

A Lean 4 formalization of the area law for three-dimensional $U(1)$ lattice gauge theory with the Wilson action

Tejasvi Singh Tomar
ORCID 0000-0003-4668-3639

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Abstract

The paper [U1] proves that the Wilson loop of three-dimensional $U(1)$ lattice gauge theory with the Wilson action obeys an area law at every coupling $\beta > 0$, uniformly in free finite boxes, with an explicit string tension for large β . We describe a formalization of this theorem in Lean 4 with Mathlib. The Lean statement of the main theorem quantifies over all rectangles in coordinate planes and all finite boxes containing them, with the perimeter constant chosen before the coupling, as in the paper. It has no hypotheses, and its proof uses only the three standard axioms of Lean and Mathlib. The formal proof follows the paper section by section. Where the paper carries a step over from the literature “with changes”, that step is written out in full in a separate document, refereed in a separate session and formalized from that document. We also formalize the corollaries for compact gauge groups containing the scalars and for the Georgi–Glashow model at infinite Higgs coupling. The formalization has 215,711 lines of Lean in 862 files, builds from a clean checkout in 22 minutes on a shared many-core server, and certifies every numerical constant that the proof uses. It also yields a list of corrections to the paper, none of which changes a main result.

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1 Introduction

Let $\Lambda \subset \mathbb{Z}^3$ be a finite box, $\beta > 0$, and let $\langle \cdot \rangle_{\Lambda, \beta}^W$ be the $U(1)$ Wilson measure on Λ with free boundary conditions. For a rectangle C in a coordinate plane, with sides R and T , write θ_C for the sum of the link angles around C . We first recall the main theorem of [U1].

Theorem 1.1 ([U1, Theorem 1.2]). *There is a constant c such that for every $\beta > 0$ there are $\sigma(\beta) > 0$ and $R_1(\beta)$ with*

$$\langle \cos \theta_C \rangle_{\Lambda, \beta}^W \leq \exp(-\sigma(\beta)RT + c(R + T))$$

for every rectangle C in a coordinate plane with $\min(R, T) \geq R_1(\beta)$ and every finite box $\Lambda \supseteq C$. Moreover, for every $K > K_* = 377.38$ there is $\beta_0(K)$ such that for $\beta \geq \beta_0(K)$ one can take $\sigma(\beta) \geq 1.35 \cdot 10^{-4} c_g c_L (K\beta)^{-1/2} e^{-2.49436K\beta}$, with $c_g = \frac{1}{4}$ and $c_L = 1.6 \cdot 10^{-19}$.

The paper also deduces an area law for compact gauge groups $G \subseteq U(n)$ that contain the scalars, with or without a Higgs field blind to the centre ([U1, Corollary 1.3]), and for the Georgi–Glashow model at infinite Higgs coupling ([U1, Corollary 1.4]).

We formalize Theorem 1.1 and both corollaries in Lean 4 [Lean4] with Mathlib [Mathlib]. Section 2 gives the Lean statements and explains why they say what the paper says.

Theorem 1.2 (formal). *The file `U1/MainTheorem.lean` proves, without hypotheses:*

- `U1.Main.thm_1_2` : `U1.AreaLaw` and `U1.Main.thm_1_2_explicit` : `U1.AreaLawExplicit`, the two assertions of Theorem 1.1;
- `U1.Main.rem_8_1`, the infinite-volume form of the area law ([U1, Remark 8.1]);
- `U1.Main.cor_1_3_rel` : `U1.Cor13`: for every closed subgroup $G \subseteq U(n)$ that contains the scalars, with or without a Higgs weight blind to the centre, the area law of the $U(1)$ model at coupling $2n\beta$ implies the area law for G at β with the same constants; with `thm_1_2` this is Corollary 1.3, with constants that do not depend on G ;
- `U1.Main.cor_1_4_rel` : `U1.Cor14`: the Georgi–Glashow limit exists in every finite box, and the area law of the $U(1)$ model passes to it with the same constants; with `thm_1_2` this is Corollary 1.4;
- `U1.Main.prop_9_1` : `U1.Prop91`, Fröhlich’s comparison (Proposition 9.1).

The file also contains `U1.Main.cor_1_3` and `U1.Main.cor_1_4`, weaker forms in which σ and R_1 are chosen after G , or for the limit model. For each of these theorems, `#print axioms` lists `propxt`, `Classical.choice` and `Quot.sound` and nothing else.

The proof in [U1] has five steps. A reduction (Section 2) first bounds the Wilson loop by an average of frozen Villain expectations over random sets of frozen plaquettes. A deterministic regularization (Sections 3–4) then turns each frozen set into a slit domain with good geometry. A contour bound (Section 5) controls the minimal cut area of the slit domain. On each slit domain, two stages of a renormalization-group analysis (Sections 6–7) prove an area law. They build on the Debye-screening expansion of Brydges and Federbush [BF80] and on the multiscale analysis of G pfert and Mack [GM82]. The chain of Section 8 of [U1] assembles these bounds, and its Section 9 deduces the corollaries.

Besides the formal proof itself, this note records the following.

- A statement of Theorem 1.1 in Lean that keeps the paper’s quantifier order and its uniformity in the box (Section 2), checked against the printed statement by the fidelity audit of Section 7.
- An architecture of explicit interfaces between the parts of the proof (Section 3). The interfaces are Lean structures or propositions. All of them are discharged by proved theorems, and an audit checks that their hypotheses can be satisfied.
- Written-out proofs of the steps that [U1] takes from the literature with changes (Section 5). They total 233 pages, and each comes with a referee report.
- Certified numerics for all the constants in the proof, with kernel-checked computations (Section 6).
- A list of corrections to [U1]: errors, gaps and unstated hypotheses, each with the fix used in the formal proof (Section 8).

