algebra with unit δ_0 , and $\|a * b\|_\omega \leq \|a\|_\omega \|b\|_\omega$.*

Proof. Sums and negatives of admissible cochains are admissible, N is admissible for $[N]$, and $\sum_p |(a * b)(p)|\omega(p) \leq \sum_{p,q} |a(q)|\omega(q) |b(p-q)|\omega(p-q)$.

Lemma 7.44 (Cell diameter). *Let Ω be a cell of the output. For all sites $x, y \in \Omega$, $d_{G'}(x, y) \leq D_{\text{cell}} := 7r_0(2t_1 + 1)$, where $t_1 := 1 + \lceil (12k_P/\chi_{\text{N3}}) \ln k_P^3 \rceil$ (k_P in coarse units). If $c_L \leq 1.99 \cdot 10^{-4} \chi_{\text{N3}}$, then $D_{\text{cell}} < L'$ for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$.*

Sketch. By (N3) of Theorem 3.20, there is no χ_{N3} -weak link. Hence, in the component $\mathfrak{K}$ of $G' \cap \hat{Q}(Q)$ with $\Omega = Q \cap \mathfrak{K}$ (Definition 3.14; $|\mathfrak{K}| \leq k_P^3$), balls with $|B_t| \leq |\mathfrak{K}|/2$ grow by the factor $1 + \chi_{N3}/(6k_P)$ per step. Balls of radius t_1 about x, y therefore meet. A coarse path of length ℓ gives a G' -path of at most $7r_0(\ell+1)$ fine steps. With $L_c \geq 35$ and $c_L \leq 1.99 \cdot 10^{-4} \chi_{N3}$, we obtain $D_{\text{cell}} < L'$ (Section S1.8).

Lemma 7.45 (Local primitive). *Let N be a long wall field, $w := 2\pi\beta_{\text{GM}}^{-1/2}N$, and let Ω be a cell with the set $\mathfrak{K}_\Omega^b$ of Proposition 6.9 (iv). Then there is $h' : \mathfrak{K}_\Omega^b \rightarrow P\mathbb{Z}$, constant on Ω , with $\nabla h' = w$ on $E(\mathfrak{K}_\Omega^b)$.*

Proof. A primitive along a spanning tree can fail only on closed walks of about twice the diameter of $\mathfrak{K}_\Omega^b$, which is less than L' . The graph $(\mathfrak{K}_\Omega^b, E(\mathfrak{K}_\Omega^b))$ is connected, of diameter $d_b \leq 2 + 0.469\beta_{\text{GM}}/\tilde{m} = 2 + 0.235L'$ (Proposition 6.9 (iv), the proof of (v), Section S1.5). We define $h'(x)$ as the sum of w along the path from $x_0 \in \Omega$ to x in a breadth-first spanning tree. Then h' is $P\mathbb{Z}$ -valued. A bond of $E(\mathfrak{K}_\Omega^b)$ off the tree closes a walk of length at most $2d_b + 1 < 2L'$ with two tree paths, on which N has period 0. Hence $\nabla h' = w$ on $E(\mathfrak{K}_\Omega^b)$. For $x \in \Omega$, the tree path from x_0 to x and the same-cell step back form a closed m3-walk of length at most $d_b + D_{\text{cell}} \leq 2 + 0.235L' + L + 0.964L' < 2L'$ (Lemma 7.44, Section S1.8; $L' \geq 16$, $L \leq L'/(4\beta_{\text{GM}})$). Therefore $h'(x) = h'(x_0)$ for each such x .

Proposition 7.46 (Long wall fields). *At $k = 0$, the bounds of Subsection 7.6 and of Appendices A and B, and Lemma 7.48, hold with $\mathcal{N}(X)$ and $\mathcal{N}_\sigma(X)$ replaced by $\mathcal{N}^{\text{long}}(X)$ and $\mathcal{N}_\sigma^{\text{long}}(X) := \{N_X \in \mathcal{N}^{\text{long}}(X) : N_X|_{E(Z)} = N_Z, Z \text{ a union of elements of } N_X\}$, for seed configurations $\sigma = (Z, N_Z)$ with $N_Z \in \mathcal{N}^{\text{long}}(Z)$, with the integrand of Lemma 7.24 and the same constants. In Lemma 7.32 (a) the label is replaced by the primitive h' of Lemma 7.45.*

Sketch. Sums over configurations enter only through the injection $N_X \mapsto (N_I)_{I \subseteq X}$ into integer interface values (Lemma 7.30 (b), steps (2), (3) of the proof of Proposition A.8), valid for all wall fields. At a fixed configuration the label enters only through the phases (Lemma 7.24), $w = 0$ inside cells, the walls in (7.28), and, in Lemma 7.32 (a), a primitive of w_Y on $E(\mathfrak{K}_\Omega^b)$ constant on Ω . In that case $N\mathbf{1}_{E(Y)}$ is long (Lemma 7.26 (a)), and Lemma 7.45 gives such a primitive (Section S2.6).

Proposition 7.47 (Absolute convergence). *There is $c < \infty$, depending on τ and the fixed constants only, such that $\|\mathcal{Z}(X)\|_\omega \leq c\mathcal{N}e^{(c+c'_4)|X^c|}$ for every union X of $\mathcal{U}$ -cells, with c'_4 from Lemma 7.30 at $c_4 = c'/2$. Moreover $\sum_Y \sum_{N_Y} |\mathcal{K}_{N_Y}^{(Z)}(Y)| \omega([N_Y]) e^{\alpha|Y|} < \infty$ for the seeds Z of Theorem 7.27 (b), (c), and $\sum_{N_X} |\mathcal{K}_{N_X}(X)| < \infty$ in Theorem 7.27 (a), the sum over the inside fields N_X at $k = 1$ with $p_1(N_X) \in \Gamma_{\text{long}}$ (the other inside fields contribute 0 to Theorem 7.27 (a), as $\mathcal{Z}_p(X) = 0$ for $p \notin \Gamma_{\text{long}}$). Hence the sums of Theorem 7.27 converge absolutely.*

Proof. In outline, each configuration pays for its walls, which carry at least the area of their class, with a fixed fraction of its F_1^{loc} . By Hölder's inequality as in Proposition 7.35 (b), (d) at $k = 0$, with Lemma 7.31 at $N = 0$ and Proposition 7.46, the term of N in $\mathcal{Z}_p(X)$ is at most $c\mathcal{N}e^{c|X^c|}e^{-c'F_1^{\text{loc}}(X^c, N)}$. Moreover, $\omega([N]) \leq e^{\sigma' \sum |N|} \leq e^{(\varepsilon_E/2)F_1^{\text{loc}}(X^c, N)}$ by Lemma 7.43 and Theorem 7.13 ($|N| \leq N^2$), $\varepsilon_E \leq c'$, and the sum of $e^{-(c'/2)F_1^{\text{loc}}(X^c, N)}$ over the wall fields on $E(X^c)$ is at most $e^{c'_4|X^c|}$ (Lemma 7.30 (b)). We bound the seeded sums by Lemma 7.48 (a), (c'), with $\varepsilon = \varepsilon_E/2$ at $k = 0$ and $\varepsilon = 0$ at $k = 1$. At $k = 1$ the terms of Theorem 7.27 (a) with $p_1(N_X) \notin \Gamma_{\text{long}}$ vanish, since $\mathcal{Z}_p(X) := 0$ for $p \notin \Gamma_{\text{long}}$ (Definition 7.25(c)). For the inside fields at $k = 1$ with $p_1(N_X) \in \Gamma_{\text{long}}$, Lemma 7.48 (a) applies as in the proof of Theorem 7.57.

Constants of the seeded sums. The constant C_9 bounds the number of interfaces within Euclidean distance r of a given interface or site by $C_9(r/L + 1)^3$. The Euclidean link range is $g := 2L' + 4D_E + 2$ (D_E of Proposition 7.19). We put $c_2 := c_S + 2 = O(\log \tau)$ and $c_{16} := c_{16}^D + \eta_{\text{seed}} \rightarrow 0$ as $\tau \rightarrow \infty$, where c_{16}^D is defined in (7.32) and c_S, η_{seed} in Section S1.8. We also set $\mathcal{P}' := \mathcal{P}'_G$ if $k = 1$ and $\mathcal{P}' := 1$ if $k = 0$. For a seed configuration $\sigma = (Z, N_Z)$, Z^Σ is the union of its wall polymers and $Z^o := Z \setminus Z^\Sigma$. For an anchor $S_0 \subseteq U$ (a nonempty union of $\mathcal{U}$ -cells, $S_0 \supseteq X_1$ if $k = 1$) and $N_X \in \mathcal{N}(X)$, $\text{sat}_{N_X}(S_0)$ is the union of the elements of N_X meeting S_0 .

Lemma 7.48 (Seeded sums at $k = 0$ and $k = 1$). *Let (U, k) be admissible, $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, and $\varepsilon \in [0, c'/2]$. (a) (Variable first set.) For every anchor $S_0 \subseteq U$,*

$$\sum_{S_0 \subseteq X \subseteq U} \sum_{N_X \in \mathcal{N}(X)} |\mathcal{K}_{N_X}^{(\text{sat}_{N_X}(S_0))}(X)| e^{\alpha|X| + \varepsilon F_1^{\text{loc}}(X, N_X)} \leq \mathcal{P}' e^{c_2|S_0|}.$$

(b) (Strict containment.) *The sum in (a) without the term $X = S_0, N_X = 0$ is at most $c_{16} \mathcal{P}' e^{c_2|S_0|}$. (c') (Prescribed seed walls, $k = 0$.) For every seed configuration $\sigma = (Z, N_Z)$ in $(U, 0)$ with $Z^o = \emptyset$, every $\varepsilon \in [0, c'/8]$ and $c_w := c'/16$,*

$$\sum_{Z \subseteq Y \subseteq U} \sum_{N_Y \in \mathcal{N}_\sigma(Y)} |\mathcal{K}_{N_Y}^{(Z)}(Y)| e^{\alpha|Y| + \varepsilon F_1^{\text{loc}}(Y, N_Y)} e^{c_w F_1^{\text{loc}}(Z, N_Z)} \leq \prod_{\mathcal{S}} e^{-(c'/2)F_1^{\text{loc}}(\mathcal{S})},$$

the product over the wall polymers $\mathcal{S}$ of N_Z .

We prove Lemma 7.48 in Appendix A. Proposition A.8 treats a fixed seed configuration, and for (a), (b) the entropy of the first set is paid from its walls.

Theorem 7.49 (The Kirkwood–Salsburg system in $\mathcal{A}_\omega$; [BF80, Lemma A4.1] on G'). *Fix a linear order on the $\mathcal{U}$ -cells; for $X \neq \emptyset$ let $\ell(X)$ be its last $\mathcal{U}$ -cell and $X^* := X \setminus \ell(X)$. For a $\mathcal{U}$ -cell Δ and a union $Y \ni \Delta$ of $\mathcal{U}$ -cells let $\mathcal{K}'_\Delta(Y) \in \mathcal{A}_\omega$ be the sequence $q \mapsto \sum \mathcal{K}_{N_Y}^{(\text{sat}(\Delta))}(Y)$, the sum over the N_Y of (7.21) with $[N_Y] = q$ and $(Y, N_Y) \neq (\Delta, 0)$. There is exactly one family $(r(X))_X$ in $\mathcal{A}_\omega$ with $r(\emptyset) = \delta_0$ and $\max_{X \neq \emptyset} e^{-\alpha|X|} \|r(X)\|_\omega < \infty$ such that, for every $X \neq \emptyset$,*

$$r(X^*) = \tilde{\mathcal{K}}(\ell(X)) r(X) + \sum_{Y \ni \ell(X), Y \cap X^* = \emptyset} \mathcal{K}'_{\ell(X)}(Y) * r(X^* \cup Y), \quad (7.33)$$

and it satisfies, with $\alpha = \ln 8$: (a) $\|r(X)\|_\omega \leq \frac{8}{31} e^{\alpha|X|}$ for every $X \neq \emptyset$; (b) $\mathcal{Z}(X) = r(X) * \mathcal{Z}(\emptyset)$ for every X , that is, $\mathcal{Z}_p(X) = \sum_q r(X)(q) \mathcal{Z}_{p-q}$.

Proof. By Theorem 7.27 (b) the system has the form of [BF80, (A4.5)] with values in $\mathcal{A}_\omega$, and we run the contraction of [BF80, (A4.6) to (A4.13)] with $\|\cdot\|_\omega$ in place of $|\cdot|$. The normalization $r(\emptyset) = \delta_0$ replaces the division by Z , which is not available for the sequence $\mathcal{Z}(\emptyset)$, and (b) follows from uniqueness in the finite box.

Inputs. (K1) $\tilde{\mathcal{K}}(\Delta) \geq \frac{1}{2}$ (Lemma 7.41). (K2) $\sum_{Y \ni \Delta} \|\mathcal{K}'_\Delta(Y)\|_\omega e^{\alpha|Y|} \leq \frac{1}{8}$. By Lemma 7.43 and Theorem 7.13, $\omega([N_Y]) \leq e^{\sigma' \sum |N_Y|} \leq e^{(\varepsilon_E/2) F_1^{\text{loc}}(Y, N_Y)}$. Hence (K2) follows from Lemma 7.48 (b) with $S_0 = \Delta$, $k = 0$ and $\varepsilon = \varepsilon_E/2$, over long fields (Proposition 7.46), and from $c_{16} e^{c_2} \leq \frac{1}{8}$ (Lemma S1.2).

Step 1 (the map). Let $\mathfrak{B}$ be the Banach space of families $s = (s(X))_{X \neq \emptyset}$ in $\mathcal{A}_\omega$ with $\|s\| := \max_{X \neq \emptyset} e^{-\alpha|X|} \|s(X)\|_\omega$, put $s(\emptyset) := \delta_0$, and let $(Ts)(X)$ be $\tilde{\mathcal{K}}(\ell(X))^{-1}$ times $s(X^*)$ minus the sum in (7.33). A family satisfies (7.33) if and only if $Ts = s$.

Step 2 (contraction). By $\|a * b\|_\omega \leq \|a\|_\omega \|b\|_\omega$, $|X^* \cup Y| = |X| - 1 + |Y|$, $\|s(\emptyset)\|_\omega = 1$, (K1) and (K2), we obtain $e^{-\alpha|X|} \|(Ts)(X)\|_\omega \leq 2e^{-\alpha} [\max(1, \|s\|) + \|s\|/8] = \frac{1}{4} \max(1, \|s\|) + \|s\|/32$, and in the same way $\|Ts - Ts'\| \leq 2e^{-\alpha}(1 + \frac{1}{8})\|s - s'\| = \frac{9}{32}\|s - s'\|$, since $(s - s')(\emptyset) = 0$. Let r be the fixed point of T .

Step 3 ((a)). If $\|r\| \geq 1$, Step 2 gives $\|r\| \leq \frac{9}{32}\|r\|$, which is impossible. Therefore $\|r\| \leq \frac{1}{4} + \|r\|/32$, that is, $\|r\| \leq \frac{8}{31}$.

Step 4 ((b)). We set $s(X) := r(X) * \mathcal{Z}(\emptyset) - \mathcal{Z}(X)$, which has $s(\emptyset) = 0$. By associativity and commutativity, $X \mapsto r(X) * \mathcal{Z}(\emptyset)$ satisfies (7.33), and so does $X \mapsto \mathcal{Z}(X)$, by (7.21) with $X_0 = X^*$ and $\Delta = \ell(X)$. Consequently s satisfies the linear part of (7.33) with $s(\emptyset) = 0$. Moreover, $\|s\| < \infty$, since there are finitely many X and $\|\mathcal{Z}(X)\|_\omega < \infty$ (Proposition 7.47). As in Step 2, $\|s\| \leq \frac{9}{32}\|s\|$, and hence $s = 0$.

Beyond the input (K1), whose proof (Lemma 7.41) uses positivity at $k = 0$, the proof uses no positivity, and the bounds are uniform in G' , Λ° and C . We obtain the invertibility of $\mathcal{Z}(\emptyset)$ in $\mathcal{A}_\omega$ as a consequence of Theorem 7.49 (b): $r(V) * \mathcal{Z}(\emptyset) = \mathcal{Z}(V) = \mathfrak{cN} \delta_0$.

7.8 Frozen flux costs area: group charge and the count

In this subsection we show that the partition function at fixed class r carries the area weight of r (Proposition 7.53), and it is only here that the handles of the insulating set and (NF_*) , which is (R10) of Theorem 3.20, enter the bounds. In a configuration of class r , a group of wall polymers realizing r is re-expanded as a seed at $X = \emptyset$, and its entropy is controlled by Proposition 7.52.

Polymers, realizations and the reference realization. Fix $r \in \Gamma_{\text{long}} \setminus \{0\}$ with $Z_r \neq 0$. For a configuration N of Z_r (Definition 7.25), $\mathcal{P}(N)$ is the set of its wall polymers, and we put $N_S := N \mathbf{1}_{E(S)}$ for $S \in \mathcal{P}(N)$ and $N_R := \sum_{S \in R} N_S$, $q_R := [N_R]$, $|R| := \|N_R\|_1$ for $R \subseteq \mathcal{P}(N)$. Each wall bond lies in exactly one wall polymer (Proposition 7.8 (a)). Hence $\sum_{S \in \mathcal{P}(N)} N_S = N$ and $q_{\mathcal{P}(N)} = r$, and $q_R \in \Gamma_{\text{long}}$ (Lemma 7.26 (a)). A family R realizes r if $q_R = r$, and is *irreducible* if no nonempty $T \subseteq R$ has $q_T = 0$. An inclusion-minimal realizing subfamily of $\mathcal{P}(N)$ is irreducible (if $\emptyset \neq T \subseteq R$ had $q_T = 0$, then $T \neq R$ as $r \neq 0$, and $R \setminus T$ would realize r). Let $\mathfrak{S}(r)$ be the set of families R , with their wall configurations, that occur as irreducible realizations of r in some configuration of Z_r . We fix $R_0 \in \mathfrak{S}(r)$ of least area (a deterministic choice), so that $|R| \geq |R_0| \geq A(r)$ for all $R \in \mathfrak{S}(r)$ (Lemma 7.43). A polymer is determined by its wall interfaces and their values (closures and links are functions of the walls). Thus R is determined by N_R . Put $\nu(R) := \sum_{I \subseteq \text{supp } N_R} N_I^2$, $c_w := c'/16$, $\lambda_E := 2c_w \sigma_F = \frac{c'}{8} \sigma_F$ and $\Theta_4(r) := \sum_{R \in \mathfrak{S}(r)} e^{-\lambda_E |R|} \prod_{S \in R} e^{-(c'/2) F_1^{\text{loc}}(S)}$. The notion of c -connectedness is that of Definition 3.18, and distances between bond sets are Euclidean.

Lemma 7.50 (Localization of realizations). *Let $R \in \mathfrak{S}(r)$. Then $N_R - N_{R_0} = \sum_k \varepsilon_k \nabla 1_{P_k}$ with $\varepsilon_k \in \{-1, 1\}$, where (i) each P_k is a nonempty c -connected cell set with c -connected complement in its component $\mathfrak{K}$ of G' ; (ii) $\partial_{G'} P_k \subseteq \text{supp } N_R \cup \text{supp } N_{R_0}$; (iii) $\sum_k |\partial_{G'} P_k| = \|N_R - N_{R_0}\|_1 \leq |R| + |R_0| \leq 2|R|$; and (iv) every P_k is joined to $\text{supp } N_{R_0}$ by a chain $P_k = P_{k_0}, \dots, P_{k_j}$ with $\text{dist}(\partial P_{k_{i-1}}, \partial P_{k_i}) \leq g$ and $\text{dist}(\partial P_{k_j}, \text{supp } N_{R_0}) \leq g$.*

Proof. Two realizations of one class differ by an exact cochain, whose cell-constant potential has level sets giving the pieces. Irreducibility forbids groups of pieces far from R_0 , which

would be sub-families of R of class zero. (1) Since $[N_R - N_{R_0}] = q_R - q_{R_0} = 0$, we have $N_R - N_{R_0} = \nabla f$ with $f : V \rightarrow \mathbb{Z}$ constant on each cell (Definition 7.25). (2) By the coarea formula, $\nabla f = \sum_t \pm \nabla 1_{U_t}$ with $\sum_t |\partial U_t| = \|\nabla f\|_1$. We split each U_t into its c-components V and write $\nabla 1_V = -\sum_i \nabla 1_{C_i}$ over the c-components C_i of $\mathfrak{K} \setminus V$, where $\mathfrak{K}$ is the component of V . Since each C_i is c-connected with c-connected complement and the cuts partition, (i) to (iii) hold. (3) For (iv), group the pieces into clusters by the relation “cuts within distance g ”. Let $\mathcal{V}$ be a cluster no cut of which is within g of $\text{supp } N_{R_0}$, and $B := \bigcup_{k \in \mathcal{V}} \partial P_k$. Within distance g of B , $N_{R_0} = 0$ and the other pieces contribute nothing. It follows that $N_R = N_{\mathcal{V}} := \sum_{k \in \mathcal{V}} \varepsilon_k \nabla 1_{P_k}$ in this region, and each wall bond of N_R within distance g of B lies in B . Two wall bonds of one polymer are joined by a chain of wall bonds of that polymer with consecutive gaps at most g (walls of one S_β have gaps less than $2L' + 2D_E$, and linked closures add at most $2D_E + 2$). Hence the polymers of R meeting B form a sub-family T with $N_T = N_R 1_B = N_{\mathcal{V}}$, the gradient of a cell-constant function, and therefore $q_T = 0$. By irreducibility $T = \emptyset$, and hence $N_{\mathcal{V}} = 0$. By (iii) the terms of $N_R - N_{R_0}$ do not cancel, and hence neither do those of the sub-sum: $\|N_{\mathcal{V}}\|_1 = \sum_{k \in \mathcal{V}} |\partial P_k|$. So $N_{\mathcal{V}} = 0$ forces $\partial P_k = \emptyset$ for every $k \in \mathcal{V}$. This is impossible, since by (i) each P_k is a nonempty proper cell set of its component $\mathfrak{K}$. Hence no such cluster exists: every cluster has a cut within g of $\text{supp } N_{R_0}$, and chaining inside the cluster gives (iv).

For a finite set B of interfaces and $s > 0$, the s -clusters of B and their number $\varphi_s(B)$ are those of Definition 3.18 (for a cut, of its set of interfaces), and $\text{MST}(B)$ denotes a minimal spanning tree of the complete graph on B with edge lengths ℓ_e the distances of interfaces. Let $j(\ell) := \#\{j \geq 0 : 2^j L < \ell\}$.

Lemma 7.51 (Kruskal’s identity). *For every $s > 0$, $\#\{e \in \text{MST}(B) : \ell_e > s\} = \varphi_s(B) - 1$, and $\sum_{e \in \text{MST}(B)} j(\ell_e) = \sum_{j \geq 0} (\varphi_{2^j L}(B) - 1)$.*

Proof. The edges of $\text{MST}(B)$ of length $\leq s$ form a spanning forest of the threshold graph $\{\ell \leq s\}$ (Kruskal’s algorithm processes them first). Thus they number $|B| - \varphi_s(B)$. For the second identity we sum $1[\ell_e > 2^j L]$ over $j \geq 0$ and e .

Let $\kappa_{\text{NF}} := 4(c_g + 4)/c_g^2$, and R_{NF} , θ_{NF} , c_3^{NF} , J as in Definition 3.18. By (R10) (Proposition 3.19), applied to the pieces P_k of Lemma 7.50,

$$J(P_k) \leq \theta_{\text{NF}} F_{\text{unit}} |\partial_{G'} P_k| / L^2, \quad (7.34)$$

with the cut measured in fine bonds (a coarse face carries at least r_0^2 of them), and J depends only on the cut.

Proposition 7.52 (The count). *For $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, every $r \in \Gamma_{\text{long}} \setminus \{0\}$ and every $\nu \geq 1$, $\#\{R \in \mathfrak{S}(r) : \nu(R) = \nu\} \leq \exp((c'/2)F_{\text{unit}} \nu)$. Consequently $\Theta_4(r) \leq \frac{1}{4}$.*

Proof. The idea is that, by Kruskal’s identity, locating the cut patches costs, per dyadic scale, the number of clusters minus one. Below R_{NF} this is at most $\log_2(R_{\text{NF}}/L)$ per interface, within the allowance $e^{(c'/2)F_{\text{unit}}}$, and above R_{NF} it is J , bounded by the cut (7.34). Fix R with $\nu(R) = \nu$. Let S, S_0 be the interface sets of $\text{supp } N_R$ and $\text{supp } N_{R_0}$, $n := |S| \leq \nu$, $n_0 := |S_0|$, $T := S \cup S_0$, and I_k the interface set of ∂P_k . An interface has between $c'_g L^2$ and $4L^2$ bonds by (R8) and since block sides are less than $2L$, and $|R_0| \leq |R| \leq 4L^2 \nu$. Hence $n_0 \leq (4/c'_g)\nu$, and by Lemma 7.50(iii) $\sum_k |\partial P_k| \leq 8L^2 \nu$, $\sum_k |I_k| \leq (8/c'_g)\nu \leq \kappa_{\text{NF}} \nu$. The last inequality follows

from $8/c'_g \leq 128/(9c_g) \leq 4(c_g + 4)/c_g^2$, which holds since $c'_g \geq \frac{9}{16}c_g$ by Section S1.1.4(e) and $c_g \leq \frac{1}{4}$. Since a bond where $N_R \neq N_{R_0}$ lies on some ∂P_k , we have $S \subseteq S_0 \cup \bigcup_k I_k \subseteq T$.

Step 1 (a spanning forest). For each k let t_k be the least j for which a chain $P_k = P_{k_0}, \dots, P_{k_j}$ as in Lemma 7.50(iv) exists. Let H be the graph on T whose edges are the edges of the $\text{MST}(I_k)$ and, for each k , one parent edge at distance at most $2g$: from an interface of I_k to one of S_0 if $t_k = 0$, and otherwise to one of $I_{k'}$, where $P_{k'}$ is the second piece of a chain of length t_k , so that $t_{k'} = t_k - 1$ (a bond and its dual coarse face are within distance $\sqrt{3}r_0 \leq g/2$). Each node of $T \setminus S_0$ lies in some I_k , and t decreases along parent edges. Hence the parent edges lead to S_0 . Consequently (add a virtual node joined to all of S_0 , and remove it from a spanning tree) H contains a spanning forest F of T with roots in S_0 . Put $B := \sum_{e \in F} j(\ell_e) \leq \sum_{e \in H} j(\ell_e)$.

Step 2 (encoding). We record the roots (at most 2^{n_0} choices), the shape of the plane forest on $|T| \leq n + n_0$ nodes (at most 4^{n+n_0}), and for each child the number j_e and the child among the at most $C_9(2^{j_e} + 1)^3 \leq 8C_9 8^{j_e}$ interfaces within distance $2^{j_e}L$ of its parent. Sequences of $|F|$ nonnegative integers with sum at most $B_{\max}$ (the bound of Step 3, a function of ν only) number at most $2^{B_{\max} + |F|}$. We then record $S \subseteq T$ (at most 2^{n+n_0}) and the values N_I with $\sum N_I^2 = \nu$ (at most 4^ν : signs, and compositions of $\sum |N_I| \leq \nu$). Since R is determined by N_R , we obtain $\ln \#\{R : \nu(R) = \nu\} \leq C(n + n_0 + \nu) + B_{\max} \ln 16$ with $C := \ln(64C_9) + 2$.

Step 3 (the bound on B). By Lemma 7.51, $\sum_{e \in \text{MST}(I_k)} j(\ell_e) = \sum_{j \geq 0} (\varphi_{2^j L}(I_k) - 1)$. The scales with $2^j L < R_{\text{NF}}$ number at most $j_* := c_3^{\text{NF}} F_{\text{unit}} / \ln 2 + 1$, each contributing at most $|I_k| - 1$, while the scales with $2^j L \geq R_{\text{NF}}$ contribute exactly $J(P_k)$ (Definition 3.18). A parent edge has $j \leq j_g := \log_2(2g/L) + 1 = O(\ln \beta_{\text{GM}})$, and there are at most $\sum_k |I_k| \leq \kappa_{\text{NF}} \nu$ pieces. With (7.34),

$$B \leq (j_* + j_g) \kappa_{\text{NF}} \nu + \theta_{\text{NF}} F_{\text{unit}} \sum_k |\partial P_k| / L^2 \leq \left(\frac{c_3^{\text{NF}} \kappa_{\text{NF}}}{\ln 2} + 8\theta_{\text{NF}} \right) F_{\text{unit}} \nu + (1 + j_g) \kappa_{\text{NF}} \nu =: B_{\max}.$$

Step 4. Since $4c_3^{\text{NF}} \kappa_{\text{NF}} = c'_0/4$ and $8\theta_{\text{NF}} \ln 16 \leq c'_0/8$ (Definition 3.18), $\ln \# \leq \frac{3}{8} c'_0 F_{\text{unit}} \nu + O((1 + \ln \beta_{\text{GM}}) \nu) \leq \frac{c'_0}{2} F_{\text{unit}} \nu$ for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, as F_{unit} is exponentially large (Section S1.8).

It remains to prove $\Theta_4 \leq \frac{1}{4}$. By Lemma 7.30(a), $F_1^{\text{loc}}(\mathcal{S}) \geq 2F_{\text{unit}} \nu(\mathcal{S})$ for each wall polymer $\mathcal{S}$. Hence the summand of Θ_4 is at most $e^{-c' F_{\text{unit}} \nu(R)}$, and $\Theta_4(r) \leq \sum_{\nu \geq 1} e^{(c'/2) F_{\text{unit}} \nu} e^{-c' F_{\text{unit}} \nu} \leq 2e^{-(c'/2) F_{\text{unit}}} \leq \frac{1}{4}$. The bound $\Theta_4 \leq \frac{1}{4}$ rests on the factor 2 of Lemma 7.30(a), that is, on the constant 1/18 of Theorem 7.13, set against the allowance of the count.

Proposition 7.53 (Frozen flux costs area). *For every $r \in \Gamma_{\text{long}}$, $Z_r \leq e^{-\sigma' A(r)} Z$.*

Proof. We extract an irreducible realization from each configuration and re-expand it as a seed, which pays for its own cells and shape from its own F_1^{loc} and returns the fraction $c'/2$. The outer factors are bounded by Theorem 7.49 and by the quantity being bounded, which closes the argument. Put $C_{\text{FF}} := \sup_{s \in \Gamma_{\text{long}}} \omega(s) Z_s / Z$. Then $1 \leq C_{\text{FF}} < \infty$ (take $s = 0$; Proposition 7.47). Let $r \neq 0$ with $Z_r \neq 0$.

(1) *Union bound.* For each configuration we choose one irreducible realization $R(N) \subseteq \mathcal{P}(N)$ by a fixed rule. All terms are nonnegative (Lemma 7.42). Hence $Z_r \leq \sum_{R \in \mathfrak{S}(r)} Z_r(R)$, with $Z_r(R)$ as in Theorem 7.27(c). Dropping the condition $R = R(N)$ only enlarges the right side.

(2) *Re-expansion.* By Theorems 7.27(c) and 7.49(b), with $\sigma := (\bigcup R, N_R)$, $Z_r(R) \leq \sum_{Y \supseteq \bigcup R} \sum_{N_Y \in \mathcal{N}_{\sigma}^{\text{long}}(Y)} |\mathcal{K}_{N_Y}^{(\bigcup R)}(Y)| \sum_q |r(Y)(q)| Z_{r - [N_Y] - q}$.

(3) *Weights.* By Lemma 7.43, $\omega(r) \leq \omega([N_Y])\omega(q)\omega(r - [N_Y] - q)$, and $\omega([N_Y]) \leq e^{(\varepsilon_E/2)F_1^{\text{loc}}(Y)}$ by Theorem 7.13. By the definition of C_{FF} and Theorem 7.49 (a) ($Y \supseteq \bigcup R \neq \emptyset$), $\sum_q |r(Y)(q)|\omega(q)\omega(r - [N_Y] - q)Z_{r-[N_Y]-q} \leq \frac{8}{31}e^{\alpha|Y|}C_{\text{FF}}Z$.

(4) *Activity.* So that the sum takes the form $\Theta_4(r)$ of Proposition 7.52, we insert $1 \leq e^{c_w F_1^{\text{loc}}(\bigcup R)}e^{-\lambda_E|R|}$, which is valid since $F_1^{\text{loc}}(\bigcup R) \geq 2\sigma_F|R|$ and $2c_w\sigma_F = \lambda_E$. Lemma 7.48 (c') over long fields (Proposition 7.46), with the seed configuration σ (all of whose $\mathcal{U}$ -cells lie in wall polymers) and $\varepsilon = \varepsilon_E/2 \leq c'/8$, bounds the sum over Y and N_Y , including $e^{\alpha|Y|}e^{(\varepsilon_E/2)F_1^{\text{loc}}(Y)}e^{c_w F_1^{\text{loc}}(\bigcup R)}$, by $\prod_{S \in R} e^{-(c'/2)F_1^{\text{loc}}(S)}$. Collecting (1) to (4), we obtain $\omega(r)Z_r/Z \leq \frac{8}{31}C_{\text{FF}}\Theta_4(r) \leq \frac{2}{31}C_{\text{FF}}$ by Proposition 7.52. The same estimate holds without the insertion: the proof of $\Theta_4 \leq \frac{1}{4}$ bounds $e^{-\lambda_E|R|}$ by 1, and Lemma 7.48 (c') without the factor $e^{c_w F_1^{\text{loc}}(Z, N_Z)} \geq 1$ follows from (c').

(5) *The supremum.* With $s = 0$, we obtain $C_{\text{FF}} \leq \max(1, \frac{2}{31}C_{\text{FF}})$. Since $C_{\text{FF}} < \infty$, this gives $C_{\text{FF}} = 1$.

No position factor (a sum over a cycle detecting r , or $|\gamma|$) appears, since the position entropy is part of the count of $\mathfrak{S}(r)$. In particular, a long handle with a tapered thin neck needs no separate treatment, since its cuts of one class are compared with R_0 by Lemma 7.50.

Lemma 7.54 (The area-weighted bound). *Let $\alpha = \ln 8$, $\varepsilon_E \in (0, c'_0/8]$, $\sigma' = \varepsilon_E\sigma_F$ and $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$. For every nonempty union X of $\mathcal{U}$ -cells and every $p \in \Gamma_{\text{long}}$, $0 \leq \mathcal{Z}_p(X)/Z \leq \frac{8}{31}e^{\alpha|X|}e^{-\sigma'A(p)}$. The bound is uniform in G' , Λ° and C .*

Proof. By Theorem 7.49 (b) and Lemma 7.43, we have $\omega(p)\mathcal{Z}_p(X)/Z \leq \sum_q |r(X)(q)|\omega(q) \cdot \omega(p - q)Z_{p-q}/Z$, and this is at most $\|r(X)\|_\omega \leq \frac{8}{31}e^{\alpha|X|}$ by Proposition 7.53 and Theorem 7.49 (a). Nonnegativity is Lemma 7.42.

7.9 The deterministic area law

In this subsection we prove the deterministic area law, Theorem 7.57.

Lemma 7.55 (Area split). *Let $X \supseteq X_1$ be a union of $\mathcal{U}$ -cells, N_X an inside field at $k = 1$ on X , and $p := p_1(N_X)$ its outer class (Definition 7.25). Then $\sum_{e \in E(X)} |N_X(e)| + A(p) \geq A_{G'}(C)$.*

Proof. Since linking numbers are additive, the inside walls and a cheapest cochain with the forced periods form a wall field at $k = 1$, which spans C modulo the frozen complex (Theorem 7.14). Let $N' \in \mathbb{Z}^E$ realize $A(p)$ (finite, as $-\xi - N_X 1_{E(X)}$ has periods p). By definition of p_1 , $N_X 1_{E(X)}$ has periods $-\text{lk}(\cdot, C) - p$, and hence $M := N_X 1_{E(X)} + N'$ has periods $-\text{lk}(\cdot, C)$. Since ξ has periods $\text{lk}(\cdot, C)$ (Lemma 7.6(d)), we get $M + \xi = \nabla m$ for some $m : V \rightarrow \mathbb{Z}$, and $M = \nabla m - \xi$ is the $k = 1$ wall field of m . By Theorem 7.14, $\sum_E |M| \geq \#\{M \neq 0\} \geq A_{G'}(C)$, and $\sum_E |M| \leq \sum_{E(X)} |N_X| + \sum_E |N'| = \sum_{E(X)} |N_X| + A(p)$.

Lemma 7.56 (Seed count). *With $C_{\mathcal{U}}$ and c_u as in (R7) and $C_X := 3C_{\mathcal{U}}/c_u$, $|X_1| \leq C_X(\tilde{m}^2 RT + 3C_{\mathcal{U}}\tilde{m}(R + T) + 9C_{\mathcal{U}}^2)$.*

Proof. The $\mathcal{U}$ -cells of X_1 meet bonds crossing Ξ (Definition 7.20). Thus they lie within Euclidean distance $D_E + 2$ of Ξ , $D_E = C_{\mathcal{U}}\tilde{m}^{-1} \geq 4$, a set of volume at most $3D_E(R + 3D_E)(T + 3D_E)$. Each of these cells has at least $c_u\tilde{m}^{-3}$ sites by (R7).

For a rectangle C of sides R, T we put $R_1(\beta) := (2r_\square + 5)L$, where L is the block scale of Definition 3.1 at $\tau = K\beta$.

Theorem 7.57 (Deterministic area law). *Fix the constants of Section S1.1.1 of the supplement. There is β_{GM}^0 such that for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, for every output $G' = G'(D)$ of the construction, for every D , and for every rectangle C with $\min(R, T) \geq R_1(\beta) = (2r_\square + 5)L$,*

$$W_{G'}(C) \leq \exp(-\sigma_V[A_{G'}(C) - \eta_S RT] + c_N(R + T)),$$

with

$$\sigma_V = \varepsilon_E \sigma_F = \frac{c'_g \tilde{m} L}{9216} \frac{\tilde{m}}{\tau} \quad (\varepsilon_E = \frac{1}{128}), \quad \eta_S = \frac{18c_2 C_X}{\varepsilon_E \pi^2 c'_g \tilde{m} L} \tilde{m} \beta_{\text{GM}} = O(\tilde{m} \beta_{\text{GM}} \log \tau),$$

and c_N as in Remark 7.58. The only geometric inputs are (R2), (R5) to (R10), Theorems 6.5, 6.8 to 6.11 and Proposition 6.17; the frame, which does not depend on D : the box (Lemma 3.3), the grid with its windows and removal cells (Definition 3.4) and the alignment (A1) to (A3) (Lemma 3.5); the coarse alignment of the holes and frozen bonds, which are unions of coarse sites and of whole coarse faces (Section 2.5); Lemma 4.4, in Lemma 6.13; and (N3) of Theorem 3.20, in Lemma 7.44. The degenerate output $\mathbb{X} = \Lambda^\circ$ ($G' = \emptyset$, $W_{G'}(C) = 1$, $A_{G'}(C) = 0$) is included.

Proof. We bound the outer factors of the loop expansion by Lemma 7.54. Schematically, a fraction ε_E of the wall energy of the seeded polymer, with the area weight, pays $\varepsilon_E \sigma_F A_{G'}(C)$ by the area split, the seed costs $e^{c_2|X_1|}$ (bulk part η_S , the rest a perimeter term), and the near cells give the perimeter factor $\mathcal{P}'_G$. If no bond of G' crosses Ξ (for instance if $G' = \emptyset$), then $X_1 = \emptyset$ and the bound holds trivially. Indeed, in this case $j = 0$ on E , which gives $Z(1, \Xi) = Z(0, \Xi)$ and $W_{G'}(C) = 1$. Moreover, each plaquette of Ξ is then a face of a hole cube or a frozen face (proof of Lemma 7.6(c)), hence lies in $\mathfrak{F}$, and $A_{G'}(C) = 0$ (Definition 2.11 with $\Sigma = \Xi$). Finally, $\sigma_V \eta_S RT + c_N(R + T) \geq 0$. Otherwise $X_1 \neq \emptyset$, and

$$\begin{aligned} W_{G'}(C) &= \sum_{X \supseteq X_1} \sum_{N_X} \mathcal{K}_{N_X}(X) \frac{\mathcal{Z}_{p_1(N_X)}(X)}{Z} \leq \frac{8}{31} \sum_{X \supseteq X_1} \sum_{N_X} |\mathcal{K}_{N_X}(X)| e^{\alpha|X|} e^{-\sigma' A(p_1(N_X))} \quad (7.35) \\ &\leq \frac{8}{31} e^{-\varepsilon_E \sigma_F A_{G'}(C)} \sum_{X \supseteq X_1} \sum_{N_X} |\mathcal{K}_{N_X}(X)| e^{\alpha|X| + \varepsilon_E F_1^{\text{loc}}(X, N_X)} \leq \frac{8}{31} \mathcal{P}'_G e^{c_2|X_1|} e^{-\sigma_V A_{G'}(C)} \\ &\leq \exp\left(-\sigma_V A_{G'}(C) + \sigma_V \eta_S RT + \left[\frac{2c_{\text{near}}}{p'} + 3c_2 C_X C_U \tilde{m} + \frac{9c_2 C_X C_U^2}{R_1(\beta)}\right](R + T)\right), \end{aligned}$$

and the bracket, $b(\beta)$, is at most c_N . The equality is Corollary 7.28 with Lemma 7.29(f). Its terms with $p_1(N_X) \notin \Gamma_{\text{long}}$ vanish, since $\mathcal{Z}_p(X) := 0$ for $p \notin \Gamma_{\text{long}}$ (Definition 7.25(c)). Consequently, in (7.35), each sum over N_X runs over the inside fields at $k = 1$ with $p_1(N_X) \in \Gamma_{\text{long}}$. The first inequality is Lemma 7.54. The second uses $F_1^{\text{loc}}(X, N_X) \geq 2\sigma_F \sum_{E(X)} |N_X|$ (X is a union of elements; Theorem 7.13), Lemma 7.55 and $\sigma' = \varepsilon_E \sigma_F$. The third is Lemma 7.48(a) with $k = 1$, $U = V$, $S_0 = X_1$, $\varepsilon = \varepsilon_E \leq c'/2$, over these fields. For these fields its proof runs as at $k = 0$ (Proposition 7.46). At a fixed configuration the proof uses the label, beyond the walls (Lemma 7.24), only in Lemma 7.32(a), as a function h' on $\mathfrak{R}_\Omega^b$, constant on Ω , with $\nabla h' = P(N_Y + \xi)$ on $E(\mathfrak{R}_\Omega^b)$, where Y is a union of elements, $N_Y := N_X \mathbf{1}_{E(Y)}$ and $\Omega \subseteq Y$ is a

cell. We now show that such an h' exists. Each wall of N_X on $E(\mathfrak{K}_\Omega^\flat)$ lies in Y (first step of the proof of Lemma 7.32(a)), and ξ vanishes off $E(X)$, as $X \supseteq X_1$. Hence $N_Y + \xi = (N_X + \xi)\mathbf{1}_{E(X)}$ on $E(\mathfrak{K}_\Omega^\flat)$, and its period on a closed m3-walk γ in $\mathfrak{K}_\Omega^\flat$ is $-p_1(N_X)(\gamma)$, which is 0 if $|\gamma| < 2L'$, since $p_1(N_X) \in \Gamma_{\text{long}}$. The breadth-first-tree argument of Lemma 7.45, whose closed walks (fundamental cycles of length at most $2d_b + 1$, same-cell closures of length at most $d_b + D_{\text{cell}}$) are shorter than $2L'$, gives h' with values in $P\mathbb{Z}$. The last inequality uses $\log(8/31) < 0$, Lemma 7.56 with $c_2 C_X \tilde{m}^2 = \sigma_V \eta_S$ and $R + T \geq 2R_1(\beta)$, and $\mathcal{P}'_G = e^{n'(V)/p'} \leq e^{2c_{\text{near}}(R+T)/p'}$ (Lemma 7.33). The values of σ_V and η_S are computed in Section S1.8.

Remark 7.58 (The perimeter constant). Put $c_N := \sup_{\beta_{\text{GM}} \geq \beta_{\text{GM}}^0} b(\beta)$ ($b(\beta)$ the bracket in (7.35), $p' = 1.3$). It is finite and depends only on the constants fixed before K : c_{near} is bounded uniformly for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ (Lemma 7.33; $\log(1/\tilde{m})/\beta_{\text{GM}} < 0.0632$), and the last two terms tend to 0 (Section S1.8). Of the constants of Section S1.1.1, Theorem 7.57, its threshold β_{GM}^0 and c_N use only items 0 to 14: K and $\beta_0(K)$ (items 15 and 16) do not enter, since β and K enter Sections 3, 6 and 7 only through $\tau = K\beta$, that is, through $\beta_{\text{GM}} = 4\pi^2\tau$, and every condition on β_{GM} used there is a lower bound in terms of items 0 to 14. This is what the absoluteness of $c = c_N + 10$ in Section 8 needs. No factor $\log(1 + \min(R, T))$ appears: the solid-angle energy never enters (Corollary 7.10(d)).

8 Proof of the main theorem

8.1 The chain

The deterministic area law holds for each frozen set D , and the min-cut lower tail is exactly the exponential moment of the area deficit $RT - A_{G'}(C)$. Multiplying the two, we obtain $e^{-\sigma_V RT} \cdot \mathbb{E}_D e^{\sigma(RT - A)}$, and the two deficits η_S and η_T add. We fix $K > K_*$ and $\beta \geq \beta_0(K)$ (Section 8.2). The conditions defining $\beta_0(K)$ are listed in Section S1.1.4. With $\tau = K\beta$, they include $\beta \geq \pi^{-2}$, $\delta < 1$ and (2.5), that is $\beta \geq \beta_1(K)$ (Lemma 2.6), as well as the hypotheses of Theorems 5.1 and 7.57 for β and $\beta_{\text{GM}} = 4\pi^2\tau$. They also include $\eta_T + \eta_S \leq \frac{1}{2}$ and $\sigma_V \leq 1$, which are true for large τ as $\eta_T \rightarrow 0$ (Theorem 5.1), $\eta_S = O(\tilde{m}\beta_{\text{GM}} \log \tau) \rightarrow 0$ (Theorem 7.57) and $\sigma_V = (c'_g \tilde{m} L / 9216) \tilde{m} / \tau \rightarrow 0$. Let C be a rectangle with $\min(R, T) \geq R_1(\beta) = (2r_\square + 5)L$ (Definition 3.1), $\Lambda^\circ = \Lambda^\circ(C)$ the box of Definition 3.2, and $\Lambda \supseteq C$ any finite box.

