

THE ALL-CONTACT CASE OF THE 24-CELL CONJECTURE

TEJASVI SINGH TOMAR

ABSTRACT. The 24-cell conjecture states that every Voronoi cell of a packing of unit balls in $\mathbb{R}^4$ has volume at least 8, the volume of the regular 24-cell. We prove it for every cell whose ball touches all balls centred within distance $2\sqrt{2}$ of its centre, with equality only for the D_4 root configuration. The proof bounds the cell from below by a third-order inclusion–exclusion expansion truncated at radius $\sqrt{2}$, which is exact at D_4 . Configurations with 24 contacts are D_4 by the uniqueness theorem of de Laat, Leijenhurst and de Muinck Keizer. For at most 23 contacts we give an exact rational certificate for the second level of the Lasserre hierarchy with value $8.03874\dots$, together with interval-arithmetic bounds for the polynomial approximations it uses. We also show that, for every packing, no point inside the ball of radius $\sqrt{2}$ lies in five of the caps cut off by the Voronoi hyperplanes, so the inclusion–exclusion expansion of the truncated cell stops at order four.

1. INTRODUCTION

Let $\mathcal{P} \subset \mathbb{R}^4$ be the set of centres of a packing of unit balls, so that distinct centres are at distance at least 2. The Voronoi cell of $c \in \mathcal{P}$ is

$$V_c = \{x \in \mathbb{R}^4 : |x - c| \leq |x - y| \text{ for all } y \in \mathcal{P}\}.$$

For the lattice D_4 , scaled to minimal distance 2, every Voronoi cell is a regular 24-cell with inradius 1 and volume 8. The *24-cell conjecture* states that $\text{vol}(V_c) \geq 8$ for every packing and every centre. It implies that the density of a packing of equal balls in $\mathbb{R}^4$ is at most $\pi^2/16$, the density of D_4 . The best proved upper bound on this density is 0.63611, from the three-point bounds of Cohn, de Laat and Salmon [4]. A proof of the conjecture has been announced in [2]. It treats cells according to their number of contacts and uses a covering estimate, a semidefinite certificate and the uniqueness of the D_4 root system. The proof below takes from [2] the truncation at radius $\sqrt{2}$. The bound [2, Lemma 7.32] and the value at D_4 [2, Prop. 7.34] are reproved in Section 2.

We prove the conjecture for cells all of whose near neighbours touch the central ball.

Theorem 1.1. *Let $c \in \mathcal{P}$, and suppose that every centre $y \in \mathcal{P}$ with $0 < |y - c| < 2\sqrt{2}$ satisfies $|y - c| = 2$. Then $\text{vol}(V_c) \geq 8$, with equality if and only if the centres at distance 2 from c form a translate of a D_4 root system scaled by $\sqrt{2}$.*

Section 2 bounds $\text{vol}(V_c)$ from below by a quantity T_3 built from caps of the ball $B = B(c, \sqrt{2})$, following [2, Lemma 7.32]. We show that, for every packing, no point of B lies in five caps (Proposition 2.1), which identifies the loss in this bound exactly. For contact configurations, T_3 depends only on the kissing configuration $C \subset S^3$ of contact directions, and $T_3(D_4) = 8$. Section 3 gives an exact certificate for the second level of the Lasserre hierarchy, in the form developed by de Laat, Leijenhurst and de Muinck Keizer [7], showing $T_3(C) > 8$ whenever $|C| \leq 23$. Their uniqueness theorem disposes of $|C| = 24$. Section 4 controls the polynomial approximations in the certificate. Section 5 assembles the proof, and Section 6 records numerical results on the same relaxation without the bound on $|C|$.