Section 4 goes through the formalization part by part, Section 7 describes the audits, and Section 9 gives the statistics and the instructions to reproduce the build.

2 The Lean statements

We state the main definitions in Lean syntax, with mathematical symbols for readability. All names are in the namespace U1.

Lattice and model. Sites are points of $\mathbb{Z}^3$. Bonds, plaquettes and cubes carry their orientations and incidence numbers, and the boundary maps satisfy $\partial\partial = 0$ (bd1_bd2). A box is a product of three integer intervals, and `Box.complex` is the finite 2-complex of its bonds and plaquettes. The angle group is $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. On $\mathbb{T}^E$, `FinComplex.wilsonExp` is the expectation for the probability measure proportional to $\prod_p e^{\beta \cos \theta_p} d\theta$, normalized so that it is 1 on constants (`wilsonExp_const`). A rectangle `Rect` lies in a coordinate plane and has sides `C.R`, `C.T`. Its boundary C is `C.chain`, traversed along the first coordinate direction first (`Rect.chain_eq_loop`). The predicate `C.InBox B` says $\Lambda \supseteq C$. Then `B.wilsonLoop  $\beta$  C.chain` is $\langle \cos \theta_C \rangle_{\Lambda, \beta}^W$ with free boundary conditions.

In Lean, the first assertion of Theorem 1.1 reads

```
def AreaLawAt ( $\beta$   $\sigma$  :  $\mathbb{R}$ ) ( $R_1$  :  $\mathbb{N}$ ) ( $c$  :  $\mathbb{R}$ ) : Prop :=
   $\forall$  C : Rect,  $R_1 \leq C.R \rightarrow R_1 \leq C.T \rightarrow \forall$  B : Box, C.InBox B  $\rightarrow$ 
    B.wilsonLoop  $\beta$  C.chain  $\leq$  Real.exp ( $-\sigma * C.R * C.T + c * (C.R + C.T)$ )
```

```
def AreaLaw : Prop :=
   $\exists c : \mathbb{R}, \forall \beta : \mathbb{R}, 0 < \beta \rightarrow \exists \sigma : \mathbb{R}, 0 < \sigma \wedge \exists R_1 : \mathbb{N}, \text{AreaLawAt } \beta \sigma R_1 c$ 
```

and the second reads

```
def AreaLawExplicit : Prop :=
   $\exists c : \mathbb{R}, \forall K : \mathbb{R}, K_{\text{star}} < K \rightarrow \exists \beta_0 : \mathbb{R}, 0 < \beta_0 \wedge$ 
   $\exists \sigma : \mathbb{R} \rightarrow \mathbb{R}, \exists R_1 : \mathbb{R} \rightarrow \mathbb{N},$ 
   $(\forall \beta, \beta_0 \leq \beta \rightarrow$ 
     $1.35\text{e-}4 * (1/4) * 1.6\text{e-}19 * (K * \beta) ^ {-(1:\mathbb{R})/2})$ 
     $* \text{Real.exp } (-2.49436 * (K * \beta)) \leq \sigma \beta) \wedge$ 
   $\forall \beta, 0 < \beta \rightarrow \text{AreaLawAt } \beta (\sigma (\max \beta \beta_0)) (R_1 (\max \beta \beta_0)) c$ 
```

with $K_{\text{star}} = 377.38$. We check four points of fidelity. In the order of the quantifiers, the perimeter constant c comes before β , and before K in the second assertion, as in [U1]; a statement with c depending on β would be much weaker. The statements are also uniform in the box: **AreaLawAt** quantifies over all rectangles with $\min(R, T) \geq R_1$ and all boxes that contain them, with free boundary conditions, so no infinite-volume limit or boundary condition enters the statement. In the explicit string tension, the constants are those of [U1, Theorem 1.2] with $c_g = \frac{1}{4}$ and $c_L = 1.6 \cdot 10^{-19}$ substituted, and for $\beta < \beta_0(K)$ the bound holds with the values chosen at $\beta_0(K)$, which is the paper’s convention. Finally, the two assertions are consistent: **AreaLawExplicit.toAreaLaw** proves that the second assertion implies the first with the same c , using $K = 378$.

For the corollaries, **U1.Cor13** states Corollary 1.3 relative to the area law: for every closed subgroup $G \subseteq U(n)$ that contains the scalars and every $\beta > 0$, the area law of the $U(1)$ model at coupling $2n\beta$ with constants (σ, R_1, c) implies $|\langle \text{Tr } U_C \rangle| \leq 2n \exp(-\sigma RT + c(R + T))$ for the G model at β . The Higgs field takes values in any probability space, enters through a homomorphism ρ of G that is trivial on the scalars, and has any nonnegative integrable weight with positive integral. Similarly, **U1.Cor14** states Corollary 1.4: in every finite box the $SU(2)$ model with a unit adjoint Higgs field converges, as the hopping coupling $\kappa \rightarrow \infty$, to the $U(1)$ Wilson model in the sense of expectations of continuous gauge-invariant functions (Proposition 9.3), and its Wilson loops inherit the area law (Corollary 9.4). **Cor13.of_areaLaw** and **Cor14.of_areaLaw** combine these with **AreaLaw**.

Two of our statements are more general than the printed ones, but the paper’s proofs cover them.

- **U1.Prop91** holds on every finite 2-complex whose plaquettes are attached along closed edge paths, for every closed edge path ℓ , and with the Higgs weight of Corollary 9.2 included. On a box with weight 1 it is Proposition 9.1, and on a box with a general weight it is the first assertion of Corollary 9.2.
- In Corollaries 1.3 and 9.2 the map ρ is any group homomorphism trivial on the scalars, where the paper has a representation, and only the composite $(U, \varphi) \mapsto h((\rho(U_e))_e, \varphi)$ needs to be measurable and integrable. The paper’s proof uses only $\rho(zU) = \rho(U)$ for central z .