Proof of Theorem 1.2. We prove the chain (1.4) to (1.8), with $\mathbb{E}_D$ the expectation over $D \sim \text{Ber}(\delta)^{\otimes (\Lambda_2^\circ \setminus \partial_2 \Lambda^\circ)}$.

Step (1.4). Lemma 3.3(e) gives $\text{dist}(C, \partial \Lambda^\circ) \geq \max(R, T) \geq 1$. Hence Proposition 2.9 applies with $\Lambda' = \Lambda^\circ$ and $\delta' = \delta$. The box Λ° depends only on C and the constants (Lemma 3.3(a)), not on Λ or D , and is therefore fixed before D is drawn.

Step (1.5). For each $D \subseteq \Lambda_2^\circ \setminus \partial_2 \Lambda^\circ$ the output $(\mathbb{X}(D), S(D))$ is a deterministic function of D with $D \subseteq \mathfrak{F}(D)$ (Theorem 3.20). Hence Proposition 2.10 applies to each D .

Step (1.6). Theorem 7.57 applies to each output $G'(D)$ and each rectangle with $\min(R, T) \geq R_1(\beta)$, pointwise inside $\mathbb{E}_D$. We need it for every D , because $\text{Ber}(\delta)$ charges each subset of $\Lambda_2^\circ \setminus \partial_2 \Lambda^\circ$ and the construction is deterministic. Its functional $A_{G'(D)}(C)$ is the min-cut area of Definition 2.11 for the same output on the same box Λ° .

Step (1.7) is the identity $-\sigma_V A = -\sigma_V RT + \sigma_V (RT - A)$.

Step (1.8). Theorem 5.1 at $\sigma = \sigma_V \in (0, 1]$ bounds the expectation in (1.7) by $\exp(\sigma_V \eta_T R T + 4|C| + 4)$, for the same random set D , box Λ° , output $G'(D)$ and functional $A_{G'(D)}(C)$ as in Steps (1.5) and (1.6).

Constants. Since $|C| = 2(R + T)$ and $R + T \geq 2$, we have $4|C| + 4 \leq 10(R + T)$, and (1.8) gives (1.2) with $\sigma(\beta) := (1 - \eta_T - \eta_S)\sigma_V$, $R_1(\beta) = (2r_\square + 5)L$, both depending on K through $\tau = K\beta$, and $c := c_N + 10$, which is absolute in the sense of Theorem 1.2 because c_N depends only on items 0 to 14 of Section S1.1.1, fixed before K (Remark 7.58). By $\eta_T + \eta_S \leq \frac{1}{2}$ and $c'_g \tilde{m} L \geq \frac{9}{16} c_g c_L$ (Section S1.1.3),

$$\sigma(\beta) \geq \frac{c_g c_L \tilde{m}}{2^{15} \tau} \geq \frac{(8\pi^2)^{1/2}}{2^{16}} c_g c_L \tau^{-1/2} e^{-\pi^2 v_{\text{Cb}}(0)\tau} \geq 1.35 \cdot 10^{-4} c_g c_L \tau^{-1/2} e^{-2.49436 \tau}, \quad (8.1)$$

by $\tilde{m} \geq m_D/2$ and $m_D = (8\pi^2 \tau)^{1/2} e^{-\pi^2 v_{\text{Cb}}(0)\tau}$ (Section 6), with the Debye exponent $\pi^2 v_{\text{Cb}}(0) = 2.4943550872 \dots$ rounded up and the prefactor rounded down (rounding rule in Section S1.1.3). This proves the second assertion of Theorem 1.2 for $\beta \geq \beta_0(K)$. Remark 8.1 gives the case $\beta < \beta_0(K)$ at the same K , and the first assertion follows with one fixed $K > K_*$, for instance $K = 378$.

Remark 8.1 (Small β ; uniformity). For fixed $K > K_*$ and $\beta < \beta_0(K)$, Theorem 2.1(b), pointwise in Λ and C , gives Theorem 1.2 at β with the three constants $\sigma(\beta_0(K))$, c , $R_1(\beta_0(K))$; only the Wilson side is monotone in β (raising β raises τ but lowers δ , so the right side of Theorem 2.5 is not). The bound depends on Λ only through Proposition 2.9, on D only inside $\mathbb{E}_D$, and on C only through R, T ; by Theorem 2.1(c) it holds for the infinite-volume limit $\langle \cos \theta_C \rangle_\beta^W$, nondecreasing in β , so the string tension $\sigma_\infty(\beta) := \liminf_{\min(R, T) \rightarrow \infty} -\frac{1}{RT} \log \langle \cos \theta_{C_{R, T}} \rangle_\beta^W \in [0, \infty]$ ($C_{R, T}$ a coordinate rectangle of sides R, T) is nonincreasing in β and at least $\sigma(\beta)$.

Remark 8.2 (Loops below the non-fragmentation scale). If $\text{diam } \Lambda^\circ \leq R_{\text{NF}}$ (for instance if both $3\sqrt{3} \max(R, T) + \sqrt{3}$ and $\sqrt{3}(r_0(\ell + 3) + 1)$ are at most R_{NF}), Phase 2 makes no firing (Proposition 3.19) and Proposition 5.11, with the deficit $\eta_T^{(1)}$, replaces Theorem 5.1 in Step (1.8); the string tension uses Theorem 5.1 in full.

8.2 The order of the constants

We fix the constants once, in the following order (values, and the statements that use each constant, in Section S1.1.1). (0) Absolute constants, among them $\pi^2 v_{\text{Cb}}(0)$, the threshold 14.9662 of Lemma 2.6 and the absolute constants quoted from [GM82, GM81a, BF80] (those that depend on the parameters C , δ_{GM} , μ , κ_S , λ_{PV} of Theorem 6.5 are fixed with them, in (12) and (13)); (1) r_0 ; (2) $c_g = \frac{1}{4}$ (all $c_g \in (0, \frac{1}{4}]$ work); (3) $\eta_w = \frac{1}{10}$; (4) $c_\rho = \frac{1}{12}$, that is $\lambda = 12/L_c$; (5) $k_P/L_c = 304$; (6) c_1 and c_{sh} ; (7) $c_s \in (0, 10]$; (8) the forcing threshold ϑ , the penalty μ_B and the thin-set constant ε_{th} ; (9) the weak-link constants $\chi = 2.0 \cdot 10^{-8}$ of (R6) and $\chi_{\text{N3}} = \chi/3$ of (N3); (10) the constants derived from these, among them C'_* , C_{free} , C_a , C_b , C_c and C_{resp} ; (11) $c_L = 1.6 \cdot 10^{-19}$, and then the scales L_c , L , k_P , ℓ , which depend on β only through $\tilde{m}$; (12) the Stage 1 and Stage 2 constants: $C \in (0, \frac{1}{4}]$ and $\delta_{\text{GM}} \leq \frac{1}{2}$ of Theorem 6.5, $\eta_H = 0.62$, the rate $c_{10} \leq \frac{7}{9} r_C$ with r_C taken at $\tilde{m}L = c_L/2$, then c''_{10} , then $\mu \geq 1 + 2c''_{10}$ and κ_S , and $c'_0 = \frac{1}{16}$; (13) λ_{PV} , and with it M_Y , $\tilde{m} = \lambda_{\text{PV}} M_Y$ and the constants of [GM82, GM81a, BF80] and of Stage 1 that depend on λ_{PV} ; (14) the cut-rule constants θ_{NF} and c_3^{NF} ; (15) $K > K_*$; (16) $\beta_0(K)$. Four placements are forced. First, k_P/L_c , ϑ and ε_{th} precede χ , because the expansion constant of Theorem 4.16(c), which bounds χ , is computed

from them. Second, χ precedes c_L , because the condition $C'_P c_L^2 \leq (1 - 4/\pi^2)/8$ that fixes c_L has $C'_P \leq 2916r_{\square}^5 C_{\text{bl}}^2/\chi^2$. Since $1 - 4/\pi^2 < \eta_H$ and $\tilde{m}L \leq c_L$, the condition that fixes c_L also gives the condition $C'_P(\tilde{m}L)^2 \leq \eta_H/8$ of Stage 2. Third, C , δ_{GM} , c_{10} , c''_{10} , μ and κ_S precede λ_{PV} , because the Debye condition of Theorem 6.5 gives $\lambda_0 = \lambda_0(\kappa_S, \mu, C, \delta_{\text{GM}})$ and each earlier constant uses λ_{PV} only through $\lambda_{\text{PV}} \leq \frac{1}{4}$. Fourth, θ_{NF} and c_3^{NF} are built from the pure number c'_0 , and no earlier constant depends on them. Each condition defining $\beta_0(K)$ (Section S1.1.4) is, at fixed K , a lower bound on $\tau = K\beta$ that persists for larger τ . Those of Section 2 are $\beta \geq \pi^{-2}$, $\beta \geq \beta_1(K)$ and $\delta < 1$, which (2.4) gives for all large β , and in all the others the left side is a fixed function of τ (built from $\tilde{m}$, τ , $\log \tau$ and the doubly exponential scale R_{NF}) tending monotonically to its limit. The value $\beta_0(K)$ is the largest of these thresholds divided by K . It is very large ($|\log \varepsilon''| \gtrsim 10^{14}$ in Theorem 5.1), and the constants are far from sharp, but the proof uses only that those fixed before K do not depend on τ .

9 Corollaries

In this section we prove the two corollaries of the introduction. The central phases are described by a $U(1)$ Wilson model with configuration-dependent couplings at most $n\beta$ (Section 1.2), also in the presence of a Higgs term invariant under the centre. Since the plaquettes of a finite 2-complex are attached along closed edge paths, U_p (the ordered product of the links around p) and θ_p are defined. We write Tr for the trace of $n \times n$ matrices.

Proposition 9.1 (Fröhlich's comparison). *Let $G \subseteq U(n)$ be a closed subgroup with $\{zI : |z| = 1\} \subseteq G$, X a finite 2-complex, and $\mu_{G,\beta}(dU) \propto \prod_p \exp(\beta \text{Re Tr } U_p) \prod_e dU_e$, dU_e Haar measure on G (free boundary conditions). For every closed edge path ℓ in X ,*

$$\left| \left\langle \text{Tr} \prod_{e \in \ell} U_e \right\rangle_{G,\beta} \right| \leq 2n \langle \cos \theta_\ell \rangle_{X,2n\beta}^W, \quad (9.1)$$

where the right side is the $U(1)$ Wilson model on the same complex.

Proof. Let ν be the law on $U(1)^{E(X)} \times G^{E(X)}$ with density proportional to $\prod_p \exp(\beta \text{Re Tr}(\xi_p U_p))$, where $\xi_e = e^{i\phi_e}$ and $\xi_p = e^{i\phi_p}$ is the product of the $\xi_e^{\pm 1}$ around p . Since $\xi_e I \in G$ is central, its image under $(\xi, U) \mapsto \xi U$ is $\mu_{G,\beta}$, and $\text{Tr} \prod_{e \in \ell} (\xi_e U_e) = e^{i\phi_\ell} \text{Tr} \prod_{e \in \ell} U_e$. Given U , ξ has the law $\langle \cdot \rangle_U$ with density proportional to $\prod_p \exp(\beta |\text{Tr } U_p| \cos(\phi_p - \psi_p))$, $\psi_p := -\arg \text{Tr } U_p$. Hence $|\langle \text{Tr} \prod_{e \in \ell} U_e \rangle_{G,\beta}| \leq n \sup_U \sup_\alpha \langle \cos(\phi_\ell - \alpha) \rangle_U$. We bound $\langle \cos(\phi_\ell - \alpha) \rangle_U$ by $\langle \cos \theta_\ell \rangle_{X,n\beta}^W$ using [MMSP78, Proposition 1 and the remark after its proof, p. 21], with couplings $n\beta \geq \beta |\text{Tr } U_p|$ on the plaquette vectors (terms with the same vector up to sign combined). Consequently, the left side of (9.1) is at most $n \langle \cos \theta_\ell \rangle_{X,n\beta}^W$, which is at most the right side, since $U(1)$ Wilson loops are nonnegative and nondecreasing in the coupling by Ginibre's inequality, on all finite complexes (as in Theorem 2.1(a), (b)). The bound (9.1) keeps the constants $2n$ and $2n\beta$ of [Frö79, (23), (24)]. The remark is stated (p. 21) for α equal to the phase of the interaction. For each α it follows from the proof of Proposition 1, since for the edge vector M of ℓ and two copies ϕ, ϕ' of the angles $\cos(M \cdot \phi) - \cos(M \cdot \phi' - \alpha) = 2 \sin(M \cdot \Phi - \frac{\alpha}{2}) \sin(M \cdot \Phi' - \frac{\alpha}{2})$, $\Phi = \frac{1}{2}(\phi' + \phi)$, $\Phi' = \frac{1}{2}(\phi' - \phi)$.

Corollary 9.2 (Corollary 1.3). *Let G be as in Proposition 9.1, with the Wilson action $\beta \text{Re Tr } U_p$ on finite boxes of $\mathbb{Z}^3$, ρ a representation of G with $\rho(zI) = 1$ for $|z| = 1$, (Φ, ρ_0) a*

probability space of Higgs field values, and $h \geq 0$ a measurable function of $((\rho(U_e))_e, (\varphi_x)_x)$, the Higgs weight, for any values of the Higgs couplings, with $0 < \int h < \infty$ ($h \equiv 1$ is the model without Higgs field). On a finite box Λ take the measure with density proportional to $h((\rho(U_e))_e, (\varphi_x)_x)$ times $\prod_p \exp(\beta \operatorname{Re} \operatorname{Tr} U_p)$ with respect to $\prod_e dU_e \prod_x \rho_0(d\varphi_x)$. Then (9.1) holds for every closed path ℓ , and the bound of Corollary 1.3 for every rectangle C with $\min(R, T) \geq R_1(2n\beta)$ and every finite box $\Lambda \supseteq C$, with σ , R_1 , c of Theorem 1.2. In particular the string tension of G at coupling β is at least $\sigma(2n\beta) > 0$, also for $U(2)$ with an adjoint Higgs field, at all gauge and Higgs couplings.

Proof. Since $\rho(\xi_e U_e) = \rho(\xi_e I) \rho(U_e) = \rho(U_e)$, the proof of Proposition 9.1 applies to the joint law with the extra factor $h((\rho(U_e))_e, \varphi) \prod_x \rho_0(d\varphi_x)$, which cancels from the conditional law of ξ . We then apply Theorem 1.2 at $2n\beta$. For $U(2)$, $\operatorname{Ad}(zU) = \operatorname{Ad}(U)$. ([Frö79, Corollary 2] states confinement of static quarks for all $U(n)$ in three dimensions, from the comparison and the results cited there. The area law is the new part.)

The Georgi–Glashow model. On a finite box Λ we take links $U_e \in SU(2)$, $U_{-e} = U_e^{-1}$, a unit adjoint Higgs field $\mathbf{n}_x \in S^2$ (the London limit), the adjoint representation $R : SU(2) \rightarrow SO(3)$, $\beta, \kappa > 0$, and

$$\mu_{\Lambda; \beta, \kappa} \propto \exp\left(\beta \sum_p \frac{1}{2} \operatorname{Tr} U_p + \kappa \sum_{e=(x,y)} \mathbf{n}_x \cdot R(U_e) \mathbf{n}_y\right) \prod_e dU_e \prod_x d\mathbf{n}_x \quad (9.2)$$

(dU_e Haar, $d\mathbf{n}_x$ uniform on S^2). The weight in (9.2) is invariant under $U_e \mapsto g_x U_e g_y^{-1}$, $\mathbf{n}_x \mapsto R(g_x) \mathbf{n}_x$ ($g \in SU(2)^\Lambda$), and $f(U, \mathbf{n})$ is *gauge invariant* if it is. Write $D(\theta)_e := \operatorname{diag}(e^{i\theta_e}, e^{-i\theta_e})$ and e_3 for the constant field $\mathbf{n}_x = e_3$.

Proposition 9.3 (Georgi–Glashow model at infinite Higgs coupling). *For every finite box Λ , every $\beta > 0$ and every bounded continuous gauge invariant f , the limit $\lim_{\kappa \rightarrow \infty} \langle f \rangle_{\Lambda; \beta, \kappa}$ equals $\langle f(D(\theta), e_3) \rangle_{\Lambda, \beta}^W$, the expectation in the $U(1)$ Wilson model at coupling β . In particular, for every closed path C , $\langle \frac{1}{2} \operatorname{Tr} \prod_{e \in C} U_e \rangle_{\Lambda; \beta, \kappa} \rightarrow \langle \cos \theta_C \rangle_{\Lambda, \beta}^W$.*

Proof. We pass to the unitary gauge. Since $R(SU(2))$ is transitive on S^2 , gauge invariance gives $\iint F d\mathbf{n} dU = \int F(U, e_3) dU$ for $F = fw$ and $F = w$, w the weight of (9.2). With U_e of diagonal entries $a_e, \bar{a}_e$ and off-diagonal entries $b_e, -\bar{b}_e$, the hopping factor is $\prod_e e^{\kappa - 2\kappa|b_e|^2}$, Haar measure is b_e uniform on the unit disc times $\arg a_e = \theta_e$ uniform, and as $\kappa \rightarrow \infty$ the weights $e^{-2\kappa|b_e|^2}$ concentrate at $b = 0$, where $U_e = D(\theta)_e$ and $\frac{1}{2} \operatorname{Tr} U_p = \cos \theta_p$. The limit is taken in each fixed box, in the London limit $|\mathbf{n}_x| = 1$, and for $\frac{1}{2} \operatorname{Tr}$ (the full trace gives $2 \cos \theta_C$).

Corollary 9.4 (Corollary 1.4). *At $\kappa = \infty$, for every $\beta > 0$, every rectangle C with $\min(R, T) \geq R_1(\beta)$ and every finite box $\Lambda \supseteq C$, the limit of $\langle \frac{1}{2} \operatorname{Tr} \prod_{e \in C} U_e \rangle_{\Lambda; \beta, \kappa}$ as $\kappa \rightarrow \infty$ has absolute value $\langle \cos \theta_C \rangle_{\Lambda, \beta}^W$, which is at most $\exp(-\sigma(\beta)RT + c(R + T))$.*

Proof. This follows from Proposition 9.3, Theorem 2.1(a) and Theorem 1.2.

Remark 9.5 (Villain action, uniformly in free boxes; scope). The theorem of [GM82] holds in the form of Definition 1.1: for $\tau \geq K\beta_0(K)$ and $\beta := \tau/K$, Lemma 2.3(c),(d) as in Proposition 2.9 give $\langle \cos \theta_C \rangle_{\Lambda, \tau}^V \leq g_{\Lambda^\circ, \tau}^{\operatorname{fix}}(C; \emptyset) \leq \mathbb{E}_D g_{\Lambda^\circ, \tau}^{\operatorname{fix}}(C; D)$, Steps (1.5) to (1.8) give the bound of Theorem 1.2 at β , and Lemma 2.3(b), pointwise in Λ , gives smaller τ . The Georgi–Glashow

model at finite Higgs and weak gauge coupling is not covered: the centre $\mathbb{Z}_2$ of $SU(2)$ contains no $U(1)$, and [Frö79, Theorem 1] lands on the pure $\mathbb{Z}_2$ gauge theory, which confines only at strong coupling. The proof uses $d = 3$ throughout.

A Brydges–Federbush §9.1 on slit domains, and a counting step

In this appendix we prove Lemmas 7.37 and 7.48, carrying [BF80, §9.1] with [Bry78, §5] over to G' . In the count of the sets we keep the tree weight $w(T)$ outermost, which cancels the moments $d_j!$ that the count produces (Proposition A.4).

Conventions. We assume that (U, k) is admissible and that $\sigma = (Z, N_Z)$ is a seed configuration in (U, k) (Section 7.6). At $k = 0$ the configuration sets may be those of long wall fields (Proposition 7.46). We write $|Y|$ for the number of $\mathcal{U}$ -cells of Y and set $\text{dist}(A, B) := \tilde{m} d_{G'}(A, B)$. The $\mathcal{U}$ -cell graph joins two $\mathcal{U}$ -cells when a bond of G' joins them, and $D_{\mathcal{U}}$ bounds its degree. The quantity $\mathcal{K}_{N_X}^{(Z)}(X)$ is the sum over $\bar{y} = (Y_1, \dots, Y_n)$, $Y_1 = Z$, of the terms of Proposition 7.22. For $2 \leq j \leq n$, *link* j is the operator $\kappa(j-1)[Y_j]$ of (C.6), which creates Y_j , differentiates s_{j-1} , and is a sum of elementary terms of the five types listed after (C.5). We also write $X_j := Y_1 \cup \dots \cup Y_j$. A *tree* is a map $T : \{2, \dots, n\} \rightarrow \mathbb{N}$ with $T(j) < j$. We put $d_i(T) := \#\{j : T(j) = i\}$. Thus $d_n(T) = 0$ and $\sum_i d_i(T) = n - 1$. The weight of [BF80, (9.115)] is

$$q(T, s) := \prod_{j=2}^n \prod_{k=T(j)}^{j-2} s_k, \quad w(T) := \int_{[0,1]^{n-1}} q(T, s) ds \quad (\text{A.1})$$

since $\frac{d}{ds_{j-1}}(s_{T(j)} \cdots s_{j-1}) = s_{T(j)} \cdots s_{j-2}$. The variable s_{n-1} does not occur.

Lemma A.1 (Tree form). *Let $X \supseteq Z$, $N_X \in \mathcal{N}_\sigma(X)$ and $\bar{y} = (Z, Y_2, \dots, Y_n)$, $n \geq 2$. Expand $\kappa(\bar{y}, s) Q_{N_X}^{\mathfrak{h}}(X)$ into elementary terms $\theta = (\theta_2, \dots, \theta_n)$, θ_j a term of link j , and assign to θ a tree $T = T(\theta)$ and $\mathcal{U}$ -cells $\Delta'_j \subseteq Y_{T(j)}$, $\Delta''_j \subseteq Y_j$: if θ_j is of type (2) to (5), it contains $\partial_{s_{j-1}} \mathbf{C}^{\bar{y}}(x, y)$ with $x \in Y_j$, $y \in Y_a$, $a < j$, and $T(j) := a$, Δ'_j , Δ''_j the $\mathcal{U}$ -cells of x , y ; if $\theta_j = (\partial_{s_{j-1}} w_s(\tau)) e_\tau$ is of type (1), $T(j) := \min\{i : \text{supp } \tau \cap Y_i \neq \emptyset\}$ and Δ'_j , Δ''_j are the first $\mathcal{U}$ -cells, in a fixed order, of $\text{supp } \tau$ in $Y_{T(j)}$ and in Y_j . Then*

$$(\theta\text{-term}) = q(T, s) R_\theta(s), \quad (\text{A.2})$$

where $R_\theta(s)$ depends on s only through the weights $w_s(\tau') \in [0, 1]$ of E -terms, through the covariance $\mathbf{C}_s^{\bar{y}}$ and through $e^{E(X, s)}$. Each θ receives exactly one tree.

Proof. This is the tree decomposition [Bry78, (5.1), (5.2)] of the expansion of Proposition 7.22, with the following modifications. (1) We interpolate the covariance as in [GM82, App. E, (E.3)]: for $x \in Y_j$, $y \in Y_a$, $\partial_{s_{j-1}} \mathbf{C}^{\bar{y}}(x, y) = s_a \cdots s_{j-2} \mathbf{C}(x, y)$. (2) For the weights $w_s(\tau)$ we use the rule of [GM82, p. 89], under which $i \in I$ if $1 \leq i \leq n-1$ and τ has a cube $a_\alpha \subset Y_j$ for some $j > i$ and a cube $a_\beta \subset \bigcup_{\ell \leq i} Y_\ell$. Under this rule the expansion is an identity (proof of Proposition 7.22), and $\partial_{s_{j-1}} w_s(\tau) = \prod_{k=T(j)}^{j-2} s_k$, as among $k \leq j-1$ the set $\text{supp } \tau$ meets both X_k and its complement exactly for $k \geq T(j)$. The rule printed in [BF80, p. 212] has Y_{i+1} in place of $\bigcup_{j>i} Y_j$. Under that rule a term meeting Y_1 and Y_3 but not Y_2 carries no factor s_1 , and no tree weight dominates it. Each s_k is differentiated once, in link $k+1$ (Table S1.3, S6).

As the bounds of Section 7.6 and Appendix B on $R_\theta(s)$ are uniform in s (Table S1.3, S1) and the weights lie in $[0, 1]$, the identity $\int q(T, s) ds = w(T)$ gives

$$|\mathcal{K}_{N_X}^{(Z)}(X)| \leq \sum_{\bar{y}} \sum_T w(T) \sum_{\theta: T(\theta)=T} \sup_s \left| \int d\mu_s e^{E(X,s)} R_\theta(s) Q_{N_X}^\natural(X) \right|_0. \quad (\text{A.3})$$

Lemma A.2 (The s -integral). *For $n \geq 2$ and $y_1, \dots, y_n \geq 0$,*

$$\sum_T \int_{[0,1]^{n-1}} q(T, s) \prod_{j=2}^n y_{T(j)} ds \leq e^{y_1 + \dots + y_{n-1}}. \quad (\text{A.4})$$

In particular $\sum_T w(T) \leq e^{n-1}$.

Proof. This is [Bry78, (5.5)] and [BF77, Lemma 5.5], for real weights. The sum over T is $\prod_{m=1}^{n-1} (y_m + P_{m-1}(s))$ with $P_m := s_m(y_m + P_{m-1})$, $P_0 := 0$, and we integrate over $s_{n-2}, \dots, s_1$ in turn using $\int_0^1 x e^{sx} ds \leq e^x$. For $n \leq 7$ the inequality has also been checked exactly (Section S3.1.5).

Lemma A.3 (Factorial moments). *For $n \geq 2$, $y_1 \geq 0$ and $b_2, \dots, b_{n-1} \in [0, 1]$,*

$$\sum_T w(T) y_1^{d_1(T)} \prod_{i=2}^{n-1} d_i(T)! b_i^{d_i(T)} \leq e^{y_1} \prod_{i=2}^{n-1} (1 - b_i)^{-1}. \quad (\text{A.5})$$

Proof. We use the identity $d! b^d = \int_0^\infty (bt)^d e^{-t} dt$ for each $i = 2, \dots, n-1$. Since $\prod_j y_{T(j)} = \prod_i y_i^{d_i(T)}$ and $d_n(T) = 0$, Tonelli's theorem and (A.4) with $y_i := b_i t_i$ ($2 \leq i \leq n-1$) bound the left side by $\int_{\mathbb{R}_+^{n-2}} e^{y_1 + \sum_i b_i t_i} e^{-\sum_i t_i} dt = e^{y_1} \prod_{i=2}^{n-1} (1 - b_i)^{-1}$.

For a finite set A of $\mathcal{U}$ -cells, $\mathfrak{Y}(A)$ is the set of nonempty unions Y of $\mathcal{U}$ -cells each of whose components in the $\mathcal{U}$ -cell graph meets A . For $a = \log(2eD_{\mathcal{U}}) + a' + a''$ with $a' \geq 4$, $a'' \geq 0$, and all integers $d \geq 0$,

$$\sum_{Y \in \mathfrak{Y}(A)} |Y|^d e^{-a|Y|} \leq e^{-a''} 2^{|A|} d! 2^{-d} : \quad (\text{A.6})$$

we root each component at its first cell in A ($2^{|A|}$ sets of roots), count connected sets by Lemma 5.8 and sizes by the 2^{N-1} compositions of $N := |Y|$, and use $N^d e^{-a'N/2} \leq d! (2/a')^d$.

Link data. The *link datum* λ_j of an elementary term θ_j consists of its type; for each of its E -terms (at most two), the number of its points and the set of $\mathcal{U}$ -cells containing them; and the cells Δ'_j, Δ''_j of Lemma A.1. The set $A(\lambda_j)$ consists of Δ''_j and the $\mathcal{U}$ -cells of its E -terms, and $\varphi \geq 0$ is a weight on link data with

$$C_\varphi := \sup_{\Delta_0} \sum_{\lambda: \Delta'=\Delta_0} \varphi(\lambda) 2^{|A(\lambda)|} < \infty. \quad (\text{A.7})$$

Each Y_j ($j \geq 2$) of a term with $\theta_j \neq 0$ lies in $\mathfrak{Y}(A(\lambda_j))$. Indeed, each element is connected in the $\mathcal{U}$ -cell graph, since a wall component is a component of $\hat{\Sigma}$, a closure adds bond neighbours, and linked closures share a $\mathcal{U}$ -cell (Definition 7.20). Moreover, each element of Y_j meets $S(\theta_j) \setminus X_{j-1}$ (Table S1.3, S5), which lies in the cells of $A(\lambda_j)$, as the point y of a covariance line lies in X_{j-1} .

Proposition A.4 (Combined count). *Let a be as in (A.6), with $2C_\varphi e^{-a''} \leq \frac{1}{2}$. For every seed configuration $\sigma = (Z, N_Z)$,*

$$\sum_{n \geq 1} \sum_T \sum_{(Y_j, \lambda_j)_{j=2}^n} w(T) \prod_{j=2}^n \varphi(\lambda_j) e^{-a|Y_j|} \leq 2e^{|Z|}, \quad (\text{A.8})$$

where $Y_1 := Z$, the sum runs over all trees T , all link data with $\Delta'_j \subseteq Y_{T(j)}$, and all $Y_j \in \mathfrak{Y}(A(\lambda_j))$ (disjointness of the Y_j and $Y_j \subseteq U$ are not required), and the term $n = 1$ equals 1.

Proof. We keep the tree outermost with its weight and count the sets backward from the last one, each set rooted at its own link and each link rooted in an earlier set. The roots in earlier sets produce the moments of Lemma A.3. We fix $n \geq 2$ and T and sum in the order $(Y_n, \lambda_n), \dots, (Y_2, \lambda_2)$, innermost first. Inside stage j we sum first over Y_j , then over λ_j with Δ'_j fixed, then over $\Delta'_j \subseteq Y_{T(j)}$. Each constraint links a variable only to variables outer to it. We claim that at stage j the summand depends on Y_j only through $|Y_j|^{d_j(T)} e^{-a|Y_j|}$ and the constraint $Y_j \in \mathfrak{Y}(A(\lambda_j))$. At stage j , (A.6) gives $\sum_{Y_j \in \mathfrak{Y}(A(\lambda_j))} |Y_j|^{d_j} e^{-a|Y_j|} \leq e^{-a''} 2^{|A(\lambda_j)|} d_j! 2^{-d_j}$. The sum over λ_j with Δ'_j fixed then gives at most $C_\varphi e^{-a''} d_j! 2^{-d_j}$ by (A.7), and the remaining sum contributes $\sum_{\Delta'_j \subseteq Y_{T(j)}} 1 = |Y_{T(j)}|$, a factor that is handled at stage $T(j) < j$. This keeps the claim. After stage 2 the factor $|Z|^{d_1(T)}$ remains, and the sum over $(Y_j, \lambda_j)_{j \geq 2}$ is at most $(C_\varphi e^{-a''})^{n-1} |Z|^{d_1(T)} \prod_{j=2}^{n-1} d_j(T)! 2^{-d_j(T)}$. Multiplying by $w(T)$ and summing over T , we obtain the bound $(C_\varphi e^{-a''})^{n-1} e^{|Z|} 2^{n-2}$ from Lemma A.3 with $y_1 = |Z|$ and $b_i = \frac{1}{2}$. Finally $1 + e^{|Z|} \sum_{n \geq 2} (C_\varphi e^{-a''})^{n-1} 2^{n-2} \leq 1 + \frac{1}{2} e^{|Z|} \leq 2e^{|Z|}$.

Remark A.5 (The printed deduction). In [Bry78, p. 329] the bound corresponding to Proposition A.4, (5.4), is deduced from (5.5), $\sum_T \int ds q(T, s, \bar{y}) \leq e^{|X|}$ (trees at fixed sets), and (5.6) (sets at fixed tree, with the supremum over trees outside). Writing the summand of (5.4) as $f(T, \bar{y}) g(T, \bar{y})$ with $f := \int ds q(T, s, \bar{y}) e^{-|X|}$, the deduction uses

$$\sum_{T, \bar{y}} f(T, \bar{y}) g(T, \bar{y}) \leq \left(\sup_{\bar{y}} \sum_T f(T, \bar{y}) \right) \left(\sup_T \sum_{\bar{y}} g(T, \bar{y}) \right),$$

which fails for general nonnegative f, g : for $T, \bar{y}$ ranging over $\{1, \dots, N\}$ and $f = g = \delta_{T, \bar{y}}$ the left side is N and the right side 1. The conclusion holds with a different per-cell constant, by Proposition A.4, or by keeping (5.6) and using $\sum_{T, \bar{y}} fg \leq (\sum_T \sup_{\bar{y}} f e^{-\epsilon|X|}) \sup_T \sum_{\bar{y}} g e^{\epsilon|X|}$, whose first factor is at most $((1 + \epsilon)/\epsilon)^{n-1}$ by Lemma A.3 and whose second is (5.6) with c_5 raised by 2ϵ . In [BF80, p. 215] the joint choice for 5Σ and 6Σ cites “(5.6) of [1] or Lemma 5.5 of [2]”; the inequality used there is [Bry78, (5.5)], which [BF77, Lemma 5.5] states at fixed weights, and the conclusion of [BF80, Lemma 9.1] holds.

For a link datum λ , $O_0(\lambda)$ is the number of points of its E -terms and $L_0(\lambda)$ the sum over its E -terms of half the dist-length of a shortest tree spanning their $\mathcal{U}$ -cells ($L_0 = O_0 = 0$ for type (2)). An E -term with cells in two components of G' has coefficient 0 (Proposition 6.3(d)) and is absent. We put

$$\varphi(\lambda) := e^{-\delta_2 \text{dist}(\Delta', \Delta'')} e^{-c_{10} L_0(\lambda)} e^{-c''_{10} O_0(\lambda)}, \quad c_\gamma := \sup_{\Delta_0} \sum_{\Delta} e^{-\gamma \text{dist}(\Delta_0, \Delta)} \quad (\gamma > 0); \quad (\text{A.9})$$

$c_\gamma < \infty$, since by (R7) at most $\frac{4\pi}{3}(r + 2C_{\mathcal{U}})^3/c_u$ $\mathcal{U}$ -cells lie within distance r of a given one.

Lemma A.6 (Link sum). *Let $c_{10}, \delta_2 > 0$ and $e^{c''_{10}} \geq 16e \max(1, c_{c_{10}/2})$. Then $C_\varphi \leq 1 + 8c_{\delta_2} < \infty$. Moreover, for every E -term $\mathcal{E}$ of λ , with $\mathbf{a}$ the set of $\mathcal{U}$ -cells of its points, and every augmented tree η^A of [BF80, Appendix 1] whose marked points lie in the cells of $\mathbf{a}$, one in each, the contribution of $\mathcal{E}$ to $L_0(\lambda)$ is at most L_{η^A} ; so $e^{c_{10}L_0(\lambda)}e^{c''_{10}O_0(\lambda)}$ is dominated, inside $|\cdot|_0$, by replacing each E -term $\mathcal{E}$ by $\sum_{\eta^A} e^{c_{10}L_{\eta^A}}e^{c''_{10}t}\mathcal{E}_{\eta^A}$, for any decomposition $\mathcal{E} = \sum_{\eta^A} \mathcal{E}_{\eta^A}$ ([BF80, (9.18), (9.18a)]). Which point of $\mathcal{E}$ lies in which cell of $\mathbf{a}$, the positions and the charges are not part of λ ; they are summed inside $|\cdot|_0$.*

Proof. This is 8Σ and 10Σ of [BF80, p. 217] (the choice $B_8B_{10} = e^{c''_{10}O_0}e^{c_{10}L_0}$), with (9.18) and the lengths [BF80, (A1.3)], adapted as follows. (1) Positions are sets of $\mathcal{U}$ -cells. We count a set of k cells once for its $(k-1)!$ orderings, and the tree sum gives $e^k c_{c_{10}/2}^{k-1}$. The terms with the same link datum differ in the assignment of the points of each E -term to the cells of its set, in their positions and in their charges (and, for $t = 2$, in the form). Proposition B.7 bounds their sum through (M4), a bound for the sum over all assignments (Lemma B.1(d)). A bound for one ordered tuple of cells would not suffice, because a set of k cells carries $k! S(t, k)$ assignments of t points (S the Stirling numbers of the second kind, and $t!$ assignments if $k = t$). (2) The factor $2^{|A(\lambda)|}$ costs 4^k per E -term with a marked cell, which raises c''_{10} (whose existence alone [BF80, p. 217] states) by $\log 2$, and Δ'' costs $2c_{\delta_2}$ in each of the four types with a covariance line. Hence $C_\varphi \leq 1 + 8c_{\delta_2}$. (3) The length L_{η^A} is an infimum over the positions of the marked points in the cells of $\mathbf{a}$ and of the auxiliary points, measured with dist , and a tree on given points has at least half the length of a shortest tree spanning them, which dominates one spanning their cells. Hence $L_0 \leq L_{\eta^A}$, and $e^{c_{10}L_0}|\mathcal{E}| \leq \sum_{\eta^A} e^{c_{10}L_{\eta^A}}|\mathcal{E}_{\eta^A}|$ as $\mathcal{E} = \sum_{\eta^A} \mathcal{E}_{\eta^A}$. The factors $e^{c_{10}L_{\eta^A}}e^{c''_{10}t}$ of κ' are paid in the *Tree weights* step of the proof of Proposition B.7, with $c_{10} \leq \frac{7}{9}r_C$ and $\mu \geq 1 + 2c''_{10}$. Section S1.8 gives the order of these constants.

The count of [GM82, (7.3) to (7.5)]. There the Mayer factors $E(a_1, \dots, a_t)$ keep the ordered tuple of cubes $(a_1, \dots, a_t)$ among the summed parameters, with a weight exponential in the least tree length $L(a_1, \dots, a_t)$ of [GM82, (1.11b)]. The count there is that of [BF80, 10Σ], which sums the positions of the cubes at a fixed augmented tree η^A , with the constant weights b_{η^A} of [BF80, (A1.4)] ([BF80, p. 214]). With the least tree length in place of these weights, the counting step needs the summed form of the coefficient bound, a bound for the sum over all positions of the points with their tree length inside the sum, which the derivation of [GM82, (4.31)] provides through the pinned norm [GM82, (4.12)]. Theorem 6.5 states (S2) in this form, and the count above uses it, with positions counted as sets of cells and the assignments summed inside $|\cdot|_0$, as in the proof of Lemma A.6.

Proposition A.7 (Lemma 7.37 on G'). *Lemma 7.37 holds, with C_φ the constant of Lemma A.6. The same proof gives the bound with $e^{\alpha|X|}$ replaced on the left by $e^{\alpha|X| + \varepsilon F_1^{\text{loc}}(X, N_X)}$, $\varepsilon \geq 0$, and $e^{c'_1 F_1^{\text{loc}}}$ replaced on the right by $e^{(c'_1 + \varepsilon) F_1^{\text{loc}}}$.*

Proof. This is [BF80, Lemma 9.1], proved by the device of [BF80, (9.12)], the bound $\sum_v |f(v)| \leq (\sum_v B(v)^{-1}) \sup_v |B(v)f(v)|$ over the data $v = (n, T, \bar{y}, (\lambda_j)_{j \geq 2}, N_X)$ of (A.3), with $B(v) := e^{a|X \setminus Z|} e^{c'_1 F_1^{\text{loc}}(X, N_X)} \prod_j \varphi(\lambda_j)^{-1}$, $a := \log(2eD_U) + 4 + c'_4 + a''$, $e^{a''} \geq 4C_\varphi$. Terms with the same link data differ only inside $|\cdot|_0$, namely in the maps l of Lemma B.2 and in the data of the E -terms listed in the proof of Lemma A.6, and their sum is bounded as there. We change the argument at the following points. (1) The count of the sets is Proposition A.4 ($a' = 4$), after summing N_X innermost by Lemma 7.30(b) and keeping only $Y_j \in \mathfrak{Y}(A(\lambda_j))$.

Thus $\sum_v w(T)B(v)^{-1} \leq e^{(1+c'_4+\log 2)|Z|} \leq e^{a|Z|}$, the supremum carries $e^{c'_2|X|}$, $c'_2 := a + \alpha$, and Lemma A.6 turns $\prod_j e^{c_{10}L_0(\lambda_j)+c'_{10}O_0(\lambda_j)}$ and R_θ into κ' . (2) The seed is fixed (seeds depending on the configuration are reduced to fixed ones in Proposition A.8) and enters only through $|Z|$ and $\mathcal{N}_\sigma(X)$, never through its connectivity. (3) No step refers to X_1 . (4) For $U = X_0^c$ the activity does not depend on the region (Table S1.3, S13). (5) We use only the distance-zero form ($W \subseteq Z$ or $W \supseteq Z$, where the restriction $X \cap W \neq \emptyset$ is void), since on G' a $\mathcal{U}$ -cell can have $d_{G'}$ -diameter of order L' and the clustering form [BF80, (9.122)] would therefore cost $e^{O(\tilde{m}L')} = e^{O(\tau)}$ per non-seed cell. As $F_1^{\text{loc}}(X, N_X)$ depends neither on ψ nor on s , we may move $e^{\varepsilon F_1^{\text{loc}}}$ into the supremum.

For an anchor S_0 , $\Sigma_{\text{sat}}(S_0)$ is the set of seed configurations $\sigma = (Z, N_Z)$ with $Z \supseteq S_0$ each element of which meets S_0 . We write $F_1^{\text{loc}}(\mathcal{S}, N_S)$ for the sum of $F_1(S_\beta)$ over the wall components of a wall polymer $\mathcal{S}$, and D_I for the largest number of interfaces within Euclidean distance g of a given interface or site. The conditions $(\beta 1)$, $(\beta 3)$, $(\beta 4)$ on β_{GM}^0 are (S1.5) of Section S1.8.

Proposition A.8 (Core estimate). *Let (U, k) be admissible, $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ and $\varepsilon \in [0, c'/2]$. For every seed configuration $\sigma = (Z, N_Z)$ in (U, k) ,*

$$\sum_{\substack{X \supseteq Z \\ X \subseteq U}} \sum_{N_X \in \mathcal{N}_\sigma(X)} |\mathcal{K}_{N_X}^{(Z)}(X)| e^{\alpha|X| + \varepsilon F_1^{\text{loc}}(X, N_X)} \leq \mathcal{P}' e^{c_S|Z^o|} e^{-(11c'/16 - \varepsilon)F_1^{\text{loc}}(Z, N_Z)}, \quad (\text{A.10})$$

and the sum restricted to $X \neq Z$ is at most c_{16}^D times the right side. Consequently Lemma 7.48 holds.

Proof. This is the per-cell bound of [BF80, §9.8] after Lemma 7.37, with $e^{-F_1^{\text{loc}}}$ booked region by region, followed by a split of the sum by the first set.

(1) *Booking.* Proposition A.7 ($c'_1 = c'/4$, with the extraction $e^{\varepsilon F_1^{\text{loc}}}$), Proposition 7.35, Lemma 7.38 and Proposition 7.39 leave $e^{-(3c'/4 - \varepsilon)F_1^{\text{loc}}(X)}$, the part on Z constant on $\mathcal{N}_\sigma(X)$. We book this factor by Table S1.4 (Section S1.8): on $X \setminus Z$ we book $c'/16$ for $|D|$, $|F'_2|$ and $c'/8$ for the wall polymers, and on Z we book $c'/32$ for $|D|$, $|F'_2|$ and $c'/32$ for the cells of Z^Σ under $(\beta 1)$. Then each non-seed cell carries at most $\varepsilon_\Delta \leq \frac{1}{2}c_{16}^D \leq \frac{1}{2}$ by $(\beta 4)$, lone seed cells carry at most e^{c_S} , and the seed retains $c' - c'/4 - \varepsilon - c'/32 - c'/32 = 11c'/16 - \varepsilon$, which is (A.10). Terms with $X \neq Z$ contain a non-seed cell and gain $\frac{1}{2}c_{16}^D$.

(2) *Splitting.* The map $(X, N_X) \mapsto (\sigma(N_X), X, N_X)$, $\sigma(N_X) := (Z, N_X|_{E(Z)})$ with $Z := \text{sat}_{N_X}(S_0)$, is a bijection onto the triples with $\sigma \in \Sigma_{\text{sat}}(S_0)$, $X \supseteq Z$, $N_X \in \mathcal{N}_\sigma(X)$, preserving the summands (Lemma 7.23).

(3) *Entropy of the first set.* As $11c'/16 - \varepsilon \geq c'/8$, the left side of Lemma 7.48(a) is at most $\mathcal{P}' e^{c_S|S_0|} \sum_\sigma \prod_{\mathcal{S}} e^{-(c'/8)F_1^{\text{loc}}(\mathcal{S}, N_S)}$. The wall polymers of σ , assigned to their first cells in S_0 and counted by their wall interfaces and values (Lemmas 5.8, 7.30(a)), give the factor $1 + \eta_{\text{seed}}$ per cell of S_0 under $(\beta 3)$. This gives (a), then (b), and (c') with $\mathcal{P}' = 1$, $Z^o = \emptyset$. We give the details in Section S2.7.