2. THE TRUNCATED CELL

Fix a packing and translate so that $c = 0$. Let $B = \{x : |x| < \sqrt{2}\}$. For a centre $y \neq 0$ put

$$H_y = \{x : \langle x, y \rangle > |y|^2/2\} = \{x : |x - y| < |x|\}, \quad K_y = H_y \cap B.$$

Since V_0 is closed and each H_y is open, $V_0 = \mathbb{R}^4 \setminus \bigcup_y H_y$, so $V_0 \cap B = B \setminus \bigcup_y K_y$. If $|y| \geq 2\sqrt{2}$, then every $x \in H_y$ has $|x| \geq \langle x, y \rangle / |y| > |y|/2 \geq \sqrt{2}$, so $K_y = \emptyset$. Write Y for the finite set of centres with $0 < |y| < 2\sqrt{2}$, and let S_k be the sum, over all k -element subsets of Y , of the volumes of the intersections of the corresponding caps.

Proposition 2.1. *No point of B lies in five of the sets K_y . Consequently*

$$\text{vol}(V_0 \cap B) = 2\pi^2 - S_1 + S_2 - S_3 + S_4.$$

Proof. Let $x \in B$ lie in $K_{y_1}, \dots, K_{y_m}$, and put $p_0 = 0$ and $p_i = y_i$. Each p_i satisfies $|p_i - x| \leq |x|$, and $|p_i - p_j| \geq 2$ for $i \neq j$. Hence

$$\langle p_i - x, p_j - x \rangle = \frac{1}{2}(|p_i - x|^2 + |p_j - x|^2 - |p_i - p_j|^2) \leq |x|^2 - 2 < 0.$$

The $m + 1$ vectors $p_i - x$ are nonzero and have pairwise negative inner products. In $\mathbb{R}^4$ there are at most five such vectors, so $m \leq 4$. Inclusion-exclusion over the caps covering a point therefore ends at order four, and $\text{vol}(B) = 2\pi^2$. $\square$

Put

$$T_3(Y) = 2\pi^2 - S_1 + S_2 - S_3.$$

Since at most four caps meet, S_4 is the volume of the set of points of B that lie in four caps.

Corollary 2.2. $\text{vol}(V_0) \geq \text{vol}(V_0 \cap B) = T_3(Y) + S_4 \geq T_3(Y)$.

The inequality $\text{vol}(V_0 \cap B) \geq T_3(Y)$ is [2, Lemma 7.32], stated there for contact configurations. Corollary 2.2 shows that its only loss is the measure of the set of points of B covered by four caps.

Proposition 2.3. *If a centre moves radially outward, or is removed, then $T_3(Y)$ does not decrease. Moving a centre of Y radially inward strictly decreases it.*

Proof. For $x \in B$ let $k(x)$ be the number of caps containing x . Then $T_3(Y) = \int_B b(k(x)) dx$ with $b(k) = 1 - k + \binom{k}{2} - \binom{k}{3}$, and $b(k) - b(k-1) = -(k-2)(k-3)/2 \leq 0$ for $k \geq 1$. Moving y outward along its ray shrinks H_y , and removing y removes it, so $k(x)$ does not increase anywhere. For an inward move of $y \in Y$ the gained region contains points near $y/2$. This point lies in no other half-space $H_{y'}$, because the packing condition gives $|y/2 - y'|^2 - |y/2|^2 = \frac{1}{2}(|y'|^2 + |y - y'|^2 - |y|^2) \geq (8 - |y|^2)/2 > 0$. There k goes from 0 to 1 and b from 1 to 0. $\square$

Contact configurations. Suppose now that every $y \in Y$ has $|y| = 2$. Then $y = 2w$ with $w \in S^3$, and $|y - y'| \geq 2$ means $\langle w, w' \rangle \leq \frac{1}{2}$. The directions form a *kissing configuration* $C \subset S^3$: a finite set of unit vectors with pairwise inner products in $[-1, \frac{1}{2}]$. All caps are cut off by hyperplanes at distance 1. Let ω_1 be the volume of one such cap of B , let $\omega_2(u)$ be the volume of the intersection of two caps whose normals have inner product u , and let $\omega_3(u, v, t)$ be the analogous volume for three caps. Then

$$T_3(C) = 2\pi^2 - \omega_1|C| + \sum_{\{w, w'\} \subset C} \omega_2(\langle w, w' \rangle) - \sum_{\{w, w', w''\} \subset C} \omega_3(\langle w, w' \rangle, \langle w, w'' \rangle, \langle w', w'' \rangle).$$