The fidelity audit of Section 7 checks these statements against the printed ones, and its reports are in the directory **audit/** of the archive [repo]. Besides the comparison, the audit

checks that no statement is true for a definitional reason: the expectations are normalized and take no junk values under the hypotheses, and the bounds are not automatic, since $|\langle \text{Tr } U_C \rangle|$ can be close to n . As the partition function is positive (`integral_wilsonWeight_pos`), no expectation degenerates. The statements involve only the definitions of the models; in particular, no parameter of the proof, such as λ_{PV} , μ , κ_S or the construction, enters a statement. For the main theorems the final audit, also described in Section 7, re-checks the statements from scratch. Throughout the library, each Lean theorem that corresponds to a paper result cites that result by its printed number in its docstring.

3 Architecture

We organize the formal proof as a chain of interfaces. An interface is a Lean structure or proposition that collects the results passed from one part of the proof to another. The passage from one interface to the next is always a proved theorem, and no axiom is added at any point. Hypotheses enter only as arguments of theorems, never as global assumptions, and the final theorems have none.

The first interface is the structure `U1.R0`. For one deterministic construction, it packages the results of [U1] on which the chain of its Section 8 relies:

- the construction: for each set D of frozen plaquettes, a box, a hole set and a set of frozen bonds, with Theorem 3.20 (R1) and Lemma 3.3(e);
- the screening mass $\tilde{m}$, with $m_D/2 \leq \tilde{m} \leq 2m_D$ for large τ ;
- Theorem 7.57, the area law on every slit domain, with its error term $\eta_S \rightarrow 0$;
- Theorem 5.1, the contour bound, for all large K , with its error term $\eta_T \rightarrow 0$.

`U1.area_law_of_R0 : U1.R0 → U1.AreaLaw` formalizes Section 8. It covers the chain (1.4)–(1.8) at one coupling, the choice of constants $\sigma = (1 - \eta_T - \eta_S)\sigma_V$ and $c = c_N + 10$, the bound $4|C| + 4 \leq 10(R + T)$ and the case of small β in Remark 8.1, by monotonicity in β . It uses no numerical value of K_* or of $v_{\text{Cb}}(0)$, only $0 < v_{\text{Cb}}(0) < \infty$.

The next interface, `U1.R1`, is `U1.R0` without the construction, which becomes data (`ConstructionMaps`) with a specification (`ConstructionSpec`). For the construction of Sections 3–4 the specification is proved (`U1.consMaps_spec`), and `U1.R1Cons` is `U1.R1` at this construction.

In `U1.R2` only the screening mass is kept. Theorems 7.57 and 5.1 for the outputs of the construction become the propositions `U1.Thm7610fRegSpec K r_0 m` and `U1.Thm510fRegSpec K`, and the properties of the construction needed in Sections 5–7 become a third, `U1.RegSpec K`: every run of the construction satisfies `Construction.RunSpec`, which collects Theorems 3.20 and 3.21 in the form the later sections require. The point `U1.R2.canon` fixes $r_0 = 1$ and the screening mass `Gas.mtCanon lamPV`, with $\lambda_{\text{PV}} = e^{-2213.5}$ (Section 6). The bounds on this mass are proved in `mtCanon_lamPV_bounds`. All three propositions are proved:

- `U1.RegSpec K` by `U1.consMaps_regSpec` (Sections 3–4);
- `U1.Thm510fRegSpec K` by `U1.MinCut.thm_5_1_of_regSpec` (Section 5), and its form for every $K > K_*$ by `U1.MinCut.thm51Kstar_of_regSpec`;
- `U1.Thm7610fRegSpec` at the canonical point by the chain described next.

Stage C is the part of the formalization in `U1/StageC/`; it covers Sections 6–7 and Appendices A–D of [U1] and the assembly described here. For the third proposition, a family of Stage C data for the outputs of the construction gives Theorem 7.57 on every output. Its fields are assembled from the results of Sections 6–7 and Appendix B into an explicit list: scalar constants, a threshold in τ , and for every valid run of the construction at large τ the per-run fields. In turn, all fields of the list are produced from one bound per run: the position bound of Step 4 of Proposition B.7, which bounds the sum over positions of the activity Q by a product of masses per cell times the fractions of the local free energy returned by Lemma B.8. This bound is proved at the canonical run (`U1.StageC.Seed1.qMassRun_HK`). The theorem `U1.area_law_of_qmass` and its variants turn it into the main theorem and its corollaries, which `U1/MainTheorem.lean` obtains by applying them.

interface	content	proved by
<code>R0</code>	construction, screening mass, Thm 7.57, Thm 5.1	<code>R1.toR0</code>
<code>R1Cons</code> <code>R2</code>	<code>R0</code> at the construction of Sections 3–4 screening mass and three propositions	<code>consMaps_spec</code> , <code>R2.toR1Cons</code> <code>R2.canon</code> , <code>mtCanon_lamPV_bounds</code>
<code>RegSpec K</code>	Theorems 3.20, 3.21 on every run	<code>consMaps_regSpec</code>
<code>Thm510fRegSpec K</code>	Theorem 5.1 on the outputs	<code>MinCut.thm_5_1_of_regSpec</code>
<code>Thm7610fRegSpec</code>	Theorem 7.57 on the outputs	<code>StageC.Assembly</code> . <code>leaf_of_qmass</code> with <code>Seed1.qMassRun_HK</code>

Table 1: The interfaces of the formal proof and the theorems that discharge them.