B Brydges–Federbush Section 9.8 bookkeeping on slit domains

In this appendix we prove Lemma 7.38 and Proposition 7.39, the bookkeeping of [BF80, §9.8, (9.81) to (9.824)] on G' . In Debye units ($b_D := \beta_{\text{GM}}\tilde{m}$) each field derivative carries $b_D^{1/2}$ and

each coefficient $1/b_D$. A lone cell is of class A and contains a link end or a Mayer cube (minimality of [GM82, (E.11)]), which gives it net weight $\geq \frac{1}{6}$. Cells of X_Σ are paid by F_1^{loc} . We control factorials by pinning. Since $\mathcal{O}((1+r)^3)$ cells lie within distance r of a cell, a sum of N distances grows like $N^{4/3}$, which outweighs $\log((N!)^{q_{\text{pin}}})$.

Setting. Cells are $\mathcal{U}$ -cells, grid cells are denoted Ω , and $\text{vol}_D(\Delta) := \tilde{m}^3 \# \Delta \in [c_u, C_u]$. Classes A, B and $X = Z \sqcup X_\Sigma \sqcup X_o$ are those of Section 7.6 (seed cells in wall polymers count with X_Σ). On a class-A cell $\vartheta = \psi$ modulo $P\mathbb{Z}$ in each phase. In the seeded activity of Section 7.6, $X_i := Y_1 \cup \dots \cup Y_i$ and $\kappa(\bar{y}, s) = \kappa(n-1) \dots \kappa(1)$ as in [GM82, (E.9), (E.10)] (Appendix C). When its brackets are expanded, $\kappa(i)$ is of one of five *kinds*: $K0 = \partial_{s_i} E^{(i)}$, $K1 = \dot{C} \partial_x \partial_y$, $K2 = \dot{C} \partial_x (\partial_y E)$, $K3 = \dot{C} (\partial_x E) \partial_y$ and $K4 = \dot{C} (\partial_x E) (\partial_y E)$. Here $\partial_x := \partial / \partial \psi(x)$, $x \in Y_{i+1}$ and $y \in X_i$. An $\mathcal{E}$ -factor is one Mayer term of $E(X, s)$ of the forms [GM82, (E.5a) to (E.5d)], with its coefficient ρ_t and its t factors. Derivatives act on the factors to the right of the differentiation, as in [GM82, (E.13) to (E.16)] (the rule of [BF80, p. 229]). The ψ -dependent *atoms* of $Q^\natural(X)$ are: (a1) r_Ω of (7.4), for each grid cell $\Omega \subseteq X$; (a2) $e^{G_2(\Omega)}$, $G_2(\Omega) := G_{2,1}(\Omega) + G_{2,2}(\Omega)$; (a3) $e^{g_z^\mathfrak{D}}$, $z \in X \cap \mathfrak{D}$; (a4) $e^{g_z^\natural}$, $z \in X \setminus \mathfrak{D}$; (a5) $e^{-F_2(X)} = e^{-\langle 1_X \psi, F_2' \rangle}$. Here $g_z^\mathfrak{D}$, $g_z^\natural$ are the z -terms of $G_{2,3}^\mathfrak{D}$, $G_{2,3}^\natural$ and, pairing m with $-m$ by (S3),

$$G_{2,1}(\Omega) = \sum_{z \in \Omega} \sum_m \rho^{\text{ref}}(m) f_2(m \beta_{\text{GM}}^{1/2} \tilde{\delta}(z)), \quad G_{2,2}(\Omega) = \sum_{z \in \Omega} \sum_m \rho^{\text{ref}}(m) v_m(\Omega) f_1(m \beta_{\text{GM}}^{1/2} \tilde{\delta}(z)), \quad (\text{B.1})$$

with $f_1(y) := e^{iy} - 1 - iy$, $f_2(y) := \cos y - 1 + \frac{1}{2}y^2$ and $v_m(\Omega) := e^{im\beta_{\text{GM}}^{1/2} \text{avg}_\Omega \vartheta} - 1$. Thus $G_{2,1}$ is of fourth order in $\tilde{\delta}$. The factors of an $\mathcal{E}$ -factor are $\varepsilon(\xi) = e^{im\beta_{\text{GM}}^{1/2}(\vartheta(x)+h(x))} - 1$ in (E.5a), $\tilde{\varepsilon}(\xi) := \varepsilon(\xi) - im\beta_{\text{GM}}^{1/2}\psi(x)$ in (E.5b) to (E.5d), and the ψ -factor $im\beta_{\text{GM}}^{1/2}\psi(x)$ in (E.5c), (E.5d).

Lemma B.1 (Debye units). *Put $d\hat{x} := \tilde{m}^3 \times$ counting measure on V , $\hat{\psi} := \tilde{m}^{-1/2}\psi$, $\delta/\delta\hat{\psi}(\hat{x}) := \tilde{m}^{-5/2}\partial/\partial\psi(x)$, $\hat{C}(\hat{x}, \hat{y}) := \tilde{m}^{-1}C'_s(x, y)$ (likewise for $\partial_{s_i}C'_s$), and $\hat{c} := \tilde{m}^{-3}c$ for a coefficient $c(x)$ per site. Then (a) $\sum_{x,y} K(x, y) \partial_{\psi(x)} \partial_{\psi(y)} = \iint d\hat{x} d\hat{y} \tilde{m}^{-1} K(x, y) \frac{\delta}{\delta\hat{\psi}(\hat{x})} \frac{\delta}{\delta\hat{\psi}(\hat{y})}$; (b) if $F(\psi) = \sum_z c(z) f(\beta_{\text{GM}}^{1/2}\psi(z))$, then $\delta F / \delta\hat{\psi}(\hat{x}) = \hat{c}(\hat{x}) b_D^{1/2} f'(b_D^{1/2}\hat{\psi}(\hat{x}))$: each derivative of a function of $\beta_{\text{GM}}^{1/2}\psi$ produces exactly $b_D^{1/2}$; (c) $|\hat{C}(\hat{x}, \hat{y})| \leq (2K_1''/\lambda_{\text{PV}}) e^{-r_{\text{cov}} d(x,y)}$, $\hat{C}(\hat{x}, \hat{x}) \leq \alpha_D K_1/\lambda_{\text{PV}}$ and $\sup_{\hat{x}} \int d\hat{y} |\hat{C}(\hat{x}, \hat{y})| e^{r_{\text{cov}} d(x,y)} \leq 2K_3$; (d) for a cell Δ , $k \geq 2$ and $C_k \leq Ck!(2\kappa_S)^{-k}$: (M1) $\int_\Delta d\hat{x} \sum_m |\hat{\rho}^{\text{ref}}(m)| |m|^k \leq C_k \text{vol}_D(\Delta)/b_D$; (M2) $\int_{\mathfrak{D} \cap \Delta} d\hat{x} \sum_m |\hat{\rho}_1 - \hat{\rho}^{\text{ref}}| |m|^k \leq C_k c_{\text{dep}}$; (M3) $\int_{\Delta \setminus \mathfrak{D}} d\hat{x} \sum_m |\hat{\rho}_1 - \hat{\rho}^{\text{ref}}| |m|^k \leq C_k \theta' \text{vol}_D(\Delta)/b_D$; (M4) for $t \geq 2$, a set $\mathbf{a}$ of cells and a cell $a_* \in \mathbf{a}$,*

$$\frac{1}{t!} \int_{\Xi_t(\mathbf{a})} d\hat{x}_1 \dots d\hat{x}_t \sum_{m_1, \dots, m_t} |\hat{\rho}_t| e^{2\kappa_S \sum_i |m_i|} \leq t K_{M4} C_{M4}^t \frac{4 \text{vol}_D(a_*)}{b_D} e^{-\mu(t-1)} e^{-(1-\delta_{\text{GM}})(A/\tilde{m}) \hat{L}(\mathbf{a})},$$

and hence, for all integers $k_1, \dots, k_t \geq 0$, the left side with $\prod_i |m_i|^{k_i}$ in place of $e^{2\kappa_S \sum_i |m_i|}$ is at most $\prod_i k_i! (2\kappa_S)^{-k_i}$ times the right side. Here $\Xi_t(\mathbf{a})$ is the set of $(x_1, \dots, x_t) \in V^t$ such that $\mathbf{a}$ is the set of the cells that contain some x_i , A is the rate of (S2), $\hat{L}(\mathbf{a}) := \tilde{m} L_{G'}(\mathbf{a})$ is the tree length of (S2) of the cells of $\mathbf{a}$, each listed once, in Debye units, $C_{M4} := (1-C)^{-1}$ per point and $K_{M4} := e^{2\kappa_S} (1-C)^{-10}$, where C is the parameter of (S2), fixed in $(0, \frac{1}{4}]$ (Section S1.1.1). The left side is the sum over all assignments of the t points to the cells of $\mathbf{a}$ that use every cell; it contains $\int_{a_1} \dots \int_{a_t}$ for each t -tuple of cells with $\{a_1, \dots, a_t\} = \mathbf{a}$.

Proof. Parts (a) and (b) are the change of variables ($\tilde{m}^{-5/2}\beta_{\text{GM}}^{1/2} = \tilde{m}^{-3}b_D^{1/2}$). Part (c) is Theorem 6.11 for $C' \leq \alpha_D C_0$ (Lemma 7.29), with $M_Y/\tilde{m} = 1/\lambda_{\text{PV}}$ and entrywise domination of C'_s . Bound (M1) follows from $\beta_{\text{GM}} \sum_m m^2 \rho^{\text{ref}}(m) = \tilde{m}^2$ and (S4'). Bounds (M2) and (M3) follow from Propositions 6.17, 6.6 and (7.26). We derive (M4) from (S2). As $e^{2\kappa_S \sum_i |m_i|} = e^{2\kappa_S} e^{2\kappa_S(\sum_i |m_i| - 1)}$, the left side is $e^{2\kappa_S} \frac{1}{t!} \sum_{x \in \Xi_t(\mathbf{a})} \sum_m |\rho_t| e^{2\kappa_S(-1 + \sum_i |m_i|)}$ (in Debye units $\int d\hat{x} |\hat{\rho}| = \sum_x |\rho|$). Each $x \in \Xi_t(\mathbf{a})$ has a coordinate in a_* . As ρ_t , the weight and $\Xi_t(\mathbf{a})$ are symmetric, the sum is at most t times its part with $x_1 \in a_*$. For $x \in \Xi_t(\mathbf{a})$, $L_{G'}(x_1, \dots, x_t) \geq L_{G'}(\mathbf{a})$. Indeed, in a tree for $x_1, \dots, x_t$ we keep one marked vertex in each cell of $\mathbf{a}$ and declare the others auxiliary. Hence the part is at most $e^{-(1-\delta_{\text{GM}})AL_{G'}(\mathbf{a})}$ times the left side of (6.5) with $s = t$ and $a_1 = a_*$, and (S2), $(1 - C)^{-t-10} = (1 - C)^{-10} C_{M4}^t$ and $|a_*| m_D^2 / \beta_{\text{GM}} \leq 4 \text{vol}_D(a_*) / b_D$ ($m_D \leq 2\tilde{m}$) give (M4). The form with $\prod_i |m_i|^{k_i}$ follows from $|m|^k \leq k!(2\kappa_S)^{-k} e^{2\kappa_S |m|}$. No factorial arises, because the sum over the assignments ($|\mathbf{a}|! S(t, |\mathbf{a}|)$ of them, with S as in the proof of Lemma A.6) is a part of the sum over all ordered tuples of points in (S2), and the decay is read off the weight on that part.

Thus the expansion is that of [BF80, §9.8] with $\tilde{l}_D = 1$, $\beta \rightarrow b_D$, on a fine lattice with a regularized covariance, and infinitely many charges only produce the $k!$ in (d). A factor f has *weight* w if $|f| \leq K b_D^w \cdot (\text{field monomial}) \cdot (\text{numerical factors})$ for all positions in their cells and all fields, with K independent of b_D , L' , τ . Here a field monomial is a product of values $|\hat{\psi}(\hat{x})|$ and averages $\text{avg}_{\Omega} |\hat{\psi}|$, paid by the Wick bound T_1 (each $|\hat{\psi}|$ costs $(\alpha_D K_1 / \lambda_{\text{PV}})^{1/2}$). A numerical factor is $|\delta \hat{D}(\hat{x})| := |\hat{D}(\hat{x}) - \text{avg}_{\Omega} \hat{D}|$ or $|\hat{F}'_2(\hat{x})|$, with $\hat{D} := \tilde{m}^{-1/2} D$, $\hat{F}'_2 := \tilde{m}^{-5/2} F'_2$. The weight of a term is the sum of those of its factors.

Lemma B.2 (Types i) to viii) and the Leibniz count). *Expand every derivative of $\kappa(\bar{y}, s)$ by the Leibniz rule, as in [BF80, (9.84), (9.85)]. (a) $\kappa(\bar{y}, s) Q^{\natural}(X)$ is a finite sum of terms, each labelled by the kinds, the $\mathcal{E}$ -factors, and a map l that sends each derivative to one atom or one factor of an $\mathcal{E}$ -factor which depends on the derivative's point and lies to its right; the term is $[\prod_i \partial_{s_i} C'(x_i, y_i | s)] \prod_a \partial^{l^{-1}(\mathbf{a})} a \prod_e [\text{coefficient of } e \cdot \prod_{\phi \in e} \partial^{l^{-1}(\phi)} \phi]$. (b) The multiple derivatives are of the types: i) $\partial^S r_{\Omega}$; ii) from an observable (absent here); iii) $\partial^S e^{-F_2} = (-1)^{|S|} \prod_{j \in S} F'_2(x_j) e^{-F_2}$; iv) $\partial^S \varepsilon$, $\partial^S \tilde{\varepsilon}$, $\partial^S (im \beta_{\text{GM}}^{1/2} \psi)$; v), vi) the Faà di Bruno blocks $\rho^{\text{ref}}(m) \partial^B[v_m f_1]$ and $\rho^{\text{ref}}(m) \partial^B[f_2]$ of $\partial^S e^{G_2(\Omega)}$; vii), viii) the Faà di Bruno blocks of $\partial^S e^{g_z^{\mathbb{D}}}$ and $\partial^S e^{g_z^{\mathbb{H}}}$. (c) With n_{α} link ends, m_{α} factors of $\mathcal{E}$ -factors and w_{α} ($= 0$ here) factors of an observable in the cell Δ_{α} , the number of maps l is at most $\prod_{\alpha} (m_{\alpha} + w_{\alpha} + 5)^{n_{\alpha}}$, and for $c_p > 0$, $n \geq 1$ and integers $m, w \geq 0$, $(m + w + 5)^n \leq C_r^{n+w} n! e^{c_p m}$ with $C_r := 4e^5 \max(1, 1/c_p)$.*

Proof. This is [BF80, (9.84) to (9.87), Lemma 9.10] with two new types, vii) and viii). A bare derivative at $x \in \Delta_{\alpha}$ can hit $r_{\Omega(x)}$, $e^{G_2(\Omega(x))}$, $e^{g_x^{\mathbb{D}}}$ or $e^{g_x^{\mathbb{H}}}$, e^{-F_2} , or one of the $m_{\alpha} + w_{\alpha}$ factors in Δ_{α} . The count therefore has +5 in place of the +3 of [BF80, (9.86)]. The undifferentiated exponentials recombine into $e^{G^{\natural}(X)}$ and e^{-F_2} , which stay in T_4 and T_2 of Proposition 7.35. We bound each r_{Ω} by Lemma 7.31, whose factor is included in T_3 . The last inequality follows from $(m + w + 5)^n \leq 2^n (m^n + (w + 5)^n)$, $m^n \leq n! c_p^{-n} e^{c_p m}$ and $(w + 5)^n \leq n! e^{w+5}$. The factor $n!$ is paid by pinning (Lemma B.4). Within a fixed l the further expansions (Faà di Bruno, the Leibniz rule inside $v_m f_1$, the two parts of $\partial \tilde{\delta}(z) / \partial \psi(x)$, the charge sums of Lemma B.1(d)) cost at most $C^{n_{\alpha}} (n_{\alpha}!)^2$ per cell, and $e^{c_p \sum_{\alpha} m_{\alpha}} = \prod_e e^{c_p t_e}$ is a constant per $\mathcal{E}$ -factor, absorbed in the *Constants* step of the proof of Proposition B.7.

Lemma B.3 (Witness). (a) Link localization. Let $1 \leq i \leq n-1$ and let $Y \in \tilde{Y}(h)$ with $Y \subseteq Y_{i+1}$. Then Y contains the point x_i of the bare derivative of $\kappa(i)$ (kinds K1, K2), or a cube of an $\mathcal{E}$ -factor of $\kappa(i)$. (b) Every lone cell is of class A, and $F'_2 = 0$ on it. (c) Let $\Delta \in X_o$ with $\Delta \subseteq Y_{i+1}$, $i \geq 1$. Then (W1) the bare derivative of $\kappa(i)$ sits at $x_i \in \Delta$ and hits an atom of kind (a1) to (a4) of Δ , or (W2) an $\mathcal{E}$ -factor of $\kappa(i)$ has a cube equal to Δ .

Proof. (a) By [GM82, (E.11)], Y_{i+1} is the smallest union of elements of $\tilde{Y}(h)$ containing $\mathcal{O} - X_i$, where $\mathcal{O}$, the support of the term of $\kappa(i)$, consists of the cubes of its $\mathcal{E}$ -factors and the cube of a bare x_i if there is one. An element $Y \subseteq Y_{i+1}$ meeting no point of $\mathcal{O} - X_i$ would make $Y_{i+1} \setminus Y$ a smaller such union. As cells are $\mathcal{U}$ -cells, “meets” is “contains”. (b) $\text{supp } D_\beta, \text{supp } F'_{2,\beta} \subseteq S_\beta$ (Proposition 7.8), and a lone cell lies in no wall polymer. (c) By (a), Δ contains x_i or a cube of an $\mathcal{E}$ -factor of $\kappa(i)$ (W2). In the first case the bare derivative hits a factor to its right depending on $\psi(x_i)$: an atom (a1) to (a4) at x_i gives (W1); (a5), and the derivatives of e^{-F_2} , give $F'_2(x_i) = 0$ by (b); a factor at x_i of the $\mathcal{E}$ -factor of the y -bracket of $\kappa(i)$ (kind K2) gives (W2); all other factors are functions of ψ on X_i , which is disjoint from Y_{i+1} .

For each $\Delta \in X_o$ we fix one witness, (W1) if available and otherwise an $\mathcal{E}$ -factor e as in (W2). Let $W(e)$ denote the set of lone cells witnessed by e . Each lone cell lies in exactly one Y_{i+1} , and $\kappa(i)$ has at most two $\mathcal{E}$ -factors. It follows that all cells of $W(e)$ lie in $Y_{i(e)+1}$, where $\kappa(i(e))$ is the link of e .

The cells of G' have polynomial growth: $\#\{\Delta : \text{dist}(\Delta_0, \Delta) \leq r\} \leq K_g(1+r)^3$ in Debye units, for the Euclidean distance between cells and $K_g := (4\pi/3) \max(1, 2C_U)^3/c_u$, by (R7) and $\text{vol}_D \geq c_u$. Links have one end in Y_{i+1} and one in X_i . The sets Y_{i+1} are pairwise disjoint.

Lemma B.4 (Pinning with Euclidean distance). (a) Let Δ_0 be a cell and $\Delta_1, \dots, \Delta_N$ cells, each repeated at most twice, with $r_k := \text{dist}(\Delta_0, \Delta_k)$ (Euclidean, in Debye units). For $q \geq 1$ and $c_p > 0$, $(N!)^q \leq c_N^N \exp(c_p \sum_k r_k)$ with $c_N := e^{c_p} \max(1, 3q/(c_p a_g))^{3q}$, $a_g := \frac{3}{4}(2K_g)^{-1/3}$. (b) Let $d := \sum_i \text{dist}(\Delta'_i, \Delta''_i)$ be the total length of the links, $\hat{L}^E(e)$ the Euclidean Steiner length, in Debye units, of the cells of the $\mathcal{E}$ -factor e (the least Steiner length of points chosen in its cells), and $\hat{L}_0 := \sum_e \hat{L}^E(e)$. Pin each $\mathcal{E}$ -factor e at a set $\Pi(e)$ of at most four of its cells (chosen in Theorem B.6), and let $N_\alpha := \#\{e : \Delta_\alpha \in \Pi(e)\}$, in either class. Then, with $c = c(q, c_p, K_g, C_U)$, $\prod_\alpha (n_\alpha!)^q \leq c \sum n_\alpha e^{c_p d}$ and $\prod_\alpha (N_\alpha!)^q \leq c^{\#\mathcal{E}} e^{4c_p(\hat{L}_0 + 2d)}$.

Proof. Part (a) is [BF80, Lemma 9.11, (9.820)], with the growth $K_g(1+r)^3$ in place of the lattice count of [BF80, Appendix 2]. In increasing order the distances satisfy $r_{(k)} \geq (k/(2K_g))^{1/3} - 1$. Summing these bounds, we obtain $\sum_k r_k \geq a_g N^{4/3} - N$. Moreover, $qN \log N - c_p a_g N^{4/3} \leq 3qN \log^+(3q/(c_p a_g))$. (b) We first treat the derivatives. Since Δ_α lies in exactly one Y_{i+1} , at most one x -end lies in Δ_α . All other ends in Δ_α are y -ends y_j of links j with $\Delta_\alpha \subseteq X_j$, whose partners x_j lie in Y_{j+1} , disjoint for distinct j . We apply (a) to the y -ends, with $n! \leq 2^n(n-1)!$. For the $\mathcal{E}$ -factors, let e be pinned at Δ_α , with link $i = i(e)$. If $\Delta_\alpha \subseteq Y_{i+1}$, there are at most two such e . Otherwise $\Delta_\alpha \subseteq X_i$ and e reaches Y_{i+1} : for K0 since $\partial_{s_i} E^{(i)}$ keeps only Mayer terms with cubes in X_i and in Y_{i+1} ; for the x -side $\mathcal{E}$ -factor of K3, K4 through a factor at x_i ; for the y -side one of K2, K4 through a factor at y_i , joined to x_i by link i . Thus $\text{dist}(\Delta_\alpha, Y_{i+1}) \leq \hat{L}^E(e) + 2C_U + d_i$, as $\hat{L}^E(e)$ is a Steiner length over points of the cubes of e and cells have Euclidean diameter at most C_U in Debye units. The partner is the cell of Y_{i+1} nearest to Δ_α . Since distinct links give disjoint Y 's and a link has at most two $\mathcal{E}$ -factors, each partner is used at most twice. We apply (a). Each e appears at most $|\Pi(e)| \leq 4$ times and each link at most twice, and the factor $e^{8c_p C_U}$ is absorbed into c .

factor ([BF80])	k	bound	weight
v) $v_m(\Omega)$, any class (9.814)	$\geq 1; 0$	$(m b_D^{1/2}/v_-)^k; 2$	$k/2; 0$
$v_m(\Omega)$, class A	0	$ m b_D^{1/2} \hat{\psi}_\Omega $	1/2
vi) $f_1(mb_D^{1/2}\hat{\delta})$ (9.815)	$0; 1; \geq 2$	$\frac{1}{2}m^2b_D \hat{\delta} ^2; m^2b_D \hat{\delta} ; (m b_D^{1/2})^k$	$1; 1; k/2$
f_1 , bounded form	1	$2 m b_D^{1/2}$	1/2
vi) $f_2(mb_D^{1/2}\hat{\delta})$	$\leq 4; \geq 4$	$m^4b_D^2 \hat{\delta} ^{4-k}; (m b_D^{1/2})^k$	$2; k/2$
f_2 , bounded form	2; 3	$2m^2b_D; m ^3b_D^{3/2}$	$1; 3/2$
iv) $\tilde{\varepsilon}$, class A (9.816)	$0; 1; \geq 2$	$\frac{1}{2}m^2b_D \hat{\psi} ^2; m^2b_D \hat{\psi} ; (m b_D^{1/2})^k$	$1; 1; k/2$
$\tilde{\varepsilon}$, class B (9.818)	0; 1	$2 + m b_D^{1/2} \hat{\psi} ; 2 m b_D^{1/2}$	$0; 1/2$
ψ -factor, any class (9.816)	0; 1	$ m b_D^{1/2} \hat{\psi} ; m b_D^{1/2}$ (zero for $k \geq 2$)	$1/2; 1/2$
ε , any class (9.817)	≥ 1	$(m b_D^{1/2})^k$	$k/2$
ε undistinguished; distinguished, class A (9.819)	0	$2; m b_D^{1/2} \hat{\psi} $	$0; 1/2$
vii) $\cos(mb_D^{1/2}\hat{\vartheta}) - 1$	≥ 1	$(m b_D^{1/2})^k$	$k/2$
viii) $f_2(mb_D^{1/2}\hat{\psi})$, class A	$\leq 4; \geq 4$	$m^4b_D^2 \hat{\psi} ^{4-k}; (m b_D^{1/2})^k$	$2; k/2$
viii), class B	$1; 2; \geq 3$	$ m b_D^{1/2} + m^2b_D \hat{\psi} ; 2m^2b_D; (m b_D^{1/2})^k$	$\frac{1}{2}; 1; k/2$
i) r_Ω (Lemma 7.31)	≥ 0	$c_7(c_7b_D^{1/6}/v_-)^k e^{c_7^2 k \log k} e^{\frac{1}{2}\eta_H \tilde{m}^2 \Sigma_\Omega \vartheta^2}$	$k/6$
iii) $\hat{F}'_2$, class B only		numerical factor	0

Table B.1: Bounds per type (Lemma B.5(a)).

With c_p of Proposition 7.36 and $r_C = r_{\text{cov}}/\tilde{m}$, the factor $e^{9c_p d}$ is paid by the link decay $e^{-r_C \text{dist}(\Delta'_i, \Delta'_i)}$ of Lemma B.1(c), leaving $e^{-(\tilde{\delta} + (r_C - \tilde{\delta})/2) \text{dist}}$. The factor $e^{4c_p \hat{L}_0}$ is paid by the Mayer decay of (M4), with the factors $e^{c_{10} L_{\eta^A}}$ of κ' (see the *Tree weights* step of the proof of Proposition B.7), as $\hat{L} \geq 3^{-1/2} \hat{L}^E$ and, on outputs, $(1 - \delta_{\text{GM}})A/\sqrt{3} \geq \frac{1}{2}\kappa_Y M_Y \geq 2r_C \tilde{m} > 4c_p \tilde{m}$ for $\delta_{\text{GM}} \leq \frac{1}{2}$ (Proposition 6.10). Finally, $c^{\#\mathcal{E}} \leq c^{2\#X}$ is a constant per cell, as a link has at most two $\mathcal{E}$ -factors and there are $n-1 < \#X$ links. In [BF80, (9.821)], N_α counts distinguished factors ε in class-A cubes. The proof in [BF80, Appendix 2] does not use the class, and (b) pins $\mathcal{E}$ -factors at cells of either class, as required by Proposition B.10. As in [BF80, (9.822)], we multiply each term by $1 = f_q \cdot \prod_\alpha (n_\alpha!)^q (N_\alpha!)^q$, $f_q := \prod_\alpha (n_\alpha!)^{-q} (N_\alpha!)^{-q}$, and bound the second factor by (b).

We state the bounds below in the variables of Lemma B.1, at fixed positions, for k derivatives at points of the factor's cell (for site factors, at the site). Here $\hat{\delta} := \tilde{m}^{-1/2} \tilde{\delta}$, $\hat{\psi}_\Omega := \text{avg}_\Omega \hat{\psi}$, and $v_\Omega \in [v_-, v_+]$ is the Debye volume of a grid cell. The chain-rule factors $\delta(\hat{\psi}_\Omega)/\delta\hat{\psi}(\hat{x}) = 1_\Omega(\hat{x})/v_\Omega$ and $\delta\hat{\delta}(\hat{z})/\delta\hat{\psi}(\hat{x}) = \hat{\delta}(\hat{x} - \hat{z}) - 1_\Omega(\hat{x})/v_\Omega$, $\hat{\delta}(\hat{x} - \hat{z}) := \tilde{m}^{-3} 1[x = z]$, are not shown. They are absorbed when the line end is integrated, or bounded by $1/v_-$. In class B, $|\hat{\delta}| \leq |\hat{\delta}\hat{\psi}| + |\delta\hat{D}|$. A row of Table B.1 with an equation number of [BF80] is that bound in Debye units, with the charges and volumes shown. The other rows are new.

Lemma B.5 (Per-type bounds). (a) With C numerical and $|m|$ the charge of the factor, the factors satisfy the bounds of Table B.1. (b) Vertex weights. A vertex is a block B on a G_2 -, vii)- or viii)-atom with its coefficient, an r_Ω with all its derivatives, an $\mathcal{E}$ -factor with its coefficient and factors, or one factor $\hat{F}'_2$. The coefficient weight is -1 for one-body, off- $\mathfrak{D}$ and Mayer coefficients ((M1), (M3), (M4)) and $-\frac{1}{6}$ for deficit coefficients on $\mathfrak{D}$ ((M2) with $c_{\text{dep}} \leq b_D^{-1/6}$ for $\tau \geq \tau_0$). For a block with $k = |B| \geq 1$: a $G_{2,1}$ -block has weight ≥ 1 in class A, ≥ 0 in class B if $k \geq 2$, and for $k = 1$ in class B it is a term of weight 0 equal to $\hat{\delta}(\hat{x})$ at the derivative point plus a term of weight $-\frac{1}{2}$ with no field or numerical factor; a $G_{2,2}$ -block has weight $\geq \frac{1}{2}$

in class A and $\geq -\frac{1}{2}$ in class B (≥ 0 if $k \geq 2$), with no numerical factor; a vii)-block has weight $\geq \frac{1}{3}$; a viii)-block has weight ≥ 1 in class A and $\geq -\frac{1}{2}$ in class B (≥ 0 if $k \geq 2$); r_Ω with k derivatives has weight $k/6$; $\hat{F}'_2$ has weight 0. So in class A every non- $\mathcal{E}$ vertex has weight ≥ 0 , and $\geq \frac{1}{6}$ if it carries a derivative; in class B every non- $\mathcal{E}$ vertex has weight $\geq -\frac{1}{2}$, and is negative only for single-derivative blocks.

Proof. (a) By Lemma B.1(b) each derivative of a function of $\beta_{\text{GM}}^{1/2}\psi = b_D^{1/2}\hat{\psi}$ brings $b_D^{1/2}$, and each derivative of $\hat{\psi}_\Omega$ brings $1/v_\Omega$. This gives the rows of [BF80]. For the new rows we use the following facts. In class A, $v_m(\Omega) = e^{im\beta_{\text{GM}}^{1/2}\text{avg}_\Omega\psi} - 1$ (as $h \in P\mathbb{Z}$) and $|e^{it} - 1| \leq |t|$. Next, $|f'_1| = |e^{iy} - 1| \leq \min(|y|, 2)$. Further, $f_2^{(j)}(0) = 0$ for $j \leq 3$ and $|f_2^{(4)}| \leq 1$, hence $|f_2^{(k)}(y)| \leq |y|^{4-k}/(4-k)!$ for $k \leq 4$, $|f_2^{(k)}| \leq 1$ for $k \geq 4$, $|f_2^{(2)}| = |1 - \cos y| \leq 2$ and $|f_2^{(3)}| = |\sin y| \leq 1$. For vii), $|\partial_t^k e^{imt}| = |m|^k$. In class A the factor viii) is $f_2(mb_D^{1/2}\hat{\psi}(\hat{z}))$, and in class B its first derivative is $-mb_D^{1/2}\sin(\cdot) + m^2b_D\hat{\psi}$ and its second $m^2b_D(1 - \cos(\cdot))$. Row i) is Lemma 7.31 with $\partial A_\Omega/\partial\psi(x) = \tilde{m}^3/v_\Omega$. For iii), $F_2 = \int d\hat{x} \hat{\psi} \hat{F}'_2$ and $\text{supp } F'_2 \subseteq \bigcup S_\beta$. (b) The coefficient weights are those of (M1) to (M4). Moreover, $c_{\text{dep}} = \mathcal{O}(\log \tau)$, while $b_D^{-1/6} = e^{0.416\tau(1+o(1))}$. For $G_{2,1}$ in class A the weight is $-1 + 2$ for $k \leq 4$ and $-1 + k/2$ for $k \geq 4$, while in class B with $k = 2, 3$ the bounded forms give $-1 + 1$ and $-1 + \frac{3}{2}$. In class B with $k = 1$, as $\sum_{z \in \Omega} \tilde{\delta}(z) = 0$ and $f'_2(y) = y - \sin y$, we have $\partial_x G_{2,1}(\Omega) = \tilde{m}^2 \tilde{\delta}(x) - \sum_m \rho^{\text{ref}}(m) m \beta_{\text{GM}}^{1/2} [\sin y_x - \text{avg}_\Omega \sin y]$, whose first term is $(b_D \sum_m m^2 \hat{\rho}^{\text{ref}}(m)) \hat{\delta}(\hat{x}) = \hat{\delta}(\hat{x})$ (weight $-1 + \frac{1}{2} + \frac{1}{2}$) and whose second has weight $-1 + \frac{1}{2}$. For $G_{2,2}$ we split $k = k_1 + k_2$ between v_m and f_1 . In class A the weight is $-1 + \frac{1}{2} + \max(1, k_2/2)$ if $k_1 = 0$, else $-1 + k_1/2 + \max(1, k_2/2)$. In class B we distinguish three cases. If $k_2 = 0$, then $\sum_{z \in \Omega} f_1(y_z) = \sum_{z \in \Omega} (e^{iy_z} - 1)$ as $\sum_z y_z = 0$, which is bounded by $2\#\Omega$ with no numerical factor, and hence the weight is $-1 + k_1/2 \geq -\frac{1}{2}$. If $k_2 = 1$, the bounded form gives $-1 + k_1/2 + \frac{1}{2}$. If $k_2 \geq 2$, the weight is $\geq -1 + k_1/2 + k_2/2 \geq 0$. For a vii)-block the weight is $-\frac{1}{6} + k/2$. For a viii)-block it is $-1 + 2$ in class A, and $-1 + \frac{1}{2}$, $-1 + 1$, $-1 + k/2$ in class B.

A factor of an $\mathcal{E}$ -factor is *gaining* if its weight is at least $\frac{1}{2}$. This is the case for all differentiated factors, in either class, for all distinguished ε 's (class A), for all $\tilde{\varepsilon}$'s and ψ -factors in class A, and for all ψ -factors in class B. Each factor has weight ≥ 0 . We select distinguished factors as in [BF80, p. 232], with the attribution to cells fixed by Rule D.

Rule D. Let e be an $\mathcal{E}$ -factor of type (E.5a), $t \geq 3$. (D-a) For each $\Delta \in W(e)$ in which e has no differentiated factor, we distinguish one undifferentiated ε of e in Δ (it exists, since e has a cube equal to Δ). (D-b) While e has fewer than three gaining factors, we distinguish another undifferentiated ε of e in a class-A cell, if there is one. All other undifferentiated ε 's are bounded by 2. For types (E.5b) to (E.5d) no choice is made. The pinning set $\Pi(e)$ consists of the cells of the factors distinguished in (D-b) (at most three) for (E.5a), or of the cells of e (at most two) for (E.5b) to (E.5d), together with the charged cell defined below, if there is one. Hence $|\Pi(e)| \leq 4$.

Let $P(e)$ be the weight of e under Rule D. An $\mathcal{E}$ -factor gives $\frac{1}{6}$ to each cell of $W(e)$. Its remainder $\varrho(e) := P(e) - |W(e)|/6$ is assigned to one chosen class-B cube of e (its *charged cell*) if e has a class-B cube, and to any cell of e otherwise. We call e *charged* if $\varrho(e) < 0$. Each non- $\mathcal{E}$ vertex is assigned to its own cell.

Theorem B.6 (Net weight per lone cell). *In every term, after this distribution: (i) every cell $\Delta \in X_o$ receives weight at least $\frac{1}{6}$; (ii) every class-A cell receives weight at least 0; (iii) every*

class-B cell receives weight at least $-N_\Delta^{\text{ch}} - \frac{1}{2}n_\Delta^{(1)}$, where N_Δ^{ch} is the number of $\mathcal{E}$ -factors charged to Δ and $n_\Delta^{(1)} \leq n_\Delta$ the number of single-derivative blocks in Δ . The value $\frac{1}{6}$ is attained, both by a lone cell whose only vertex of positive weight is r_Ω with one derivative, and by an $\mathcal{E}$ -factor of type (E.5a) with $t = 3$ whose three factors are undifferentiated ε 's in three cells of $W(e)$.

Proof. We first treat the $\mathcal{E}$ ledger. If all cubes of e are of class A, then $P(e) \geq |W(e)|/6$ and $P(e) \geq 0$. In general $P(e) \geq -1 + |W(e)|/2$, and hence $\varrho(e) \geq -1$. Indeed, suppose first that all cubes are of class A. For (E.5a), (D-b) stops only when e has $g(e) \geq 3$ gaining factors, and the cells of $W(e)$ contain distinct gaining factors by (D-a), hence $g(e) \geq \max(3, |W(e)|)$ and $P(e) \geq -1 + g(e)/2 \geq g(e)/6 \geq |W(e)|/6$. For (E.5b), $P(e) \geq -1 + 1 + 1$, and for (E.5c), (E.5d), $P(e) \geq -1 + 1 + \frac{1}{2}$, both $\geq \frac{2}{6}$. In general the cells of $W(e)$ are lone, hence of class A (Lemma B.3(b)), each contains a gaining factor of e , and all other factors have weight ≥ 0 . (ii) A class-A cell receives its non- $\mathcal{E}$ vertices (weight ≥ 0 by Lemma B.5(b)), credits $\frac{1}{6}$, and remainders $\varrho(e) \geq 0$ of $\mathcal{E}$ -factors with no class-B cube. (i) Let $\Delta \in X_o$ with its witness (Lemma B.3(c)). In case (W1) a derivative hits an atom of Δ . If this atom is r_Ω , its vertex has weight $\geq \frac{1}{6}$, and otherwise the Faà di Bruno block containing the derivative has weight $\geq \frac{1}{3}$ (class A). In case (W2), $\Delta \in W(e)$ receives $\frac{1}{6}$ from e . All other items received are ≥ 0 , as in (ii). (iii) Class-B non- $\mathcal{E}$ vertices have weight $\geq -\frac{1}{2}$, with negative weight only for single-derivative blocks. Charged remainders are ≥ -1 , and uncharged ones are ≥ 0 . As for attainment, the first example is (W1) with Lemma B.5(b). In the second, $P(e) = -1 + \frac{3}{2} = 3 \cdot \frac{1}{6}$, with nothing left over.

Each field factor of an $\mathcal{E}$ -factor lies in a cell of $\Pi(e)$ or is the single (D-a) factor of a lone cell, and a non- $\mathcal{E}$ vertex carries at most four field factors per derivative. Hence the number n_j of field factors in the cell Δ_α satisfies $n_j \leq 1 + 4(n_\alpha + N_\alpha)$.

Proposition B.7 (Per-cell form of (9.824)). *Fix the constants of Section 7.6, with $c_{10} \leq \frac{7}{9}r_C$ in Lemma 7.37, $q_{\text{pin}} := q_0 + 2$ with $q_0 := 8 + c_7''$, and c_p of Proposition 7.36. For every term κ' of the activity of a polymer X with seed Z (kinds and link-end cells fixed, and for each $\mathcal{E}$ -factor e its number t_e of points and the set $\mathbf{a}(e)$ of cells containing them; κ' contains, inside $|\cdot|_0$, the sum over all configurations of the points of e with this set of cells, that is, over the assignments of the t_e points to the cells of $\mathbf{a}(e)$ that use every cell, over their positions in these cells and over their charges, and over the forms [GM82, (E.5b) to (E.5d)] if $t_e = 2$), every h_X and every s ,*

$$\int d\mu_{C'_s} |e^{E(X,s)} \kappa' Q^\natural(X)|_0 \leq \mathcal{P}'_G e^{C_{\mathcal{D}} c_{\text{dep}} N_{\mathcal{D}}(X)} e^{-c' F_1^{\text{loc}}(X) + \frac{c'}{16} F_1^{\text{loc}}(X \setminus Z) + \frac{c'}{32} F_1^{\text{loc}}(Z)} \\ \times \prod_i e^{-(\bar{\delta} + (r_C - \bar{\delta})/2) \text{dist}(\Delta'_i, \Delta''_i)} \prod_e \mathbf{m}_e \prod_{\Delta \subseteq X} \Psi_\Delta,$$

where $\mathbf{m}_e := e^{-(\mu - 1 - 2c_{10}'')(t_e - 1)} e^{-r_C \hat{L}^E(e)}$ is the leftover Mayer weight, $\hat{L}^E(e)$ the Euclidean Steiner length of the cells of $\mathbf{a}(e)$ (Lemma B.4(b)) ($\mathbf{m}_e \leq 1$ for $\mu \geq 1 + 2c_{10}''$), and $\Psi_\Delta \leq K_0^B$ for every class-A cell, $\Psi_\Delta \leq K_0^B b_D^{1/6}$ for $\Delta \in X_o$, and $\Psi_\Delta \leq K_0^B e^{c_B(b_D)}$ for every class-B cell, with c_B as in Proposition B.10. The constant K_0^B depends only on $\lambda_{\text{PV}}, C, \mu, \kappa_S, \delta_{\text{GM}}, q_{\text{pin}}, c_p, c',$ the constants of Theorem 6.11, Lemma 7.31 and the Wick bound, and the geometry constants $c_u, C_u, C_U, D_U, v_\pm$; it does not depend on b_D, L' or τ .

We give the proof after Lemma B.9.

The numerical factors $\delta\hat{D} = \hat{D} - \text{avg}_\Omega \hat{D}$ and $\hat{F}'_2$ sit at line ends integrated freely over their cells. For a term we put $\mathcal{N} := \prod_j \int_{\Delta_j} |N_j(\hat{x})| d\hat{x}$, the product over its numerical factors N_j , and $n_\Delta^N \leq n_\Delta$ denotes the number of them in the cell Δ .

Lemma B.8 (Numerical factors). *All numerical factors lie in class-B cells. For every union Y of elements of $\tilde{Y}(h)$ and every $c > 0$, the numerical factors located in Y satisfy*

$$\prod_{j: \Delta_j \subseteq Y} \int_{\Delta_j} |N_j| d\hat{x} \leq \prod_{\Delta \subseteq Y} \left[\left(\frac{C_u}{2c} \right)^{n_\Delta^N/2} (n_\Delta^N!)^{1/2} \right] e^{3cF_1^{\text{loc}}(Y)}.$$

In particular, with $3c = c'/16$ on $X \setminus Z$ and $3c = c'/32$ on Z , $\mathcal{N} \leq \prod_\Delta [\dots] e^{\frac{c'}{16} F_1^{\text{loc}}(X \setminus Z) + \frac{c'}{32} F_1^{\text{loc}}(Z)}$.

Proof. First, $\text{supp } D_\beta, \text{supp } F'_{2,\beta} \subseteq S_\beta$ (Proposition 7.8(b), (e)), so all numerical factors lie in class-B cells, and S_β lies in one element of $\tilde{Y}(h)$. By the Cauchy–Schwarz inequality on each cell and $\|\delta\hat{D}\|_{L^2(\Delta)} \leq \|\hat{D}\|_{L^2(\Delta)}$, the left side is at most $\prod_\Delta (C_u a_\Delta)^{n_\Delta^N/2}$ with $a_\Delta := \|\hat{D}\|_{L^2(\Delta)}^2 + \|\hat{F}'_2\|_{L^2(\Delta)}^2$, and $a^{n/2} \leq (2c)^{-n/2} (n!)^{1/2} e^{ca}$. Finally $\sum_{\Delta \subseteq Y} a_\Delta \leq (2 + \varepsilon(L')) F_1^{\text{loc}}(Y) \leq 3F_1^{\text{loc}}(Y)$ by Corollary 7.10, Theorem 7.16 and Corollary 7.18.

Lemma B.9 (Factorials). *All factorials produced in a term are at most $\prod_\alpha C^{n_\alpha + N_\alpha} (n_\alpha!)^{q_0} (N_\alpha!)^{q_0}$ with $q_0 := 8 + c''_7$. Consequently the pinning factor $f_{q_{\text{pin}}}$ with $q_{\text{pin}} = q_0 + 2$ absorbs them and leaves $\prod_\alpha (n_\alpha!)^{-2} (N_\alpha!)^{-2}$.*

Proof. We count the factorials per cell, up to factors $C^{n_\alpha + N_\alpha}$. The Wick bound $T_1 \leq c_T^{\sum n_j} \prod_j n_j!$ with $n_j \leq 1 + 4(n_\alpha + N_\alpha)$ gives $(n_\alpha!)^4 (N_\alpha!)^4$, the expansions inside the types and Lemma B.2(c) give $(n_\alpha!)^3$, Lemma 7.31 gives $(k!)^{c''_7}$ (since $k \log k \leq \log k! + k$), and Lemma B.8 gives $(n_\alpha!)^{1/2}$. The total exponent is $7.5 + c''_7 \leq q_0$.

Proof of Proposition B.7. We follow the argument of [BF80, (9.88) to (9.823)] on G' , make it quantitative, and add the payment of the tree weights of κ' .

Assignments. For each $\mathcal{E}$ -factor e we fix its form and a surjection $\pi_e : \{1, \dots, t_e\} \rightarrow \mathfrak{a}(e)$, the cell of each of its points. We carry out the steps headed *Count*, *Pinning*, *Hölder and positions* up to the integration of the points of the $\mathcal{E}$ -factors, *Numerical factors and factorials* and *Weights* for the terms with these assignments. In these steps the maps l , Rule D, the pinning sets $\Pi(e)$, the witnesses and the weights $P(e)$ are those of these terms. The bounds these steps give do not depend on the assignments. The numbers $m_\alpha, n_\alpha, N_\alpha, n_j$ enter only through Lemmas B.2(c), B.4(b), B.9 and the Wick bound, whose right sides are uniform, and $\hat{L}^E(e)$ depends only on $\mathfrak{a}(e)$. The positions and charges of the points of e enter only through $|\hat{\rho}_{t_e}|$ and $\prod_i |m_i|^{k_i}$, k_i the number of derivatives on the i th factor. By $|m_i|^{k_i} \leq k_i! (2\kappa_S)^{-k_i} e^{2\kappa_S |m_i|}$, the factorials $k_i!$ are among those of Lemma B.9 (the charge sums in the proof of Lemma B.2), and the weight $e^{2\kappa_S \sum_i |m_i|}$ does not depend on the assignment. We therefore bound the sum over the assignments, the positions and the charges of the points of e at once by (M4) with $\mathfrak{a} = \mathfrak{a}(e)$. The sum over the forms costs at most a factor 3 (see the *Constants* step).