Lemma 2.4. *If k caps of a contact configuration share a point of B , then their normals have pairwise positive inner products. In particular $\omega_2(u) = 0$ for $u \leq 0$. For three of the normals, with Gram matrix G and pairwise inner products $u, v, t \leq \frac{1}{2}$, we have $\omega_3(u, v, t) > 0$ if and only if*

$$q(u, v, t) = 4uvt - (u + v + t - 1)^2 = 2 \det G - \mathbf{1}^\top \operatorname{adj}(G) \mathbf{1} > 0.$$

Proof. If $x \in B$ satisfies $\langle x, w_i \rangle > 1$, then the angle between x and w_i has cosine greater than $1/|x| > 1/\sqrt{2}$. Any two of the w_i are therefore at angle less than $\pi/2$. Now let three caps meet in B . By the first part their inner products u, v, t lie in $(0, \frac{1}{2}]$, so $\det G = 1 + 2uvt - u^2 - v^2 - t^2 \geq \frac{1}{4}$. The entries of $\operatorname{adj}(G)\mathbf{1}$ are $(1-t)(1+t-u-v)$ and its cyclic analogues, all positive. Let $x_0 \in \operatorname{span}\{w_1, w_2, w_3\}$ satisfy $\langle x_0, w_i \rangle = 1$ for each i . Then $x_0 = \sum_i \lambda_i w_i$ with $\lambda = G^{-1}\mathbf{1} > 0$, and $|x_0|^2 = \mathbf{1}^\top G^{-1}\mathbf{1}$. Every x in the intersection of the three closed half-spaces satisfies

$$|x|^2 \geq |x_0|^2 + 2\langle x - x_0, x_0 \rangle = |x_0|^2 + 2 \sum_i \lambda_i (\langle x, w_i \rangle - 1) \geq |x_0|^2,$$

so $\mathbf{1}^\top G^{-1}\mathbf{1} < 2$. Since $\operatorname{adj} G = (\det G) G^{-1}$, we get $q = \det G (2 - \mathbf{1}^\top G^{-1}\mathbf{1}) > 0$. Conversely, let $q > 0$. If $\det G = 0$, then $\operatorname{adj} G$ is positive semidefinite and $q = -\mathbf{1}^\top \operatorname{adj}(G)\mathbf{1} \leq 0$. Hence $\det G > 0$ and $|x_0|^2 = \mathbf{1}^\top G^{-1}\mathbf{1} < 2$, and for small $\varepsilon > 0$ the point $(1 + \varepsilon)x_0$ lies in all three caps. The identity for q is a direct expansion. $\square$

Lemma 2.5 ([2, Prop. 7.34]). *For the D_4 root system, normalised to unit vectors, $T_3 = 8$.*

Proof. The Voronoi cell of $\sqrt{2}D_4$ is the regular 24-cell with inradius 1, circumradius $\sqrt{2}$ and volume 8, so $\operatorname{vol}(V_0 \cap B) = 8$. The lattice $\sqrt{2}D_4$ has no vectors of length strictly between 2 and $2\sqrt{2}$, so Y consists of the 24 contacts. The inner products of distinct roots lie in $\{-1, -\frac{1}{2}, 0, \frac{1}{2}\}$. The eight roots at inner product $\frac{1}{2}$ with a fixed root form the vertices of a cube, two of them being at inner product $\frac{1}{2}$ exactly when they are adjacent. A cube has no triangles, so no four roots have all six inner products equal to $\frac{1}{2}$. By Lemma 2.4 no point of B lies in four caps, so $S_4 = 0$, and Corollary 2.2 gives $T_3 = 8$. $\square$

Numerically,

$$\omega_1 \approx 0.7460119958, \quad \omega_2(\tfrac{1}{2}) \approx 0.0760762771, \quad \omega_3(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \approx 0.0118567031,$$

and Lemma 2.5 reads $2\pi^2 - 24\omega_1 + 96\omega_2(\frac{1}{2}) - 96\omega_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 8$.