We use interfaces because they make the dependency graph of the proof explicit and checkable. We formalize the chain of Section 8 and the corollaries against `U1.R0`, while the parts of the proof are formalized independently against fixed statements. However, an interface carries a risk: if its hypotheses are unsatisfiable, all theorems derived from it are vacuous. The non-vacuity audit of Section 7 addresses this risk.

4 The formalization, part by part

We follow the paper’s sections. For each part we give the Lean directory, the main results and the points where the formal proof differs from the printed one. Table 2 gives the size of each directory.

4.1 Sections 2, 8 and 9: the reduction, the chain and the corollaries

The lattice, the models and the expectations are defined in `U1/Basic/` and `U1/Model/`, and the statements of Section 2 are in `U1/Reduction/`. Section 9 and the correlation inequalities of Section 2 (Theorem 2.1, Lemma 2.3) are both in `U1/Correlation/`; the inequalities are proved through positive kernels and the doubling map $(a, b) \mapsto (a + b, a - b)$ on compact abelian groups. The modified Bessel functions of real order and the mixture of Hartman and Watson [HW74] behind Lemma 2.2 are in `U1/Special/`, together with the lattice Coulomb and Yukawa kernels. Lemma 2.4 and Theorem 2.5 are in `U1/Annealed/`, the dual representation and the frozen limit (Lemmas 2.7, 2.8) in `U1/Dual/`, and Lemma 2.6 and the threshold K_* in `U1/Numerics/`

directory of U1/	content	files	lines
Basic, Model	lattice, models, expectations, statements of Section 9	14	2,315
Reduction	statements of Section 2	6	506
top-level files and U1.lean	statements, interfaces, Section 8, main theorems	13	2,728
Correlation, Special, Annealed, Dual	Sections 2 and 9	64	14,813
Cubical	cubical chains (Sections 2 and 5)	50	14,310
Regularization, Geometry	Sections 3–4	81	20,044
MinCut	Section 5	48	12,957
StageC	Sections 6–7, Appendices A–D, the assembly	502	122,178
Gaussian, Trees, Cluster	Gaussian calculus, tree graphs, cluster expansions	49	12,827
Numerics	certified constants	35	13,033
total		862	215,711

Table 2: Files and lines of Lean at the audited commit. Every file is in the import closure of the root module U1.

(Section 6). Cubical chains on boxes, used for Propositions 2.9 and 2.10 and in Section 5, are in U1/Cubical/.

The chain of Section 8 is formalized in U1/Chain.lean, together with the variants described in Section 3 above. In Section 9, Proposition 9.1 holds for every phase of the interaction, since the comparison applies one plaquette at a time. Its proof gives the constants n and $n\beta$, but the statement keeps the printed $2n$ and $2n\beta$, which follow by monotonicity in β . Proposition 9.3 uses a unitary gauge chosen for each fixed Higgs configuration, so no measurable section is needed. The limit $\kappa \rightarrow \infty$ is a concentration argument on $SU(2)$ that uses only the invariance of Haar measure, its full support and Fubini’s theorem.

4.2 Sections 3–4: the regularization

In Lean, the construction of Section 3 is a function from frozen sets to slit domains (U1/Regularization/, and U1/Geometry/ for the geometry). Its grid is a product of one-dimensional partitions, and (G1)–(G5) are products of one-dimensional facts. Both Phase 1 and Phase 2 are iterations to a fixed point, with explicit bounds on the number of steps. For the submodular functionals, the smallest minimizers are intersections of all minimizers. We formalize the analysis of Section 4 for every grid that satisfies (G1)–(G5). The isoperimetric bounds of Theorem 4.5 and Lemma 4.11 use the discrete Loomis–Whitney inequality [LW49, Theorem 2], which we prove by slicing and the Cauchy–Schwarz inequality (U1/Geometry/LoomisWhitney.lean). In the cut rule of Definition 3.18, the selection of a fragmented pair is a parameter of the construction (see the entry on Definition 3.18 in Section 8). Finally, U1.consMaps_regSpec proves Theorems 3.20 and 3.21 for every run of the construction.

4.3 Section 5: the contour bound

Theorem 5.1 is formalized in `U1/MinCut/`. Its cubical lemmas are in `U1/Cubical/` and its counts of lattice animals in `U1/Trees/`. For the outputs of the construction it is proved by `U1.MinCut.thm_5_1_of_regSpec`, with the sharper counting factor of Section 8. We formalize the separators, the Leaf Lemma, the charges and the classes of Subsections 5.4–5.6 for every selection of the cut rule.

4.4 Section 6: Stage 1

The first stage of the renormalization-group analysis (`U1/StageC/S6/`) controls the Debye-screened Coulomb gas on the slit domain. Its ingredients are the expansion with the forest formula and its coefficients, the screening mass, heat-kernel bounds on the slit graph, the depletion estimates, and the Yukawa and Coulomb kernels of `U1/Special/`. We state and prove the heat-kernel bound (HK2) of Theorem 6.8 for the tail mass at a fixed time, a form implied by the printed exit-time form. Appendix D, the only place where (HK2) is applied, needs only this form.

4.5 Section 7: Stage 2

The second stage follows the paper’s subsections.

- Subsections 7.0–7.5 and Appendix C (`U1/StageC/S70/`): the expansion of Gōpfert and Mack [GM82, App. E] on the slit domain, walls and their tension, the collar, wall fields and classes, Proposition 7.22 and Theorem 7.27.
- Subsection 7.6 and Appendix B (`U1/StageC/S76/`): the bounds at a fixed configuration, with the per-cell form of [BF80, (9.824)] (Proposition B.7).
- Subsections 7.7–7.9 and Appendix A (`U1/StageC/S77/`): the tree form of the activities, the link sum, the Kirkwood–Salsburg system in a Beurling algebra, the proof that frozen flux costs area, and the deterministic area law (Theorem 7.57).
- The assembly of Section 3 (`U1/StageC/Assembly/`, with the position bound in `U1/StageC/Seed1/`).
- Gaussian calculus (`U1/Gaussian/`); tree-graph bounds, Kruskal’s identity, the Beurling algebra and the polymer gas (`U1/Trees/`, `U1/Cluster/`).