Count. The sum over the maps l and the further expansions is at most the number of terms (Lemma B.2(c)) times the largest term (the “Sup” of [BF80, (9.88)]).

Pinning. We insert $f_{q_{\text{pin}}}$ and bound $\prod_{\alpha} (n_{\alpha}!)^{q_{\text{pin}}} (N_{\alpha}!)^{q_{\text{pin}}}$ by Lemma B.4(b), paid as stated after that lemma. This leaves $e^{-(\bar{\delta} + (r_C - \bar{\delta})/2)\text{dist}}$ per link. The Mayer part is paid in the next step.

Tree weights. In κ' (Lemma 7.37) an $\mathcal{E}$ -factor e appears as $\sum_{\eta^A} e^{c_{10}L_{\eta^A}} e^{c''_{10}t_e} \mathcal{E}_{\eta^A}$, where L_{η^A} is the length [BF80, (A1.3)], measured with dist and with auxiliary points in V , of an augmented tree η^A whose marked points lie in the cells of $\mathbf{a}(e)$, one in each (Lemma A.6), and $\mathcal{E} = \sum_{\eta^A} \mathcal{E}_{\eta^A}$. Since only $\sum_{\eta^A} \mathcal{E}_{\eta^A} = \mathcal{E}$ is used, we may put $\mathcal{E}_{\eta_e^A} := \mathcal{E}$ for one such tree η_e^A of least length and $\mathcal{E}_{\eta^A} := 0$ otherwise. Then e carries the number $e^{c_{10}L^d(e)} e^{c''_{10}t_e}$, $L^d(e) := L_{\eta_e^A}$, which depends only on $\mathbf{a}(e)$, that is, on the link datum. No sum over positions at a fixed η^A occurs (the property used in [BF80, p. 214] for constant weights b_{η^A}), because positions are counted as sets of cells in Lemma A.6 and the assignments are summed by (M4). A minimizer exists, since V is finite and an auxiliary vertex of degree at most 2 can be removed without increasing the length. If $L_{G'}(\mathbf{a}(e)) = \infty$, every configuration with the cells $\mathbf{a}(e)$ has points in two components of G' , hence ρ_{t_e} vanishes on it (Proposition 6.3(d)) and $\mathcal{E} = 0$. Such a factor is absent (its coefficient is 0, hence no product $0 \cdot \infty$ arises), as before Lemma A.6. In what follows, $L_{G'}(\mathbf{a}(e)) < \infty$. Since $\text{dist} = \tilde{m}d_{G'}$ and $L_{G'}$ is computed with $3^{-1/2}d_{G'}$, we have $L^d(e) = \sqrt{3}\tilde{m}L_{G'}(\mathbf{a}(e)) = \sqrt{3}\hat{L}(\mathbf{a}(e))$. Thus the Mayer decay of (M4) is $e^{-\rho_S L^d(e)}$ with $\rho_S := (1 - \delta_{\text{GM}})A/(\sqrt{3}\tilde{m}) \geq 2r_C$ (the chain after Lemma B.4). Moreover, $L^d(e) \geq \hat{L}^E(e)$, as $d_{G'}$ dominates the Euclidean distance (both lengths are minima over points of the cells of e). With the pinning factor $e^{4c_p \hat{L}^E(e)}$ of Lemma B.4(b) we obtain

$$e^{c_{10}L^d(e)} e^{4c_p \hat{L}^E(e)} e^{-\rho_S L^d(e)} \leq e^{-(\rho_S - c_{10} - 4c_p)\hat{L}^E(e)} \leq e^{-r_C \hat{L}^E(e)},$$

because $\rho_S - c_{10} \geq 0$ and $\rho_S - c_{10} - 4c_p \geq 2r_C - \frac{7}{9}r_C - \frac{2}{9}(r_C - \bar{\delta}) = r_C + \frac{2}{9}\bar{\delta}$. The payment uses L^d , not $\hat{L}^E$, because on a slit graph $d_{G'}$ is not bounded by a multiple of the Euclidean distance, and hence $e^{-r_C \hat{L}^E(e)}$ alone cannot pay $e^{c_{10}L^d(e)}$. Finally, $t_e \geq 2$ for each $\mathcal{E}$ -factor ([GM82, (E.5a) to (E.5d)] and Appendix C). Hence $e^{c''_{10}t_e} \leq e^{2c''_{10}(t_e-1)}$, which cancels against the factor $e^{-2c''_{10}(t_e-1)}$ of $e^{-\mu(t_e-1)}$ and leaves $e^{-(\mu-2c''_{10})(t_e-1)}$.

Hölder and positions. At fixed positions we bound T_1 by the Wick bound, $T_2 e^{-F_1} T_3 T_4$ by Proposition 7.35(d), and the factors r_{Ω} by Lemma B.5. As $0 \leq \partial_{s_i} p(x, y|s) \leq 1$ in [GM82, (E.3)], $|\partial_{s_i} \hat{C}| \leq \tilde{m}^{-1}|C'|$. We bound each line by its supremum over its pair of end cells (Lemma B.1(c)) and integrate its ends against a vertex point, a cell average, an r_{Ω} -derivative (cost at most v_+/v_-) or a numerical factor (kept in $\mathcal{N}$). The points of each $\mathcal{E}$ -factor e , summed over the assignments, positions and charges, are integrated against its coefficient by (M4) with $\mathbf{a} = \mathbf{a}(e)$, and the other internal points by (M1) to (M3) (Lemma B.1(d)). This gives at most $\prod_i (2K_1''/\lambda_{\text{PV}}) e^{-r_C \text{dist}(\Delta'_i, \Delta''_i)} \prod_{\text{vertices}} [\text{mass} \cdot b_D^w] (1 + v_+/v_-)^{2(n-1)} \mathcal{N}$.

Numerical factors and factorials. We bound these by Lemmas B.8 and B.9.

Weights. We use Theorem B.6 and $b_D \leq 1$. A cell with weight $\geq \frac{1}{6}$ gets at most $b_D^{1/6}$, and one with weight ≥ 0 at most 1. A class-B cell gets $b_D^{-N_{\Delta}^{\text{ch}} - n_{\Delta}^{(1)}/2}$ with $(N_{\alpha}!)^{-2} (n_{\alpha}!)^{-2}$ from Lemma B.9 and $N_{\Delta}^{\text{ch}} \leq N_{\alpha}$ (a charged $\mathcal{E}$ -factor is pinned at its charged cell), and Proposition B.10 bounds the product by $e^{c_B(b_D)}$.

Constants. Each remaining constant is per line (at most two per cell, as $\sum_{\alpha} n_{\alpha} = 2(n-1) \leq 2\#X$), per cell, or per $\mathcal{E}$ -factor. Only $\mathcal{E}$ -factors whose cells lie in one component of G' occur (see the *Tree weights* step). The per- $\mathcal{E}$ -factor constants that grow with t_e are 2 for each undistinguished ε (Rule D), C_{M4} for each point and the factor t_e of (M4), and e^{c_p} for

each factor (Lemma B.2(c)). Their product is at most $t_e(2C_{M4}e^{c_p})^{t_e}$. As $C \leq \frac{1}{4}$, we have $2C_{M4} \leq \frac{8}{3}$. Moreover, $c_p < r_C/18 < \frac{1}{540}$ ($r_C \leq \kappa_0 \leq (\varepsilon_1^{\text{HK}})^{1/2} < \frac{1}{30}$, Theorem 6.11). Therefore $q := 2C_{M4}e^{c_p}/e \leq 0.9829$, $K_t := \sup_{t \geq 1} tq^{t-1} \leq 22$ and $2C_{M4}e^{c_p}K_t = e \sup_{t \geq 1} tq^t \leq 58$. Hence $t_e(2C_{M4}e^{c_p})^{t_e} \leq 2C_{M4}e^{c_p}K_t e^{t_e-1}$, and one factor $e^{-(t_e-1)}$ absorbs e^{t_e-1} . This turns the factor $e^{-(\mu-2c_{10}'')(t_e-1)}$ left by the *Tree weights* step into $e^{-(\mu-1-2c_{10}'')(t_e-1)}$. The per- $\mathcal{E}$ -factor constants independent of t_e (K_{M4} , $4C_u$, $2C_{M4}e^{c_p}K_t \leq 58$, the factor 3 for the forms when $t_e = 2$, and the pinning constant of Lemma B.4) amount to a constant per cell, as $\#\mathcal{E} \leq 2\#X$. Summing the assignments thus changes K_0^B only by the numerical factor $K_t \leq 22$ per $\mathcal{E}$ -factor (at most 22^2 per cell), and the forms by at most 3^2 per cell, so the dependence of K_0^B stated in the proposition is unchanged.

Proposition B.10 (Class-B cost). *Let $q_1 \geq 1$ be the exponent of the leftover pinning factor $\prod_\alpha (n_\alpha!)^{-q_1} (N_\alpha!)^{-q_1}$ ($q_1 = 2$ for $q_{\text{pin}} = q_0 + 2$). The b_D -dependent part of the factor on a class-B cell is at most $e^{c_B(b_D)}$, where $c_B(b_D) := q_1(C_\mathcal{E}/b_D)^{1/q_1} + q_1(C_G b_D^{-1/2})^{1/q_1} \leq q_1(C_\mathcal{E}^{1/q_1} + C_G^{1/q_1})b_D^{-1/q_1}$, $C_\mathcal{E}$ being the constant of one charged $\mathcal{E}$ -factor and C_G that of one single-derivative block; both are independent of L' . For $q_1 = 2$, $c_B(b_D) \leq 2(C_\mathcal{E}^{1/2} + C_G^{1/2})b_D^{-1/2}$. The bound gives $c_B(b_D) = o(\Phi_U)$ as $\tau \rightarrow \infty$ if and only if $q_1 > 1$.*

Proof. By Theorem B.6(iii) and the *Weights* step of the proof of Proposition B.7, the cell carries at most $(C_\mathcal{E}/b_D)^{N^{\text{ch}}} (C_G b_D^{-1/2})^{n^{(1)}} (N_\alpha!)^{-q_1} (n_\alpha!)^{-q_1}$ with $N^{\text{ch}} \leq N_\alpha$, $n^{(1)} \leq n_\alpha$. We may assume $C_\mathcal{E}/b_D, C_G b_D^{-1/2} \geq 1$ and raise the exponents to N_α, n_α . For $q, a \geq 1$, $\sup_N a^N/(N!)^q \leq (\sum_N a^{N/q}/N!)^q = e^{qa^{1/q}}$. For the last assertion, we use $\Phi_U \asymp b_D^{-1} \beta_{\text{GM}}^{-3}$ (Proposition 7.19, as $\tilde{m}L' = K_{\text{col}}\beta_{\text{GM}}$) and $b_D = e^{-2.4944\tau(1+o(1))}$. If $q_1 > 1$, then $c_B/\Phi_U \leq C b_D^{-1/q_1} \beta_{\text{GM}}^3 \rightarrow 0$. If $q_1 = 1$, the bound is of order b_D^{-1} and the ratio grows like β_{GM}^3 . The order b_D^{-1/q_1} cannot be improved within the bounds of Lemma B.5. Indeed, a class-B cell can contain an $\mathcal{E}$ -factor all of whose factors are undifferentiated class-B ε 's, of weight exactly -1 , since $\varepsilon = e^{im\beta_{\text{GM}}^{1/2}(\psi+D+h)} - 1$ with $D \neq 0$ has no small bound. As Lemma B.4 holds for all q , the choice $q_{\text{pin}} = q_0 + 2$ is admissible.

Remark B.11 (One or two derivatives on G_2 in class-A cells). With the bounds [BF80, (9.814), (9.815)] as printed on [BF80, p. 231], $(c\beta^{1/2})^n \cdot 2$ for the n th derivative of a term from v), and $c\beta(|\delta|^{2-n} + |g-h|^{2-n})$ if $n < 2$, $(c\beta^{1/2})^n$ if $n \geq 2$, for either term from vi), a G_2 -block with one or two derivatives in a class-A cell carries no net power of b_D : a $G_{2,2}$ -block whose single derivative falls on the factor vi) has weight $-1 + 0 + 1 = 0$, and a $G_{2,1}$ -block with one or two derivatives has weight $-1 + 1 = 0$. A lone cell whose only witness is such a block then receives no power of b_D ; the class-A bounds of Lemma B.5(a) supply it: in a class-A cell $|v_m(\Omega)| \leq |m|\beta_{\text{GM}}^{1/2}|\text{avg}_\Omega \psi|$ as $h \in P\mathbb{Z}$, and by charge symmetry $G_{2,1}$ is of fourth order in $\tilde{\delta}$; both give weight at least $\frac{1}{2}$ (Lemma B.5(b)). Without charge symmetry the third-order form of the second term of vi) already gives $-1 + \frac{3}{2} = \frac{1}{2}$.

Proof of Lemma 7.38. We insert Proposition B.7 into Lemma 7.37. With $e^{c_2^L|X|}$ and $e^{c_{13}|X|}$, its factors are distributed cell by cell as stated, with $c_o := c_2^L + c_{13} + \log K_0^B + 2\log(2K_1''(1 + v_+/v_-)/\lambda_{\text{PV}})$ (the share of the line constants, at most two lines per cell) and c_B of Proposition B.10 with $q_1 = 2$. Since a class-B cell meets $\bigcup_\beta S_\beta$, it lies in a wall polymer, in X_Σ or in the seed. Factors on class-A cells carry in addition b_D^w , $w \geq 0$ (Theorem B.6(ii)), with

constants independent of L' , as are $C_{\mathcal{E}}$, C_G (Section S1.7.6; no class-A cell carries D , F'_2 or a quantity depending on $\hat{\Sigma}$, Proposition 7.8(b), (e)). Thus L' enters only as stated, $\varepsilon(L')$ through Lemma B.8.

Proof of Proposition 7.39. We insert Proposition B.7 into Lemma 7.37 with $c'_1 = c'/4$ and book F_1^{loc} by Table S1.4: $c'/16$ for the numerical factors (Lemma B.8) and $c'/8$ for the wall polymers on $X \setminus Z$, $c'/32$ for the numerical factors on Z . The share $c'/8$ gives each cell of a wall polymer of $X \setminus Z$ the factor $e^{-(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}$, since $F_1(S_{\beta}) \geq \Phi_{\mathcal{U}} \# S_{\beta}$ (Proposition 7.19(c)), a closure adds at most $D_{\mathcal{U}}$ cells per cell, and $F_1^{\text{loc}}(\mathcal{S}) \geq \sum_{S_{\beta} \subseteq \mathcal{S}} F_1(S_{\beta})$ (by the definition of F_1^{loc} in Corollary 7.10). The link decay of Proposition B.7 pays $e^{\delta_2 d}$ ($\delta_2 < \bar{\delta}$). The Mayer cubes and the per- $\mathcal{E}$ -factor constants growing with t_e are paid inside Proposition B.7, and $\mathfrak{m}_e \leq 1$ as $\mu \geq 1 + 2c''_{10}$ (item 12 of Section S1.1.1). With $K_B := K_0^B(2K_1''(1+v_+/v_-)/\lambda_{\text{PV}})^2$, independent of b_D , L' , τ , the per-cell factors and the line-constant shares give ε_{Δ} (Section S1.7.6). The factor $e^{C_{\mathfrak{D}} c_{\text{dep}}}$ is present only on cells meeting $\mathfrak{D}$, and the rest of the factor on $X \setminus Z$ is at most 1. The exponent $p_0 = \frac{1}{6}$ is attained by Theorem B.6.

C Göpfert–Mack Appendix E on slit domains

In this appendix we prove Proposition 7.22. The expansion of [GM82, Appendix E] is an algebraic identity (Bernoulli interpolation of the couplings between consecutive sets, the fundamental theorem of calculus, Gaussian integration by parts). Its only analytic input, absolute convergence, needs on G' the Poincaré inequality on cells, as the Peierls identity removes the mass in the cell-fluctuation directions (Lemma C.2). On block cubes this is the bound of [BF80, (9.640) to (9.642)].

Setting. Fix h . Only its walls and the integrand of Lemma 7.21 enter, and therefore everything holds verbatim for wall fields (Lemma 7.24). The covariance $\mathbf{C}$ is $C = (C_0^{-1} + \nu)^{-1}$ of Proposition 7.7 or C' of Lemma 7.29, positive definite on V with $\mathbf{C} \leq \alpha C_0$, $\alpha = 2$ for C and $\alpha = \alpha_D \leq \frac{7}{6}$ for C' . The integrand is $e^E Q_h$, $Q_h = e^G e^{-F_1^{\text{loc}}(V)} e^{-F_2}$, with $G^{\natural}$ for G when $\mathbf{C} = C'$. We write $\partial_x := \partial/\partial\psi(x)$. A *sequence* is $\bar{y} = (Y_1, \dots, Y_m)$ of nonempty admissible sets (Definition 7.20), pairwise disjoint, with the first set Y_1 fixed before the expansion starts. Its union is $X(\bar{y}) := \bigcup_i Y_i$.

Interpolation. For pairwise disjoint admissible $Y_1, \dots, Y_n$ with union X , let $Y_{n+1} := V \setminus X$, $\chi_i := \mathbf{1}_{Y_i}$ and, for $s \in [0, 1]^n$,

$$p(x, y \mid s) := \sum_{1 \leq i < j \leq n+1} s_i \cdots s_{j-1} [\chi_i(x) \chi_j(y) + \chi_i(y) \chi_j(x)] + \sum_{i \leq n+1} \chi_i(x) \chi_i(y), \quad (\text{C.1})$$

and $\mathbf{C}(s)(x, y) := p(x, y \mid s) \mathbf{C}(x, y)$, [GM82, (E.2), (E.3)] with $\chi_{n+1} = \mathbf{1}_{V \setminus X}$. For $x \in Y_a$, $y \in Y_b$, $a \leq b$, $p(x, y \mid s) = \prod_{a \leq k < b} s_k$, and hence $\partial_{s_k} p \in [0, 1]$ vanishes unless x, y lie on different sides of the cut between $\bigcup_{i \leq k} Y_i$ and $\bigcup_{i > k} Y_i$. For $\varepsilon \in \{0, 1\}^n$ let $\pi(\varepsilon)$ be the partition of $\{1, \dots, n+1\}$ into the maximal runs of consecutive indices joined by the k with $\varepsilon_k = 1$, $Y_J := \bigcup_{i \in J} Y_i$ (admissible), and $\lambda_{\varepsilon}(s) := \prod_k s_k^{\varepsilon_k} (1 - s_k)^{1 - \varepsilon_k}$. Then

$$\mathbf{C}(s) = \sum_{\varepsilon} \lambda_{\varepsilon}(s) \mathbf{C}_{\varepsilon}, \quad \mathbf{C}_{\varepsilon} := \sum_{J \in \pi(\varepsilon)} \mathbf{1}_{Y_J} \mathbf{C} \mathbf{1}_{Y_J}, \quad (\text{C.2})$$

since $\sum_\varepsilon \lambda_\varepsilon(s) \mathbf{C}_\varepsilon(x, y)$ is $\mathbf{C}(x, y)$ times the probability, for independent Bernoulli(s_k) variables ε_k , that a and b lie in one run. Hence $\lambda_{\min}(\mathbf{C}) \leq \mathbf{C}(s) \leq \|\mathbf{C}\| \leq \alpha \tilde{m}^{-2}$, $\mathbf{C}(s)(x, x) = \mathbf{C}(x, x)$, $|\mathbf{C}(s)(x, y)| \leq |\mathbf{C}(x, y)|$ (so each entrywise monotone norm of $\mathbf{C}(s)$ is at most that of $\mathbf{C}$), and $\|\mathcal{M}^{1/2} \mathbf{C}(s) \mathcal{M}^{1/2}\| \leq \|\mathcal{M}^{1/2} \mathbf{C} \mathcal{M}^{1/2}\|$ for each symmetric $\mathcal{M} \geq 0$ commuting with all χ_i . For a sequence $\bar{y}$ of length n , we write $\mathbf{C}^{\bar{y}}(s, s_n)$ for (C.1) built from $\bar{y}$ ($s = (s_1, \dots, s_{n-1})$), $d\mu_{ss_n}^{\bar{y}}$ for its Gaussian measure and $d\mu_s^{\bar{y}}$ for the measure at $s_n = 0$.

The weights of E . By (S1) and [GM82, (E.5a) to (E.5d)], E is a sum $\sum_\tau e_\tau$ of terms of two kinds: for $t \geq 3$ particles, $e_\tau = \frac{1}{t!} \rho_t(\xi_1, \dots, \xi_t) \varepsilon_k(\xi_1) \cdots \varepsilon_k(\xi_t)$, and for $t = 2$ the three terms $\frac{1}{2} \rho_2 \tilde{\varepsilon}_k(\xi_1) \tilde{\varepsilon}_k(\xi_2)$, $\frac{1}{2} \rho_2 \tilde{\varepsilon}_k(\xi_1) i m_2 \beta_{\text{GM}}^{1/2} \psi(x_2)$ and $\frac{1}{2} \rho_2 i m_1 \beta_{\text{GM}}^{1/2} \psi(x_1) \tilde{\varepsilon}_k(\xi_2)$. Here $x_i \in V$, $m_i \neq 0$, $\varepsilon_k(\xi) = e^{im(\beta_{\text{GM}}^{1/2}(\psi+g)(x)+kf(x))} - 1$ and $\tilde{\varepsilon}_k(\xi) := \varepsilon_k(\xi) - i m \beta_{\text{GM}}^{1/2} \psi(x)$. With $\text{supp } \tau := \{x_1, \dots, x_t\}$, $I(\tau)$ is the set of $k \leq n$ such that $\text{supp } \tau$ meets both $\bigcup_{i \leq k} Y_i$ and $\bigcup_{i > k} Y_i$. We put $w_s(\tau) := \prod_{k \in I(\tau)} s_k$, $E(Y, s) := \sum_{\text{supp } \tau \subseteq Y} w_s(\tau) e_\tau$ and $E(Y) := E(Y, 1)$, and $E^{\bar{y}}(Y, (s, s_n))$ denotes $E(Y, \cdot)$ built from $\bar{y}$. This is $E(\bar{X}, s)$ of [GM82, p. 89] and $E(\Lambda, ss_n)$ of [GM82, p. 90, (v)], with sites for the unit cubes a_i .

Absolute convergence. Let P_V be the cell-fluctuation projection, $(P_V \varphi)|_\Omega := \varphi|_\Omega - \text{avg}_\Omega \varphi$, and $\|\nu\|_0 := \sup_x \sum_y |\nu(x, y)|$. The fine Poincaré inequality on cells (Theorem 3.20 (R6), Proposition 6.9 (iv)), summed with the overlap N_{ov} , gives $\|P_V \varphi\|^2 \leq C'_P L^2 \|\nabla \varphi\|^2$, $C'_P := C_W^f(k_\square/L)^2 N_{\text{ov}}$, and (L8-2) of Subsection 7.6 gives $C'_P(\tilde{m}L)^2 \leq \eta_H/8 < \frac{1}{7}$. The constants c_V , b_V , C_α below, which we use only to prove convergence, may depend on G' .

Lemma C.1 (Growth of $\ln Q_h$). *Let $\vartheta := \psi + D$. For every admissible Y , every sequence, every s and real ψ : (a) $E(Y, s)$ is real, and $E(Y, s) = B_{Y,s}(\psi) + \frac{1}{2} \langle \psi, \nu_s^Y \psi \rangle$ with $|B_{Y,s}(\psi)| \leq b_V$ and $\nu_s^Y(x_1, x_2) := \mathbf{1}_Y(x_1) \mathbf{1}_Y(x_2) w_s(x_1, x_2) \nu(x_1, x_2)$, where $w_s(x_1, x_2) := \prod_{a \leq k < b} s_k$ for $x_1 \in Y_a$, $x_2 \in Y_b$, $a \leq b$; moreover $\|\nu_s^Y\|_{2 \rightarrow 2} \leq \|\nu\|_0 \leq \varepsilon_\nu \tilde{m}^2$, and $|\partial^\alpha E(Y, s)| \leq C_\alpha(1 + \|\psi\|)$ for $|\alpha| \geq 1$; (b) $G_{2,1}(Y) \leq \frac{1}{2}(1 + C e^{-2\kappa_S}) \tilde{m}^2 \|\mathbf{1}_Y P_V \vartheta\|^2$; (c) $G_1(Y) \leq c_V$, $|G_{2,2}(Y)| \leq c_V(1 + \|\psi\|_1)$ and $|G_{2,3}(Y)| \leq c_V$; for the terms $G_{2,3}^{\mathfrak{D}}$, $G_{2,3}^{\mathfrak{H}}$ of $G^{\mathfrak{H}}$ (Lemma 7.29), $G_{2,3}^{\mathfrak{D}}(Y) \leq c_V$ and $G_{2,3}^{\mathfrak{H}}(Y) \leq c_V + \frac{1}{2} \theta' \tilde{m}^2 \|\psi\|^2$; (d) $F_1^{\text{loc}}(Y) \geq 0$ does not depend on ψ , $F_2(Y)$ is linear in ψ , and $|\partial^\alpha \ln Q_h(Y)| \leq C_\alpha(1 + \|\psi\|)$ for $|\alpha| \geq 1$.*

Sketch. (a) By (S3), the terms pair into complex conjugates. With $\varepsilon_k = \tilde{\varepsilon}_k + i m \beta_{\text{GM}}^{1/2} \psi$ the terms with $t = 2$ give $\frac{1}{2} \rho_2 \varepsilon_k \varepsilon_k$ plus a part with charge sum $\frac{1}{2} \nu$. Using (S2), we bound $B_{Y,s}$ by $\sum_{t \geq 2} \frac{2^t}{t!} \sum |\rho_t|$, and the Schur test with $0 \leq w_s \leq 1$ gives the norm bound. (b) Here $\tilde{\delta} = P_V \vartheta$, and by (S3) $G_{2,1}$ is a sum of terms $\rho^{\text{ref}}(m) [\cos y - 1 + \frac{1}{2} y^2]$ of modulus at most $\frac{1}{2} |\rho^{\text{ref}}(m)| y^2$, since $0 \leq \cos y - 1 + \frac{1}{2} y^2 \leq \frac{1}{2} y^2$, with $\beta_{\text{GM}} \sum_m m^2 |\rho^{\text{ref}}(m)| \leq (1 + C e^{-2\kappa_S}) \tilde{m}^2$. Part (c) is explicit on the finite set V , with Proposition 7.5 off $\mathfrak{D}$. Part (d) follows from Corollary 7.10, Proposition 7.8, and the fact that $\ln r_\Omega$ is smooth and periodic in A_Ω .

Lemma C.2 (Margin). *Assume the standing hypothesis (7.8), and let κ_S be so large that $(1 + C e^{-2\kappa_S}) \eta_H/8 \leq \frac{1}{7}$, so that (L8-2) gives $(1 + C e^{-2\kappa_S}) C'_P(\tilde{m}L)^2 \leq \frac{1}{7}$. Put $\mathcal{M} := (1 + C e^{-2\kappa_S}) \tilde{m}^2 P_V + (\varepsilon_\nu + \theta') \tilde{m}^2$ for $\mathbf{C} = C'$ and $\mathcal{M} := (1 + C e^{-2\kappa_S}) \tilde{m}^2 P_V + \varepsilon_\nu \tilde{m}^2$ for $\mathbf{C} = C$. Then $\mathcal{M} \geq 0$ is symmetric, commutes with $\mathbf{1}_\Omega$ for every cell Ω , $\mathcal{M} \geq (1 + C e^{-2\kappa_S}) \tilde{m}^2 P_V + \|\nu\|_0 + \theta' \tilde{m}^2 \mathbf{1}_{[\mathbf{C}=C']}$, and*

$$\langle \varphi, \mathcal{M} \varphi \rangle \leq q_M \langle \varphi, \mathbf{C}^{-1} \varphi \rangle, \quad q_M \leq \frac{1}{6} \quad (\mathbf{C} = C'), \quad q_M \leq \frac{2}{7} \quad (\mathbf{C} = C). \quad (\text{C.3})$$

Moreover $\|\mathcal{M}^{1/2}\mathbf{C}(s)\mathcal{M}^{1/2}\| \leq q_M$ for every sequence of admissible sets and every s , and the same holds for $\mathbf{1}_U\mathbf{C}\mathbf{1}_U$ on $\mathbb{R}^U$ whenever U is a union of cells.

Proof. We control the cell fluctuations by the gradient through the Poincaré inequality on cells, and the constant part by the mass. Both are dominated by $C_0^{-1} = M_Y^{-2}\Delta^2 - \Delta + \tilde{m}^2 \geq \tilde{m}^2 - \Delta$, and $C_0^{-1} \leq \alpha\mathbf{C}^{-1}$. The projection P_V is block diagonal over cells, and thus $\mathcal{M}$ commutes with each $\mathbf{1}_\Omega$. Moreover, $\|\nu\|_0 \leq \varepsilon_\nu \tilde{m}^2$ by (S5). By $\|P_V\varphi\|^2 \leq C'_P L^2 \|\nabla\varphi\|^2$ (see the paragraph headed *Absolute convergence*),

$$\langle \varphi, \mathcal{M}\varphi \rangle \leq (1 + Ce^{-2\kappa s})C'_P(\tilde{m}L)^2 \|\nabla\varphi\|^2 + (\varepsilon_\nu + \theta')\tilde{m}^2 \|\varphi\|^2 \leq \frac{1}{7} \langle \varphi, C_0^{-1}\varphi \rangle,$$

which is at most $\frac{\alpha}{7} \langle \varphi, \mathbf{C}^{-1}\varphi \rangle$, with $\alpha = \alpha_D \leq \frac{7}{6}$ for C' and $\alpha = 2$ for C (where θ' is absent). We now turn to the last statement. Inequality (C.3) says that $\|\mathcal{M}^{1/2}\mathbf{C}\mathcal{M}^{1/2}\| \leq q_M$. Each Y_i and $V \setminus X$ is a union of cells, and hence $\mathcal{M}$ commutes with each χ_i and (C.2) applies. Since $\mathbf{1}_U$ commutes with $\mathcal{M}$, we also get $\|\mathcal{M}^{1/2}\mathbf{1}_U\mathbf{C}\mathbf{1}_U\mathcal{M}^{1/2}\| \leq q_M$, and (C.2) applies on $\mathbb{R}^U$. Any $q_M < 1$ suffices here.

Lemma C.3 (Absolute convergence). *Call $F : \mathbb{R}^V \rightarrow \mathbb{C}$ admissible if F is C^2 and, for $|\alpha| \leq 2$, $|\partial^\alpha F(\psi)| \leq K_F(1 + \|\psi\|)^{d_F} \exp(\frac{1}{2}\langle \psi, \mathcal{M}\psi \rangle + k_F\|\psi\|_1)$. Under the hypotheses of Lemma C.2: (a) for every admissible Y and s' , and every finite composition $\mathfrak{L}$ of operators ∂_x and of multiplications by ψ -derivatives of $E(\cdot, s')$, the function $e^{E(Y, s')} \mathfrak{L}Q_h(Y)$ is admissible, uniformly in s' ; (b) if $t \mapsto \mathbf{C}(t)$ is one of the covariances (C.1) with one s -variable moving and F is admissible, then $\int d\mu_{\mathbf{C}(t)}|F|$ is bounded uniformly in t and in the other variables, $t \mapsto \int d\mu_{\mathbf{C}(t)}F$ is C^1 on $[0, 1]$, and*

$$\partial_t \int d\mu_{\mathbf{C}(t)}F = \frac{1}{2} \sum_{x, y \in V} \partial_t \mathbf{C}(t)(x, y) \int d\mu_{\mathbf{C}(t)} \partial_x \partial_y F; \quad (\text{C.4})$$

(c) if F also depends on t , is C^1 in t , and $\partial_t F$ is admissible uniformly in t , then $\partial_t \int d\mu_{\mathbf{C}(t)}F = \int d\mu_{\mathbf{C}(t)}\partial_t F$ plus the right side of (C.4).

Sketch. (a) We collect the exponents of Lemma C.1 and use the lower bound on $\mathcal{M}$ of Lemma C.2. (b) By Lemma C.2 and (C.2), $\mathbf{C}(t)^{-1} \geq (\mathcal{M} + 2\epsilon)/q'$ with $q' := q_M + 2\epsilon\|\mathbf{C}\| < 1$ for small $\epsilon > 0$. Therefore $|F|g_t$ (g_t the Gaussian density) is integrable uniformly in t . Since $\partial_t g_t = \frac{1}{2} \sum_{x, y} \partial_t \mathbf{C}(t)(x, y) \partial_x \partial_y g_t$, two integrations by parts give (C.4), the step “By partial integration” of [GM82, p. 91]. (c) This follows from the product rule under the same domination.

The operators κ . Fix $\bar{y}$ of length n with $X := X(\bar{y}) \subsetneq V$ and write $E := E^{\bar{y}}(V, (s, s_n)) = \sum_\tau w(\tau)e_\tau$, $E_x := \partial_x E$. As $e^{-E}\partial_x \partial_y (e^E F) = (\partial_x + E_x)(\partial_y + E_y)F$ is symmetric in x, y , and $\partial_{s_n} \mathbf{C}^{\bar{y}}(x, y) = 0$ unless x, y lie on different sides of X , Lemma C.3 (c) gives $\partial_{s_n} \int d\mu'_{ss_n} e^E F = \int d\mu'_{ss_n} e^E \tilde{\kappa}(n)F$ for F not depending on s_n with $e^E F$ admissible, where

$$\tilde{\kappa}(n) := \partial_{s_n} E + \sum_{x \in V \setminus X} \sum_{y \in X} \partial_{s_n} \mathbf{C}^{\bar{y}}(x, y) (\partial_x + E_x)(\partial_y + E_y), \quad (\text{C.5})$$

[GM82, (E.15), (E.16)] (the factor $\frac{1}{2}$ of (E.16) cancels by symmetry, p. 92). Inserting $E = \sum_\tau w(\tau)e_\tau$, we write $\tilde{\kappa}(n) = \sum_\theta \mathfrak{t}_\theta$ with elementary terms of five types: (1) $(\partial_{s_n} w(\tau))e_\tau$;

(2) $\partial_{s_n} \mathbf{C}(x, y) \partial_x \partial_y$; (3) $\partial_{s_n} \mathbf{C}(x, y) w(\tau) (\partial_x \partial_y e_\tau)$; (4) $\partial_{s_n} \mathbf{C}(x, y) w(\tau) (\partial_y e_\tau) \partial_x$, respectively $(\partial_x e_\tau) \partial_y$; (5) $\partial_{s_n} \mathbf{C}(x, y) w(\tau) w(\tau') (\partial_x e_\tau) (\partial_y e_{\tau'})$. The support $S(\theta)$ is $\text{supp } \tau$ in type (1), and $\{x, y\}$ with the supports of the τ 's involved otherwise. If $\mathbf{t}_\theta \neq 0$, then $S(\theta) \not\subseteq X$ (in type (1) $n \in I(\tau)$, otherwise $x \in V \setminus X$). Put $Y_{n+1}(\theta) := \text{sat}_h(S(\theta) \setminus X)$ and

$$\kappa(n)[Y'] := \sum_{\theta: Y_{n+1}(\theta)=Y'} \mathbf{t}_\theta, \quad \tilde{\kappa}(n) = \sum_{Y'} \kappa(n)[Y'], \quad (\text{C.6})$$

as in [GM82, (E.11), (E.20)]. With $\bar{y}' := (\bar{y}, Y')$ and $X' := X \cup Y'$, each θ in $\kappa(n)[Y']$ has $S(\theta) \subseteq X'$, coefficients depending on $\psi|_{X'}$ only, derivatives at $x \in Y'$, $y \in X$, and E -terms of $E^{\bar{y}'}(X', (s, s_n))$ with the same weights. Thus $\kappa(n)[Y']$ depends on $s_1, \dots, s_n$ only. Moreover, $\partial_{s_n} \mathbf{C}^{\bar{y}}(x, y) = s_a \cdots s_{n-1} \mathbf{C}(x, y) = \partial_{s_n} \mathbf{C}^{\bar{y}'}(x, y)$ for $x \in Y'$, $y \in Y_a$. For $\bar{y} = (Y_1, \dots, Y_m)$ the ordered product $\kappa(\bar{y}, s) := \kappa(m-1) \circ \cdots \circ \kappa(1)$, with $\kappa(i) := \kappa(i)[Y_{i+1}]$ built from $(Y_1, \dots, Y_i)$ and each operator acting on everything to its right, is $\kappa(\bar{y}, s)$ of [GM82, (E.9), (E.10)].

Proof of Proposition 7.22. The proof is that of [GM82, Lemma E.1, Proposition E.2, (E.2) to (E.24)], an induction in n . Each step is the fundamental theorem of calculus [GM82, (E.17)] in a new variable s_{n+1} , started from the measure and the E of the previous step by the consistency and decoupling relations [GM82, (E.21a), (E.21b)], which hold on G' by (C.1) and the definition of $w_s(\tau)$. Lemma C.3 justifies each exchange of derivatives and integrals. The following changes are needed on G' . (1) *Atoms are sites* (GM: unit cubes, [GM82, p. 89]): since each Y_j is admissible, $w_s(\tau)$ depends only on which Y_j contain the points. (2) *One finite matrix with mass at every site*, and $\chi_{n+1} = \mathbf{1}_{V \setminus X}$, so no region outside Λ occurs ([GM82, p. 92]). Termination occurs at $m \leq e_h + 1$, since each Y_j , $j \geq 2$, contains an element outside Y_1 ([GM82, p. 93]). (3) *First sets may depend on h and be disconnected* (GM: a fixed union of unit cubes, [GM82, p. 90, (iii)]): the step $n = 1$ uses only $Q_h = Q_h(Y_1)Q_h(V \setminus Y_1)$ (Lemma 7.21), the selection rule only that each $\bigcup_{j \leq i} Y_j$ is admissible, and nothing compares two values of h . Type (2) terms reach the element of x at any distance at which $\mathbf{C}(x, y) \neq 0$, and therefore the sets $X(\bar{y})$ are disconnected in general. (4) *On $U = X_0^c$ the covariance is $\mathbf{1}_U \mathbf{C} \mathbf{1}_U$* : marginals of Gaussian measures are compressions, and hence $\mathcal{K}_{\bar{y}}$ involves the covariance only through the block $p(x, y | s) \mathbf{C}(x, y)$, $x, y \in X(\bar{y})$. Lemma C.2 covers $\mathbf{1}_U \mathbf{C} \mathbf{1}_U$. (5) *The selection rule: $S(\theta)$ contains the bare point x of each covariance term*, so that all derivatives of a term of $\kappa(n)[Y']$ act in $X \cup Y'$. By contrast, [GM82, (E.11)] is stated for the supports of the E -terms, and for the term without E -terms [GM82, p. 91; (E.19), p. 93] take $Y_{n+1} \ni x$. (6) “ $\chi C \chi$ ” is an exact block-sum decomposition: the convex combination of [GM82, p. 88] is (C.2), with compressions to admissible blocks. This decomposition gives the bounds on $\mathbf{C}(s)$ used in Lemmas C.2, C.3 and in Section 7 (Table S1.3). The integrals $\int d\mu_{\mathbf{C}} e^{E(Y)} Q_h(Y)$ are positive ($E(Y)$, G , F_1^{loc} , F_2 are real, Lemma C.1) and finite (Lemma C.3 (b)).

D Heat kernel, coarse-to-fine transfer and covariance bounds on slit domains

In this appendix we prove Proposition 6.9 and Theorem 6.8 and sketch the proofs of Proposition 6.10, Theorem 6.11 and Lemma D.2. Throughout, $G' = (V, E)$ is an output, $d = d_{G'}$, $\mathcal{E}(f) := \sum_{\{x, y\} \in E} (f(x) - f(y))^2$, and $\partial U = \partial_{G'} U$ is the set of bonds of E with exactly one end in U .

D.1 Coarse-to-fine transfer: proof of Proposition 6.9

A large fine set either fills many coarse cells, whose boundary (R2) controls, or is spread thinly inside cells, where box isoperimetry applies (canonical paths and Loomis–Whitney). A face count joins the two scales. The coarse cells, the sites $c \in V_c$ of the coarse slit graph $\Gamma = (V_c, E_c)$ of §6.3, are boxes B_c with sides in $[r_0, 2r_0 - 1]$ (Definition 3.4), and the box graph of $B_c \cup B_{c'}$, $\{c, c'\} \in E_c$, lies in G' (§2.5). Thus a coarse bond between a *heavy* cell ($|A \cap B_c| \geq |B_c|/2$) and a light one carries at least $r_0^2/4$ bonds of ∂A in $B_c \cup B_{c'}$, and $|\partial A| \geq (r_0^2/24)|\partial_\Gamma H|$ for the set H of heavy cells. This gives (i) with (R2) (Theorem 4.5(i), (ii)), and (iii) with the expansion of K in place of (R2). We deduce (ii) from (i) as in Theorem 4.5(iv), and from Theorem 4.5(ii) and $r_0^3 V_* = 2V_*^f$. Part (iv) is (iii) for the blow-up of the set $\mathfrak{K}_c^b$ of (R6) (Lemma 4.18) with Cheeger’s inequality, the overlap count inherited and $|\Omega| \geq c_1 L^3$ from (R5). Finally, (v) holds because balls in a connected graph of degree at most 6 and expansion $h \leq 1$ grow by the factor $1 + h/6$ per step up to half the vertices, so the diameter is at most $2 + (24/h) \log |\mathfrak{K}^b| < L'$ under (L8-2). The constants are given in §S1.5.

D.2 Nash’s inequality from Faber–Krahn, and the on-diagonal bound

Lemma D.1 (Nash inequality). *For $f \geq 0$ on V with $\sum f = 1$, and every $r \geq 1$ with $2r^3 \leq V_*^f$, $\|f\|_2^2 \leq A_N r^2 \mathcal{E}(f) + r^{-3}$, where $A_N := 24 \cdot 2^{2/3}/c_F^2$.*

Proof. Put $s := 1/(2r^3)$. Then $f^2 \leq 2(f - s)_+^2 + 2\min(f, s)^2$ and $\sum \min(f, s)^2 \leq s$. The function $(f - s)_+$ vanishes off $\Omega_s := \{f > s\}$, which has $|\Omega_s| < 1/s = 2r^3 \leq V_*^f$, and $\mathcal{E}((f - s)_+) \leq \mathcal{E}(f)$. By Proposition 6.9(ii), $\sum (f - s)_+^2 \leq (12/c_F^2)(2r^3)^{2/3} \mathcal{E}(f)$.

Proof of (HK0) and (HK1). (HK0) is Davies’ perturbation argument and needs only the degree bound: for $\psi := -\alpha d(x, \cdot)$,

$$\langle g, e^\psi \Delta_{G'} e^{-\psi} g \rangle = -\mathcal{E}(g) + 2 \sum_{\{x, y\} \in E} (\cosh(\psi(x) - \psi(y)) - 1) g(x) g(y) \leq 6(\cosh \alpha - 1) \|g\|^2,$$

so $\|e^\psi P_t e^{-\psi}\|_{2 \rightarrow 2} \leq e^{6(\cosh \alpha - 1)t}$ and $p_t(x, y) = e^{\psi(y) - \psi(x)} \langle \delta_x, e^\psi P_t e^{-\psi} \delta_y \rangle \leq e^{-\alpha d + 6(\cosh \alpha - 1)t}$. At $\alpha = \frac{1}{4}$, $6(\cosh \frac{1}{4} - 1) < \frac{1}{4} - \frac{1}{17}$. The number N_t of jumps up to time t is dominated by a Poisson variable of mean $6t$, and $d(x, X_t) \leq N_t$.

(HK1). The function $\psi(t) := \|P_t \delta_x\|^2 = p_{2t}(x, x)$ is nonincreasing, $\psi \leq 1$ and $\psi' = -2\mathcal{E}(P_t \delta_x)$. With $r(t)$ given by $r^{-3} = \psi/2$, the condition $2r^3 \leq V_*^f$ becomes $\psi \geq 4/V_*^f$, which holds on an interval $[0, t_*]$. On this interval Lemma D.1 gives $-\psi' \geq \psi^{5/3}/(2^{2/3} A_N) = c_N \psi^{5/3}$, hence $\psi(t) \leq (\frac{2}{3} c_N t)^{-3/2}$, while $\psi < 4/V_*^f$ after t_* . Hence $p_s(x, x) \leq \max(C_H s^{-3/2}, 4/V_*^f)$, $p_s(x, y) \leq (p_s(x, x) p_s(y, y))^{1/2}$, and $4/V_*^f = \frac{64}{27} R_{\text{HK}}^{-3}$ (§S1.5). The zero modes lie below the plateau, since $p_t(x, x) \rightarrow 1/|\mathfrak{K}_x| < 1/(2V_*^f)$ (Proposition 6.9(ii)).

D.3 Barlow’s argument: proof of Theorem 6.8

From now on we assume $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, so that $\tilde{m} \leq K_0(2C_H)^{-1/3}$, $R_{\text{HK}} \geq K_0/\tilde{m} \geq (2C_H)^{1/3}$, and $p_t \leq C_H t^{-3/2}$ on $[1, R_{\text{HK}}^2]$ by (HK1). (HK1') is [Bar04, Lemma 3.3(b), Prop. 3.4], (HK2) is [Bar04, Lemmas 3.5, 3.6, Prop. 3.7], and (HK3) is the fixed-time form of [Bar04, Thm 3.8], with the following adjustments, each forced by G' :

(a) [Bar04, Prop. 3.1, Cor. 3.2] (on-diagonal decay in a very good ball for $t \leq cR^2/\log R$) are replaced by (HK1), which holds at all centres. Every ball is a very good ball with $N_B = 1$ [Bar04, Def. 1.7], there is no localization and no $\log R$ loss, and the window is $t \leq \varepsilon_1^{\text{HK}} R_{\text{HK}}^2$. The exit-time bound $\frac{1}{2} + 4C_{\text{mo}}^2 t/r^2$ of [Bar04, Lemma 3.5] holds at all centres for $t \leq R_{\text{HK}}^2/2$, and exceeds 1 for larger t when $r \leq 2C_{\text{mo}} R_{\text{HK}}$.