3. A LEVEL-TWO CERTIFICATE

We use the second level of the Lasserre hierarchy for spherical codes in the form of [7], which built on [5]. The dual objects are a kernel K on $\mathcal{I}_2$, the subsets of S^3 of size at most two, invariant under the orthogonal group and positive semidefinite in the sense of [7]. For a finite set S of at most four points put

$$A_2 K(S) = \sum_{\substack{J, J' \in \mathcal{I}_2 \\ J \cup J' = S}} K(J, J').$$

Positive semidefiniteness of K , applied to the indicator vector of the subsets of a finite code C , gives

$$(1) \quad 0 \leq \sum_{J, J' \subseteq C, |J|, |J'| \leq 2} K(J, J') = K(\emptyset, \emptyset) + \sum_{\emptyset \neq S \subseteq C, |S| \leq 4} A_2 K(S).$$

In the symmetry-reduced form of [7], $A_2K(S)$ is an $S_{|S|}$ -invariant polynomial in the pairwise inner products of S . We use the degree parameters $(d_1, \delta) = (14, 16)$ of [7], and with these every such polynomial has degree at most 16.

The objective is encoded by right-hand sides on pieces of the configuration spaces. Write $D = [-1, \frac{1}{2}]$, and call a finite set of points *admissible* if its pairwise inner products lie in D and its Gram matrix is positive semidefinite. Let p_2 be a polynomial of degree at most 16 in one variable, and let p_3 be an S_3 -invariant polynomial of degree at most 16 in three variables. The pieces are:

- singletons, with right-hand side $-\omega_1^+$, where ω_1^+ is a rational number with $\omega_1 \leq \omega_1^+$;
- pairs with $u \in [-1, 0]$, right-hand side 0, and pairs with $u \in [0, \frac{1}{2}]$, right-hand side $p_2(u)$;
- admissible triples with $q \leq 0$, right-hand side 0, and admissible triples with $q \geq 0$, right-hand side $-p_3$;
- admissible quadruples, right-hand side 0.

On each piece we require an identity $A_2K(S) + \Sigma(S) = c(S)$, where Σ is a sum of terms $g \cdot h$ with h a sum of squares and g a polynomial that is nonnegative on the piece. Put $s(x) = (x + 1)(\frac{1}{2} - x)$. For pairs on $[a, b]$ the polynomials g are 1 and $(u - a)(b - u)$. For triples they are 1, the elementary symmetric functions of $s(u), s(v), s(t)$, the Gram determinant, and $-q$ or q on the two triple pieces. For quadruples they are 1, the elementary symmetric functions of the values of s at the six inner products, the Gram determinant, and the elementary symmetric functions of the four 3×3 principal minors of the Gram matrix. In the singleton identity we add a variable $\mu \geq 0$, giving $A_2K(\{x\}) + r - \mu = -\omega_1^+$ with a constant $r \geq 0$.