4.6 The appendices and the supplement

Appendices A, B and C, which treat Brydges–Federbush Section 9.1, Section 9.8 of [BF80] and Gōpfert–Mack Appendix E on slit domains, are formalized with the subsections of Section 7 listed above. Appendix D (heat kernel, coarse-to-fine transfer and covariance bounds on slit domains) is formalized with Section 6 of [U1], and the constants of the supplement are in `U1/Numerics/`, which Section 6 of this paper describes.

5 Written-out proofs and their refereeing

Several steps of [U1] are carried over from the literature “with changes”: the Debye-screening analysis of Brydges and Federbush [BF80, Section 9] and the multiscale expansion of G pfert and Mack [GM82, Appendix E] are applied on slit domains, with constants made explicit. A formal proof cannot follow such a step until the changes are written out. We therefore write out each of these parts as a self-contained document, which the formalization then follows.

All the documents have the same form. Each states its inputs, the results it takes from other parts, precisely and in the form in which it uses them, and its outputs, the results it supplies to other parts. Every step is marked as taken unchanged from a source, proved in full, or an input; a step taken from a source is cited by equation and page, and equations quoted from the sources carry page references. If a step could not be completed, it is marked as a gap, with the missing piece stated. Throughout, paper results are cited by their printed numbers.

document	part of [U1]	pages
Sections 3–4	the regularization: the omitted steps	26
Section 5	the contour bound	33
Section 6	Stage 1, with Appendix D and the supplement	29
Proposition 7.22	the expansion of [GM82, App. E] on slit domains	28
Subsections 7.0–7.5	walls, collars, wall fields and classes	23
Subsection 7.6	bounds at a fixed configuration	26
Appendix B	the per-cell form of [BF80, (9.824)]	27
Subsections 7.7–7.9, App. A	Kirkwood–Salsburg, frozen flux, the area law	33
Section 9	the corollaries, with explicit measure theory	8
total		233

Table 3: The written-out proofs and their length in pages. Each was checked by a referee, and the documents and the referee reports are archived with the formalization [repo].

For each document there is a referee report, written in a separate session, that is, a run of the AI tools that took no part in writing the document or code it checks (see the statement on the use of AI tools). The referee re-derives all proofs line by line and recomputes all constants, checks the transcriptions against the sources, and checks that each input is proved elsewhere or assigned to another part; the report lists its findings by severity. We archive the final version of each document after the review, together with the corrections made later. The current versions, with the corrections of the findings list applied, form Part S4 of [U1]. Findings that concern [U1] itself go to the list of Section 8.

In Lean, the modules of Sections 3–7 follow the documents statement by statement. Their interfaces are the documents’ inputs and outputs, and their docstrings cite the document’s statement and the paper result. Two entries of Section 8 show what the documents add to the paper: the distance factor of display (A.9), which the documents drop for links of the first type, since these have no covariance line, and the seeded sums of Lemma 7.48, which they state for a class of wall fields defined by local primitives.

6 Certified numerics

The proof uses numerical constants in three ways: the threshold $K_* = 377.38$ and the explicit string tension of the second assertion; Watson’s integral $v_{\text{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0, 0)$; and the many inequalities between constants in the supplement of [U1] and in the written-out proofs. We certify all of them in Lean (`U1/Numerics/`).

The threshold. We certify the values of $c_\delta^{\text{sp}}(K)/(2K)$ quoted in Lemma 2.6 through a parametrization $K = e^t/(2t)$, rational brackets for the saddle point and Taylor bounds on `exp` checked by `norm.num`. In particular $c_\delta^{\text{sp}}(K)/(2K) \geq 14.9663$ for $K \geq 377.38$, and the root of $c_\delta^{\text{sp}}(K)/(2K) = 14.9662$ is unique and lies in $[377.36345, 377.36355]$. By contrast, the qualitative assertion of Theorem 1.1 uses no value of K_* .

Watson’s integral. The bounds

$$0.2527310098586 \leq v_{\text{Cb}}(0) \leq 0.2527310098631$$

reduce to two inequalities between rational numbers: a head of the series expansion of $v_{\text{Cb}}(0)$ and a bound on its tail. Both are decided by the kernel (`decide +kernel`). The resulting theorem `U1.Numerics.vCb0_bounds` uses only the standard axioms. It gives $2.494355087 \leq \pi^2 v_{\text{Cb}}(0) \leq 2.494355088$ (`U1.Numerics.pi_sq_vCb0_bounds`). The explicit string tension and the statements for large β use these bounds.

A second certificate, the fast twin, decides the same two inequalities by compiled code (`native_decide`). It is the file `U1/Numerics/Fast/WatsonFast.lean`, the only file with `native_decide`. With the pinned toolchain each use of `native_decide` adds a fresh axiom, which is generated in this file (see Section 7). The root module imports the file, so a full build checks it. The import closure of `U1.MainTheorem` (824 modules), however, does not contain it. Hence the generated axiom lies outside the proof of every main theorem.