(b) The walk is the variable-speed walk with counting measure on a finite, possibly disconnected graph, and [Bar04, (3.8) to (3.10)] hold verbatim with $a_{xy} = 1$, $\mu \equiv 1$, as finite sums.

(c) The volume bound [Bar04, (1.1)] enters only [Bar04, Lemma 3.3(b)], through $\sum_y e^{-a d(x,y)} \leq c_4 a^{-3}$ for $0 < a \leq 2$, which follows from $\#\{y : d(x,y) = k\} \leq 4k^2 + 2$, as $d_{G'} \geq \|\cdot\|_1$.

(d) [Bar04, Lemma 1.1] is replaced by (HK0), (HK1) and the Poisson domination of N_t . For $M(t) := \mathbb{E}_x d(x, X_t)$, (HK1) gives $M(t) \geq \mathbb{P}_x(X_t \neq x) \geq 1 - C_H t^{-3/2} \geq \frac{1}{2}$ for $t \geq (2C_H)^{2/3}$, N_t gives $M(t) \leq 6t$ before, and for $\rho > A_1 t/4$ (A_1 of §S1.5) the event $\tau(x, \rho) < t$ forces $N_t > \rho$.

(e) (HK4) is not in [Bar04], but it follows from (HK2). With $D := d(w, X_s)$, we split the sum in $\mathbb{E} e^{\theta D} - 1 \leq \theta \sum_{k \geq 1} e^{\theta k} \mathbb{P}(D \geq k)$ at $k = s$ into a Gaussian part, summed with $\theta \leq s^{-1/2}$, and a Poisson part, summed with $\theta \leq c_{\text{ex}}/2$. For $s < 1$, $D \leq N_s$.

For (HK3), Chapman–Kolmogorov gives $p_t(x, y) \leq 2 \sup_{z,w} p_{t/2}(z, w) \sup_w \mathbb{P}_w(d(w, X_{t/2}) \geq d/2)$, and (HK1), (HK2) bound the two factors. The constants $C_{\text{mo}}, c_{\text{ex}}, A_1, C_{\text{gs}}, c_{\text{gs}}, C_M$ depend only on C_H (§S1.5).

D.4 Resolvent decay and covariance bounds

For Proposition 6.10, (HK4) gives $\mathbb{E}_w e^{\theta_M d(w, X_s)} \leq \frac{3}{2}$ for $s \leq T_1 := \min(M^{-2}, \varepsilon_1^{\text{HK}} R_{\text{HK}}^2)$, blocks of length T_1 give $\mathbb{E}_x e^{\theta_M d(x, X_t)} \leq \frac{3}{2} e^{M^2 t/2}$, and the mass term outweighs this growth in $\sum_y v_{G'}^{(M)}(x, y) e^{\theta_M d(x, y)} = \int_0^\infty e^{-M^2 t} \mathbb{E}_x e^{\theta_M d(x, X_t)} dt$. For Theorem 6.11 we write $C_0 = \int_0^\infty k(t) P_t dt$ with $0 \leq k(t) \leq \min(M_Y^2 t, \varpi) e^{-r-t}$ (§S1.5). With this representation, (H1), (H1'), (H2) follow from (HK0), (HK1), (HK3), (6.13) and the blocking bound by splitting the time integral, and (H_C) follows from the Schur test and a Neumann series in $\|\cdot\|_{r_{\text{cov}}}$. The full proofs are in §S2.4.

D.5 Bounds on the lattice Yukawa kernel

Lemma D.2 (Lattice Yukawa kernel). *Let $0 < M \leq \frac{1}{5}$ and $v^{(M)} := (-\Delta_{\mathbb{Z}^3} + M^2)^{-1}$. There is an absolute constant C_L such that for all $x \in \mathbb{Z}^3$ and all unit vectors e ,*

$$v^{(M)}(x) \leq C_L (1 + |x|)^{-1} e^{-3M|x|/4}, \quad |v^{(M)}(x+e) - v^{(M)}(x)| \leq C_L (1 + |x|)^{-2} e^{-3M|x|/4}. \quad (\text{D.1})$$

If $\Lambda' = \prod_\mu [1, n_\mu]$ is a box with $\min_\mu n_\mu \geq M^{-1}$ and $u_x := v_{\Lambda'}^{(M)}(\cdot, x)$ is the kernel of the box with all its bonds, then for $x, y, y+e \in \Lambda'$, with C'_L absolute,

$$(\text{L1}) \quad u_x(y) \leq C'_L (1 + |y-x|)^{-1} e^{-M|y-x|/2},$$

$$(\text{L2}) \quad |u_x(y+e) - u_x(y)| \leq C'_L (1 + |y-x|)^{-2} e^{-M|y-x|/2}.$$

Sketch. Exponential tilts of the three one-dimensional heat kernels and log-concavity give (D.1) after the time integral, the rate $3M/4$ coming from $M^2 t + \frac{9}{64}|x|^2/t \geq \frac{3}{4}M|x|$. The box kernel is the image sum $v_{\Lambda'}^{(M)}(y, x) = \sum_{g \in \mathcal{G}} v^{(M)}(y - gx)$ over the group $\mathcal{G}$ generated by the

reflections in the face planes. Since $\min_{\mu} n_{\mu} \geq M^{-1}$, the images cost a factor $(1 + M|y - x|)^3$, which the loss of rate from $3M/4$ to $M/2$ absorbs. The full proof is in §S2.4.

Declarations

Use of AI tools. Generative AI tools were used to find and prove the results, to write the programs, and to draft the text.

Conflict of interest. The author declares no conflict of interest.

Data availability. No data were generated or analysed. The scripts behind the numerical checks of Section S3.1 of the supplementary material are in the file `scripts.zip` of the Zenodo record of this paper, <https://doi.org/10.5281/zenodo.23445738>. No proof depends on them.

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Supplementary material

S1 Constants

S1.1 Constants of Sections 2 and 8: the ordered list, the threshold K_* and the conditions defining $\beta_0(K)$

The order of the constants and the reasons for it are in Section 8.2. Here we give the values. They are checked by a computer evaluation in high-precision arithmetic and enter the proofs only through the inequalities stated.

S1.1.1 The ordered list

Each constant is fixed after all constants above it, and no constant in items 1 to 14 depends on β , τ , K , the loop C or D , except that $\bar{r}_0$ of item 1 is r_0 or $2r_0 - 1$ according to $\mathfrak{G}(C)$. The scales of item 11 depend on β only through $\tilde{m}$, and the derived quantities c'_g and c_u depend on β only through the rounding of L_c and lie between two constants of this kind. We give illustrative values at $(c_g, c_s, r_0) = (1/4, 1, 1)$. We use coarse units for the regularization and fine units in Sections 6 and 7.

0. *Absolute constants.* $c_I = 2^{2/3}$; the Debye exponent $\pi^2 v_{\text{Cb}}(0) = 2.49435509\dots$, where $v_{\text{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0, 0) = \frac{\sqrt{6}}{192\pi^3} \Gamma(\frac{1}{24}) \Gamma(\frac{5}{24}) \Gamma(\frac{7}{24}) \Gamma(\frac{11}{24}) = 0.25273101\dots$, half of Watson's integral [Wat39] in the normalization of [Zuc11], whose closed form is [Zuc11, (3.6)], with $\log(1/m_D) = \pi^2 v_{\text{Cb}}(0) \tau - \frac{1}{2} \log(8\pi^2 \tau)$ (rounding rule in Section S1.1.3); $C_{\text{bl}} = 384$ (Proposition 6.9); the window count 151^3 (Definition 3.4); $\alpha = \ln 8$ and $8/31$ (Theorem 7.49, Lemma 7.54); $c_T = 2 + \log 96$ and the threshold $14.9662 \geq 6 \cdot 2.4943551 = 14.96613$ (Lemma 2.6, Section 5); the absolute constants among those of Göpfert–Mack, Brydges–Federbush and Stage 1 quoted in Sections 6, 7 and Appendices A to C. Their constants that depend on the parameters C , δ_{GM} , μ , κ_S , λ_{PV} of Theorem 6.5 are fixed with these parameters, in items 12 and 13.
1. $r_0 \geq 1$, an integer, and $\bar{r}_0 := 2r_0 - 1$ ($\bar{r}_0 := r_0$ if $\mathfrak{G}(C)$ has no exceptional layer). Used in Definition 3.4, Lemma 5.7 and η_T .
2. $c_g := 1/4$. Any $c_g \in (0, 1/4]$ works, since the proofs use only $0 < c_g \leq 1/4$. Used in (S-face) (Definition 3.17), which the construction gives for all $c_g \leq 1 - 134\vartheta$ (Theorem 4.16).
3. $\eta_w = 1/10$, the hull-weight deficit of the frozen bonds (Definition 3.8).
4. $c_\rho = 1/12$: $\rho_\lambda = L_c/12$, $\lambda = 12/L_c$ (the equality case of Lemma 4.6).
5. $k_P/L_c = 304$: windows have sides less than $151 \cdot 2L_c < k_P$ (Definition 3.4, (G4)). We set $r_\square := (k_P + 3)/L_c \leq 305$. It precedes χ and c_L .
6. $c_1 := \min(c_g/2, ((1 - \eta_w)c_I c_\rho/2)^3)$, which equals $2.109 \cdot 10^{-4}$ for $c_g \geq 4.22 \cdot 10^{-4}$. We set $c_{\text{sh}} := c_1/216$. Used in (R5) and Lemma 5.15.
7. $c_s \in (0, 10]$ (illustration $c_s = 1$), with $w_* := (1 + c_s)|\partial\Delta|$ (Definition 3.13) and $\lambda_U := 2c_s/\ell$ (Definition 3.11).

8. The forcing threshold $\vartheta = 3 \cdot 10^{-5}$ and the penalty $\mu_B = 25$ (Definition 3.11), and the thin-set constant $\varepsilon_{\text{th}} = 0.008$ (Section 4.2). They satisfy $24.6 < \mu_B$ (Lemma 4.9), $128\vartheta < \varepsilon_{\text{th}}$, $2\varepsilon_{\text{th}} + 12\vartheta \leq 0.0164$ and $2/\sqrt{0.0164} > 14$ (Lemma 4.13), and $254.4\vartheta \leq \varepsilon_{\text{th}}$ (Theorem 4.16). The arithmetic is in Section S1.2.
9. The weak-link constant $\chi := 2.0 \cdot 10^{-8}$ and $\chi_{N3} := \chi/3 = 6.67 \cdot 10^{-9}$ (Definition 3.15). Its value, which depends only on items 3, 5 and 8 (item 3 through the factor $12/\eta_w$ of Proposition 4.15, which enters the expansion bound), is constrained by Theorem 4.16(c), which gives the expansion $h_\star = 2.2 \cdot 10^{-11}/L_c$ on each component of $G' \cap \hat{Q}$ at rest, and $3k_P h_\star = 2.006 \cdot 10^{-8} \geq \chi$, hence (N3) holds with χ_{N3} and (R6) with χ (Section S1.2).
10. *Consequences, fixed with the constants above and independent of L_c and ℓ .* $C_1 := 7 + 27 \cdot 24.6/\vartheta = 2.214 \cdot 10^7$, $C_{\text{seal}} := 726$, $C'_\star := C_1 + 2C_{\text{seal}} = 2.2141 \cdot 10^7$ and $C_{\text{free}} := C'_\star(2 + 1/(2c_s) + 1/3696) = 5.536 \cdot 10^7$ (Propositions 4.15, 4.24); $C_a := 4362 + 5808/c_s$, $C_b := 2 + 4/c_s$, $C_c := 1 + 2/c_s$, equal to 10170, 6, 3 (Lemma 5.20); the response load $C_{\text{resp}} := 375C_a + 125C_c C'_\star = 8.307 \cdot 10^9$ (Section 5.6); $C_{\text{rm}} = 20$; $C_{\mathfrak{B}} := 30C'_\star + 170C_{\text{resp}} + 30 = 1.413 \cdot 10^{12}$; $C_3 := C_{\text{free}} + 20(2C'_\star + 4C_{\text{resp}} + 1) = 6.655 \cdot 10^{11}$ (Proposition 5.24, Theorem 5.1).
11. c_L : $C'_P c_L^2 \leq (1 - 4/\pi^2)/8$, which holds if $2916r_\square^5 C_{\text{bl}}^2 c_L^2 / \chi^2 \leq (1 - 4/\pi^2)/8$, since $C'_P = C_W^f r_\square^2 N_{\text{ov}} \leq 2916r_\square^5 C_{\text{bl}}^2 / \chi^2$ ($C_W^f = 108C_{\text{bl}}^2 / \chi^2$ of Proposition 6.9(iv) and $N_{\text{ov}} \leq 27r_\square^3$); and $c_L \leq \frac{1}{2}$. The admissible choice is $c_L \leq 8.0 \cdot 10^{-12}\chi$ (the bound is $8.09 \cdot 10^{-12}\chi$ at $r_\square = 305$), that is $c_L = 1.6 \cdot 10^{-19}$ at $\chi = 2.0 \cdot 10^{-8}$. This choice implies $\tilde{m}L \leq \frac{1}{2}$ (Lemma 4.21) and $c_L \leq 1.99 \cdot 10^{-4}\chi_{N3} = 1.33 \cdot 10^{-12}$ (Lemma 7.44). Then $L_c := c_L \tilde{m}^{-1}/r_0$ rounded down to an even integer, $L := r_0 L_c$, $k_P := 304L_c$, $\ell_D \approx \tilde{m}^{-1}$ a multiple of L , and $\ell := \ell_D/r_0 \geq k_P + 4L_c$ (Definition 3.1). Consequences: $c'_g = c_g(1 + 1/L_c)^{-2}$; $c_u := r_0^3 V_\star \tilde{m}^3 = ((1 - \eta_w)c_I \tilde{m}L/24)^3$, which lies in $[((1 - \eta_w)c_I c_L/48)^3, ((1 - \eta_w)c_I c_L/24)^3]$ once $c_L \tilde{m}^{-1} \geq 4r_0$ (then $\tilde{m}L \geq c_L/2$); $R_{\text{HK}} = (1 - \eta_w)c_I c_\rho L/3$ and the constant $K_0 := (1 - \eta_w)c_I c_\rho c_L/6 = 0.01984 c_L$. Since $c_L/2 \leq \tilde{m}L \leq c_L$, we have $K_0 \leq \tilde{m}R_{\text{HK}} \leq 2K_0$ (each use needs only the lower bound).
12. *Stage 1 and Stage 2 constants* (Sections 6, 7, Appendices A to C). The parameters $C \in (0, \frac{1}{4}]$ and $\delta_{\text{GM}} \in (0, \frac{1}{2}]$ of Theorem 6.5, where this C is not the loop. The bound $C \leq \frac{1}{4}$ gives $2C_{M4} = 2/(1 - C) \leq \frac{8}{3}$, hence $q := 2C_{M4}e^{c_p}/e \leq 0.9829$ since $c_p < \frac{1}{540}$, so that one factor $e^{-(t_e-1)}$ absorbs the per- $\mathcal{E}$ -factor constants $t_e(2C_{M4}e^{c_p})^{t_e-1}$ in Proposition B.7, up to the factor $\sup_{t \geq 1} tq^{t-1} \leq 22$, which contributes one factor per cell. The factor t_e is that of (M4), Lemma B.1(d), which bounds the sum over all assignments of the points to their cells. The bound $\delta_{\text{GM}} \leq \frac{1}{2}$ is used for (S5) on outputs and in Lemma B.4 and Proposition B.7. Further $\eta_H = 0.62$; $(p_3, p', q', p_1, p_2, p_4) = (1.2, 1.3, 15.6, 20, 20, 15)$; $c_\lambda = 1/24$; $\theta_0 = 0.01$; the link-sum rate c_{10} with $0 < c_{10} \leq \frac{7}{9}r_C$, r_C evaluated at $\tilde{m}L = c_L/2$ (Proposition B.7: for $\delta_{\text{GM}} \leq \frac{1}{2}$ the rate of (S2) in Debye units is at least $2r_C$ for all $\lambda_{\text{PV}} \leq \frac{1}{4}$, by Proposition 6.10, and pinning uses $4c_p < \frac{2}{9}r_C$ of this rate for every $\bar{\delta}$), where $r_C = \min(\kappa_0/2, (2c_{\text{gs}})^{1/2})$ (Theorem 6.11) depends only on items 0 to 11: it is computed with the constant K_0 of item 11, the value of $\tilde{m}R_{\text{HK}}$ at $\tilde{m}L = c_L/2$, the worst case of $\tilde{m}L \in [c_L/2, c_L]$. Hence c_{10} is fixed independently of β , λ_{PV} and $\bar{\delta}$. We then fix c''_{10} with $e^{c''_{10}} \geq 16e \max(1, c_{c_{10}/2})$ (Lemma A.6), and then the parameters μ and κ_S of Theorem 6.5, both large: $\mu \geq 1 + 2c''_{10}$, after c_{10} and c''_{10}

(the factor $e^{-\mu(s-1)}$ of (S2) pays $e^{c'_{10}t} \leq e^{2c'_{10}(t-1)}$, Proposition B.7), and κ_S so large that $(1 + Ce^{-2\kappa_S})\eta_H/8 \leq \frac{1}{7}$ (Lemma C.2). This last condition is satisfiable: the constant C comes from the tail bound (6.10), that is, from (S2) at $s = 1$ with $a_1 = \{x\}$ and from (6.7), whose right sides involve C and μ but not κ_S , and hence C can be chosen independently of κ_S and λ_{PV} . Further $\varepsilon_\nu + \theta' \leq 1/7$ and $c_E \leq 1/70$; the depletion constant $c_{\text{dep}} := A_0^D + A_1^D \log(C'\tau/\theta_0)$ of Proposition 6.17, which depends on τ but enters only through lower bounds on τ (Theorem 7.40); $K_{\text{col}} = 2 > 27 \cdot 2.4944/(4\pi^2) = 1.706$ and $\varepsilon_0 = 0.003125$ (Corollary 7.18); $0 < \delta_2 < \bar{\delta} < r_C$ and $c'_{\text{rate}} = \frac{1}{2}(r_C - \bar{\delta})$ (Proposition 7.36); the pure number $c'_0 := \frac{1}{2}(1 - 1/p_3 - c_\lambda) = 1/16$, a lower bound for the fraction c' of Proposition 7.35; $\varepsilon_E = c'_0/8 = 1/128$; $c_w = c'/16$; $\alpha = \ln 8$.

13. $\lambda_{PV} \leq \min(\frac{1}{4}, \lambda_0)$, with $\lambda_0 = \lambda_0(\kappa_S, \mu, C, \delta_{\text{GM}})$ of Theorem 6.5 (the Debye condition in its proof). Here M_Y is the smallest fixed point of (6.11) and $\tilde{m} := \lambda_{PV} M_Y$ (Section 6.2). It comes after c_L and after μ, κ_S of item 12, and a constant above uses λ_{PV} only through the a priori bound $\lambda_{PV} \leq \frac{1}{4}$. The constants of Göpfert–Mack, Brydges–Federbush and Stage 1 that depend on $(\lambda_{PV}, \kappa_S, \mu, C, \delta_{\text{GM}})$, among them the threshold β_{GM}^0 of Theorem 6.5, are fixed here, after λ_{PV} .
14. *The cut-rule constants* (Definition 3.18). $\theta_{\text{NF}} := c'_0 c_g / (90(c_g + 4)) = 4.08 \cdot 10^{-5}$, $c_3^{\text{NF}} := c'_0 c_g^2 / (64(c_g + 4)) = 1.44 \cdot 10^{-5}$, $\kappa_{\text{NF}} := 4(c_g + 4)/c_g^2 = 272$ at $c_g = 1/4$. They belong to the regularization but come after item 12, because they are built from the pure number c'_0 , and no constant above depends on θ_{NF} or c_3^{NF} .
15. $K > K_* = 377.38$, which gives $c_\delta^{\text{sp}}(K)/(2K) > 14.9662$, and the root of $c_\delta^{\text{sp}}(K)/(2K) = 14.9662$ is 377.3635 (Section S1.1.2). The same threshold is used in Proposition 5.11 and Theorem 5.1.
16. $\beta_0(K)$, from the conditions of Section S1.1.4.

The scales determined by these constants are: $\tilde{m}^{-1}$ with $\tilde{m} \in [m_D/2, 2m_D]$; L_c, L, k_P, ℓ as in item 11; $V_* = (1 - \eta_w)^3 \rho_\lambda^3 / 2$; $V_{\mathcal{U}} = \tilde{m}^{-3}$; $R_* = (J + 1)w' + 6L_c = O((\ell/c_s) \log(\ell/c_s))$ (Theorem 4.23) and $r' = 2R_* + 2$; $R_{\text{resp}} = 10\ell$ and $R' = R_* + 2R_{\text{resp}}$; $L' = 2(\beta_{\text{GM}}/a)\tilde{m}^{-1}$; $R_1(\beta) = (2r_\square + 5)L$; $F_{\text{unit}} = \sigma_F L^2 = (\pi^2 c'_g / 18)(\tilde{m}L)^3 / (\tilde{m}\beta_{\text{GM}})$ (Definition 3.18); $R_{\text{NF}} = Le^{c_3^{\text{NF}} F_{\text{unit}}}$; the box Λ° of side $M'_\Lambda \in \{M_\Lambda, M_\Lambda + 1\}$, $M_\Lambda = \max(3 \max(R, T), r_0(\ell + 3))$.

S1.1.2 The threshold K_* of Lemma 2.6

Lemma 2.6 gives, with a_* the solution of $a = K \log(2a)$,

$$c_\delta(K) = \frac{|\log \delta|}{\beta} \geq c_\delta^{\text{sp}}(K) - \frac{1}{4a_*} - O\left(\frac{\log \beta}{\beta}\right), \quad c_\delta^{\text{sp}}(K) := a_* \left(\frac{1}{2} \log(2a_*) - 1 \right) + 1. \quad (\text{S1.1})$$

The function $K \mapsto c_\delta^{\text{sp}}(K)/(2K)$ is increasing. Indeed $da_*/dK = a_* \log(2a_*)/(a_* - K)$ and $a_* - K = K(\log(2a_*) - 1)$, which give

$$\frac{dc_\delta^{\text{sp}}}{dK} = \left(\frac{1}{2} \log(2a_*) - \frac{1}{2} \right) \frac{da_*}{dK} = \frac{1}{2} \log^2(2a_*),$$

and $K \cdot \frac{1}{2} \log^2(2a_*) - c_\delta^{\text{sp}}(K) = K \log(2a_*) - 1 > 0$. Hence $\frac{d}{dK} \frac{c_\delta^{\text{sp}}(K)}{K} > 0$. The threshold is 14.9662, the number $6 \cdot 2.4943551 = 14.96613$ rounded up (Lemma 2.6, (2.5)). The equation

$c_\delta^{\text{sp}}(K)/(2K) = 14.9662$ has the unique root $K = 377.3635$, and $K_* := 377.38$ lies above it. Hence $c_\delta^{\text{sp}}(K)/(2K) > 14.9662$ for all $K \geq K_*$. Numerically, at $K = K_*$,

$$\frac{c_\delta^{\text{sp}}(K_*)}{2K_*} = 14.96639, \quad a_* = 3321.43, \quad \frac{1}{4a_*} = 7.5 \cdot 10^{-5}.$$

The computation behind (2.4) (proof of Lemma 2.6). We put $\nu := a_*\beta$ in Lemma 2.4(iv). Then $\nu^2\varepsilon/2 = a_*^2\beta/(2K)$. Let $h(\mu) := (\beta/2)^\mu/\Gamma(\mu+1)$ and $\mu_n := (n^2 + \nu^2)^{1/2} \geq \nu$. From the series $I_\mu(\beta) = \sum_{k \geq 0} (\beta/2)^{\mu+2k}/(k!\Gamma(\mu+k+1))$ and $\Gamma(\mu+k+1) = \Gamma(\mu+1) \prod_{i=1}^k (\mu+i) \geq \Gamma(\mu+1)(\mu+1)^k$, we get $I_\mu(\beta) \leq h(\mu) \sum_{k \geq 0} (\beta^2/(4(\mu+1)))^k/k! = h(\mu)e^{\beta^2/(4(\mu+1))} \leq h(\mu)e^{\beta/(4a_*)}$ for $\mu \geq \nu$. Since $\log \Gamma$ is convex, its difference quotients increase. Thus for $\mu \geq \nu$

$$\log \Gamma(\mu+1) - \log \Gamma(\nu+1) \geq (\mu - \nu)(\log \Gamma(\nu+1) - \log \Gamma(\nu)) = (\mu - \nu) \log \nu,$$

hence $h(\mu) \leq h(\nu)(\beta/(2\nu))^{\mu-\nu} \leq h(\nu)2^{-(\mu-\nu)}$, because $\nu = a_*\beta \geq \beta$. As $\mu_n - \nu \geq 0$ for $|n| \leq \nu$ and $\mu_n - \nu \geq |n| - \nu$ otherwise,

$$\sum_{n \in \mathbb{Z}} I_{\mu_n}(\beta) \leq e^{\beta/(4a_*)} h(\nu) \left(2\nu + 1 + 2 \sum_{m > \nu} 2^{-(m-\nu)} \right) \leq (2a_*\beta + 5) e^{\beta/(4a_*)} h(\nu).$$

Stirling's inequality $\Gamma(x+1) \geq (x/e)^x$ gives $\log h(\nu) \leq -a_*\beta(\log 2a_* - 1)$. Finally, for $\beta \geq \pi^{-2}$, using $\cos \theta \geq 1 - \theta^2/2$ on $[0, \beta^{-1/2}] \subseteq [0, \pi]$, we obtain $I_0(\beta) = \frac{1}{\pi} \int_0^\pi e^{\beta \cos \theta} d\theta \geq \frac{1}{\pi\sqrt{\beta}} e^{\beta-1/2}$. Collecting these bounds, we obtain

$$\log \delta \leq -\beta \left[a_*(\log 2a_* - 1) - \frac{a_*^2}{2K} + 1 - \frac{1}{4a_*} \right] + \log(2a_*\beta + 5) + \log(\pi\sqrt{\beta}) + \frac{1}{2},$$

and $a_*^2/(2K) = \frac{1}{2}a_*\log 2a_*$ by the definition of a_* , which gives (2.4), with exactly the function $c_\delta^{\text{sp}}(K)$ and the correction $1/(4a_*)$ of (S1.1). A high-precision evaluation confirms the step and the values: $\Gamma(\mu+1)(\mu+1)^k/\Gamma(\mu+k+1) \leq 1$ and $I_\mu(\beta) \leq h(\mu)e^{\beta^2/(4(\mu+1))}$ on a grid of $\mu \in [0, 10^4]$, $k \leq 59$, $\beta \in [0.1, 100]$; $a_*(K_*) = 3321.434$, $c_\delta^{\text{sp}}(K_*)/(2K_*) = 14.966392$, the root 377.363524 (377.363533 with the correction $-1/(4a_*)$), and $1/(8K_*a_*) = 9.97 \cdot 10^{-8}$; and at $K = K_*$ and $\beta = 0.2, 1, 3$ the bound on $\log \delta$ before the last two steps lies below the right side of (2.4).

The correction $1/(4a_*)$ in (S1.1) moves $c_\delta/(2K)$ by 10^{-7} and the root by less than 10^{-4} , and the term $O(\log \beta/\beta)$ is absorbed in the term $O(\log \tau + \log r_0)$ of Lemma 5.12. The margins $c_\delta^{\text{sp}}(K)/(2K) - 14.9662$ that enter η_T are:

K	377.38	10^3	$3 \cdot 10^3$	10^4
$c_\delta^{\text{sp}}(K)/(2K)$	14.9664	19.521	25.289	32.372
margin	$1.9 \cdot 10^{-4}$	4.55	10.32	17.41

The entropy exponent of Lemma 5.12 is $6\pi^2 v_{\text{Cb}}(0) < 6 \cdot 2.4943551 = 14.96613 < 14.9662$. Hence the threshold is not affected by rounding the Debye exponent up to 2.4944 elsewhere: that rounding would give the threshold 14.9664, whose root 377.3807 exceeds K_* .

S1.1.3 The string tension and the rounding rule

The slit-domain tension is $\sigma_V = \varepsilon_E \sigma_F = (c'_g \tilde{m} L / 9216) \tilde{m} / \tau$ (Theorem 7.57) with $c'_g = c_g(1 + 1/L_c)^{-2}$ and $\tilde{m} L > c_L - 2r_0 \tilde{m}$ (Section 3.1). For $L_c \geq 35$ (Section S1.1.4(e)) we have $c_L \tilde{m}^{-1} \geq 35r_0$, and hence $c'_g \tilde{m} L \geq (36/35)^{-2}(1 - 2/35) c_g c_L \geq 0.89 c_g c_L \geq (9/16) c_g c_L$, the inequality used in (8.1). There $18432 \cdot 16/9 = 32768 = 2^{15}$ exactly and $(8\pi^2)^{1/2}/2^{16} = 1.3559 \cdot 10^{-4}$. At $c_g = 1/4$ and $c_L = 8.0 \cdot 10^{-12} \cdot 2.0 \cdot 10^{-8} = 1.6 \cdot 10^{-19}$ (items 9 and 11), (8.1) gives $\sigma(\beta) \geq 5.4 \cdot 10^{-24} (K\beta)^{-1/2} e^{-2.49436K\beta}$. Any smaller admissible c_L also works.

The rounding rule and its uses (Section 8.1 and throughout). The Debye exponent is $\pi^2 v_{\text{Cb}}(0) = 2.49435509 \dots < 2.4943551$. In a lower bound on $\tilde{m}$ (hence on a tension) the exponent is rounded up, to 2.49436, and prefactors are rounded down, to $1.35 \cdot 10^{-4}$. In an upper bound on a scale the exponent is rounded up to 2.4944: since $\tilde{m} \geq m_D/2$, $\log(1/\tilde{m}) \leq 2.4943551\tau - \frac{1}{2} \log(8\pi^2\tau) + \log 2 \leq 2.4944\tau$ for all $\tau \geq 1$. This is the form used for $K_{\text{col}} > 27 \cdot 2.4944/(4\pi^2)$, in the logarithmic step of the proof of Lemma 6.18, and in the bound $\log(1/\tilde{m})/\beta_{\text{GM}} \leq 2.4944/(4\pi^2) = 0.063184 < 0.0632$ of Remark 7.58. In the bounds that determine K_* the exponent is not rounded: the entropy $6 \log R_*$ of a root and a part in Lemma 5.12 (with $\ell \approx \tilde{m}^{-1}/r_0$, R_* , r' and R' of Section 5) is bounded with $\pi^2 v_{\text{Cb}}(0) < 2.4943551$, as in Section S1.1.2 and Section S1.4. The effect of this rounding on the threshold is given at the end of Section S1.1.2. Where the exponent enters as a lower bound on $\log(1/\tilde{m})$, it is rounded down: the hypothesis of Lemma 6.18 has 2.494τ , because it is used to bound $\tilde{m} \leq 2m_D \leq 17.8 \tau^{1/2} e^{-2.494\tau}$ from above.

S1.1.4 The conditions defining $\beta_0(K)$

We fix the constants of items 0 to 15. Each condition below is a lower bound on $\tau = K\beta$ that remains true for all larger τ : in each, the left side is a fixed function of τ that tends to its limit monotonically for large τ (powers of $\tilde{m}$, of τ , of $\log \tau$, and of the doubly exponential scale R_{NF}), and hence the set of admissible τ contains a half-line. Then $\beta_0(K)$ is τ_0/K for the largest of the thresholds τ_0 .

- (a) *Section 5* (Proposition 5.9, Lemma 5.10, Proposition 5.11, Lemma 5.17, Proposition 5.19, Lemma 5.23, Proposition 5.24, Lemmas 5.25, 5.26, Theorem 5.1):

$$\begin{aligned} \varepsilon_c \leq \frac{1}{2}, \quad \eta_T^{(1)} \leq 1, \quad \varepsilon'' \leq \frac{1}{2}, \quad \eta_T \leq 1, \quad 2^{29} \cdot 26e \cdot \varepsilon'^{1/(3C_{\text{B}})} \leq \frac{1}{2}, \\ 2(2[R'/L_c] + 5)^3 \varepsilon''^{1/(48\varepsilon_D(1+8L_c)(C'_*+3C_{\text{resp}}+6))} \leq \frac{1}{2}, \\ 8\varepsilon_D(1+8L_c)(C'_*+3C_{\text{resp}}+6)(48C_{\text{resp}}+2) \leq 1 \quad (\text{that is, (5.9)}), \\ R_{\text{NF}} \geq \max(3456\bar{r}_0 L_c, 2048\bar{r}_0 L_c/(c_{\text{sh}}\theta_{\text{NF}}F_{\text{unit}})). \end{aligned}$$

Here $\varepsilon_D = 3.2768 \cdot 10^7 \bar{r}_0 L_c / (c_{\text{sh}}\theta_{\text{NF}}F_{\text{unit}}R_{\text{NF}})$ is doubly exponentially small. The factor 2^{29} contains the factor 2^b for the sizes of the components in Lemma 5.26. The binding condition is the first of (5.10), $|\log \varepsilon''| \geq 75.16 C_{\text{B}} \approx 1.1 \cdot 10^{14}$, while $\eta_T \leq 1$ needs about $1.7 \cdot 10^{13}$ and $\eta_T^{(1)} \leq 1$ about $3.6 \cdot 10^8$ (Section S1.4).

- (b) *Section 6*. $\tilde{m} \leq K_0(2C_H)^{-1/3}$; $M_Y \leq m_W$ and $\tilde{m} \leq \min(1/68, m_W/2)$, with $m_W = \min(1, c_{\text{ex}}/2)$ of Proposition 6.10 (Theorem 6.11); $M_Y \leq \frac{1}{5}$ (Lemma D.2 on Λ° , in the proof of Theorem 6.15); $m_D/2 \leq \tilde{m} \leq 2m_D$; the coupling condition of Lemma 6.18, $\tau \geq 2$ and $2.494\tau \geq \ln(C_*^{\text{dep}}/\theta_0) + 2.5 \ln \tau$ (for the rounding of the exponent in this condition, see Section S1.1.3); and the threshold $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0(\lambda_{\text{PV}}, \kappa_S, \mu, C, \delta_{\text{GM}})$ of Theorem 6.5.

- (c) *Section 7 and Appendices A to C.* The slack in $\varepsilon_\nu + \theta' \leq 1/7$ and $c_E \leq 1/70$, which needs the terms $\beta_{\text{GM}} M_Y$ and $e^{-\mu}$, $e^{-2\kappa_S}$ to be small; $\varepsilon(L') \leq \varepsilon_0$ (Corollary 7.18); (β_1) and (β_3) of (S1.5); the cell-diameter conditions of Lemma 7.44 and Proposition 6.9, of the form $2 + C(k_\square/\chi) \log k_\square < L'$; Proposition 7.39 and Theorem 7.40, the latter a condition $0.416\tau \geq C_{\mathfrak{D}} A_1^D \log(C'\tau/\theta_0) + \alpha + O(\log \tau)$; $c_{16} e^{c_2} \leq 1/8$ (Lemma S1.2); Lemma 7.48; $\Theta_4 \leq \frac{1}{4}$ (Proposition 7.53); $F_{\text{unit}} \geq (8C'/c')(1 + \ln \beta_{\text{GM}})$ (Proposition 7.52).
- (d) *Section 8.* $\eta_T + \eta_S \leq \frac{1}{2}$. Both deficits are explicit upper bounds that are eventually decreasing in τ : $\eta_T = O_K(1/\tau)$, because $|\log \varepsilon''| \geq \gamma_T(K)\tau - O(\log \tau + \log r_0)$ with $\gamma_T(K) > 0$ for $K > K_*$ (Lemma 5.12), and $\eta_S = O(\tilde{m}\beta_{\text{GM}} \log \tau)$.
- (e) *Rounding of L_c* (Section 3). $L_c \geq 4$, which gives $c'_g \geq (16/25)c_g \geq (9/16)c_g$ (Lemma 4.19); $L_c \geq 35$, used in Lemma 7.44 and for $c'_g \tilde{m}L \geq (9/16)c_g c_L$ (Section S1.1.3, Section 8.1); and $c_L \tilde{m}^{-1} \geq 4r_0$, which gives $\tilde{m}L \geq c_L/2$ (the bounds on c_u and on $C_{\mathcal{U}}$, Lemma 4.21).
- (f) *Section 2.* $\beta \geq \pi^{-2}$ (Lemma 2.6); (2.5), that is $\beta \geq \beta_1(K)$; and $\delta < 1$, used in Step (1.4) ($\delta' = \delta$ in Proposition 2.9), which (2.4) gives for all large β .

In addition, the range of Remark 8.2 (an empty Phase 2) is nonempty once $\sqrt{3}(r_0(\ell+3)+1) \leq R_{\text{NF}}$, a condition that Theorem 1.2 does not use and that holds for all large τ because $r_0\ell \approx \tilde{m}^{-1}$ is exponential and R_{NF} doubly exponential in τ .

All these thresholds are finite but very large (Section 8.2).

S1.2 Constants of Section 3 and Subsections 4.1 and 4.2

We group the constants by the main-text statement in which they enter.

Definition 3.1: the scales. The screening mass satisfies $m_D/2 \leq \tilde{m} \leq 2m_D$, so that $\log(1/\tilde{m}) = \pi^2 v_{\text{Cb}}(0) \tau + O(\log \tau)$ with $\pi^2 v_{\text{Cb}}(0) = 2.49435509\dots$, and $\tilde{m}^{-1}$ is the Debye length. The scales are those of Definition 3.1. Their place in the order of constants is item 11 of Section S1.1.1. Since $c_L \leq 1/308$, $\ell \geq \tilde{m}^{-1}/r_0 \geq L_c/c_L$ gives (3.1). The conditions $L_c \geq 4$ and $c_L \tilde{m}^{-1} \geq 4r_0$ are those of Section S1.1.4(e). The first gives $c'_g := c_g(1 + 1/L_c)^{-2} \geq \frac{9}{16}c_g$. The second gives $L > c_L \tilde{m}^{-1} - 2r_0 \geq c_L \tilde{m}^{-1}/2$, that is, $\tilde{m}L \geq c_L/2$.

Definition 3.4: windows. A window is the union of the blocks at block distance at most 75 from a block. It therefore has at most 151 blocks per direction, each of coarse side at most $2L_c - 1$, and its sides are at most $151(2L_c - 1) < 302L_c < k_P$. A block lies in the windows of the at most $151^3 = 3.44 \cdot 10^6$ blocks at block distance at most 75 from it, and $151^3 < 27 \cdot 304^3 = 7.6 \cdot 10^8 = 27(k_P/L_c)^3$, the overlap bound of (R6).

Definition 3.11, Lemma 4.9, Theorem 4.17: the cost of a fill. A block Q has $|\partial Q| \leq 6(2L_c - 1)^2 < 24L_c^2$ and $|Q| < 8L_c^3$. Hence, with $\lambda_U = 2c_s/\ell$, $c_s \leq 10$ and $\ell \geq 308L_c$, we have $\lambda_U|Q| < 16c_s L_c^3/\ell \leq 0.52L_c^2$ and $|\partial Q| + \lambda_U|Q| < 24.6L_c^2 < \mu_B L_c^2 = 25L_c^2$. Also $\lambda_U = 2c_s/\ell \leq 20/(308L_c) < \lambda = 12/L_c$. A removal cell Δ with sides $a, b, c \in [\ell, 3\ell]$ has $|\Delta|/|\partial\Delta| = abc/(2(ab + bc + ca)) \leq (abc)^{1/3}/6 < \ell/2$ by the arithmetic-geometric mean inequality, so $\lambda_U|\Delta| < c_s|\partial\Delta|$ and $|\partial\Delta| + \lambda_U|\Delta| < w_*$. The number of forced blocks is at most $27|\mathbb{M}|/(\vartheta L_c^2)$,

since a mark lies in at most 27 sets $3Q$. Hence filling them costs at most $27 \cdot 24.6/\vartheta = 664.2/\vartheta$ per mark.

Lemma 4.4: constants that depend on the hull weight. Frozen bonds have the hull weight $2 - \eta_w$ (Definition 3.8). The hull weight enters in two ways. Through Lemma 4.4 it gives $(a_\eta, b_\eta) = (1 + 1/(1 - \eta_w), 1/(1 - \eta_w))$ and the factor $1/(1 - \eta_w)$ in the bound on $N_2(W)$. In the cell analysis it enters as the deficit η_w of a short segment through a frozen bond (Lemma 4.8(b)), which gives the factor $120 = 12/\eta_w$ of Proposition 4.15. Through this factor η_w enters $C_{\text{seal}} = 726$, the 134 of Theorem 4.16(b) and the 158 of the expansion bound behind $h_\star$ in Theorem 4.16(c) (Section S1.2), and hence the constants built from them, such as $C'_\star$, C_{free} and C_{resp} , and the constraint on the weak-link constant χ (items 9 and 10 of Section S1.1.1). Table S1.1 lists the constants that depend on η_w through Lemma 4.4, with their formula in η_w and their value at $\eta_w = 1/10$. No other constant depends on η_w through Lemma 4.4. In particular, the constant c_0 of the density bound, the bound on c_L , $K_\star$, $\sigma(\beta)$ and $R_1(\beta)$ do not.

Table S1.1: Constants depending on the hull weight; values at $\eta_w = 1/10$, $c_I = 2^{2/3}$, $\rho_\lambda = L_c/12$.

quantity	formula	value at $\eta_w = 1/10$	enters
(a_η, b_η)	$(1 + 1/(1 - \eta_w), 1/(1 - \eta_w))$	$(19/9, 10/9)$	Lemma 4.4, (R2)
isoperimetry for $ W \leq V_\star$	$(c_I/(2a_\eta)) W ^{2/3}$	$(9c_I/38) W ^{2/3}$	= Theorem 4.5(i)
$V_\star$	$(1 - \eta_w)^3 \rho_\lambda^3/2$	$0.3645 \rho_\lambda^3$	Definition 3.1, (R2)
volume growth constant	$(c_I/(48a_\eta))^3$	$3.844 \cdot 10^{-6}$	Theorem 4.5(iii)
saturation radius of (iii)	$24a_\eta(1 - \eta_w)\rho_\lambda$	$45.6\rho_\lambda$	Theorem 4.5(iii)
Faber–Krahn constant	$c_I^2/(48a_\eta^2)$	$c_I^2/213.93$	Theorem 4.5(iv)
$N_2(W)$ factor	$1/(1 - \eta_w)$	$10/9$	Lemmas 4.4, 6.13
c_1 (second entry)	$((1 - \eta_w)c_I/24)^3$	$2.109 \cdot 10^{-4}$	(R5)
c_u	$((1 - \eta_w)c_I\tilde{m}L/24)^3$	$(0.0595\tilde{m}L)^3$	(R7)
R_{HK}, K_0	$(1 - \eta_w)c_I\rho_\lambda/3, K_0 = (1 - \eta_w)c_Ic_\rho c_L/6$	$\tilde{m}R_{\text{HK}} = 0.0397\tilde{m}L \in [K_0, 2K_0], K_0 = 0.01984c_L$	Theorem 6.8
c_F (fine isoperimetry)	$(c_I/(2a_\eta))/152.4$	$2.467 \cdot 10^{-3}$	Proposition 6.9
$C_{\mathcal{U}}, N_{\text{near}}, A_2$	through $1/c_1$ and $1/c_u$, at most the factor $(1 - \eta_w)^{-3}$	factor at most 1.372	(R7), Section 7
c_{sh}	$c_1/216$	$9.77 \cdot 10^{-7}$	Lemma 5.15

At $\eta_w = 1/10$, c_1 equals its second entry, with the factor $(1 - \eta_w)^3 = 0.729$, when $c_g \geq 4.22 \cdot 10^{-4}$, and $c_1 = c_g/2$ otherwise. Hence the factor $(1 - \eta_w)^{-3} = 1.372$ in the table is an upper bound. The Nash constant $768/c_I^2$ of Appendix D does not depend on η_w , although in its proof we need $2 \cdot 2^{2/3} \cdot 48a_\eta^2/c_I^2$, which is $679.2/c_I^2 \leq 768/c_I^2$ at $\eta_w = 1/10$.

Theorem 4.5. We have $V_* = (c_I \rho_\lambda / (2b_\eta))^3$, since $(c_I/2)^3 = \frac{1}{2}$. Hence $b_\eta \lambda V_*^{1/3} = c_I/2$. In (iii), $\alpha := c_I/(12a_\eta) = 0.0626 \dots \leq \frac{1}{4}$, and the two expressions in the minimum are equal at $r = 24(a_\eta/b_\eta)\rho_\lambda = 45.6\rho_\lambda$.

Lemma 4.21.

$$C_U := 8\sqrt{3}(2 + 6/c_g)/(c_1 c_L^2),$$

which bounds $2\sqrt{3}(2 + 6/c_g)/(c_1(\tilde{m}L)^2)$ when $\tilde{m}L \geq c_L/2$.

Proposition 4.26: the constant c_0 . The fine plaquettes dual to the insulating bonds with an end in a removal cell Δ , those of frozen bonds counted twice, number at most $\bar{r}_0^2(3+c_s)|\partial\Delta|$, since a coarse bond is dual to at most $\bar{r}_0^2$ fine plaquettes. Hence they number at most $c_0/(z\tau)$ with $c_0 := z\tau \bar{r}_0^2(3+c_s) \max_\Delta |\partial\Delta|$, where z is the monopole activity of Section 6, $m_D^2 = 2\beta_{\text{GM}}z$ and $z\tau = m_D^2/(8\pi^2)$. By $\bar{r}_0 < 2r_0$, $r_0\ell = \ell_D < \tilde{m}^{-1} + L \leq (1+c_L)\tilde{m}^{-1}$, $m_D \leq 2\tilde{m}$, $|\partial\Delta| < 54\ell^2$ (G3) and $c_L \leq \frac{1}{2}$,

$$c_0 < \frac{4\tilde{m}^2}{8\pi^2} \cdot 4 \cdot 54(3+c_s)(1+c_L)^2\tilde{m}^{-2} \leq \frac{243}{\pi^2}(3+c_s) < 25(3+c_s),$$

independently of β . On a grid without exceptional layers and with cubic removal cells, $c_0 = 6(3+c_s)(m_D\ell_D)^2/(8\pi^2)$.

Lemma 4.10, Proposition 4.12, Lemmas 4.13, 4.14, Proposition 4.15, Theorem 4.16: the cell analysis. With $\vartheta = 3 \cdot 10^{-5}$ and $\varepsilon_{\text{th}} = 0.008$:

- *Giant.* $12 + 6.4 \cdot 9 \cdot 2 = 127.2 < 128$, and $(1 - 3\vartheta)L_c^3 > 384\vartheta L_c^3 = 3 \cdot 128\vartheta L_c^3$ since $384\vartheta = 0.01152$.
- *Thin sets.* $2\varepsilon_{\text{th}} + 12\vartheta = 0.01636 \leq 0.0164 \leq \frac{1}{4}$, and $2/\sqrt{0.0164} = 15.62 \geq 15.6 > 14$. Non-giant components are thin since $128\vartheta = 0.00384 < \varepsilon_{\text{th}}$.
- *Seal.* Lemma 4.8(b) with $\text{supp} \leq 6(m_Q + m_{Q'})$ and $\eta_w = 1/10$ gives $120(m_Q + m_{Q'})$ lines, plus $m_Q + m_{Q'}$ lines with a mark, in total 121 per unit of $m_Q + m_{Q'}$, and $C_{\text{seal}} = 6 \cdot 121 = 726$.
- *Interfaces.* $1 + 121 + 12 = 134$ non-good lines per unit of $m_Q + m_{Q'} < \vartheta L_c^2$, so $\text{good}(I) \geq (1 - 134\vartheta)|F'| = 0.99598|F'| \geq c_g|F'|$.
- *Expansion.* For two large sides, $\varepsilon_{\text{th}} - 127.2\vartheta = 0.004184 \geq \varepsilon_{\text{th}}/2$, so $\text{cut} \geq \varepsilon_{\text{th}}L_c^2/25.6 = 3.125 \cdot 10^{-4}L_c^2$. For adjacent in/out blocks, $(1 - 158\vartheta - 2\varepsilon_{\text{th}})/3 = 0.32642$. Both are at least $3.1 \cdot 10^{-4}L_c^2$, and $|A| < k_P^3/2 = 1.4047 \cdot 10^7 L_c^3$. Hence $h_\star \geq 3.1 \cdot 10^{-4}/(1.4047 \cdot 10^7 L_c) = 2.207 \cdot 10^{-11}/L_c \geq 2.2 \cdot 10^{-11}/L_c$. Then $3k_P h_\star = 912 \cdot 2.2 \cdot 10^{-11} = 2.006 \cdot 10^{-8} \geq \chi = 2.0 \cdot 10^{-8}$, and $\chi_{N3} = \chi/3 = 6.67 \cdot 10^{-9} \leq k_P h_\star = 6.69 \cdot 10^{-9}$.

The construction gives c_1 -type volume $(1 - 3\vartheta)L_c^3$ per cell, one interface per block face and six neighbouring cells per cell (Theorem 4.16). We keep the constants c_1 , $1/c_g$ and $6/c_g$ of (R5), (R8), which these imply.

S1.3 Constants of Sections 4.3 and 4.4

We first list the sizes on the loop-adapted grid and then the constants, grouped by main-text statement.

Sizes on the loop-adapted grid. Table S1.2 lists the sizes on $\mathfrak{G}(C)$, with the values on the uniform grid (all coarse cells of side r_0 , all block spacings L_c) for orientation only. The constant c_1 is that of item 6 of Section S1.1.1. The opacity property (S-face) is normalized per face (threshold $c_g|F(I)|$, Definition 3.17). Hence the good fraction and the cell degree do not change with the face area.

Table S1.2: Sizes on the loop-adapted grid $\mathfrak{G}(C)$, in coarse units unless stated, with the values on the uniform grid for orientation and the statements in which each size enters.

quantity	on $\mathfrak{G}(C)$	uniform grid	enters
block side	$[L_c, 2L_c - 1]$	L_c	(G1)
block and cell volume	$< 8L_c^3$	L_c^3	(R5); Lemma 4.21 ($\tilde{m}L \leq \frac{1}{2}$)
face area $ F' $	$[L_c^2, 4L_c^2)$	L_c^2	(S-face), Lemma 4.9
block diameter	$< 2\sqrt{3}L_c$	$\sqrt{3}L_c$	Cor. 4.7
window side; win- dows per site	$< k_P; \leq 151^3$	$151L_c; 151^3$	(G4), (R6)
removal cell side	$[\ell, 3\ell)$	ℓ	Lemma 5.21
$ \partial\Delta $	$[6\ell^2, 54\ell^2)$	$6\ell^2$	(R9), c_0 of Sec- tion S1.2
reach R_*	Theorem 4.23	same	Theorem 3.21
k_P	$304L_c$	same	(G4), (R6)
c_1 ; good fraction; cell degree	$2.109 \cdot 10^{-4}; c_g; 6/c_g$	same	(R5), (R8)
fine good fraction	$c_g(1 + 1/L_c)^{-2}$	c_g	Lemma 4.19
q_m	$3(\bar{r}_0 + 1)^3\delta^{1/2} \leq 24r_0^3\delta^{1/2}$	$3(r_0 + 1)^3\delta^{1/2}$	Lemma 5.7
fine area of a coarse face	$\leq \bar{r}_0^2 < 4r_0^2$	r_0^2	Section 5 ($\bar{r}_0^2$ in η_T)
c_I for $ W \leq V_*$	$2^{2/3}$	$2^{2/3}$	Theorem 4.5

Lemma 4.22(c) and Theorem 4.23: the reach. In Lemma 4.22(c), $\lambda^{-1} + \lambda_U^{-1} = L_c/12 + \ell/(2c_s) \leq \ell/c_s$ because $c_s \leq 10 < 6\ell/L_c$, and $w \geq 120\ell/c_s$ gives the factor $30\ell/(c_s w) \leq \frac{1}{4}$. In Theorem 4.23, $w' \leq 121\ell/c_s + 2L_c \leq 122\ell/c_s$ (as $2L_c \leq \ell/c_s$), and for $\rho \geq 4L_c$, $g(\rho) \leq 54\rho^2 + 54c_s\rho^3/\ell$. Therefore $g((J+1)w') \leq 10^8(J+1)^3(\ell/c_s)^2$, and J is at most the least integer with $5^J > 10^8(J+1)^3(\ell/c_s)^2$, that is $J = 2\log_5(\ell/c_s) + O(\log \log(\ell/c_s))$. Hence

$$R_* = (J+1)w' + 6L_c \leq 122(J+1)\ell/c_s + 6L_c = O((\ell/c_s) \log(\ell/c_s));$$

with $\log \ell = 2.4944\tau + O(\log \tau)$ this is $R_* \approx 380\tau\ell/c_s$, and $\log R_* = \log(1/\tilde{m}) + O(\log \tau)$, which is the only property of R_* that the threshold K_* uses (Lemma 5.12). In the clearing-out step, each site of $\mathbb{X}^1$ weighs at least λ_U in f (as $\lambda \geq \lambda_U$), $|B_{w'}(x)| \geq \ell^3$ since $w' \geq \ell$ and each side of Λ° is at least ℓ , and $\lambda_U\ell^3 = 2c_s\ell^2 > 1$.

Proposition 4.24: the area per part. The competitor costs at most $(6 + \lambda)N(P)$ for the marks, $24.6L_c^2$ for each of at most $27N(P)/(\vartheta L_c^2)$ forced blocks, and 2 for each of at most $C_{\text{seal}}N(P) = 726N(P)$ seal bonds. We therefore set

$$C_1 := 7 + 27 \cdot 24.6/\vartheta = 7 + 664.2/\vartheta = 2.214 \cdot 10^7, \quad C'_\star := C_1 + 2C_{\text{seal}} = C_1 + 1452 = 2.2141 \cdot 10^7$$

at $\vartheta = 3 \cdot 10^{-5}$, and $C_{\text{free}} := C'_\star(2 + 1/(2c_s) + 1/3696) = 5.536 \cdot 10^7$ at $c_s = 1$, where $1/(\lambda\ell) = L_c/(12\ell) \leq 1/3696$ by (3.1). The marks of a set $3Q$ are within distance $3(2L_c - 1) < 6L_c$ of each other, and a bond of $\bar{S}^1$ has both ends within $4L_c - 3$ of a mark of its two blocks, so $R_\star + 4L_c < r' = 2R_\star + 2$.

Lemma 4.25: release. The comparison base is $\mathbb{X}^1 \cup R_t$, not the base $B_t = \mathbb{M} \cup \mathbb{U}^1 \cup R_t$: a bond from a block of $\mathbb{U}^1$ to hull material of $\mathbb{X}^1$ lies in $\partial\mathbb{U}^1$ but not in $\text{ins}^1 = \partial\mathbb{X}^1$, and such bonds are not bounded by the marks. Hence only the comparison with $\mathbb{X}^1 \cup R_t$ gives a bound by ins^1 .

S1.4 Constants of Section 5

Theorem 5.1. The constants of (5.2) are

$$c_T'' := 1 + \log(64\kappa_{\text{pl}}), \quad \kappa_{\text{pl}} := 12e, \quad C_3 := C_{\text{free}} + 20(2C'_\star + 4C_{\text{resp}} + 1) \quad (\text{Proposition 5.24}),$$

with $C'_\star$, C_{free} of Proposition 4.24 and $C_{\text{resp}} := 375C_a + 125C_cC'_\star$ of Section 5.6. The quantity ε'' is defined in (5.8), with $R' = R_\star + 2R_{\text{resp}} = R_\star + 20\ell$ and $r' = 2R_\star + 2$. At the illustrative values $(c_g, c_s, r_0) = (1/4, 1, 1)$ we have $C'_\star = 2.2141 \cdot 10^7$, $C_{\text{free}} = 5.536 \cdot 10^7$, $C_{\text{resp}} = 8.307 \cdot 10^9$, $C_3 = 6.655 \cdot 10^{11}$ and $C_{\mathfrak{B}} = 1.413 \cdot 10^{12}$. The conditions used in the proof are those of item (a) of Section S1.1.4. They hold for large τ because $|\log \varepsilon''| \geq \gamma_T(K)\tau - O(\log \tau + \log r_0)$ with $\gamma_T(K) > 0$ for $K > K_\star$ (Lemma 5.12); C_3 and $C_{\mathfrak{B}}$ are free of β and K ; ε_D , M_m and R_{NF}^{-1} are doubly exponentially small in τ ; and $\log d_0 = O(\log \tau)$. The constants C_{resp} , $C_{\mathfrak{B}}$ and C_3 are fixed before K and hence change $\beta_0(K)$ and not $K_\star$. As for the order of η_T , we have $C_{\text{free}} = O(C'_\star(1 + 1/c_s))$, hence $C_3 = O((C'_\star + C_{\text{resp}})(1 + 1/c_s))$, and $\gamma_T(K)$ is of the order of $\log^2 K$ as $K \rightarrow \infty$ (Lemma 2.6).

Lemma 5.12. Since $R_\star \geq w' \geq 12\ell$ (Theorem 4.23), we have $2R_\star + 1 \leq 3R_\star$ and $2r' + 1 = 4R_\star + 5 \leq 5R_\star$. With $q_m \leq 24r_0^3\delta^{1/2}$ (Lemma 5.7) we obtain

$$\begin{aligned} \log \varepsilon_c &\leq \log(8e) + 3 \log 3 + 3 \log 5 + \log 24 + 6 \log R_\star + 3 \log r_0 - \frac{1}{2} |\log \delta| \\ &\leq 14.4 + 6 \log R_\star + 3 \log r_0 - \frac{1}{2} |\log \delta|. \end{aligned}$$

By Theorem 4.23, $R_\star = O((\ell/c_s) \log(\ell/c_s))$ (Section S1.3), and $\ell \leq \ell_D = O(\tilde{m}^{-1})$ (Definition 3.1). With $\log(1/\tilde{m}) = \pi^2 v_{\text{Cb}}(0)\tau + O(\log \tau)$ and $\pi^2 v_{\text{Cb}}(0) < 2.4943551$ these bounds give $6 \log R_\star \leq 14.9662\tau + O(\log \tau)$, because $6 \cdot 2.4943551 = 14.96613 \dots < 14.9662$. Also $\frac{1}{2} |\log \delta| = c_\delta(K)\beta/2 = \tau c_\delta(K)/(2K)$. This is the first inequality of (5.6). By Lemma 2.6, $c_\delta(K) \geq c_\delta^{\text{sp}}(K) - 1/(4a_\star) - O(\log \beta/\beta)$. The function $\gamma_T(K) := (c_\delta^{\text{sp}}(K) - 1/(4a_\star))/(2K) - 14.9662$ is increasing in K and vanishes at $K = 377.3635$ (to the digits shown; Section S1.1.2, which also gives the margins). Hence $\gamma_T(K) > 0$ for $K > K_\star$. The $O(\log \beta/\beta)$ term contributes $O(\log \tau)$ after multiplication by $\tau/(2K)$. For ε'' we use $R_{\text{resp}} = 10\ell$ (Lemma 5.21), $L_c \leq \ell$ and $R_\star = O(\ell\tau)$. Hence $\log(2R' + 4L_c + 1) \leq \log(1/\tilde{m}) + O(\log \tau)$, and the same computation applies.

Lemma 5.17. In (c), with $\rho = \rho(s)$: $|x_c - x_Q|_\infty \leq \lceil \rho \rceil / 2 \leq \rho/2 + 1$; in (ii) the ends of the other cut bond are at distance more than $73\rho - 1$ from x_c by (U2), hence more than $73\rho - \rho/2 - 2 > 9\rho$ from x_Q ; in (iii) two bonds of $B_\infty(x_c, 10\rho)$ have ends within coarse distance 20ρ , and $\sqrt{3}\bar{r}_0(20\rho + 1) < 128\bar{r}_0\rho = s$, hence (U1) applies; for the last claim the two chosen bonds have ends within coarse distance $\rho + 2$, and $\sqrt{3}\bar{r}_0(\rho + 3) \leq s$. The range $27L_c \leq \rho \leq \text{side}_{\min}/24$ holds. The lower bound follows from $R_{\text{NF}} \geq 3456\bar{r}_0L_c$. For the upper bound, $\varphi_s \geq 2$ forces s below the lattice diameter of Λ° , which is at most $\sqrt{3}\bar{r}_0 \text{side}_{\max} \leq 2\sqrt{3}\bar{r}_0 \text{side}_{\min}$ (Lemma 3.3(d)), and hence $\rho < 2\sqrt{3} \text{side}_{\min}/128$.

Lemmas 5.20 and 5.21. In the proof of Lemma 5.20, summing $121(m_q + m_{q'})$ over the six faces of each block q of Δ_0 gives at most $121(6m(\Delta_0) + 6m(U)) \leq 1452m$ seal bonds with an end in Δ_0 , and over the blocks of U at most $121(6m(U) + 6m(U_2(\Delta_0))) \leq 1452m$ seal bonds with both ends in U . With $u_3 \leq c_s^{-1}(i + 2904m + 2n)$ we obtain

$$\text{units}(\Delta_0) \leq 1452m + 2u_3 + i + 2 \cdot 1452m + 2n + 6m \leq \left(4362 + \frac{5808}{c_s}\right)m + \left(2 + \frac{4}{c_s}\right)n + \left(1 + \frac{2}{c_s}\right)i,$$

since $1452 + 2904 + 6 = 4362$ and $2 \cdot 2904 = 5808$. At $c_s = 1$, $(C_a, C_b, C_c) = (10170, 6, 3)$, and the loads of Lemma 5.21 are $125C_a = 1.27 \cdot 10^6$, $125C_b = 750$ and $125C_c = 375$. Thus

$$C_{\text{resp}} := 375C_a + 125C_cC'_\star = 3.81 \cdot 10^6 + 375 \cdot 2.2141 \cdot 10^7 = 8.307 \cdot 10^9;$$

the term $125C_cC'_\star$, the load of the Phase-1 insulation owned by a part, dominates. The range is $R_{\text{resp}} = 10\ell \geq 9\ell + 1$.

Proposition 5.22. A touched Y_Δ costs at most $2|\partial\Delta| + 2|\text{ins}^1 \cap E(\Delta)| + 4.5|\partial\Delta| = 6.5|\partial\Delta| + 2|\text{ins}^1 \cap E(\Delta)|$. A touched $Y = Y_{\Delta'}$ costs $2|\partial Y| + |Y|/\ell$, which by the bound on $|\partial Y|$ is at most $4\#\{\text{bonds of types (a) to (c) with an end in } \Delta'\} + 13|Y|/\ell$, and $13|Y|/\ell \leq (26L_c/\ell)m(\Delta')$. Since a bond has two ends, it is charged by at most two cells of the third kind, at most 4 each. A unit (Δ, b) is charged at most 6.5 by Y_Δ and at most 4 by the cell of the other end of b . A bond of ins^1 inside a removed cell is charged at most 2 by Y_Δ . All coefficients are at most 20.

Lemma 5.23 and Proposition 5.24. The constants are

$$C_{\mathfrak{B}} := 30C'_\star + 170C_{\text{resp}} + 30, \quad M_m := 16\varepsilon_D(1 + 8L_c)(C'_\star + 3C_{\text{resp}} + 6), \\ C_3 := C_{\text{free}} + 20(2C'_\star + 4C_{\text{resp}} + 1).$$

$C_{\mathfrak{B}}$. In the proof of Lemma 5.23(c), the bonds of ins^1 number at most $C'_\star N_{\mathcal{Q}}$ (as in (b)), the cut bonds at most $A_{\mathcal{F}}$, the units at most $2C_{\text{resp}}(N_{\mathcal{Q}} + A_{\mathcal{F}})$ by (a), each charged by at most two blocks, and the marks involved, at most $N_{\mathcal{Q}}$, belong to parts of $\mathcal{Q}_{\text{att}}$, each charged by at most seven blocks. Summing, we obtain

$$|\mathfrak{B}_{\mathcal{F}}| \leq 2(C'_\star N_{\mathcal{Q}} + A_{\mathcal{F}} + 2C_{\text{resp}}(N_{\mathcal{Q}} + A_{\mathcal{F}})) + 7N_{\mathcal{Q}} \leq (2C'_\star + 4C_{\text{resp}} + 10)N_{\mathcal{Q}}$$

by Lemma 5.23(b), and $2C'_\star + 4C_{\text{resp}} + 10 \leq C_{\mathfrak{B}}$.

M_m . By Lemma 5.23(d), (b), $m \leq 2A_{\mathcal{F}} \leq 16\varepsilon_D(1 + 8L_c)(C'_\star + 3C_{\text{resp}} + 6)N_{\mathcal{Q}} = M_m N_{\mathcal{Q}}$.

The size of C_{resp} . We have $C_{\text{resp}} \geq 125C_cC'_\star \geq 125C'_\star$, in particular $C_{\text{resp}} \geq C'_\star + 20$.

C_3 . In the proof of Proposition 5.24, $20(u_{\text{touch}} + c_{\text{touch}} + i_{\text{touch}}) \leq 20(C_{\text{resp}} + C'_\star)N_{\mathcal{Q}} + 20(C_{\text{resp}} + 1)A_{\mathcal{F}}$ and $26L_c/\ell \leq 26/308 < 1/11$. With $A_{\mathcal{F}} \leq N_{\mathcal{Q}}/(48C_{\text{resp}} + 2)$, all terms with $A_{\mathcal{F}}$ together are at most $N_{\mathcal{Q}}$, and

$$\#(\Xi' \cap \mathfrak{F})/\bar{r}_0^2 \leq (C_{\text{free}} + 20C'_\star + 20C_{\text{resp}} + 2)N_{\mathcal{Q}} \leq C_3N_{\mathcal{Q}}.$$

The binding condition. The first condition of (5.10) reads $|\log \varepsilon''| \geq 3C_{\mathfrak{B}} \ln(2^{30} \cdot 26e) = 75.16 C_{\mathfrak{B}}$, that is $|\log \varepsilon''| \geq 1.06 \cdot 10^{14}$ at the illustrative values; $\eta_T \leq 1$ needs $|\log \varepsilon''| \geq 3c_T''\bar{r}_0^2 C_3 = 1.73 \cdot 10^{13} \bar{r}_0^2$, with $c_T'' = 8.644$; and $\eta_T^{(1)} \leq 1$ needs $|\log \varepsilon_c| \geq c_T \bar{r}_0^2 C_{\text{free}} = 3.63 \cdot 10^8 \bar{r}_0^2$. Each is a lower bound on τ , by Lemma 5.12.

S1.5 Constants of Section 6 and Appendix D

Masses (Section 6). The Coulomb kernel at the origin is

$$v_{\text{Cb}}(0) = \frac{1}{2}W_S = (\sqrt{6}/(192\pi^3))\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24}) = 0.2527310098\dots,$$

where $W_S = \pi^{-3} \int_{[0,\pi]^3} (3 - \sum_i \cos x_i)^{-1} dx$ is Watson's integral [Wat39], in the Gamma-function form of [Zuc11, (3.6)]. Numerically, $\pi^2 v_{\text{Cb}}(0) = 2.4943550872\dots$. Since $m_D = (8\pi^2\tau)^{1/2} e^{-\pi^2 v_{\text{Cb}}(0)\tau}$ and $2(8\pi^2)^{1/2} = 17.77153\dots$, it follows that

$$\tilde{m} \leq 2m_D \leq 17.7716 \tau^{1/2} e^{-2.494\tau} \quad (m_D/2 \leq \tilde{m} \leq 2m_D). \quad (\text{S1.2})$$

The exponent 2.494 is a lower bound for $\pi^2 v_{\text{Cb}}(0)$. This is why 2.494, and not 2.4944, stands on the left of the hypothesis of Lemma 6.18.

Theorem 6.5: the Debye condition. In [GM82, (4.24)], at $A = (1 - \delta)M$, [GM82] use $\|v^2\|_A \leq 2(\delta M)^{-2}$ [GM82, Lemma 5, (3.26)], and the Debye condition [GM82, p. 34] reads $1 \leq \frac{1}{4}\delta^2 C^3(1 - C)^8 e^{-2\kappa'} M^2/m_D^2$, where $\kappa' = \kappa_S + \frac{1}{2}\mu$ by the choice made on [GM82, p. 35]. On G' the bound $\|v_{G'}^2\|_A \leq 3M_Y^{-2}$ (Proposition 6.10 on outputs for $A \leq \sqrt{3}\kappa_Y M_Y$, and $\|v_{G'}^2\|_0 \leq M_Y^{-2}$ on all slit graphs) replaces $2(\delta M)^{-2}$, that is, $\frac{1}{2}\delta^2$ by $\frac{1}{3}$. Hence $\frac{1}{4}\delta^2$ becomes $\frac{1}{6}$:

$$1 \leq \frac{1}{6}C^3(1 - C)^8 e^{-2\kappa_S - \mu} M_Y^2/m_D^2.$$

Since $M_Y \geq m_D/(2\lambda_{\text{PV}})$ on I_Y , $M_Y^2/m_D^2 \geq 1/(4\lambda_{\text{PV}}^2)$, and the condition holds for all $M_Y \in I_Y$ once

$$\lambda_{\text{PV}}^2 \leq C^3(1 - C)^8 e^{-2\kappa_S - \mu}/24. \quad (\text{S1.3})$$

In [GM82, (4.23)] the bound $\|v_{G'}^1\|_A \leq 3M_1^{-2}$, with $M_1 = -4\pi\beta_{\text{GM}}^{-1} \ln(1 - C)$, is a power of β_{GM} , which $e^{\beta_{\text{GM}}\varepsilon_1}$ absorbs for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, and neither condition depends on G' .

The scales of §6.3: the rounding of L . By Definition 3.1, $L = r_0 L_c$ with L_c the largest even integer $\leq c_L \tilde{m}^{-1}/r_0$, and $c_L \tilde{m}^{-1} \geq 4r_0$. Hence

$$c_L/(2\tilde{m}) \leq L \leq c_L/\tilde{m}. \quad (\text{S1.4})$$

In fine units $r_0 \rho_\lambda = L/12 = c_\rho L$ ($c_\rho = 1/12$), $c_I = 2^{2/3}$, $V_* = (1 - \eta_w)^3 \rho_\lambda^3/2$ (coarse) and $V_*^f = r_0^3 V_*/2$, so $R_{\text{HK}} = (1 - \eta_w) c_I r_0 \rho_\lambda/3$ and $4R_{\text{HK}}^3/V_*^f = 64/27$ exactly. The constant

$$K_0 := (1 - \eta_w) c_I c_\rho c_L/6 \quad (= 0.2381 c_\rho c_L = 0.01984 c_L \text{ at } \eta_w = 0.1)$$

satisfies $K_0 \leq \tilde{m} R_{\text{HK}} \leq 2K_0$ by (S1.4). We use K_0 only through the lower bound $\tilde{m} R_{\text{HK}} \geq K_0$, in the following places.

- (a) Proposition 6.10: $(\varepsilon_1^{\text{HK}})^{1/2} M_Y R_{\text{HK}} \geq (\varepsilon_1^{\text{HK}})^{1/2} K_0 / \lambda_{\text{PV}}$, so the rate of (6.13) at $M = M_Y$ is at least $\kappa_Y M_Y$.
- (b) Theorem 6.11, (H2): at the mass $r_-^{1/2} \geq \tilde{m}$ the rate of (6.13) is at least $\kappa_W \tilde{m} \min(1, (\varepsilon_1^{\text{HK}})^{1/2} \tilde{m} R_{\text{HK}}) \geq \kappa_0 \tilde{m}$, using $(\varepsilon_1^{\text{HK}})^{1/2} K_0 < 1$.
- (c) Theorem 6.11, (H1), (H1'): $R_{\text{HK}}^{-3} r_-^{-1} \leq \tilde{m}^{-2} R_{\text{HK}}^{-3} \leq \tilde{m} K_0^{-3} = \lambda_{\text{PV}} K_0^{-3} M_Y$.
- (d) Lemma 6.14: $R_{\text{HK}}^{-1} + M_Y^{-2} R_{\text{HK}}^{-3} \leq \tilde{m} K_0^{-1} + \lambda_{\text{PV}}^2 \tilde{m} K_0^{-3} = \eta_G$.
- (e) Appendix D: $\tilde{m} \leq K_0 (2C_H)^{-1/3}$ gives $R_{\text{HK}} \geq (2C_H)^{1/3}$.

By (S1.4), $M_Y r_0 \rho_\lambda = c_\rho M_Y L \in [c_\rho c_L / (2\lambda_{\text{PV}}), c_\rho c_L / \lambda_{\text{PV}}]$. We will use the lower bound in (f) to (h) below.

Theorem 6.8. We set $c_N := c_F^2 / (24 \cdot 2^{4/3}) = 1.006 \cdot 10^{-7}$, $C_H := (c_N/3)^{-3/2} = 1.628 \cdot 10^{11}$ and $\varepsilon_1^{\text{HK}} := (\log 2)^2 / 512 = 9.38 \cdot 10^{-4}$. In the proof (Appendix D), the constant of Lemma D.1 is $A_N := 24 \cdot 2^{2/3} / c_F^2$, so that $c_N = 1 / (2^{2/3} A_N)$, and (HK0) uses $6(\cosh \frac{1}{4} - 1) = 0.1885 < \frac{1}{4} - \frac{1}{17} = 0.1912$. The constants $C_{\text{mo}} \geq 1$ of the moment bound, $c_{\text{ex}} := \min((\log 2) / (2A_1), \frac{1}{2})$ with $A_1 := 512 C_{\text{mo}}^2 / (\log 2)^2$ (so $\varepsilon_1^{\text{HK}} = C_{\text{mo}}^2 / A_1$), $C_{\text{gs}} := 8C_H e^{c_{\text{ex}}/2}$, $c_{\text{gs}} := c_{\text{ex}}/8$ and C_M , which contains the factor $e^{1/(2c_{\text{ex}})}$, depend only on C_H . Hence $\kappa_W = \min(1, 1/(2C_M))$ of Proposition 6.10 is extremely small, but it is fixed before β_{GM} , and only its independence of τ , K , C and D is used.

Proposition 6.9. The box inequalities of the sketch in §D.1 are as follows. For a box graph B with sides in $[n_{\min}, n_{\max}]$, $\varrho := n_{\max}/n_{\min}$, and $U \subseteq B$, $|\partial_B U| \geq 4|U||B \setminus U|/(n_{\max}|B|) \geq 2 \min(|U|, |B \setminus U|)/n_{\max}$ (canonical paths) and $|\partial_B U| \geq 3|U|^{2/3} - 3|U|/n_{\min}$ (Loomis–Whitney), hence $|\partial_B U| \geq c_{\text{box}}(\varrho)|U|^{2/3}$ for $|U| \leq |B|/2$, with $c_{\text{box}}(\varrho) := 6/(3\varrho + 2)$ (by minimizing over $|U|^{1/3} = t n_{\min}$). In the face count, for a heavy cell c and a light neighbour c' , the set $D := B_c \cup B_{c'}$ has $n_{\max}(D) < 4r_0$ and $|U||D \setminus U|/|D| \geq r_0^3/4$ for $U := A \cap D$, so D carries at least $\gamma_D r_0^2$ bonds of ∂A , $\gamma_D = \frac{1}{4}$. Summing over $\partial_\Gamma H$ ($\Gamma = (V_c, E_c)$ the coarse slit graph of §6.3) counts the bonds inside a coarse cell at most six times. Hence $|\partial A| \geq (\gamma_D r_0^2/6)|\partial_\Gamma H|$. With $a := 9c_I/38$ (the constant of (R2)), $c_{\text{box}}(\varrho) > \frac{3}{4}$ for $\varrho < 2$ and $\gamma_D = \frac{1}{4}$, the two cases of (i) give $|\partial A| \geq 2^{-2/3} c_{\text{box}} |A|^{2/3} \geq 0.47|A|^{2/3}$ and $|\partial A| \geq (\gamma_D a/6) 16^{-2/3} |A|^{2/3} = (a/152.39 \dots) |A|^{2/3}$, so $c_F := a/152.4 = 2.467 \cdot 10^{-3}$. The constant $C_{\text{bl}} := 384$ comes from $\gamma_D r_0^3 / (12v_{\max}) \geq 1/384$ with $v_{\max} < 8r_0^3$. Further, $C_W^f := 108C_{\text{bl}}^2/\chi^2$, $k_\square := r_0(k_P + 3)$, $h^f := \chi/(3C_{\text{bl}}k_\square)$, $12/(h^f)^2 = C_W^f k_\square^2$ and $N_{\text{ov}} \leq 27(k_P/L_c)^3$. For (v), (L8-2) reads $C_W^f r_\square^2 N_{\text{ov}} (\tilde{m}L)^2 \leq \eta_H/8$ and $C_W^f r_\square^2 (\tilde{m}L)^2 = 12\tilde{m}^2/(h^f)^2$, so $1/h^f \leq 96^{-1/2} \tilde{m}^{-1} \leq 0.103 \tilde{m}^{-1}$ ($\eta_H < 1 \leq N_{\text{ov}}$). Moreover $\log k_\square \leq \log(1/\tilde{m}) \leq 2.4944 \tau = 0.0632 \beta_{\text{GM}}$. Hence the diameter bound $2 + (24/h^f) \cdot 3 \log k_\square \leq 2 + 24 \cdot 0.103 \cdot 3 \cdot 0.0632 \beta_{\text{GM}} / \tilde{m} < 2\beta_{\text{GM}}/\tilde{m} = L'$ holds.

Theorem 6.11. Let $\varpi := (1 - 4\lambda_{\text{PV}}^2)^{-1/2} \in [1, 1.16]$ and $r_\pm := \frac{1}{2} M_Y^2 (1 \pm (1 - 4\lambda_{\text{PV}}^2)^{1/2})$, so that $r_- = 2\tilde{m}^2/(1 + (1 - 4\lambda_{\text{PV}}^2)^{1/2}) \in [\tilde{m}^2, 1.08\tilde{m}^2]$ for $\lambda_{\text{PV}} \leq \frac{1}{4}$. The constants, which depend

only on $(c_F, c_\rho, c_L, \lambda_{\text{PV}})$, are

$$\begin{aligned} K_1 &:= \frac{1}{2} + 2C_H(1 + \varpi) + \varpi C_H \lambda_{\text{PV}} K_0^{-3}, & K_3 &:= 3\varpi \tilde{m}^2 / r_- \leq 3.5, \\ K'_1 &:= \max\{eK_1, \varpi(313 + C_{\text{gs}}(2\pi/c_{\text{gs}})^{1/2} + 4C_H(\varepsilon_1^{\text{HK}})^{-3/2} \lambda_{\text{PV}} K_0^{-3})\}, \\ K''_1 &:= \max(eK_1, K'_1). \end{aligned}$$

The number 313 comes from the bound $de^{-3d/68} \leq M_Y d^2 e^{-3d/68} \leq (136/3)^2 e^{-2} M_Y < 279 M_Y$ for $d \geq M_Y^{-1}$, and $279 \leq 313$.

Section 6.4: the depletion constants. Both A_0^{AD} and $C_F := C_{\text{Gr}} C'_L$ are upper-bound constants, taken ≥ 1 . Put $\varepsilon_0 := \theta_0 / (2\pi^2 \tau)$, $\varepsilon_F := \varepsilon_0 / (4C_F)$,

$$L_g := 3 + \log_2^+(8\lambda_{\text{PV}} / (c_\rho c_L)), \quad C_R := 256\pi^2 A_0^{\text{AD}} C_F L_g, \quad C_S := C A_0^{\text{AD}} (1 + \lambda_{\text{PV}} / (c_\rho c_L)),$$

so that $32A_0^{\text{AD}} L_g / \varepsilon_F = C_R \tau / \theta_0$ and $R_{\text{dep}} = \lceil C_R \tau / \theta_0 \rceil$. Moreover, $2/\varepsilon_F = 16\pi^2 C_F \tau / \theta_0$ and $R_{\text{dep}} \leq 2C_R \tau / \theta_0$.

(f) Theorem 6.15: the shells $2^k \leq r(y) < 2^{k+1}$ with $2^{k+1} \leq r_0 \rho_\lambda$ contribute at most $8A_0^{\text{AD}}$ each to $S_2(x)$ (Lemma 6.13), hence $C A_0^{\text{AD}} \log(2/M_Y)$ up to $r \leq M_Y^{-1}$, the shells beyond M_Y^{-1} are summed with $e^{-M_Y r/2}$, and the growth term gives $\sum_k (16A_0^{\text{AD}} / (r_0 \rho_\lambda)) 2^k e^{-M_Y 2^{k/2}} \leq C A_0^{\text{AD}} / (M_Y r_0 \rho_\lambda) \leq 2C A_0^{\text{AD}} \lambda_{\text{PV}} / (c_\rho c_L)$. Since $\log(2/M_Y) \geq 1$, $S_2(x) \leq C_S \log(2/M_Y)$.

(g) Theorem 6.16, step (ii): with $R \leq r_0 \rho_\lambda$ the far pairs contribute at most $16A_0^{\text{AD}}/R + (16A_0^{\text{AD}} / (r_0 \rho_\lambda))(2 + \log_2^+(4/(M_Y r_0 \rho_\lambda))) \leq (16A_0^{\text{AD}}/R)(3 + \log_2^+(8\lambda_{\text{PV}} / (c_\rho c_L))) = 16A_0^{\text{AD}} L_g / R$, which is at most $\varepsilon_F / 2$ for $R = R_{\text{dep}}$. (The second sum: the shells with $2^{k+1} R > r_0 \rho_\lambda$ give $\sum_{j \geq 0} e^{-a 2^j}$ with $a > M_Y r_0 \rho_\lambda / 4$, which is at most $2 + \log_2^+(1/a)$.)

(h) Lemma 6.18: see below.

The number N_{cov} of Proposition 6.17 depends on R_{dep} only through $R_{\text{dep}} \leq \ell_D$, hence not on θ_0 . It is very large but fixed, and it raises β_{GM}^0 only through the condition of Theorem 7.40.

Proof of Lemma 6.18. The hypothesis says $C_*^{\text{dep}} \tau^{5/2} e^{-2.494\tau} \leq \theta_0$. We use (S1.2).

We first prove $R_{\text{dep}} \leq r_0 \rho_\lambda$. Since $A_0^{\text{AD}}, C_F \geq 1$ and $L_g \geq 3$, $C_R \geq 768\pi^2 > 7.5 \cdot 10^3$. Hence $y := C_R \tau / \theta_0 \geq 1.5 \cdot 10^6$ and $R_{\text{dep}} = \lceil y \rceil \leq (1 + 10^{-6})y$. By (S1.4), $r_0 \rho_\lambda = c_\rho L \geq c_\rho c_L / (2\tilde{m})$, so it suffices that $2(1 + 10^{-6})C_R \tau \tilde{m} \leq c_\rho c_L \theta_0$. Using $\tau \geq 2$,

$$\begin{aligned} 2(1 + 10^{-6})C_R \tau \tilde{m} &\leq 2(1 + 10^{-6}) 17.7716 C_R \tau^{3/2} e^{-2.494\tau} \leq 17.8 C_R \tau^{5/2} e^{-2.494\tau} \\ &\leq c_\rho c_L C_*^{\text{dep}} \tau^{5/2} e^{-2.494\tau} \leq c_\rho c_L \theta_0. \end{aligned}$$

Next, $R_{\text{dep}} \leq \ell_D$ because $r_0 \rho_\lambda = c_\rho L < L \leq \ell_D$.

Finally, for (F), $\log(2/M_Y) = \ln(2\lambda_{\text{PV}} / \tilde{m}) \leq \ln(1/m_D) = \pi^2 v_{\text{Cb}}(0) \tau - \frac{1}{2} \ln(8\pi^2 \tau) \leq 2.4944 \tau$, using $\lambda_{\text{PV}} \leq \frac{1}{4}$ and $\tilde{m} \geq m_D/2$. With $\eta_G = \tilde{m}(K_0^{-1} + \lambda_{\text{PV}}^2 K_0^{-3})$,

$$C_F \eta_G C_S \log(2/M_Y) \leq C_F C_S (K_0^{-1} + \lambda_{\text{PV}}^2 K_0^{-3}) \cdot 17.7716 \tau^{1/2} e^{-2.494\tau} \cdot 2.4944 \tau,$$

and this is at most $\varepsilon_0 / 4 = \theta_0 / (8\pi^2 \tau)$ because $8\pi^2 \cdot 17.7716 \cdot 2.4944 = 3500.2 \leq 3.6 \cdot 10^3$ and $C_*^{\text{dep}} \geq 3.6 \cdot 10^3 C_F C_S (K_0^{-1} + \lambda_{\text{PV}}^2 K_0^{-3})$.

S1.6 Constants of Subsections 7.0 to 7.5 and Appendix C

Proposition 7.5. Off $\mathfrak{D}$, (S4') gives $|\rho_1(\pm 1, x) - \rho^{\text{ref}}(1)| \leq \rho^{\text{ref}}(1)(1 - e^{-\theta_0} + Ce^{-\mu})$, and $\sum_{|m| \geq 2} (|\rho_1(m, x)| + |\rho^{\text{ref}}(m)|)m^2 \leq 2Ce^{-2\kappa_S}m_D^2/\beta_{\text{GM}}$. Since $2\beta_{\text{GM}}\rho^{\text{ref}}(1) = \tilde{m}^2 - \beta_{\text{GM}}\sum_{|m| \geq 2} m^2\rho^{\text{ref}}(m) \leq (1 + Ce^{-2\kappa_S})\tilde{m}^2$ and $m_D \leq 2\tilde{m}$, we obtain, after enlarging C ,

$$\sum_m |\rho_1(m, x) - \rho^{\text{ref}}(m)|m^2 \leq \theta' \tilde{m}^2/\beta_{\text{GM}} \quad (x \notin \mathfrak{D}),$$

with θ' of (7.7). On $\mathfrak{D}$ the same tails, $|\rho_1(\pm 1, x)| \leq \rho^{\text{ref}}(1)(1 + Ce^{-\mu})$, $\rho_1(\pm 1, x) - \rho^{\text{ref}}(1) \leq Ce^{-\mu}\rho^{\text{ref}}(1)$ (since $e^{-\Phi} \leq 1$) and $2\beta_{\text{GM}}\rho^{\text{ref}}(1) \leq 2\tilde{m}^2$ give the constants of part (b). At $\theta_0 = 0.01$ we have $1 - e^{-\theta_0} = 0.00995\dots$, and hence the standing hypothesis (7.8) holds once $\varepsilon_\nu + C(e^{-\mu} + e^{-2\kappa_S}) \leq 0.1329$.

The partial fractions (7.16) and Lemma 7.15. If $\lambda_{\text{PV}} \leq \frac{1}{4}$, then $\varpi = (1 - 4\lambda_{\text{PV}}^2)^{-1/2} \in [1, 1.155]$, $c_+ \leq \varpi$, $c_- \leq 1.26\lambda_{\text{PV}}^2$, $\tilde{m}^2 \leq r_- \leq 1.08\tilde{m}^2$ and $r_+ \geq 0.932M_Y^2$. In part (b), with $\alpha = \tilde{m}/3$, the bound $\cosh x - 1 \leq \frac{x^2}{2}\cosh x \leq 0.5281x^2$ for $x \leq \frac{1}{3}$ gives $6(\cosh \alpha - 1) \leq 0.3521\tilde{m}^2$. With this α , part (a) gives $\|e^\rho R_{r_-} e^{-\rho}\| \leq (r_- - 0.3521\tilde{m}^2)^{-1} \leq 1.544\tilde{m}^{-2}$ and $\|e^\rho R_{r_+} e^{-\rho}\| \leq (r_+ - 0.3521\tilde{m}^2)^{-1} \leq 1.1M_Y^{-2} = 1.1\lambda_{\text{PV}}^2\tilde{m}^{-2}$, and hence $\|e^\rho \mathcal{G} e^{-\rho}\| \leq (1.155 \cdot 1.544 + 1.26 \cdot 1.1\lambda_{\text{PV}}^4)\tilde{m}^{-2} \leq 1.80\tilde{m}^{-2}$. The matrix $e^\rho \nabla^* e^{-\rho}$ has entries $\pm e^{\rho(x) - \rho(b)}$ with $|\rho(x) - \rho(b)| \leq \frac{1}{3}$, at most 6 per row and 2 per column. Hence its norm is at most $\sqrt{12}e^{1/3} \leq 4.835$, and $1.80 \cdot 4.835 \leq 8.71$. In part (c), $\|[\Delta, \chi]\| \leq 6\Lambda_\chi$ and $\|\Delta\| \leq 12$, so $\|[C_0^{-1}, \chi]\| \leq M_Y^{-2} \cdot 2 \cdot 12 \cdot 6\Lambda_\chi + 6\Lambda_\chi = \Lambda_\chi(144M_Y^{-2} + 6) \leq 150\Lambda_\chi M_Y^{-2}$, by $M_Y \leq 1$.

Theorem 7.16. The constant is $C_\varepsilon := 3.8 \cdot 10^9$. With $d(A', V_B) \geq L'/2 - 2$, $\Lambda_\chi = 4/L'$, $M_Y^{-4} = \lambda_{\text{PV}}^4\tilde{m}^{-4}$ and $\tilde{m} \leq 1$, Lemma 7.15 (b), (c) give $\|F'_{2,\beta}\|^2 \leq (150 \cdot \frac{4}{L'}M_Y^{-2})^2(8.71\tilde{m}^{-2})^2e^{4/3}e^{-\tilde{m}L'/3}\|w_\beta\|^2 \leq 1.71 \cdot 10^6 \cdot 16e^{4/3}\tilde{m}^2(\tilde{m}L')^{-2}\lambda_{\text{PV}}^4\tilde{m}^{-8}e^{-\tilde{m}L'/3}\|w_\beta\|^2$. Theorem 7.13 and $c'_g \geq \frac{9}{16}c_g$ give

$$\sum_{\beta: S_\beta \subseteq X} \|w_\beta\|^2 \leq 18 F_1^{\text{loc}}(X)/(c'_g\tilde{m}^2L/2) \leq 32 F_1^{\text{loc}}(X)/(c_g\tilde{m}^2L/2).$$

The product of the numerical factors is $1.71 \cdot 10^6 \cdot 16 \cdot 32e^{4/3} \leq 3.4 \cdot 10^9 \leq C_\varepsilon$.

Corollary 7.18. The Hölder exponents and the constant of Subsection 7.6 are $p_2 = 20$, $p_3 = 1.2$, $p' = 1.3$, $q' = 15.6$ and $c_\lambda = \frac{1}{24}$, so $\varepsilon_0 = \min\{(1 - \frac{1}{1.2} - \frac{1}{24})/40, 2(1.3/15.6)^2\} = \min\{0.003125, 0.0139\} = 0.003125$. Dropping $(\tilde{m}L')^{-2} \leq 1$ in (7.17), we obtain $\varepsilon(L') \leq \varepsilon_0$ as soon as

$$\tilde{m}L' \geq 27 \log(1/\tilde{m}) + 3 \log(C_\varepsilon \lambda_{\text{PV}}^4/(c_g \varepsilon_0 \tilde{m}L/2)).$$

The second term is bounded, since $c_L/2 \leq \tilde{m}L \leq c_L$ (Definition 3.1). By Definition 3.1 and $\tilde{m} \geq m_D/2$, $\log(1/\tilde{m}) \leq 2.4944\tau = (2.4944/4\pi^2)\beta_{\text{GM}}$ for $\tau \geq \tau_0$, so the right side is at most $1.706\beta_{\text{GM}} + O(1)$, while $\tilde{m}L' = K_{\text{col}}\beta_{\text{GM}} = 2\beta_{\text{GM}}$.