The symbolic constants. The paper keeps some constants symbolic, among them the parameters μ and κ_S of Sections 6 and 7 (item 12 of [U1, S1.1.1]) and the Pauli–Villars parameter λ_{PV} (item 13), and states the conditions they must satisfy. In the formal proof we fix

$$C = \frac{1}{4}, \quad \mu = 4401, \quad \kappa_S = 7, \quad \lambda_{\text{PV}} = \lambda_\star(\kappa_S, \mu) := e^{-(\kappa_S + \mu/2 + 6)} = e^{-2213.5},$$

and keep $\theta_0 = \frac{1}{100}$, $\eta_H = 0.62$, $p_4 = 15$ and $c_\lambda = \frac{1}{24}$ of the paper. We choose the constants in an order in which each condition concerns only constants already chosen: the absolute constants and the heat-kernel constants; then $c_{10} = \frac{7}{9}r_C$, with $r_C = \kappa_0/2 \approx 2.8 \cdot 10^{-183}$, and $c''_{10} = 2200$, which is certified admissible (the least admissible value is about 2137.3); then $\mu = 1 + 2c''_{10}$; then κ_S ; then λ_{PV} ; and finally β large. This order is not circular: no condition bounds μ or κ_S from above, and c''_{10} involves none of μ , κ_S , λ_{PV} and β . Table 4 lists the conditions and their certificates.

The constants of the written-out proofs. In the six written-out proofs of Sections 6 and 7 and their appendices, every numerical claim is inventoried with its location, its kind (arithmetic, parametric over a stated range, definitional or analytic) and whether it holds as written. All of them are also checked independently in 50-digit arithmetic. The arithmetic and parametric

condition	Lean certificate
$\mu \geq 1 + 2c''_{10}$ (Proposition B.7); $e^{c''_{10}} \geq 16e \max(1, c_{c_{10}/2})$ (Lemma A.6) (7.30), with the binding condition on κ_S (I6) = (L8-1): $\varepsilon_\nu + \theta' \leq \frac{1}{7}$	<code>c10pp_certificate</code> , <code>exp_neg_c10pp_le</code> , <code>tsum_cube_exp_le</code> <code>T730_global</code> <code>I6_global</code> , <code>margin_aRef_le</code> , <code>margin_I6_one</code> , <code>margin_I6_two</code>
item 12 of [U1, S1.1.1]: $(1 + C_{\text{ref}}e^{-2\kappa_S})\eta_H/8 \leq \frac{1}{7}$	<code>U1.Const.kappaS_condition_of_Cref</code> , <code>U1.Const.Cref_lt</code>
$1 + 8C_{6.10}e^{-2\kappa_S} \leq \frac{4}{3}$ the moment hypothesis of (7.26)	<code>crho_cond</code> <code>moment_hyp_rho_half</code> , <code>one_sub_exp_neg_hundredth_le</code>
the mass thresholds of Theorem 6.5 $\bar{\eta} < \eta_H = 0.62$	<code>mt_bounds_tau_thresholds</code> <code>C3_067</code>
the conditions on λ_{PV} : the Debye forms $12\lambda^2$, $24\lambda^2$, $36\lambda^2$; the dominated window; $\lambda \leq \frac{1}{4}$; $\lambda \leq \lambda_0$ of Theorem 6.5; the form used for the screening mass	<code>lamStar_debye12</code> , <code>lamStar_debye24</code> , <code>lamStar_debye36</code> , <code>lamStar_dominated</code> , <code>lamStar_le_quarter</code> , <code>lamStar_le_lam0</code> , <code>lamStar_mtCanon_le</code>

Table 4: The conditions on the symbolic constants and their kernel-checked certificates (namespace `U1.Numerics.StageC` unless another namespace is shown). The labels (I6) and (L8-1) are conditions of the written-out proofs.

claims are certified as Lean lemmas in `U1/Numerics/StageC/`, while the analytic ones are proved where they are used. Of the 1677 claims, nine fail as written: seven roundings in the last digits, one wrong digit and one equality that holds only as an inequality. Six more cannot be checked from the documents alone. All fifteen are corrected or resolved in the documents, and none affects a conclusion.

document	claims	arithmetic	parametric	definitional	analytic	certified
Section 6	358	21	149	51	137	143
Proposition 7.22	151	6	54	16	75	58
Subsections 7.0–7.5	219	14	46	35	124	60
Subsection 7.6	258	12	81	47	118	92
Appendix B	365	20	141	53	151	139
Subsections 7.7–7.9, App. A	326	25	45	44	212	71
total	1677	98	516	246	817	563

Table 5: The inventory of numerical claims in the written-out proofs of Sections 6 and 7 and their appendices.

7 Audits

A formal proof can fail in three ways that the proof checker does not detect: a statement can say less than the paper, a hypothesis can be unsatisfiable, and a build can depend on files that are not in the repository. All three are checked by audits, carried out in sessions separate from those in which the formal proofs were written. Their reports are in `audit/`.

For the first, the fidelity audit compares each Lean statement of a paper result with the

printed statement, clause by clause, comparing quantifiers and their order, ranges of parameters, boundary conditions, normalizations and the objects quantified over. Deliberate generalizations are recorded as such (Section 2). The reports cover the statements of Theorem 1.2, of Sections 2 and 8 of [U1], of the interfaces and of the corollaries. For the main theorems, the final audit re-checks the statements from scratch against Sections 1, 2, 8 and 9 of [U1]. It finds that Definition 1.1, both assertions of Theorem 1.2 (with $K_* = 377.38$), the infinite-volume form of the area law (Remark 8.1), Corollary 1.4 and Proposition 9.1 (stronger than printed) are faithful, and that Corollary 1.3 in the paper’s form is `U1.Main.cor_1.3.rel` together with `thm_1.2`.

For the second, the non-vacuity audit checks each hypothesis structure of Section 3 and of the parts for satisfiability with the paper’s objects, by inspection of every field and, where a field is suspect, by a Lean proof that it can or cannot hold. The following kinds of defect make a structure unsatisfiable or stronger than the paper, and the final structures are free of them.

- *Degenerate outputs.* A family of Stage C data cannot be required for every output of the construction. For outputs in which no bond of the slit graph crosses the rectangle, the paper’s seed set X_1 is empty, and a Lean proof shows that the data would force $W_{G'}(C) \leq 0$, while $W_{G'}(C) > 0$. We therefore require the family only for crossing outputs, following the paper’s own case split in the proof of Theorem 7.57. The degenerate case is proved directly: $W_{G'}(C) = 1$ and $A_{G'}(C) = 0$. For crossing outputs, a lemma proves that the seed set is nonempty.
- *Missing conjuncts.* The predicate for the rooted minimizers of the contour bound includes the conditions that make them minimizers, since without them the theorems that use the predicate would be formally stronger than the paper’s statements. These conditions are supplied by the theorem that proves the predicate.
- *Surjectivity.* The map from sites to $\mathcal{U}$ -cells on the output of the construction takes values in the set of $\mathcal{U}$ -cells itself, so that it is onto, as required by a theorem that uses it. One extra point in the codomain would never be hit.
- *Quantifier scope.* We require the proposition for Theorem 7.57 on the outputs only at the one screening mass where the chain uses it, not for every mass in a band.
- *Missing guards.* The list of per-run fields of Section 3 is required only for frozen sets inside the box and for crossing outputs, because without these guards a Lean proof shows that the list cannot be inhabited.
- *Uniformity in τ .* The constants of the per-run bounds must not grow with τ . Hence each factor of the per-cell constant is checked to be independent of τ once the canonical point is fixed, and the position bound binds its constants outside the quantifiers over τ and over configurations.
- *Unbounded quantifiers.* The hypotheses on the path of Proposition B.7 are swept for universal quantifiers over all natural numbers or all finite sets, degenerate skeletons, strict inequalities at boundary values and conflicting fields.

Since the main theorems have no hypotheses, every interface on the chain is inhabited by the proofs themselves. These checks matter for the intermediate interfaces, where a vacuous structure would hide a missing argument.

Axioms and sorry. At each audit the command `#print axioms` is run on every main theorem, and a theorem is counted as proved only if it lists the three axioms named in Theorem 1.2 and nothing else. The source contains no `sorry` and no `axiom` declaration, and `native_decide` occurs in one file, the fast twin of Section 6. At the kernel level, a scan of every constant of the 862 modules finds no constant that uses `sorryAx`, one declared axiom, generated by that `native_decide`, and three constants that depend on it, all in that file. The proofs of `thm_1.2`, `thm_1.2_explicit`, `rem_8.1`, `cor_1.3` and `cor_1.4` involve about 87,000 constants each, of which 17,600 to 17,900 are from the library. Those of `prop_9.1`, `cor_1.3_rel` and `cor_1.4_rel` involve about 40,000 to 42,000 constants each, of which 215, 435 and 496 are from the library. The last two are small because they take the area law of the $U(1)$ model as a hypothesis. None of them uses `sorryAx`, the generated axiom or a constant of the fast twin. A second check replays the compiled library through the kernel, by the method of the toolchain’s `leanchecker`: with Lean core, Batteries and Mathlib as the trusted base, every constant stored in the compiled files of the 862 modules is sent to the kernel again. All 25,940 constants are accepted, in 165 seconds; the count is one more than the scan’s because one congruence lemma for a Mathlib structure is stored in a module of the library but attributed to Mathlib by the scan. A third check works from a text export instead of the compiled files: the eight main theorems and every declaration they depend on, including those of Lean core and Mathlib, are exported with `lean4export` 3.1.0. This export has 89,002 declarations, among them 69,093 theorems and 16,035 definitions, and its only axioms are the same three. It is accepted by two independent kernel implementations shipped with the pinned toolchain, `nanoda` 0.4.17 and `lean4lean` (built against Lean 4.35.0-nightly-2026-09-06), and by Lean’s own kernel reading the export (`leanchecker --from-export`). The export of every named constant of the library with its dependencies (95,420 declarations) is accepted as well, once `nanoda` is allowed the one axiom generated by the `native_decide` of the fast twin; that axiom lies outside the export of the main theorems. As a control, all three kernels reject a small export in which one theorem has been given the proof of another.

Clean-checkout builds. For the third, a clean build starts from a fresh export of the commit under audit (`git archive`), with the pinned toolchain (Lean `v4.35.0-rc3`, Mathlib at commit `c55e6e786f49`) and no build products of the project. The build log is scanned for `sorry` warnings and errors, and the axiom scan is repeated on the result. For the audited commit, the clean build compiles all 862 modules with exit status 0 in 1,343 seconds of wall-clock time, using Lake’s default parallelism, on a shared many-core server that also carries other work. Table 6 gives its peak memory and the compile times of the modules.

8 Findings about the paper

We record every place where a statement or proof of [U1] is wrong, incomplete or relies on something unstated. Each entry gives a proposed fix and replacement text for [U1], and the formal proof uses the fix. The list has 70 entries: 5 errors (a false statement, or a proof step that is false as written), 11 gaps (a step that is missing or only asserted), 11 unstated hypotheses (true under the paper’s standing choices but not said), 20 points of presentation, 10 of notation, and 13 simplifications (a shorter or more elementary argument). No entry changes Theorem 1.1 or its corollaries. Where an entry changes a printed statement, the proofs already use the new statement. The list, with its replacement text, is archived with the formalization [repo].

The following entries show the kinds of issue.