Proposition 7.19. The constant of part (c) is

$$\Phi_{\mathcal{U}} := \frac{\pi c_g c_u}{24} \frac{(\tilde{m}L)^3}{(\tilde{m}\beta_{\text{GM}})(\tilde{m}L')^3(11/10)^3}.$$

Dividing (a) by (b) gives the share $\frac{\pi c'_g c_u}{12} (\tilde{m}L)^3 / ((\tilde{m}\beta_{\text{GM}})(\tilde{m}L')^3(1 + \eta_{\text{col}})^3)$, which is at least $\Phi_{\mathcal{U}}$ because $c'_g \geq \frac{9}{16}c_g \geq \frac{1}{2}c_g$ and $\eta_{\text{col}} \leq \frac{1}{10}$ for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ (L and D_E are $O(\tilde{m}^{-1})$, while $L' = 2\beta_{\text{GM}}\tilde{m}^{-1}$). With $\tilde{m}L \geq c_L/2$ and $\tilde{m}L' = 2\beta_{\text{GM}}$, $\Phi_{\mathcal{U}} \geq \pi c_g c_u c_L^3 / (24 \cdot 64 \cdot 1.331 \tilde{m}\beta_{\text{GM}}^4)$, and $\tilde{m}\beta_{\text{GM}}^4 \rightarrow 0$ because $\log(1/\tilde{m}) = \pi^2 v_{\text{Cb}}(0)\tau + O(\log \tau)$ and $\beta_{\text{GM}} = 4\pi^2\tau$.

Structural facts of the expansion (Appendix C). The bounds on the activities (Subsections 7.6 and 7.7, Appendices A and B) use the expansion of Proposition 7.22 only through the facts of Table S1.3, and none of them uses translation invariance, an infinite lattice, or the shape of the atoms.

	fact	used in	proof
S1	$\mathbf{C}_s(x, x) = \mathbf{C}(x, x)$, $ \mathbf{C}_s(x, y) \leq \mathbf{C}(x, y) $, weighted norms and $\ \mathbf{C}_s\ _{2 \rightarrow 2}$ bounded by those of $\mathbf{C}$	Lemma 7.29, App. B	Prop. 7.22, change 6
S2	$\mathbf{C}_s = \sum_{\varepsilon} \lambda_{\varepsilon} \mathbf{C}_{\varepsilon}$, $\mathbf{C}_{\varepsilon}$ block diagonal with admissible blocks	Lemma 7.33	Prop. 7.22, change 6
S3	$\log \int d\mu_{\mathbf{C}} e^{\frac{1}{2}\langle \psi + D, M(\psi + D) \rangle}$ is convex in $\mathbf{C}$ for $M \geq 0$	Lemma 7.33	[BF80, (9.67) to (9.611)]; Lemma 7.33, Step 3
S4	the integrand of $\mathcal{K}_{\bar{y}}$ depends on $\psi _{X(\bar{y})}$ only; covariance derivatives have both ends in $X(\bar{y})$	Lemma 7.37, Thm. 7.27(b), (c)	Prop. 7.22
S5	each Y_{i+1} is a union of elements, each meeting the support of its link	Lemma (App. A)	selection rule (C.6)
S6	link i carries $\prod_{k=T(i+1)}^{i-1} s_k$ (covariance link from $Y_{T(i+1)}$) or $\prod_{k=j_{\min}}^{i-1} s_k$ (E -term link); all other s -dependence is through weights in $[0, 1]$ and through $\mathbf{C}_s$	Lemma 7.37 (App. A)	(C.1), (C.5)
S7	five types of terms per link	App. A	(C.5)
S8	link ends $x \in Y_{i+1}$, $y \in Y_{T(i+1)}$, counted in $\mathcal{U}$ -cells	App. A	selection rule (C.6)
S9	sequences of disjoint admissible sets; finitely many	Lemma 7.37, Prop. 7.47	Prop. 7.22, change 2
S10	$Z(Y) > 0$; one-set activities are positive	Lemmas 7.41, 7.42, Prop. 7.53	Prop. 7.22
S11	the first set may depend on h	Thm. 7.27, Lemma 7.48	Prop. 7.22, change 3
S12	the first set may be disconnected	Thm. 7.27(c), Lemma 7.48	Prop. 7.22, change 3
S13	expansion on $U = X_0^c$ with $\mathbf{1}_U \mathbf{C} \mathbf{1}_U$; activities independent of U	Thm. 7.27(b)	Prop. 7.22, change 4
S14	absolute convergence of every term	all	Lemmas C.2, C.3

Table S1.3: Structural facts of the expansion used in Section 7.

S1.7 Section 7.6 and Appendix B

S1.7.1 The Hölder exponents and (L8-1)

Section S1.1.1 fixes $\eta_H = 0.62$, $p_1 = p_2 = 20$, $p_3 = 1.2$, $p_4 = 15$, $p' = 1.3$ and $q' = 15.6$. These values satisfy $\sum_i 1/p_i = 1$ and $1/p' + 1/q' = 1/p_3$. With $\alpha_D \leq 7/6$ (Lemma 7.29(c)), $\hat{\eta} = \alpha_D p' \eta_H \leq \frac{7}{6} \cdot 1.3 \cdot 0.62 = 0.94033 \dots < 0.941$, so (L8-1) holds. Moreover, $\eta_* \leq 1 - 4/\pi^2 + C_S e^{-2\kappa_S} < 0.62$ for κ_S large, since $1 - 4/\pi^2 = 0.5947 \dots$

S1.7.2 Lemma 7.32(c): the shell constant $C_{\text{sh}} = 92$

With $r_* = \frac{1}{2} + \frac{\sqrt{3}}{2}$, $\int_{1/2}^D (r + r_*)^2 r^{-3} dr \leq \log(2D) + 4r_* + 2r_*^2$ and $(r + r_*)/r \leq 1 + 2r_*$ for $r \geq \frac{1}{2}$, the shell count of the proof gives, for $D \geq 1$,

$$\sum_{d(b,C) \leq D} d(b,C)^{-2} \leq 6\pi|C| \left[\log(2D) + 4r_* + 2r_*^2 + \frac{1}{2}(1 + r_*)^2 \right] + 16\pi[(1 + r_*)^3 + 2(1 + 2r_*)^3] D.$$

The first bracket is at most $\log(2D) + 12$, since $4r_* + 2r_*^2 + \frac{1}{2}(1 + r_*)^2 = 11.9951 \dots$, and $16\pi[(1 + r_*)^3 + 2(1 + 2r_*)^3] = 16\pi \cdot 117.207 \dots = 5891.4 \dots \leq 5892$. At $D = D_{\mathcal{N}} = \sqrt{3}k_{\square} + 1 \leq 2k_{\square}$ and $|C| \geq 8k_{\square} \geq 4D_{\mathcal{N}}$ the sum is at most $6\pi|C|(\log k_{\square} + \log 4 + 12 + 1473/(6\pi)) \leq 6\pi|C|(\log k_{\square} + 92)$, since $\log 4 + 12 + 1473/(6\pi) = 91.53 \dots$. Multiplying by $(2\pi)^2$ gives $S(D_{\mathcal{N}}) \leq 24\pi^3|C|(\log k_{\square} + 92)$.

S1.7.3 Lemma 7.33: the constants c_8 and c_{near}

$$c_8 := \frac{\hat{\eta}(1 + 4/\eta_H)K_1 C_u}{2\lambda_{\text{PV}}(1 - \nu_0)}, \quad c_{\text{near}} := \frac{2\theta_P C_R (\log k_{\square} + C_{\text{sh}})}{\beta_{\text{GM}}} + \frac{1024\pi^2 \hat{\eta} (2r_{\square} + 5)^2 c_L^2}{\eta_H c_1 \beta_{\text{GM}}}.$$

Uniformity of c_{near} . Since $k_{\square} = r_{\square}L \leq r_{\square}c_L/\tilde{m}$, $\log k_{\square} \leq \log(r_{\square}c_L) + \log(1/\tilde{m})$ with $\log(r_{\square}c_L) < 0$. With $\tilde{m} \geq m_D/2$, $m_D = (8\pi^2\tau)^{1/2}e^{-\pi^2 v_{\text{Cb}}(0)\tau}$, $\pi^2 v_{\text{Cb}}(0) < 2.4943551$ and $\beta_{\text{GM}} = 4\pi^2\tau$,

$$\frac{\log(1/\tilde{m})}{\beta_{\text{GM}}} \leq \frac{2.4943551}{4\pi^2} - \frac{\frac{1}{2}\log(8\pi^2\tau) - \log 2}{4\pi^2\tau} < 0.0632 \quad (\tau \geq 1).$$

Hence $2\theta_P C_R \log k_{\square}/\beta_{\text{GM}} < 0.1264\theta_P C_R$, which gives the bound on c_{near} in Lemma 7.33. The coefficient θ_P in that bound satisfies $\theta_P \leq (4\hat{\eta}/\eta_H)C'_P c_L^2 \leq \frac{1}{2}$ by (L8-2), a bound that does not depend on β_{GM} .

S1.7.4 Lemma 7.34: the condition (7.30)

At $p_4 = 15$, $p_3 = 1.2$ (so $c_{\lambda} = \frac{1}{4}(1 - 1/p_3) = \frac{1}{24}$), $\alpha_D \leq \frac{7}{6}$ and $c_E \leq \frac{1}{70}$, we have $2\alpha_D p_4 c_E \leq 2 \cdot \frac{7}{6} \cdot \frac{15}{70} = \frac{1}{2}$ and $p_4 \alpha_D c_E^2 / (1 - 2\alpha_D p_4 c_E) + c_E \leq \frac{15 \cdot (7/6)}{4900 \cdot (1/2)} + \frac{1}{70} = 0.00714 \dots + 0.01428 \dots = 0.02142 \dots \leq \frac{1}{24}$. In Proposition 7.35(d), $c'_0 = \frac{1}{2}(1 - 1/p_3 - c_{\lambda}) = \frac{1}{2}(\frac{1}{6} - \frac{1}{24}) = \frac{1}{16}$.

S1.7.5 Theorem 7.40: the threshold τ_0

With $\pi^2 v_{\text{Cb}}(0) = 2.49435509 \dots$ and $p_0 = \frac{1}{6}$, the lone-cell condition (proof of Theorem 7.40) reads

$$0.4157\tau \geq C_{\mathfrak{D}} A_1^D \log(C'\tau/\theta_0) + \alpha + O(\log \tau),$$

where $c_{\text{dep}} = A_0^D + A_1^D \log(C'\tau/\theta_0)$ (Proposition 6.17). The threshold τ_0 , of which we use only the existence, is explicit but large, because A_1^D is large (θ_0 is small and the covering number of Proposition 6.17 is large). For the wall-polymer cells, Proposition 7.19 gives $\Phi_{\mathcal{U}} \geq (\pi c_g c_u c_L^3/24)/(64\beta_{\text{GM}}^3(11/10)^3 b_D)$.

S1.7.6 Lemma 7.38 and Proposition 7.39: constants and fractions

Constants on class-A cells. They depend only on the following quantities:

- c_7, c'_7, c''_7 of Lemma 7.31 (through $v_{\pm}, \eta_H, \kappa_S$, that is, on the grid scale L);
- $C_k, \theta', c_{\text{dep}}$ of Lemma B.1(d);
- $\mu, \kappa_S, \delta_{\text{GM}}, C$ and the Mayer decay;
- $K_1'', K_3, \alpha_D K_1, \lambda_{\text{PV}}, r_C$ of the lines;
- $C_r(c_p)$ and $c(q_{\text{pin}}, c_p, K_g, C_{\mathcal{U}})$ of the count and the pinning;
- c_T of the Wick bound;
- c_{13} and $C_{\mathfrak{D}} c_{\text{dep}}$ of the Hölder split;
- the geometry constants $c_u, C_u, C_{\mathcal{U}}, D_{\mathcal{U}}, v_{\pm}, K_g$.

The constants $C_{\mathcal{E}}, C_G$ of Proposition B.10 are built from the same list, and none depends on L' .

Fractions of F_1^{loc} . They are those of Table S1.4 (Section S1.8). Proposition 7.39 uses the entries for the numerical factors ($c'/16$ on $X \setminus Z$, Lemma B.8; $c'/32$ on Z) and for the wall polymers ($c'/8$ on $X \setminus Z$).

Cells. Put $K_B := K_0^B (2K_1''(1 + v_+/v_-)/\lambda_{\text{PV}})^2$. Each cell carries the per-cell constants $e^{c_2^L + c_{13}}$ and at most the share $(2K_1''(1 + v_+/v_-)/\lambda_{\text{PV}})^2$ of the line constants. Since $C_u \geq 1$, $c_2^L + c_{13} \geq 0$ and $\alpha > 0$, it follows that $e^{c_2^L + c_{13}} \leq e^{\alpha} e^{(c_2^L + c_{13})C_u}$. With $K_{\text{cell}} = K_B e^{(c_2^L + c_{13})C_u} \max(1, \lambda_{\text{PV}}^{1/6})$ and $b_D^{1/6} = \lambda_{\text{PV}}^{1/6} (b_D/\lambda_{\text{PV}})^{1/6}$, Proposition B.7 gives the following bounds: a non-seed lone cell carries at most $K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}}} (b_D/\lambda_{\text{PV}})^{1/6}$; a non-seed cell of a wall polymer at most $K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}} + c_B(b_D)} e^{-(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}$; a seed cell at most $K_{\text{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\text{dep}}}$, times $e^{c_B(b_D)}$ if it is of class B.

S1.8 Constants of Sections 7.7 to 7.9 and Appendix A

Lemma 7.44 (cell diameter). Since $t_1 \leq 2 + (36k_P/\chi_{\text{N3}}) \ln k_P$, we have $D_{\text{cell}} \leq 35r_0 + 504r_0 k_P \ln k_P/\chi_{\text{N3}}$. In fine units $r_0 k_P = 304L \leq 304c_L/\tilde{m}$ and $k_P < 1/\tilde{m}$, and therefore $\ln k_P \leq \ln(1/\tilde{m}) < 0.0632 \beta_{\text{GM}}$ (as in the proof of Lemma 7.33). Hence $D_{\text{cell}} \leq 35r_0 + 504 \cdot 304 \cdot 0.0632 (c_L/\chi_{\text{N3}}) \beta_{\text{GM}}/\tilde{m} \leq 35r_0 + 1.927 \beta_{\text{GM}}/\tilde{m}$ for $c_L \leq 1.99 \cdot 10^{-4} \chi_{\text{N3}}$, which is less than $L' = 2\beta_{\text{GM}}/\tilde{m}$ for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$, since $35r_0 \leq L$ ($L_c \geq 35$). The choice of c_L (through $C'_p c_L^2 \leq (1 - 4/\pi^2)/8$) places it far below the threshold $1.99 \cdot 10^{-4} \chi_{\text{N3}}$.

Constants of the seeded sums (before Lemma 7.48). The constant C_9 exists because blocks have side at least L , a block has fewer than $8/c_1$ cells and a cell at most $6/c_g$ interfaces.

$$c_S := \log K_{\text{cell}} + \alpha + C_{\mathfrak{D}} c_{\text{dep}} + 1 + c'_4 + \log 2, \quad \eta_{\text{seed}} := 6D_I e^{-(c'/4)F_{\text{unit}}}.$$

The summand $1 + c'_4 + \log 2$ is already contained in $\log K_{\text{cell}}$. Indeed, in the proof of Proposition A.7 the factor $e^{(1+c'_4+\log 2)|Z|}$ is bounded by $e^{a|Z|}$, which enters $c_2^L = a + \alpha$ and hence K_{cell} . Counting it again in c_S only enlarges c_S , which is harmless. Since the constants $\log K_{\text{cell}}$, D_U and α are fixed numbers and $c_{\text{dep}} = O(\log \tau)$ (Proposition 6.17), we have $c_S = O(\log \tau)$ and $c_2 = c_S + 2 = O(\log \tau)$. Since $g/L = O(\tau)$, $\log D_I = O(\log \tau)$, while F_{unit} is exponentially large in τ . Hence η_{seed} is doubly exponentially small.

Proposition 7.52 (the count), Step 4. By Definition 3.18, $c_3^{\text{NF}} \kappa_{\text{NF}} = c'_0/16$ and $\theta_{\text{NF}} = c'_0 c_g / (90(c_g + 4))$, and hence

$$\frac{\ln 16}{\ln 2} c_3^{\text{NF}} \kappa_{\text{NF}} = 4c_3^{\text{NF}} \kappa_{\text{NF}} = \frac{c'_0}{4}, \quad 8\theta_{\text{NF}} \ln 16 = \frac{8 \ln 16}{90} \frac{c_g}{c_g + 4} c'_0 \leq \frac{c'_0}{8}.$$

With $n \leq \nu$, $n_0 \leq (4/c'_g)\nu$ and $j_g = O(\ln \beta_{\text{GM}})$, the terms $C(n + n_0 + \nu)$ and $(1 + j_g)\kappa_{\text{NF}}\nu \ln 16$ of Steps 2 and 3 are at most $C'(1 + \ln \beta_{\text{GM}})\nu$ with C' depending only on C_9 , c_g and κ_{NF} . Hence $\ln \# \leq \frac{3}{8}c'_0 F_{\text{unit}}\nu + C'(1 + \ln \beta_{\text{GM}})\nu \leq \frac{c'}{2}F_{\text{unit}}\nu$ as soon as $F_{\text{unit}} \geq (8C'/c'_0)(1 + \ln \beta_{\text{GM}})$, which holds for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$ because $F_{\text{unit}} = (\pi^2 c'_g/18)(\tilde{m}L)^3/(\tilde{m}\beta_{\text{GM}}) \geq (\pi^2 c'_g/144)c_L^3/(\tilde{m}\beta_{\text{GM}})$ ($\tilde{m}L \geq c_L/2$) is exponentially large. In addition, $c' \geq c'_0$.

Lemma 7.56 and Theorem 7.57. Under the weight $2 - \eta_w$ of the hull functional the constant c_u of (R7) is the value fixed in Section S1.1.1, and only the value of $C_X = 3C_U/c_u$ depends on it. Since small components of G' are $\mathcal{U}$ -cells of Debye volume at least c_u , they are counted in Lemma 7.56. In Theorem 7.57, with $\sigma_F = (\pi^2 c'_g/18)(\tilde{m}L)\tilde{m}/\beta_{\text{GM}}$ (Definition 3.18) and $\beta_{\text{GM}} = 4\pi^2\tau$,

$$\sigma_V = \varepsilon_E \frac{\pi^2 c'_g \tilde{m}L}{18} \frac{\tilde{m}}{\beta_{\text{GM}}} = \varepsilon_E \frac{c'_g \tilde{m}L}{72} \frac{\tilde{m}}{\tau} = \frac{c'_g \tilde{m}L}{9216} \frac{\tilde{m}}{\tau} \quad \text{at } \varepsilon_E = \frac{1}{128} = \frac{c'_0}{8}, \quad \eta_S = \frac{c_2 C_X \tilde{m}^2}{\sigma_V},$$

which is the displayed expression, $O(\tilde{m}\beta_{\text{GM}} \log \tau)$ because $c_2 = O(\log \tau)$.

Remark 7.58 (the perimeter constant). By Lemma 7.33, $c_{\text{near}} \leq 0.1264\theta_P C_R + O(1/\beta_{\text{GM}})$ with $C_R = 24\pi^3$, uniformly for $\beta_{\text{GM}} \geq \beta_{\text{GM}}^0$. The coefficient $0.1264\theta_P C_R$ decreases as c_L decreases, as $\theta_P \propto c_L^2$. Since $c_2 = O(\log \tau)$, $\tilde{m} = e^{-2.4943551\tau(1+o(1))}$ and $R_1(\beta) = (2r_{\square} + 5)L \geq (2r_{\square} + 5)c_L/(2\tilde{m})$, the terms $3c_2 C_X C_U \tilde{m}$ and $9c_2 C_X C_U^2/R_1(\beta)$ of c_N tend to 0.

Lemma A.6 and the order of c_{10} , c''_{10} , μ . We fix c_{10} first, independently of $\bar{\delta}$, with $c_{10} \leq \frac{7}{9}r_C$. Then c''_{10} , which depends only on c_{10} and on the growth constants C_U , c_u , is fixed, with $e^{c''_{10}} \geq 16e \max(1, c_{c_{10}/2})$, and finally the Mayer parameter μ , with $\mu \geq 1 + 2c''_{10}$.

Proposition A.7 (the constant c_2^L). The constant $c_2^L = a + \alpha$, $a = \log(2eD_U) + 4 + c'_4 + a''$ with $e^{a''} \geq 4C_\varphi$, is built from D_U , c'_4 , C_φ and α . Here c'_4 depends on τ but is doubly exponentially small and decreasing in τ . Hence it is bounded uniformly for $\tau \geq \tau_0$. With this reading, c_2^L is uniform in L' , β_{GM} and τ . The threshold τ_0 of Lemma S1.2 grows with c_2^L , but it is finite, and only its existence is used.

Proposition A.8: the conditions on β_{GM}^0 . The conditions on β_{GM}^0 used in Proposition A.8 are

$$\begin{aligned} (\beta 1) \quad & \frac{c'}{32} \Phi_U \geq (1 + D_U)(c_S + c_B(b_D)), \\ (\beta 3) \quad & 3eD_I e^{-(c'/4)F_{\text{unit}}} \leq \frac{1}{2}, \quad (\beta 4) \quad c_{16}^D \leq 1. \end{aligned} \tag{S1.5}$$

Proposition A.8: the booking of F_1^{loc} . In the proof of the core estimate (A.10), we book the factor $e^{-(3c'/4-\varepsilon)F_1^{\text{loc}}(X)}$ left by Proposition 7.35 and the extraction as follows (Table S1.4).

- *Type-iii factors, per region.* The factors $|D|$ and $|F'_2|$ at derivatives in a region involve D and F'_2 on that region only, because $\text{supp } D_\beta$ lies in the element of S_β . For a wall component with N factors $|D|$, n_Δ of them in the cell Δ , let $a_\Delta := \|D\|_{L^2(\Delta)}$, so that $\sum_\Delta a_\Delta^2 \leq 2F_1^{\text{loc}}/\tilde{m}^2$. The weighted arithmetic-geometric mean inequality and the identity $\sup_{F>0} (2F/(\tilde{m}^2 N))^{N/2} e^{-\delta F} = (e\delta\tilde{m}^2)^{-N/2}$ give, for every $\delta > 0$,

$$\prod_\Delta a_\Delta^{n_\Delta} e^{-\delta F_1^{\text{loc}}} \leq (\delta\tilde{m}^2)^{-N/2} \prod_\Delta (n_\Delta!)^{1/2},$$

and F'_2 is handled in the same way. These constants enter K_{cell} and the pinning factor (Proposition B.7, Lemma B.8). We take $\delta = c'/16$ on $X \setminus Z$ and $\delta = c'/32$ on Z .

- *(M - Σ) on $X \setminus Z$.* We use $c'/8$ of $F_1^{\text{loc}}(X \setminus Z)$. A wall polymer $\mathcal{S}$ is the union of the closures of its components, $|\text{cl}(S_\beta)| \leq (1 + D_U)|S_\beta|$, and $F_1(S_\beta) \geq \Phi_U|S_\beta|$ (Proposition 7.19(c)). Hence $F_1^{\text{loc}}(\mathcal{S}) \geq \Phi_U|\mathcal{S}|/(1 + D_U)$, and each $\mathcal{U}$ -cell of a wall polymer of $X \setminus Z$ receives the factor $e^{-(c'/8)\Phi_U/(1+D_U)}$ of ε_Δ in Proposition 7.39. Lone cells of $X \setminus Z$ carry $\varepsilon_\Delta = K_{\text{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\text{dep}}}(b_D/\lambda_{\text{PV}})^{1/6}$. The remaining fraction $9c'/16 - \varepsilon \geq c'/16$ of $F_1^{\text{loc}}(X \setminus Z)$ is bounded by 1.
- *Seed.* We use $c'/32$ of $F_1^{\text{loc}}(Z, N_Z)$ for the $\mathcal{U}$ -cells of Z^Σ . Each carries at most $K_{\text{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\text{dep}}+c_B(b_D)} \leq e^{c_S+c_B(b_D)}$ (Proposition 7.39), and $F_1^{\text{loc}}(Z^\Sigma) \geq \Phi_U|Z^\Sigma|/(1 + D_U)$ pays it under $(\beta 1)$. Lone seed cells carry at most $K_{\text{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\text{dep}}} \leq e^{c_S}$. What remains is

$$e^{-(c'-c'/4-\varepsilon-c'/32-c'/32)F_1^{\text{loc}}(Z, N_Z)} = e^{-(11c'/16-\varepsilon)F_1^{\text{loc}}(Z, N_Z)}.$$

Each non-seed cell carries at most $\varepsilon_\Delta \leq \frac{1}{2}c_{16}^D \leq \frac{1}{2}$ by $(\beta 4)$. In (a) and (b) of Lemma 7.48 the retained fraction $11c'/16 - \varepsilon$ pays $c'/8$ for the entropy of the first set, and at least $c'/16$ is left unused. In (c') of that lemma the retained fraction pays $\varepsilon \leq c'/8$ and $c_w = c'/16$ and leaves at least $c'/2$.

Remark S1.1 (The F_1 ledger). Table S1.4 lists, for the proofs of Lemma 7.48 and Proposition 7.39, the fractions of F_1^{loc} spent on the non-seed elements ($X \setminus Z$) and on the seed elements (Z). In each column the entries add up to 1, and no fraction is used twice. The entry

for the h -entropy on Z is not needed: the walls on Z are fixed on $\mathcal{N}_\sigma(X)$ and no wall lies on an interface between Z and $X \setminus Z$, so the weight $e^{c'_1 F_1^{\text{loc}}(X \setminus Z)}$ suffices in the sum over N_X in the proof of Proposition A.7; omitting it leaves $c'/4$ of slack on the seed. It is kept so that Lemma 7.37 can be stated with $e^{c'_1 F_1^{\text{loc}}(X, N_X)}$.

item	$X \setminus Z$ (non-seed)	Z (seed)	where
Hölder split and [GM82, Lemma 13] (T_2, T_3, T_4)	$1 - c'$	$1 - c'$	Prop. 7.35
h -entropy, c'_1	$c'/4$	$c'/4$ (optional)	Prop. A.7
extraction ε ($\varepsilon \leq c'/2$; the chain uses $\varepsilon \leq \varepsilon_E$)	ε	ε	Lemma 7.48
type-iii factors $ D , F'_2 $	$c'/16$	$c'/32$	Prop. A.8
(M- Σ): cells of wall polymers	$c'/8$	none	Prop. 7.39
seed cells of Z^Σ	none	$c'/32$	($\beta 1$)
left over	$9c'/16 - \varepsilon \geq c'/16$ (discarded)	$11c'/16 - \varepsilon$ (re- tained)	

Table S1.4: The F_1 ledger: fractions of F_1^{loc} per region.

Lemma S1.2 (Constant check). *There is τ_0 such that for $\tau \geq \tau_0$ the conditions (S1.5) hold and $c_{16}e^{c_2} \leq \frac{1}{8}$.*

Proof. Each condition compares an energy that is exponentially large in τ with a cost that is logarithmic in τ or of smaller exponential order. Since $\tilde{m} \leq 2m_D$ and $m_D = (8\pi^2\tau)^{1/2}e^{-\pi^2 v_{\text{Cb}}(0)\tau}$, $\log(1/b_D) \geq \pi^2 v_{\text{Cb}}(0)\tau - \frac{3}{2}\log \tau - C$, with $\pi^2 v_{\text{Cb}}(0) = 2.49435509\dots$. As the constants $\log K_{\text{cell}}$, D_U and α are fixed numbers and $c_{\text{dep}} = O(\log \tau)$ (Proposition 6.17), $c_S = O(\log \tau)$ and $c_2 = c_S + 2 = O(\log \tau)$. By Proposition 7.19, $\Phi_U \geq cb_D^{-1}\beta_{\text{GM}}^{-3}$ with $c > 0$ fixed, while $c_B(b_D) \leq Cb_D^{-1/2}$ (Lemma 7.38). Moreover, $F_{\text{unit}} = (\pi^2 c'_g/18)(\tilde{m}L)^3(\tilde{m}\beta_{\text{GM}})^{-1} \geq (\pi^2 c'_g/144)c_L^3(\tilde{m}\beta_{\text{GM}})^{-1}$ (since $\tilde{m}L \geq c_L/2$), while $\log D_I = O(\log \tau)$ because $g/L = O(\tau)$. Hence ($\beta 1$) and ($\beta 3$) hold for $\tau \geq \tau_0$, and ($\beta 4$) is Theorem 7.40. For the product, the first branch of c_{16}^D gives

$$\log(2K_{\text{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\text{dep}}}(b_D/\lambda_{\text{PV}})^{1/6}e^{c_2}) = O(\log \tau) - \frac{1}{6}\pi^2 v_{\text{Cb}}(0)\tau,$$

and the second gives $O(\log \tau) + c_B(b_D) - (c'/8)\Phi_U/(1 + D_U)$, which tends to $-\infty$ faster. Consequently, $c_{16}^D e^{c_2} = \exp(O(\log \tau) - 2.494p_0\tau(1 + o(1)))$ with $p_0 = 1/6$. Also $\eta_{\text{seed}}e^{c_2} = 6D_I e^{c_S+2}e^{-(c'/4)F_{\text{unit}}}$ is doubly exponentially small. Both are at most $\frac{1}{16}$ for $\tau \geq \tau_0$.

S2 Full proofs of steps sketched in the main text

S2.1 Full proofs for Section 2

Proof of Lemma 2.3. The Villain weight is the limit of N -fold convolution powers of Wilson weights at coupling Nt , and a convolution power is the marginal of a chain of Wilson interactions, to which Ginibre's inequality applies. Positivity and monotonicity pass to the limit by dominated convergence in the dual representation.

Step 1 (approximants). For $b > 0$ let $w_b(\theta) := e^{b \cos \theta}/I_0(b)$, whose Fourier coefficients are $c_n(b) := I_{|n|}(b)/I_0(b) \in (0, 1]$. The N -fold convolution w_b^{*N} (on $\mathbb{T}$ with $d\theta/2\pi$) has coefficients $c_n(b)^N$.

Step 2 (convergence and a uniform majorant). Write $r_k(x) := I_{k+1}(x)/I_k(x)$. By [Amo74, (16)], for $x \geq 0$ and $k \geq 0$,

$$\frac{x}{k + \frac{1}{2} + (x^2 + (k + \frac{3}{2})^2)^{1/2}} \leq r_k(x) \leq \frac{x}{k + \frac{1}{2} + (x^2 + (k + \frac{1}{2})^2)^{1/2}}.$$

Using $x \leq (x^2 + a^2)^{1/2} \leq x + a^2/(2x)$ and $y - y^2/2 \leq \log(1 + y) \leq y$, we obtain $|\log r_k(x) + (k + \frac{1}{2})/x| \leq (k + \frac{3}{2})^2/(2x^2)$. Since $c_n(x) = \prod_{k=0}^{|n|-1} r_k(x)$ and $\sum_{k=0}^{|n|-1} (k + \frac{1}{2}) = n^2/2$,

$$\left| \log c_n(x) + \frac{n^2}{2x} \right| \leq \frac{(|n| + 1)^3}{x^2}, \quad \text{hence} \quad \lim_{N \rightarrow \infty} c_n(Nt)^N = e^{-n^2/(2t)}$$

for all n and $t > 0$. The upper bound also gives $r_k(x) \leq x/(x + \frac{1}{2})$, so $c_n(x) \leq (1 + 1/(2x))^{-|n|}$ and, by $\log(1 + y) \geq y/(1 + y)$,

$$c_n(Nt)^N \leq \left(1 + \frac{1}{2Nt}\right)^{-N|n|} \leq e^{-|n|/(2t+1)} \quad (N \geq 1). \quad (\text{S2.1})$$

Step 3 (extended system and Ginibre). We give each plaquette p auxiliary angles $\phi_{p,1}, \dots, \phi_{p,N-1}$, put $\phi_{p,0} := \theta_p$ and $\phi_{p,N} := 0$, and take the weight $\prod_p \prod_{i=1}^N \exp(Nt_p \cos(\phi_{p,i-1} - \phi_{p,i}))$. Integrating out the auxiliary angles gives $\prod_p I_0(Nt_p)^N w_{Nt_p}^{*N}(\theta_p)$. Each argument $\phi_{p,i-1} - \phi_{p,i}$ is an integer combination of the variables, and the couplings Nt_p are nonnegative, so (2.1) applies (removed plaquettes enter with coupling 0). Hence the expectation $\langle \cos \theta_C \rangle_N$ under the density $\propto \prod_p w_{Nt_p}^{*N}(\theta_p)$ is nonnegative, and it is nondecreasing in each t_p , because raising t_p raises the N couplings of p .

Step 4 (dual form and the limit). Let f_p be even functions on $\mathbb{T}$ with absolutely summable Fourier coefficients $\hat{f}_p$. Expanding $\prod_p f_p(\theta_p) = \sum_{n \in \mathbb{Z}^{P(X)}} \prod_p \hat{f}_p(n_p) e^{i(\partial n) \cdot \theta}$ and integrating over θ gives

$$\frac{\int e^{i\theta_C} \prod_p f_p(\theta_p) d\theta}{\int \prod_p f_p(\theta_p) d\theta} = \frac{\sum_{n: \partial n = C} \prod_p \hat{f}_p(n_p)}{\sum_{n: \partial n = 0} \prod_p \hat{f}_p(n_p)}, \quad (\text{S2.2})$$

where we used $\hat{f}_p(-m) = \hat{f}_p(m)$. The numerator runs over $n_0 + Z_2(X)$ if $C = \partial n_0$. For $\hat{f}_p(m) = c_m(Nt_p)^N$, the majorant (S2.1) is summable over $\mathbb{Z}^{P(X)}$, and the denominators are at least 1 (the term $n = 0$). By dominated convergence, (S2.2) for the approximants converges to its value for $\hat{f}_p(m) = e^{-m^2/(2t_p)}$, which is $\langle \cos \theta_C \rangle_{X,t}^V$. Nonnegativity and monotonicity pass to the limit, and $\langle \sin \theta_C \rangle = 0$ by the symmetry $\theta \mapsto -\theta$. This proves (a) and (b), and (c) follows in the same way, the limit $t_p \rightarrow 0$ being dominated convergence with $\hat{f}_p(m) = e^{-m^2/(2t_p)} \rightarrow \mathbf{1}_{m=0}$.

Step 5 (frozen limits). For fixed couplings off D , the expectation is a nondecreasing function of the couplings on D with values in $[0, 1]$, and therefore has a limit. This limit inherits monotonicity in the other couplings. If $D \subseteq D'$ and $s \geq \tau$, then $t^{\tau, D, s} \leq t^{\tau, D', s}$ coordinatewise, so $\langle \cos \theta_C \rangle_{t^{\tau, D, s}}^V \leq \langle \cos \theta_C \rangle_{t^{\tau, D', s}}^V \leq g_{X, \tau}(C; D')$, and $g_{X, \tau}(C; D) \leq g_{X, \tau}(C; D')$. Monotonicity in τ passes to the limit in the same way.

Proof of Lemma 2.8. Write $Q_s(k) := \frac{1}{2\tau} |\mathbf{1}_{Fc}(n_0 + \partial k)|^2 + \frac{1}{2s} |\mathbf{1}_F(n_0 + \partial k)|^2$, so that by Lemma 2.7 the left side of (2.7) at finite s is N_s/D_s with $N_s := \sum_k e^{-Q_s(k)}$ and D_s the same sum for $n_0 = 0$. Write $k = w' + w$ with $w' \in W'$, $w = \sum_{\mathfrak{R}} a_{\mathfrak{R}} \mathbf{1}_{\mathfrak{R}} \in W$. Since $\mathbf{1}_{Fc} \partial w = 0$,

$$N_s = \sum_{w' \in W'} e^{-\frac{1}{2\tau} |\mathbf{1}_{Fc}(n_0 + \partial w')|^2} S_s(w'), \quad S_s(w') := \sum_{a \in \mathbb{Z}^{K_F}} e^{-\frac{1}{2s} |b + Ba|^2},$$

with $b := \mathbf{1}_F(n_0 + \partial w') \in \mathbb{R}^F$ and $Ba := \partial(\sum_{\mathfrak{R}} a_{\mathfrak{R}} \mathbf{1}_{\mathfrak{R}}) \in \mathbb{R}^F$. The matrix $M := B^\top B$ is positive definite because ∂ is injective. Completing the square, we get $|b + Ba|^2 = (a - a_0) \cdot M(a - a_0) + |Pb|^2$ with $a_0 := -M^{-1}B^\top b$ and $P := 1 - BM^{-1}B^\top$ the orthogonal projection onto $(\text{im } B)^\perp$. Poisson summation in a gives

$$S_s(w') = (2\pi s)^{|\mathcal{K}_F|/2} (\det M)^{-1/2} e^{-|Pb|^2/(2s)} (1 + r_s(w')),$$

$$|r_s(w')| \leq \rho_s := \sum_{0 \neq m \in \mathbb{Z}^{\mathcal{K}_F}} e^{-2\pi^2 s |m|^2 / \|M\|},$$

uniformly in w' , and $\rho_s \rightarrow 0$ exponentially. The prefactor is the same in N_s and D_s and cancels. In the remaining sums the factor $e^{-|Pb|^2/(2s)}(1 + r_s)$ tends to 1, is bounded by 2 for large s , and differs from 1 by at most $|\mathbf{1}_F(n_0 + \partial w')|^2/(2s) + \rho_s$. If $\mathbf{1}_{F^c} \partial w' = 0$ for some $w' \in W' \otimes \mathbb{R}$, then w' is constant on each component of the unfrozen graph and vanishes on the component of the ground, so $w' \in (W \otimes \mathbb{R}) \cap (W' \otimes \mathbb{R}) = \{0\}$. Hence $|\mathbf{1}_{F^c}(n_0 + \partial w')|^2 \geq c|w'|^2 - C$ for some $c > 0$, and $e^{-\frac{1}{2\tau} |\mathbf{1}_{F^c}(n_0 + \partial w')|^2}$ times any polynomial in w' is summable over W' . Dominated convergence gives (2.7), and the bound on the factor gives the rate $O(1/s)$. Since the summands of (2.7) are W -invariant functions of $k \in \mathbb{Z}^{\Lambda_3}$, the sums do not depend on the complement.

S2.2 Section 3: the loop-adapted grid

Construction of $\mathfrak{G}(C)$ (Definition 3.4). With $\Lambda^\circ = \prod_\mu [a_\mu, b_\mu]$ as in Definition 3.2, we construct the objects in the following order.

1. *Special planes.* In direction μ the special planes are the lattice planes $\{x_\mu = s\}$, $s \in \mathcal{S}_\mu$, with $\mathcal{S}_1 = \{a_1, r_1, r_1 + R, b_1\}$, $\mathcal{S}_2 = \{a_2, t_1, t_1 + T, b_2\}$, $\mathcal{S}_3 = \{a_3, z_0, b_3\}$. A *special interval* is the fine interval between two consecutive special planes of one direction.
2. *Sites.* A special interval of fine length D' is divided into $\lfloor D'/r_0 \rfloor$ consecutive intervals: $\lfloor D'/r_0 \rfloor - 1$ of length r_0 , followed, at the end with the larger coordinate, by one of length $r_0 + (D' \bmod r_0) \in [r_0, \bar{r}_0]$. The sites (coarse cells) are the boxes $\prod_\mu I_\mu$ with I_μ such intervals. Their fine sides lie in $[r_0, \bar{r}_0]$, and equal r_0 outside at most one exceptional layer per special interval. The sites form, combinatorially, a box of $\mathbb{Z}^3$. We denote by E its nearest-neighbour bonds, and the coarse face dual to a bond is the common face of its two sites.
3. *Blocks.* In coarse index units, a special interval of coarse length n is divided into $p := \lfloor n/L_c \rfloor$ consecutive intervals, and with $r := n - pL_c$ the first $(r \bmod p)$ of them have length $L_c + \lceil r/p \rceil$ and the others $L_c + \lfloor r/p \rfloor$. Their endpoints are the *block planes*. Blocks are the boxes between consecutive block planes of all three directions, and a *block face* is the common face of two adjacent blocks.
4. *Removal cells.* In each direction the block intervals, in increasing order, are grouped greedily into consecutive groups of total coarse length $\geq \ell$, and a final remainder of length $< \ell$ is merged into the last group. Removal cells Δ are the boxes $\prod_\mu J_\mu$ with J_μ such groups, and $|\partial\Delta|$ is the number of bonds of $\mathbb{Z}^3$ (in coarse index units) with exactly one end in Δ , bonds leaving Λ° included.

5. *Windows.* We index the blocks by their position in each direction. The block distance is the ℓ^∞ distance of these indices. For each block Q , $\hat{Q}(Q)$ is the union of the blocks at block distance at most 75 from Q , a block-aligned box, and $\mathcal{Q}$ is the set of windows.
6. *Tube zones and tubes.* Let F' be a block face between blocks Q, Q' , in the plane P' . Its tube zone $Z(F')$ is the set of sites of $Q \cup Q'$ among the $L_c/2$ layers nearest to P' on each side. For a bond b across F' , its tube τ_b is the segment of the line of b formed by the L_c sites of that line in $Z(F')$. We write $|F'|$ for the number of coarse bonds across F' .

The choices left open by the rules (the exceptional coarse layer at the end with the larger coordinate, and the longer block intervals first) depend only on C and the constants, and the proofs hold for any such choice.

Proof of Lemma 3.3(a) to (d). (a) The quantity M_Λ uses only R, T, r_0 and ℓ , the centre uses only C , and the grid is built from Λ° and the planes of C by rules that use only the constants.

(b) By (e) and $m \geq \min(R, T) \geq R_1 \geq L$, the special planes a_μ, b_μ are at fine distance $\geq L$ from the planes of C , and $R, T \geq R_1 \geq L$.

(c) Step 2 cuts a special interval of fine length D' into $\lfloor D'/r_0 \rfloor$ coarse intervals. Hence a side of Λ° made of at most three special intervals has at least $M'_\Lambda/r_0 - 3 \geq \ell$ coarse cells. Step 4 tiles each side of coarse length $\geq \ell$ by groups of block intervals, so Λ° is a union of removal cells. Removal cells are unions of blocks by construction. The bound $\ell \geq k_P + 4L_c$ is (3.1).

(d) Each n_μ lies in $[M'_\Lambda/r_0 - 3, M'_\Lambda/r_0]$, and $M'_\Lambda/r_0 \geq \ell + 3 \geq 6$. Hence $\max_\mu n_\mu \leq M'_\Lambda/r_0 \leq 2(M'_\Lambda/r_0 - 3) \leq 2\min_\mu n_\mu$.

Proof of (G1) to (G5). (G1) By Lemma 3.3(b), a special interval has fine length $D' \geq L = r_0 L_c$, hence coarse length $n = \lfloor D'/r_0 \rfloor \geq L_c$ and $p \geq 1$. Since $r < L_c$, each block interval has length in $[L_c, 2L_c - 1]$. A block lies in one special interval in each direction, so it meets at most one exceptional layer per direction, and its fine side lies in $[r_0 L_c, (2L_c - 1)r_0 + (r_0 - 1)] = [L, 2L - 1]$.

(G2) The construction is a tensor product, and the endpoints of special intervals are block planes.

(G3) A greedy group closes as soon as its length reaches ℓ , and a block interval has length at most $2L_c - 1$. Hence a group has length in $[\ell, \ell + 2L_c - 2]$. Adding a final remainder of length at most $\ell - 1$ keeps it at most $2\ell + 2L_c - 3 < 3\ell$, by (3.1). Every side of Λ° is $\geq \ell$ by Lemma 3.3(c), so there is at least one group per direction. Then $|\partial\Delta| = 2(s_1 s_2 + s_2 s_3 + s_3 s_1)$ with $s_\mu \in [\ell, 3\ell]$.

(G4) A window has at most 151 blocks per direction, each of coarse side at most $2L_c - 1$ by (G1), so its sides are at most $151(2L_c - 1) < k_P$. If $u \in Q$ and $|y - u|_\infty \leq 75L_c$, the segment from u to y in each direction crosses at most $1 + \lfloor (75L_c - 1)/L_c \rfloor = 75$ block planes, since consecutive parallel block planes are at least L_c apart; so the block of y is at block distance at most 75 from Q , and $Q_\infty(u, 75L_c) \cap \Lambda^\circ \subseteq \hat{Q}(Q)$. An adjacent block is at block distance 1. A block lies in $\hat{Q}(Q)$ only for the at most 151^3 blocks Q at block distance at most 75 from it.

(G5) Blocks have at least L_c layers, and parallel block planes are at least L_c apart.

S2.3 Section 5

The dual paths of Lemma 5.2. In fine coordinates with $\Xi = [0, R] \times [0, T] \times \{0\}$, the dual paths are the closed paths of fine cubes in the planes $y = j + \frac{1}{2}$ around the rectangles of cube centres $[i + \frac{1}{2}, 2R - i - \frac{1}{2}] \times [i - R + \frac{1}{2}, R - i - \frac{1}{2}]$ in (x, z) , $0 \leq j < T$, $0 \leq i < R$. These paths are disjoint, and in these coordinates the path of (i, j) crosses Ξ once, at $x = i + \frac{1}{2}$.

Proof of Lemma 5.8. By [GK04, Lemma 2.1 and the sentence after it, p. 141], the first number is at most the number $\binom{Dn}{n} / ((D-1)n+1) = \binom{Dn}{n-1} / n$ of n -vertex subtrees containing the root of the infinite rooted D -ary tree, $D := D_{\max}$, and $\binom{Dn}{n-1} / n \leq D^{n-1} n^{n-1} / n! \leq (eD)^{n-1}$ as $n! \geq e(n/e)^n$. For the second, we choose the first plaquette at an edge of C of each component among at most $4|C|$ plaquettes ($2^{4|C|}$ choices), the sizes of the components in the order of these roots (2^a), and each component by the first count with $D_{\max} = 12$.

Proof of Lemma 5.10. We put $\lambda := \log(2^{k+1} \kappa_{\text{pl}})$ and $\omega := |\log \varepsilon| / F_* \geq c_T(k) / \eta \geq c_T(k) = 1 + \lambda$. By Lemma 5.8, $\mathbb{P}_D(E) \leq 2^{4|C|+j} \sum_{a \geq RT} e^{-a(\omega-\lambda)+\omega(1-\eta)RT} \leq 2^{4|C|+j} \frac{e}{e-1} e^{RT(\lambda-\eta\omega)} \leq 2^{4|C|+j+2} e^{-RT}$, as $\omega - \lambda \geq 1$. Since $j \leq 3$, $\mathbb{E}_D e^{\sigma X} \leq e^{\sigma \eta RT} + e^{\sigma RT} \mathbb{P}_D(E) \leq e^{\sigma \eta RT} (1 + 2^{4|C|+5}) \leq e^{\sigma \eta RT + 4|C| + 4}$.

Proof of Lemma 5.14(a), (c) to (e). We fix the cut j and write $U := U_j$, $U' := U'_j$, with U^α and $\mathcal{W}(\alpha)$ as in Definition 5.13.

(a) Let (u, v) be a bond of $\mathfrak{S}_j^*$ with $u \in U^\alpha$ and $v \in W$. If $v \in U$, then v is box-adjacent to u and lies in U^α , which is impossible. Hence $v \notin U$. If (u, v) is a G'_{t_j} -bond, then $v \in \mathfrak{K}_j$ because $\mathfrak{K}_j$ is a component, so $v \in U'$ and the bond is a cut bond. Otherwise $v \in \mathbb{X}_{t_j}$ (a hole face) or the bond is frozen.

(c) Let $W \in \mathcal{W}(\alpha)$ and $z \in W \cap U'$. We take a G'_{t_j} -path in $\mathfrak{K}_j$ from z to U^α and let y be the site just before its first entry into U^α . The segment from z to y is a box path avoiding U^α , so $y \in W$. Also $y \in \mathfrak{K}_j$, and $y \notin U$ because a site of U box-adjacent to U^α lies in U^α . Thus $y \in U'$, and the bond from y into U^α is a cut bond in $\partial(U^\alpha, W)$. A bond (u, v) with $u \in U$, $v \notin U$ determines α (the box-component of u in U) and W (the box-component of v in $\Lambda^\circ \setminus U^\alpha$), so distinct pieces are disjoint. Each piece contains one of the A_j cut bonds.

(d) For a cut bond (u, z) with $u \in U^\alpha$, $z \in U'$, the box-component of z in $\Lambda^\circ \setminus U^\alpha$ belongs to $\mathcal{W}(\alpha)$. The same argument with U and U' exchanged gives the piece of $\mathfrak{S}_j^{**}$.

(e) Let u^* be the last site of U on the path, $u^* \in U^{\alpha^*}$. The rest of the path lies in $\Lambda^\circ \setminus U \subseteq \Lambda^\circ \setminus U^{\alpha^*}$ and is box-connected. Hence it lies in one box-component W^* of $\Lambda^\circ \setminus U^{\alpha^*}$, and $W^* \in \mathcal{W}(\alpha^*)$ because the path ends in U' . The bond leaving u^* lies in $\partial(U^{\alpha^*}, W^*)$.

Proof of Lemma 5.15. Shells. We partition Λ° into *shell boxes* as a tensor product, cutting each coordinate axis of Λ° into intervals of lengths in $[6L_c, 12L_c]$ (possible since every coarse side of Λ° is at least $\ell \geq 308L_c$, Lemma 3.3(c)). We index the boxes by $\iota \in \prod_\mu [0, n_\mu^{\text{sh}})$ and put $\delta(b, b') := |\iota - \iota'|_\infty$ and $n_{\min} := \min_\mu n_\mu^{\text{sh}}$. For a box b_0 and $i \geq 1$ the *shell* $\text{Sh}_i(b_0)$ is the union of the boxes b with $\delta(b, b_0) \in \{3i, 3i+1, 3i+2\}$.

Shell claim: if $\text{Sh} = \text{Sh}_i(b_0)$ with $3i+2 \leq (n_{\min}-1)/2$ contains a whole cell of U_j and a whole cell of U'_j (cells at time t_j), then each of $\mathfrak{S}_j^*$ and $\mathfrak{S}_j^{**}$ has at least $(c_1/24)L_c^2$ bonds with both ends in Sh . We write $U := U_j$. Let A' be the set of sites joined to U by a box path

with no bond of $\mathfrak{S}_j^*$, and $B' := \Lambda^\circ \setminus A'$. Then $U_j' \subseteq B'$ by Lemma 5.14(e), and every box bond between A' and B' lies in $\mathfrak{S}_j^*$. We put $W := A' \cap \text{Sh}$. The sets W and $\text{Sh} \setminus W$ contain a cell each, hence at least $c_1 L_c^3$ sites each, and it suffices to find $(c_1/24)L_c^2$ box bonds inside Sh between them. The boxes of Sh form a face-connected family: since $3i+2 \leq (n_{\min}-1)/2$, each index sphere of radius $3i$, $3i+1$, $3i+2$ around b_0 keeps, in each direction, at least one of its two faces inside the index box, so its clipped faces form a connected surface, and consecutive spheres are face-adjacent. Put $f_b := |W \cap b|/|b|$.

Case 1: $f_b \leq \frac{1}{2}$ for all boxes b of Sh . The box Cheeger inequality of the proof of Lemma 5.15 in each box (sides less than $12L_c$) gives at least $|W|/(24L_c) \geq c_1 L_c^2/24$ bonds.

Case 2: $f_b \geq \frac{1}{2}$ for all b . The same argument applies to $\text{Sh} \setminus W$.

Case 3: otherwise. There are face-adjacent boxes b_1, b_2 with $f_{b_1} \geq \frac{1}{2} > f_{b_2}$. In the box $b_1 \cup b_2$ (sides less than $24L_c$) both W and its complement have at least $\frac{1}{2}(6L_c)^3 = 108L_c^3$ sites, and the same inequality gives at least $108L_c^3/(48L_c) > 2L_c^2 > (c_1/24)L_c^2$ bonds.

The same argument with U and U_j' exchanged gives the bound for $\mathfrak{S}_j^{**}$.

Counting shells. Let b_0 be the shell box of x . A c-path in U_j from the U_j -end of the cut bond of Lemma 5.15(i) (within $\rho/2 + 2$ of x , so at index distance less than $\rho/(12L_c) + 2$ from b_0) to the far site of (ii) (at index distance more than $3\rho/(4L_c) - 1$) changes the index distance by at most 1 per step, and cells lie in one block, of side less than the box sides. Hence each shell $\text{Sh}_i(b_0)$, $i \in I(\rho) := \{i \in \mathbb{Z} : \rho/(36L_c) + 1 \leq i \leq \rho/(4L_c) - 1\}$, contains a whole cell of U_j , and likewise of U_j' . These shells are disjoint, lie within 9ρ of x , satisfy $3i+2 \leq (n_{\min}-1)/2$, and number $|I(\rho)| \geq \rho/(9L_c)$ for $\rho \geq 27L_c$. In this range the shell claim in each shell gives $(\rho/(9L_c))(c_1/24)L_c^2 = c_{\text{sh}}\rho L_c$ bonds of $\mathfrak{S}_j^*$, and as many of $\mathfrak{S}_j^{**}$.

Proof of Lemma 5.26(b). Along the breadth-first spanning tree from o (fixed orders) of the graph defining anchoring, the block set is fixed by:

- the sizes of the components (2^b choices);
- the anchors of the children of o in $Z(\Xi')$, $|Z(\Xi')| \leq 2a$, and of the children of each component C among its blocks (2^{2a+k_o} and $2^{|C|+k_C}$ choices, $k_o + \sum_C k_C = m$);
- the positions of the root blocks relative to their anchors $((2d_0+1)^3$ choices each);
- the components, $*$ -connected sets of given sizes containing their root blocks $((26e)^{|C|}$ choices, Lemma 5.8 with $D_{\max} = 26$).

With $\sum_C |C| = b$ the product is $2^b \cdot 2^{2a+b+m} (2d_0+1)^{3m} (26e)^b$.

S2.4 Full proofs for Section 6 and Appendix D

We use the conventions of Appendix D.

Proof of Proposition 6.10. In outline, diffusive displacement is needed only up to the time $T_1 = \min(M^{-2}, \varepsilon_1^{\text{HK}} R_{\text{HK}}^2)$. Beyond this time the walk is controlled by blocks of length T_1 , each of which moves it ballistically at speed of order $T_1^{-1/2}$, and the mass term outweighs the resulting growth. We write $\sum_y v_{G'}^{(M)}(x, y) e^{\theta d(x, y)} = \int_0^\infty e^{-M^2 t} \mathbb{E}_x e^{\theta d(x, X_t)} dt$. If $T_1 = M^{-2}$ then $\theta_M = \kappa_W M = \kappa_W T_1^{-1/2}$. If $T_1 = \varepsilon_1^{\text{HK}} R_{\text{HK}}^2 < M^{-2}$ then $\theta_M = \kappa_W M^2 T_1^{1/2}$. In both cases

$\theta_M \leq \min(T_1^{-1/2}, c_{\text{ex}}/2)$ (as $M \leq m_W$) and $C_M \theta_M T_1^{1/2} \leq C_M \kappa_W M^2 T_1 \leq \frac{1}{2} M^2 T_1 \leq \frac{1}{2}$. Hence (HK4) gives $\sup_w \mathbb{E}_w e^{\theta_M d(w, X_s)} \leq q := 1 + C_M \theta_M T_1^{1/2} \leq \frac{3}{2}$ for $s \leq T_1$. For $t = kT_1 + s$ with $0 \leq s < T_1$, $d(x, X_t) \leq \sum_{i \leq k} d(X_{(i-1)T_1}, X_{iT_1}) + d(X_{kT_1}, X_t)$, and the Markov property gives

$$\mathbb{E}_x e^{\theta_M d(x, X_t)} \leq q^{k+1} \leq q \exp(C_M \theta_M t / T_1^{1/2}) \leq q e^{M^2 t / 2}.$$

Hence the integral is at most $2q/M^2 \leq 3M^{-2}$, which is (6.13). At $M = M_Y$, $(\varepsilon_1^{\text{HK}})^{1/2} M_Y R_{\text{HK}} \geq (\varepsilon_1^{\text{HK}})^{1/2} K_0 / \lambda_{\text{PV}}$ by (6.12), which gives (6.14). The bound for $v_{G'}^2$ follows from $0 \leq v_{G'}^2 \leq v_{G'}^2$ (Lemma 6.2(iii)), and the bound for $v_{G'}^1$ from (6.13) at $M = M_1$, since $M \mapsto \theta_M$ is nondecreasing. Finally, $\kappa_0 \leq \kappa_W$ because $(\varepsilon_1^{\text{HK}})^{1/2} K_0 < 1$. If $(\varepsilon_1^{\text{HK}})^{1/2} K_0 \geq \lambda_{\text{PV}}$, then $\kappa_Y M_Y = \kappa_W \tilde{m} / \lambda_{\text{PV}} \geq \kappa_0 \tilde{m} / \lambda_{\text{PV}}$. Otherwise $\kappa_Y M_Y = \kappa_0 \tilde{m} / \lambda_{\text{PV}}^2$. In both cases $\kappa_Y M_Y \geq 4\kappa_0 \tilde{m}$, because $\lambda_{\text{PV}} \leq \frac{1}{4}$.

Proof of Theorem 6.11. Since C_0 is a difference of two resolvents, and hence a positive mixture of heat kernels, every bound on it reduces to the heat kernel. With $r_{\pm} := \frac{1}{2} M_Y^2 (1 \pm (1 - 4\lambda_{\text{PV}}^2)^{1/2})$, the roots of $s^2 + M_Y^2 s + M_Y^2 \tilde{m}^2$ are $-r_{\pm}$, and partial fractions give

$$\begin{aligned} C_0 &= F(-\Delta_{G'}), \quad F(s) = \frac{M_Y^2}{(s + r_-)(s + r_+)} = \varpi \left(\frac{1}{s + r_-} - \frac{1}{s + r_+} \right), \\ C_0 &= \varpi(v_{G'}^{(r_-^{1/2})} - v_{G'}^{(r_+^{1/2})}) = \int_0^\infty k(t) P_t dt, \end{aligned}$$

with $k(t) = \varpi(e^{-r_- t} - e^{-r_+ t})$ and $0 \leq k(t) \leq \min(M_Y^2 t, \varpi) e^{-r_- t}$. Since $F(0) = \tilde{m}^{-2}$, $C_0 1_{\mathfrak{R}} = \tilde{m}^{-2} 1_{\mathfrak{R}}$, that is, the zero modes are part of C_0 . Put $R := R_{\text{HK}}$, so that $R^{-3} r_-^{-1} \leq \tilde{m}^{-2} R^{-3} \leq \tilde{m} K_0^{-3} = \lambda_{\text{PV}} K_0^{-3} M_Y$ by (6.12).

(H1). With $p_t \leq 1$ on $[0, 1]$ and $p_t \leq C_H t^{-3/2} + C_H R^{-3}$ on $[1, \infty)$ (HK1),

$$\begin{aligned} C_0(x, x) &\leq \int_0^1 M_Y^2 t dt + C_H \int_1^\infty \min(M_Y^2 t, \varpi) t^{-3/2} dt + C_H R^{-3} \int_0^\infty \varpi e^{-r_- t} dt \\ &\leq \frac{1}{2} M_Y + 2C_H(1 + \varpi) M_Y + \varpi C_H \lambda_{\text{PV}} K_0^{-3} M_Y, \end{aligned}$$

splitting the middle integral at $t = M_Y^{-2}$, whatever the order of M_Y^{-2} and R^2 .

(H2). Entrywise, $C_0 \leq \varpi v_{G'}^{(r_-^{1/2})}$, because $v_{G'}^{(r_+^{1/2})} \geq 0$ (Lemma 6.2). Since $\tilde{m} \leq m_W/2$, the mass $r_-^{1/2} \leq 1.04 \tilde{m}$ is at most m_W , and (6.13) at this mass has $\theta \geq \kappa_W \tilde{m} \min(1, (\varepsilon_1^{\text{HK}})^{1/2} \tilde{m} R) \geq \kappa_0 \tilde{m}$, since $\tilde{m} R \geq K_0$ and $(\varepsilon_1^{\text{HK}})^{1/2} K_0 < 1$. Hence the row sums of $C_0 e^{\kappa_0 \tilde{m} d}$ are at most $3\varpi/r_- = K_3 \tilde{m}^{-2}$, and C_0 is symmetric.

(H1'). First, $C_0(x, y) \geq 0$ because $k \geq 0$ and $p_t \geq 0$. If $d := d(x, y) < M_Y^{-1}$, then $C_0(x, y) \leq (C_0(x, x) C_0(y, y))^{1/2} \leq K_1 M_Y \leq e K_1 M_Y e^{-\kappa'_0 \tilde{m} d}$, since $\kappa'_0 \tilde{m} d \leq \lambda_{\text{PV}} \leq 1$. If $d \geq M_Y^{-1}$, we use $C_0(x, y) \leq \varpi \int_0^\infty e^{-r_- t} p_t(x, y) dt$ and split the time axis. (i) $t \leq d$: by (HK0) the contribution is at most $\varpi d e^{-d/17} \leq 313 \varpi M_Y e^{-d/68} \leq 313 \varpi M_Y e^{-\kappa'_0 \tilde{m} d}$, because $d e^{-3d/68} \leq M_Y d^2 e^{-3d/68} < 279 M_Y$ for $d \geq M_Y^{-1}$, and $\kappa'_0 \tilde{m} \leq 1/68$. (ii) $d \leq t \leq T_2 := 2\varepsilon_1^{\text{HK}} R^2$: by (HK3) (here $t \geq d \geq 1$) and $r_- t + c_{\text{gs}} d^2 / (2t) \geq (2c_{\text{gs}} r_-)^{1/2} d$, the contribution is at most

$$\begin{aligned} \varpi C_{\text{gs}} e^{-(2c_{\text{gs}} r_-)^{1/2} d} \int_0^\infty t^{-3/2} e^{-c_{\text{gs}} d^2 / (2t)} dt &= \varpi C_{\text{gs}} (2\pi / c_{\text{gs}})^{1/2} d^{-1} e^{-(2c_{\text{gs}} r_-)^{1/2} d} \\ &\leq \varpi C_{\text{gs}} (2\pi / c_{\text{gs}})^{1/2} M_Y e^{-\kappa'_0 \tilde{m} d}. \end{aligned}$$

(iii) $t \geq T_2$: as in the proof of (HK3), $p_t(x, y) \leq 2 \sup p_{t/2} \cdot \sup_w \mathbb{P}_w(d(w, X_{t/2}) \geq d/2)$. Here $\sup p_{t/2} \leq C_H(\varepsilon_1^{\text{HK}})^{-3/2} R^{-3}$ by (HK1), and Markov's inequality with the blocking bound of the proof of Proposition 6.10 (§S2.4) at the mass $r_-^{1/2}$, with its rate $\theta \geq \kappa_0 \tilde{m}$, gives $\mathbb{P}_w(d(w, X_{t/2}) \geq d/2) \leq q e^{-\theta d/2} e^{r_- t/4}$. Thus the contribution is at most $4\varpi C_H(\varepsilon_1^{\text{HK}})^{-3/2} R^{-3} r_-^{-1} e^{-\kappa_0 \tilde{m} d/2} \leq 4\varpi C_H(\varepsilon_1^{\text{HK}})^{-3/2} \lambda_{\text{PV}} K_0^{-3} M_Y e^{-\kappa'_0 \tilde{m} d}$. The three parts give K'_1 .

(H_C). By the Schur test, $\|\nu\|_{2 \rightarrow 2} \leq \|\nu\|_0 \leq \varepsilon_\nu \tilde{m}^2$, and $C_0^{-1} \geq \tilde{m}^2$. Hence $C_0^{-1} + \nu \geq (1 - \varepsilon_\nu) C_0^{-1} > 0$, C exists, is positive definite, and $C \leq (1 - \varepsilon_\nu)^{-1} C_0$, which gives $C(x, x) \leq 2K_1 M_Y$ ($\varepsilon_\nu \leq 1/(2K_3) < \frac{1}{2}$). Since $r_{\text{cov}} \leq \kappa_0 \tilde{m}$, $\|C_0\|_{r_{\text{cov}}} \leq K_3 \tilde{m}^{-2}$, and the Neumann series $C = \sum_{n \geq 0} (-1)^n C_0 (\nu C_0)^n$ converges in $\|\cdot\|_{r_{\text{cov}}}$ with $\|C\|_{r_{\text{cov}}} \leq K_3 \tilde{m}^{-2} \sum_n (K_3 \varepsilon_\nu)^n \leq 2K_3 \tilde{m}^{-2}$. For the pointwise bound we write $C = C_0 - C_0 \nu C$ and use $e^{r_{\text{cov}} d(x, y)} \leq e^{r_{\text{cov}} d(x, z)} e^{r_{\text{cov}} d(z, w)} e^{r_{\text{cov}} d(w, y)}$ and $\sup_z C_0(x, z) e^{r_{\text{cov}} d(x, z)} \leq K'_1 M_Y$ (by (H1), (H1') and $r_{\text{cov}} \leq \kappa'_0 \tilde{m}$). This gives $|C(x, y)| e^{r_{\text{cov}} d(x, y)} \leq K'_1 M_Y (1 + \|\nu\|_{r_{\text{cov}}} \|C\|_{r_{\text{cov}}}) \leq K''_1 M_Y (1 + 2K_3 \varepsilon_\nu) \leq 2K''_1 M_Y$. A matrix C_s dominated entrywise by $|C|$, with $C_s(x, x) \leq C(x, x)$, inherits the three bounds.

Proof of Lemma D.2. The heat kernel of $\mathbb{Z}^3$ factorizes into three one-dimensional kernels, and an exponential tilt of each turns the Gaussian bound into a bound with exponential decay. The time integral then produces the factor $(1 + |x|)^{-1}$ from times of order $|x|^2$ and the decay from the mass. Let $q_t(k) := e^{-2t} I_k(2t)$ be the transition probability of the continuous-time walk on $\mathbb{Z}$ that jumps to each neighbour at rate 1, so that the kernel of $e^{t\Delta_{\mathbb{Z}^3}}$ is $p_t(x) = \prod_\mu q_t(x_\mu)$ and $v^{(M)}(x) = \int_0^\infty e^{-Mt} p_t(x) dt$. For $\lambda \geq 0$, $q_t^\lambda(k) := q_t(k) e^{\lambda k - 2t(\cosh \lambda - 1)}$ is the law at time t of $N_+ - N_-$ with independent Poisson variables of means te^λ and $te^{-\lambda}$. Conditioning on N_- , $\sup_k q_t^\lambda(k) \leq \sup_j \mathbb{P}(N_+ = j) \leq \sqrt{2}(1+t)^{-1/2}$, by Stirling's formula. The law of $N_+ - N_-$ is log-concave, hence unimodal, so $\sum_k |q_t^\lambda(k+1) - q_t^\lambda(k)| = 2 \sup_k q_t^\lambda(k)$. Writing $q_t^\lambda = q_{t/2}^\lambda * q_{t/2}^\lambda$, we obtain $\sup_k |q_t^\lambda(k+1) - q_t^\lambda(k)| \leq 8(1+t)^{-1}$. Hence, for $k \geq 0$,

$$q_t(k) \leq \sqrt{2}(1+t)^{-1/2} e^{-\lambda k + 2t(\cosh \lambda - 1)},$$

$$|q_t(k+1) - q_t(k)| \leq (8(1+t)^{-1} + \sqrt{2}\lambda(1+t)^{-1/2}) e^{-\lambda k + 2t(\cosh \lambda - 1)},$$

the second because $q_t(k+1) - q_t(k) = e^{-\lambda k + 2t(\cosh \lambda - 1)} [e^{-\lambda} q_t^\lambda(k+1) - q_t^\lambda(k)]$ and $1 - e^{-\lambda} \leq \lambda$. Negative k follow by symmetry, at the cost of a factor e^λ in the difference bound.

We let $r := |x| \geq 1$, choose $\lambda_\mu := s|x_\mu|/r$ with $s \in [0, \frac{1}{2}]$, and use $\sum_\mu 2(\cosh \lambda_\mu - 1) \leq \sum_\mu \lambda_\mu^2 \cosh \lambda_\mu \leq s^2 \cosh s$ and $\sum_\mu \lambda_\mu |x_\mu| = sr$. Then $p_t(x) \leq 2^{3/2} (1+t)^{-3/2} \exp(-sr + ts^2 \cosh s)$. For $t \leq r$ the choice $s = \frac{1}{2}$ makes the exponent at most $-r/2 + r \cosh(\frac{1}{2})/4 \leq -0.218r$. For $t \geq r$ the choice $s = r/(2t) \leq \frac{1}{2}$ makes the exponent at most $-r^2/(4t)(2 - \cosh \frac{1}{2}) \leq -0.218r^2/t$, and

$$M^2 t + 0.218 r^2/t \geq (M^2 t + \frac{9}{64} r^2/t) + 0.077 r^2/t \geq \frac{3}{4} M r + 0.077 r^2/t.$$

Hence

$$\begin{aligned} v^{(M)}(x) &\leq 2^{3/2} e^{-0.218r} \int_0^r (1+t)^{-3/2} dt + 2^{3/2} e^{-3Mr/4} \int_0^\infty t^{-3/2} e^{-0.077r^2/t} dt \\ &\leq 2^{5/2} e^{-0.218r} + \frac{2^{3/2} \sqrt{\pi}}{\sqrt{0.077} r} e^{-3Mr/4}, \end{aligned}$$

and $e^{-0.218r} \leq e^{-3Mr/4} e^{-0.068r}$ because $\frac{3}{4}M \leq 0.15$. This gives the first bound in (D.1). For the difference in direction $e = \pm e_\mu$ only the factor $q_t(x_\mu)$ changes, and the same choices of λ give $|p_t(x+e) - p_t(x)| \leq C(1+t)^{-3/2}((1+t)^{-1/2} + s) \exp(-sr + ts^2 \cosh s)$. For $t \leq r$ this is bounded as before. For $t \geq r$, $s = r/(2t)$, and $\int_0^\infty (t^{-2} + rt^{-5/2}) e^{-0.077r^2/t} dt \leq Cr^{-2}$. For $r < 1$ both bounds are trivial.

For the box, the reflections in the six face planes (at half-integer positions) generate a group $\mathcal{G}$ that acts coordinatewise, and $v_{\Lambda'}^{(M)}(y, x) = \sum_{g \in \mathcal{G}} v^{(M)}(y - gx)$. Indeed, the right side is $\mathcal{G}$ -invariant, so across each face its two values agree and the Neumann condition holds, and the orbit of x meets Λ' only at x . Coordinatewise $|y_\mu - (gx)_\mu| \geq |y_\mu - x_\mu|$ for $x_\mu, y_\mu \in [1, n_\mu]$, so $|y - gx| \geq r_{yx} := |y - x|$, and $\#\{g : |y - gx| \leq \rho\} \leq 8(\rho M + 1)^3$ because $n_\mu \geq M^{-1}$. For a decreasing f , summation by parts gives $\sum_g f(|y - gx|) \leq 8[(1 + Mr_{yx})^3 f(r_{yx}) + \int_{r_{yx}}^\infty 3M(1 + M\rho)^2 f(\rho) d\rho]$. With $f(\rho) = C_L(1 + \rho)^{-j} e^{-3M\rho/4}$, $j \in \{1, 2\}$, this is at most $C(1 + Mr_{yx})^3(1 + r_{yx})^{-j} e^{-3Mr_{yx}/4} \leq C'(1 + r_{yx})^{-j} e^{-Mr_{yx}/2}$. Since the differences $u_x(y + e) - u_x(y)$ are the sums of the differences of the images, (L1) and (L2) follow.

S2.5 Full proofs for Subsection 7.5

Proof of Lemma 7.24. The integrand of Proposition 7.7 depends on the configuration only through the field $\psi + g + k\beta_{\text{GM}}^{-1/2}f$, read modulo $P\mathbb{Z}$ at each site and in each cell average, and through the walls. We write $\vartheta := \psi + D$.

(a) Let $h \in \mathcal{H}$ and $k \in \{0, 1\}$. By (7.10), $g + k\beta_{\text{GM}}^{-1/2}f = h + D$. Since h takes values in $P\mathbb{Z} = 2\pi\beta_{\text{GM}}^{-1/2}\mathbb{Z}$, $e^{im\beta_{\text{GM}}^{1/2}h(x)} = 1$, so each phase $e^{im(\beta_{\text{GM}}^{1/2}(\psi+g)(x)+kf(x))}$ of E' , ε_k , $G_{2,2}$ and $G_{2,3}$ equals $e^{im\beta_{\text{GM}}^{1/2}\vartheta(x)}$. In $G_1 = \sum_\Omega \ln r_\Omega(A_\Omega)$ the function r_Ω of (7.4) depends on A_Ω only through $A_\Omega + k\beta_{\text{GM}}^{-1/2}\bar{f}_\Omega$ and is P -periodic in it. This quantity is $\text{avg}_\Omega \vartheta + h|_\Omega$, as h is constant on Ω . Hence $r_\Omega(A_\Omega)$ is r_Ω at $k = 0$ evaluated at $\text{avg}_\Omega \vartheta$. Likewise $\tilde{\delta} = (\psi + g + k\beta_{\text{GM}}^{-1/2}f) - \text{avg}_\Omega(\psi + g + k\beta_{\text{GM}}^{-1/2}f) = \vartheta - \text{avg}_\Omega \vartheta$ on Ω . The terms added in $G^\natural$ (Lemma 7.29) are functions of ϑ and ψ through the same phases. The term $F_1(S_\beta)$ is given by (7.13) and $F_2(Y) = \langle \mathbf{1}_Y \psi, F'_2 \rangle$ with $F'_2 = \sum_\beta F'_{2,\beta}$, $F'_{2,\beta} = -[C_0^{-1}, \chi_\beta] C_0 s_\beta$ (Proposition 7.8 (e)), both functions of w alone. Hence the two definitions of $Q_N(Y)$ and of $E(Y, s)$ agree.

(b) After the replacement, the action is a sum of three kinds of terms: cell terms $\ln r_\Omega(\text{avg}_\Omega \vartheta)$; site terms, which are the terms of $G_{2,1}$, $G_{2,2}$, $G_{2,3}$, of $G^\natural$ and of E , functions of ψ and ϑ at the sites of their support (the E -terms with weights $w_s(\tau)$); and wall-component terms $F_1(S_\beta)$ and $\langle \psi, F'_{2,\beta} \rangle$, supported in S_β (Proposition 7.8 (e)). On a union Y of wall components and of $\mathcal{U}$ -cells outside $\hat{\Sigma}$, $D|_Y = \sum_{S_\beta \subseteq Y} D_\beta|_Y$ depends on N only through $N|_{E(Y)}$ (Proposition 7.8 (f)). These are the only facts used in the proof of Lemma 7.21. An admissible set contains each cell and each wall component entirely or not at all, which gives the factorization and the locality for every wall field. Definition 7.20 and the proof of Lemma 7.23 use only the wall sites, the wall components and Proposition 7.8 (a), all of which are defined from N . On $U \neq V$ we take closures in U , which is consistent because the wall polymers of N have their closures in U . Proposition 7.22 is proved in Appendix C from the factorization, the admissibility of the sets of a sequence and absolute convergence. Absolute convergence rests on Lemma C.1, whose bounds are pointwise in (ψ, ϑ) , and on the Poincaré inequality on cells (Lemma C.2), which does not involve the configuration. Here Lemma C.1 (b) is a bound on $G_{2,1}$ by $\|\mathbf{1}_Y P_V \vartheta\|^2$. In Theorem 7.13, the bound (7.14) of Corollary 7.10 (c) with $g - H = D$

and $\nabla g + k\beta_{\text{GM}}^{-1/2}R_f = w + \nabla D$ reads $F_1^{\text{loc}}(Y) \geq \frac{1}{2}\tilde{m}^2\|D\|_Y^2 + \frac{1}{2}\|w + \nabla D\|_{E(Y)}^2$, which follows from (7.13) by dropping the term with M_Y^{-2} , the supports of different β being disjoint. In the same theorem, the tube argument uses only that w vanishes on the bonds inside cells and is constant on interfaces (Lemma 7.12). The proof of Proposition 7.5 bounds $\cos(mu) - 1$ at each site, with the real number $u = \beta_{\text{GM}}^{1/2}(\varphi + k\beta_{\text{GM}}^{-1/2}f - h)(x)$, which after the translation $\varphi = \psi + g$ equals $\beta_{\text{GM}}^{1/2}\vartheta(x)$, and that proof never uses that u comes from a height.

S2.6 Full proofs for Subsections 7.7 to 7.9

Proof of Proposition 7.46. The bounds of Subsection 7.6 and Appendices A, B are stated for the configurations $\mathcal{N}(X)$, $\mathcal{N}_\sigma(X)$ produced by labels. We check that each step uses the label only in ways that survive for long wall fields. To this end we fix $k = 0$ and a configuration $N_X \in \mathcal{N}^{\text{long}}(X)$ (or $\mathcal{N}_\sigma^{\text{long}}(X)$), with the integrand of Lemma 7.24.

(i) *Sums over configurations.* They enter only through the following. Lemma 7.30 (b) maps N_X to its interface values $(N_I)_{I \subseteq X} \in \mathbb{Z}^{n_I(X)}$ and drops the realizability constraints. A wall field is constant on interfaces and vanishes off them. Hence the map is injective on $\mathcal{N}^{\text{long}}(X)$, and the bound holds verbatim. In the proof of Proposition A.8, step (2) splits by the first set $\text{sat}_{N_X}(S_0)$ through Lemma 7.23, valid in wall form (Lemma 7.24 (b)), and the seed configuration it produces is long (Lemma 7.26 (a)). Step (3) of that proof counts wall polymers with their walls through their wall interfaces and values at the Euclidean link scale g (Lemma 5.8), again a count of integer functions on interfaces, with $F_1^{\text{loc}}(\mathcal{S}) \geq 2F_{\text{unit}} \sum_I N_I^2$ (Lemma 7.30 (a)). The sum over link data and sets in Propositions A.4 and A.7 does not involve the configuration beyond these.

(ii) *Bounds at a fixed configuration.* Each is a bound on a Gaussian integral whose integrand is a function of ψ , D , w and F'_2 , that is, of ψ and N_X (Lemma 7.24). The label enters these proofs only in the following places.

- Appendix B uses the label in the phases of class-A cells, through $e^{im\beta_{\text{GM}}^{1/2}h(x)} = 1$. Under Lemma 7.24 the phase is $e^{im\beta_{\text{GM}}^{1/2}\vartheta(x)}$ by definition. The same holds for the cell function $\hat{R}_v$ of Lemma 7.31, evaluated at $\text{avg}_\Omega(\psi + D)$, and for Lemma 7.41, where $D = 0$ on the wall-free cell Δ .
- Theorem 7.13, used through Lemma 7.30 (a): valid for every wall field (Lemma 7.24 (b)).
- Lemma 7.33, Step 1: the identity (7.28) is stated in terms of D_Y , w_Y and s_Y for unions of elements and involves no extension of h . At $k = 0$ the near-cell term vanishes.
- Lemma 7.29 and the functions $G^\natural$: functions of $\vartheta = \psi + D$ and ψ , bounded pointwise.
- Lemma 7.32 (a), used in Step 2 of Lemma 7.33. Its proof first shows $w = w_Y$ on $E(\mathfrak{K}_\Omega^\flat)$, using (L8-4) and Proposition 7.8 only, and then puts $a_\Omega := h$ on $\mathfrak{K}_\Omega^\flat$, using $P_\Omega h = 0$ and $\nabla h = w$ on $E(\mathfrak{K}_\Omega^\flat)$. For a long configuration, the field $N_X \mathbf{1}_{E(Y)}$ on the union Y of elements is long (Lemma 7.26 (a)), and Lemma 7.45 gives $h' : \mathfrak{K}_\Omega^\flat \rightarrow P\mathbb{Z}$, constant on Ω , with $\nabla h' = w_Y$ on $E(\mathfrak{K}_\Omega^\flat)$. We apply Proposition 6.9 (iv) to $\vartheta + h'$. Since $\nabla(\vartheta + h') = \nabla\vartheta + w_Y$ on $E(\mathfrak{K}_\Omega^\flat)$ and $P_\Omega(\vartheta + h') = P_\Omega\vartheta$, we obtain $\|P_\Omega\vartheta\|^2 \leq C_W^f k_\square^2 \|\nabla\vartheta + w_Y\|_{E(\mathfrak{K}_\Omega^\flat)}^2$, which is (a) at $k = 0$. Step 2 of Lemma 7.33 then runs verbatim.

Absolute convergence at a fixed configuration (Proposition 7.22) rests on Lemma C.1 and the Poincaré inequality on cells, which are pointwise in (ψ, D) . Lemma 7.32 (a) is the only place where the bounds use that the configuration is long: for a wall field with a nonzero period on a short cycle of $\mathfrak{K}_\Omega^b$, or on a same-cell closure through Ω , no primitive of w_Y constant on Ω exists. With (i) and (ii), the proofs of Propositions 7.35, 7.39, Lemmas 7.37, 7.38 and Proposition A.8 hold with the same constants, and so does Lemma 7.48.

S2.7 Full proofs for Appendix A

Proof of Proposition A.8, step (3). The notation is that of Lemma 7.48 and Proposition A.8. Since $Z^o \subseteq S_0$ and $11c'/16 - \varepsilon \geq c'/8$, we put $w(\mathcal{S}) := e^{-(c'/8)F_1^{\text{loc}}(\mathcal{S}, N_S)}$, and by steps (1) and (2) the left side of Lemma 7.48(a) is at most $\mathcal{P}' e^{c_S|S_0|} \sum_\sigma \prod_{\mathcal{S}} w(\mathcal{S})$, the sum over $\sigma \in \Sigma_{\text{sat}}(S_0)$ and the product over the wall polymers $\mathcal{S}$ of σ . Assigning the disjoint wall polymers of σ to their first cells in S_0 bounds the sum by $\prod_{\Delta \subseteq S_0} (1 + \sum_{\mathcal{S} \ni \Delta} w(\mathcal{S}))$, and

$$\sum_{\mathcal{S} \ni \Delta} w(\mathcal{S}) \leq 6D_I e^{-(c'/4)F_{\text{unit}}} = \eta_{\text{seed}}$$

under $(\beta 3)$. Indeed, a wall polymer with its walls is determined by its wall interfaces and their values (Lemmas 7.6(a), 7.23), a set connected at Euclidean scale g containing one of the D_I interfaces within distance g of Δ , so the lattice-animal count of Lemma 5.8 and $F_1^{\text{loc}}(\mathcal{S}) \geq 2F_{\text{unit}} \sum_I N_I^2$ (Lemma 7.30(a)) give $\sum_{n \geq 1} D_I (eD_I)^{n-1} (3e^{-(c'/4)F_{\text{unit}}})^n$. This gives (a). In (b), the terms with $\sigma = (S_0, 0)$ and $X \neq S_0$ gain c_{16}^D , and those whose σ contains a wall polymer contribute $(1 + \eta_{\text{seed}})^{|S_0|} - 1 \leq \eta_{\text{seed}} e^{2|S_0|}$, so the sum without the term $X = S_0$, $N_X = 0$ is at most $(c_{16}^D + \eta_{\text{seed}}) \mathcal{P}' e^{(c_S+2)|S_0|} = c_{16} \mathcal{P}' e^{c_2|S_0|}$. In (c'), $\mathcal{P}' = 1$, $Z^o = \emptyset$, and (A.10) with $\varepsilon \leq c'/8$ and $c_w = c'/16$ leaves $e^{-(c'/2)F_1^{\text{loc}}(Z, N_Z)}$.

S3 Numerical checks

S3.1 Numerical checks for Sections 2, 6, 7, 8 and Appendix A

In this subsection we report computations that illustrate statements of the paper, and no proof depends on them. Each number comes from a script of the archive released with the paper (file names in brackets), in multiprecision arithmetic (30 to 60 significant digits) for the Bessel-function checks, in exact rational arithmetic for the tree-graph integrals, and in double precision for the lattice computations.

S3.1.1 Bessel functions and the Hartman–Watson law

Complete monotonicity (Lemma 2.2) [`hw.tail.py`]. For $g(\lambda) := I_{\sqrt{2\lambda}}(\beta)/I_0(\beta)$ and the step $h = \beta^{-2}$, the forward differences satisfy $(-1)^k \Delta_h^k g(0) \geq 0$ for all orders $k \leq 21$, at $\beta = 2, 10$ and 40, as [HW74, Theorem 1.3] requires.

The ratio bound (proof of Lemma 2.3) [`bessel.turan.py`]. The inequality $I_1(x)/I_0(x) \leq x/(x + \frac{1}{2})$, a consequence of [Amo74, (16)], holds at the twelve points $x = 10^e$, $e \in \{-3, -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, \dots, \frac{7}{2}\}$.

The Villain coupling matched to the Wilson weight [`dual.weights.py`]. With $t_V(\beta) := 1/(2 \log(I_0(\beta)/I_1(\beta)))$, the Villain weight $e^{-n^2/(2t_V)}$ has the same ratio at $n = 1$ as $I_n(\beta)/I_0(\beta)$.

One finds $\beta - t_V(\beta) = 0.6106, 0.5650, 0.5246, 0.5112, 0.5054$ at $\beta = 2, 5, 10, 20, 40$, in line with $t_V = \beta - \frac{1}{2} + O(1/\beta)$. At each of these β the Wilson weight exceeds the matched Villain weight already at $n = 2$, the first sign of its heavier tail. At the matched coupling the bound of [GM82] is of order $e^{-2.494\beta}$, against the exponent $2.49436K\beta$ with $K > K_*$ of (1.3): the constants of Theorem 1.2 are far from sharp.

S3.1.2 Villain monotonicity

Monotonicity in each coupling [guth_check.py]. We test Lemma 2.3(b) on the complex of two adjacent unit cubes (eleven plaquettes), with C the boundary of an outer face, using the dual representation (2.6) truncated at $|k| \leq 40$. For 400 random choices of the eleven couplings in $[0.2, 6]$, raising any one coupling by 30% never lowered $\langle \cos \theta_C \rangle$ (0 decreases in 4400 trials).

S3.1.3 The density of frozen plaquettes

The exponent of δ (Lemma 2.6) [delta_table.py]. Table S3.1 compares $-\beta^{-1} \log$ of the bound of Lemma 2.4(iv), minimized over ν without restriction (golden-section search on $[\frac{1}{2}a_*\beta, \frac{3}{2}a_*\beta]$), with the lower bound $c_\delta^{\text{sp}}(K) - 1/(4a_*)$ of Lemma 2.6. The minimizer is $\nu = a_*\beta$ to the displayed precision, and the finite- β values approach $c_\delta^{\text{sp}}(K) - 1/(4a_*)$ from below at the rate of the $O(\log \beta/\beta)$ term of (2.4). The threshold values of Lemma 2.6 [kstar.py] are recorded in Section S1.1.2.

Table S3.1: The exponent $-\beta^{-1} \log(\min_\nu e^{\nu^2 \varepsilon/2} \sum_n I_{\sqrt{n^2 + \nu^2}}(\beta)/I_0(\beta))$ at $\varepsilon = 1/(K\beta)$.

K	a_*	$c_\delta^{\text{sp}}(K) - \frac{1}{4a_*}$	$\beta = 10$	$\beta = 20$	$\beta = 40$
5	17.886	15.090	14.948	15.002	15.037
10	44.998	57.236	57.105	57.153	57.186
15	75.199	114.295	114.170	114.215	114.247
20	107.393	181.935	181.814	181.857	181.887

The Debye exponent [watson.py]. The closed form of $v_{\text{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0,0)$ given in item 0 of Section S1.1.1 and the integral $\int_0^\infty e^{-6t} I_0(2t)^3 dt$ agree. Both give $v_{\text{Cb}}(0) = 0.2527310098587$, hence $\pi^2 v_{\text{Cb}}(0) = 2.4943550872$ and $(8\pi^2)^{1/2}/2^{16} = 1.35586 \cdot 10^{-4}$, the numbers used in (8.1).

S3.1.4 The free-space field of the near-loop estimate

The solid-angle primitive (Lemma 7.32) [lemmaN_rectangle.py, slit_nearcell.py]. For the square loop $C = \partial([0, 64]^2 \times \{0\})$ and the flat surface Ξ it bounds, we evaluate the free-space solid angle $f^C = \frac{1}{2}\Omega_\Xi$ on the dual lattice of a box with margin 24 around C (602,112 dual sites). Its largest modulus is $3.0974 < \pi$. The residual R^C satisfies the bound $|R^C(b)| \leq 2\pi/d(b, C)$ of Lemma 7.32(b), with exact segment-to-segment distances, on all 1,031,184 dual bonds within distance 21 of C . Over these bonds the largest ratio $|R^C(b)| d(b, C)/(2\pi)$ is 0.3415. The bound $|R^C(b)| \leq \frac{1}{2} \sum_e d(b, e)^{-2}$, summed over the unit edges e of C , holds on the same bonds with largest ratio 0.8645. In a near box of side 32 centred on a side of C , we build eighteen slit geometries (cracks parallel and perpendicular to Ξ , holes, windows, frozen pieces of Ξ and combs of frozen sheets) by deleting dual bonds and sites. On the eighteen slit geometries and

the free box this bound holds with largest ratio 0.8303, and on each slit geometry the energy of f^C on the remaining bonds, as well as the minimum of $\sum_b (\nabla a(b) - j(b))^2$ over fields a on the slit graph, is at most the energy of f^C on the free box. Since freezing only removes terms from sums of fixed free-space quantities, slit edges do not enter the near-loop estimate.

S3.1.5 Further checks

The tree-graph s-integral (Lemmas A.2 and A.3) [`s_integral/tree_s_integral_check.py`]. In exact rational arithmetic, with the s -integrals evaluated as exact polynomial integrals, for $n = 2, \dots, 7$: $\sum_T w(T) = n^{n-2}/(n-1)!$; for random rational $y \geq 0$ the left side of (A.4) equals the volume of the polytope $\{0 \leq u_m \leq y_m + u_{m-1}, 1 \leq m \leq n-1\}$ ($u_0 := 0$) and is at most $e^{y_1 + \dots + y_{n-1}}$; Lemma A.3 holds for random y_1 and $b_i \in [0, 0.9]$; and $w(T) = 1/(n-2)!$ for the tree with $T(2) = 1$ and $T(j) = 2$ for $j \geq 3$.

Corner sites (Remark 6.7) [`corner_pair.py`, `corner_pair_hp.py`]. With $v(x) = \int_0^\infty e^{-6t} \prod_\mu I_{x_\mu}(2t) dt$ (30 digits), we obtain $v(1, 1, 0) = 0.0551914336877$ and $v(1, 1, 1) = 0.0435783543977$. The corner image sum is 0.4673457, which gives $\Phi = 9.22503\tau$, and $E_{\text{pair}} = 0.52927 < v_{G'}(x, x) = 0.72008$ in the limit $M_Y \rightarrow 0$ ($0.52919 < 0.70782$ at $M_Y = 0.02$).

The field near the loop (Lemma 7.32) [`biot_savart_shell.py`]. On five rectangles (10^5 sample points), $J(y) = \oint |dl|/|y-l|^2$ satisfies $J(y) d(y, C)/(4\pi) \leq 0.495$. On a 64×64 rectangle the dual-bond shell sums of $d(b, C)^{-2}$ are at most 0.19 times the bound of Lemma 7.32(c) at $D = 4, 8, 16$.

Kruskal's identity (Lemma 7.51) [`kruskal_check.py`]. On 9000 random point sets the identity holds without exception.