- *An order of supremum and sum* (display (A.3); error). The printed inequality takes the supremum over the interpolation parameters inside the sum over elementary terms. Proposition A.7 combines all terms with one link datum and bounds that combination uniformly in the parameters, which needs the supremum outside that sum. The corrected form has the same proof and the same constants.
- *A sphere count* (Section D.3(c); error). The proof bounds the number of points at graph distance k in a slit graph by $4k^2 + 2$, the size of an ℓ^1 -sphere of $\mathbb{Z}^3$. On a slit graph this fails: in a three-dimensional H-tree the sphere of radius 189 about the root has $2^{18} = 262144$ points, while $4 \cdot 189^2 + 2 = 142886$. What the proof uses is the bound $\sum_y e^{-a d(x,y)} \leq 64a^{-3}$ for $0 < a \leq 2$, which follows directly from $d \geq \|\cdot\|_1$.
- *An unpaid factor* (display (A.9), Lemma 7.37, Proposition 7.39; gap). The bound puts a distance factor $e^{\delta_2 \text{dist}(\Delta', \Delta'')}$ on every link of the tree expansion, but links of the first type have no covariance line, and Proposition B.7 obtains its decay per link only from the covariance bound. The fix drops the factor for these links, and since the count of link data (Lemma A.6) never uses it, all constants are unchanged.
- *A counting step* (Lemma A.6; unstated hypothesis). “The tree sum gives $e^k c^{k-1}$ ” rests on Cayley’s formula, which is not stated. Rooted plane trees give the bound $(4c)^{k-1}$ directly, for every choice of spanning tree.
- *Unstated conditions* (supplement S1.1.4 and Theorem 5.1; unstated hypotheses). The clearing-out argument of Theorem 4.23 needs $2c_s \ell^2 > 1$, which holds for every $c_s > 3.3 \cdot 10^{-7}$ but is not among the conditions on τ . Similarly, the tail bound of Theorem 5.1 needs the dilution $\delta > 0$, which follows from Lemma 2.2 but is not stated.
- *A class of fields* (Lemma 7.48; presentation). The seeded sums are stated for label configurations and applied to long wall fields and to inside fields. They hold for the class of wall fields with local primitives, which contains both.
- *An unspecified choice* (Definition 3.18; presentation). The cut rule fires on “the first fragmented pair in a fixed lexicographic order” that is not specified. Any order fixed before the frozen set gives the same theorems.
- *A sharper constant* (Theorem 5.1; simplification). The counting factor $2^{4|C|+5}$ of the tail bound can be replaced by $2^{4|C|+2}$, which is the form that the formal proof gives.

Findings about the sources. The paper already records its corrections to [B78] and [BF80] (in the proof of Lemma A.1, in Remark A.5 and in Proposition A.7), and the formal proof uses the corrected forms. One more correction concerns the order of the sums in [BF80, pp. 213–217]: the sums are listed in the inverse order in which they are performed, so the constraint of the step 3Σ , that each set contain one of the cubes “mentioned in 10Σ ”, refers to a sum that is performed earlier. The count of Proposition A.4 instead uses a valid order, in which each constraint links a variable only to variables outer to it.

9 Statistics and reproduction

How to reproduce. The Lean archive of [repo] pins the Lean toolchain (`lean-toolchain`) and the Mathlib commit (`lake-manifest.json`). With `elan` installed, the commands

```
lake exe cache get
lake build U1
```

run in the root of the unpacked archive fetch the Mathlib build for the pinned commit and build the library. At the top level of the archive, the file `PrintAxioms.lean` is not part of the library. Running

```
lake env lean PrintAxioms.lean
```

prints the statements of the main theorems (`#check`) and their axioms, through the lines

```
import U1.MainTheorem
#print axioms U1.Main.thm_1_2
#print axioms U1.Main.thm_1_2_explicit
#print axioms U1.Main.rem_8_1
#print axioms U1.Main.cor_1_3_rel
#print axioms U1.Main.cor_1_3
#print axioms U1.Main.cor_1_4_rel
#print axioms U1.Main.cor_1_4
#print axioms U1.Main.prop_9_1
```

Each of the eight reports [`propext`, `Classical.choice`, `Quot.sound`]. The map from paper results, by printed number, to Lean names is `blueprint/BUEPRINT.md`. The written-out proofs, their referee reports and the list of findings are in the same record.

quantity	value
Lean; Mathlib	v4.35.0-rc3; commit c55e6e786f49
files; lines of Lean	862; 215,711 (import closure of the main theorems: 824; 207,686)
constants (kernel level)	17,331 named constants, 12,918 of them theorems; 25,939 with the private and auxiliary ones (20,898 theorems, 4,592 definitions, 448 other)
axioms of the main theorems	propext, Classical.choice, Quot.sound
sorry; axiom in the source	0; 0
constants using sorryAx	0
kernel replay (method of leanchecker)	all 25,940 stored constants accepted, 165s, peak memory 5.7 GiB
export of the closure of the main theorems	89,002 declarations, accepted by nanoda 0.4.17, by lean4lean and by Lean’s kernel (leanchecker --from-export)
native_decide	one use, in one file outside the import closure of the main theorems; its generated axiom is used by the three constants of that file only
clean-checkout build	exit status 0, 1,343 s wall clock on a shared many-core server, peak memory 3.7 GiB; module times add up to 7,529 s
slowest module	67 s; two modules over 60 s
written-out proofs	9 documents, 233 pages, 9 referee reports
numerical claims in the written-out proofs	1677 inventoried; 563 certified as Lean lemmas
findings in [U1]	70 entries: 5 errors, 11 gaps, 11 unstated hypotheses

Table 6: Verification statistics of the formalization at commit 6f88ee6, whose library files are those of the archive [repo].

Use of AI tools

Generative AI tools were used to write the Lean code, the written-out proofs and their referee reports, the audits and the drafts of this text, under the author’s direction. Each referee report and each audit was written in a separate session from the document or code it checks.

References

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- [repo] The formalization and its working records, Zenodo, <https://doi.org/10.5281/zenodo.23480810>: the Lean sources at commit `02a34e3`, whose library files are those of the audited commit `6f88ee6`; the written-out proofs and their referee reports; and the list of findings.