Proposition 3.1. *Suppose the identities hold, with $\mu \geq 0$ and $K(\emptyset, \emptyset) + 23\mu = \kappa$. Then every kissing configuration C with $|C| \leq 23$ satisfies*

$$F(C) := 2\pi^2 - \omega_1^+|C| + \sum_{\substack{\{w, w'\} \subset C \\ u \geq 0}} p_2(u) - \sum_{\substack{\text{triples} \\ q \geq 0}} p_3 \geq 2\pi^2 - \kappa.$$

Proof. Assign each pair with $u \geq 0$ to the second pair piece and every other pair to the first. Assign each triple with $q \geq 0$ to the second triple piece and every other triple to the first. Quadruples form a single piece. On the assigned piece $\Sigma \geq 0$, so $A_2K(S) \leq c(S)$. For singletons, $A_2K(\{x\}) \leq \mu - \omega_1^+$. Substituting into (1) gives $0 \leq K(\emptyset, \emptyset) + |C|(\mu - \omega_1^+) + \sum p_2 - \sum p_3$. Since $|C|\mu \leq 23\mu$, this becomes $F(C) \geq 2\pi^2 - K(\emptyset, \emptyset) - 23\mu = 2\pi^2 - \kappa$. $\square$

Theorem 3.2. *There is a solution of the identities above in exact rational arithmetic with $\mu > 0$ and $\kappa = 1828197/156250$. Hence $F(C) \geq 2\pi^2 - 1828197/156250 = 8.0387480021787\dots$ for every kissing configuration with at most 23 points.*

The certificate was computed with the clustered low-rank solver of [8], then rounded to exact rationals. It is in the supplement [11]. It has 160 positive semidefinite blocks: 60 kernel blocks, of size up to 26, 98 sum-of-squares blocks, of size up to 395, and the 1×1 blocks r and μ . It also has one identity for each sample point of each piece, 4122 in all. Table 1 lists what the verification establishes. The sample sets are unisolvent for the relevant spaces of invariant polynomials of degree at most 16. Exact equality at the samples therefore implies each identity as a polynomial identity, given that both sides lie in those spaces. To remove the last assumption, each identity was also checked on the principal lattice $\{a/16 : a \in \mathbb{Z}_{\geq 0}^m, |a| \leq 16\}$ for $m = 3$ and $m = 6$, which is unisolvent for all polynomials of degree at most 16 in m variables [10, 3]. The degree bound itself follows from how the zonal matrices and the sum-of-squares terms are built: each zonal matrix entry, indexed as in [7] by a signature $\lambda = (\lambda_1, \lambda_2)$,

was checked to have degree $|\lambda| = \lambda_1 + \lambda_2$, and the sum-of-squares terms have degree at most 16 by construction.

TABLE 1. Verification of the certificate in Theorem 3.2. All arithmetic is exact rational, or certified ball arithmetic [1] where stated.

item	result
identities at the sample points	all 4122 hold exactly, as does the identity fixing κ
right-hand sides	equal to $-\omega_1^+$, p_2 , $-p_3$ or 0 at every sample point
positive definiteness	all 160 blocks, by Cholesky factorisation in ball arithmetic with a certified relative margin of at least 2^{-70}
unisolvence of the samples	17 points for pairs; 204 and 3679 points, equal to the dimensions of the S_3 - and S_4 -invariant polynomials of degree ≤ 16 , with full-rank evaluation matrices modulo $2^{61} - 1$
identities without invariance	exact on the principal lattices, 969 points for each triple piece and 74613 points for quadruples
zonal matrices	all 490 entries recomputed with the code of [7] and found identical
identities regenerated from the construction scripts	identical to the stored ones

4. THE POLYNOMIAL APPROXIMATIONS

The function F uses ω_1^+ , p_2 and p_3 in place of ω_1 , ω_2 and ω_3 . The following bounds are proved in the supplement by interval arithmetic. The volume ω_2 has a closed form. For ω_3 the partial derivative $\partial\omega_3/\partial u$ has a closed form in terms of a one-dimensional integral, and ω_3 is enclosed by certified one-dimensional integration [1]. The same formula bounds the derivatives of $\omega_3 - p_3$ over each box of a cover of the triple piece.

Lemma 4.1. *Let $\varepsilon_2 = 2.8111 \cdot 10^{-12}$ and $\varepsilon_3 = 10^{-6}$. Then*

- (1) $\omega_1 \leq \omega_1^+$;
- (2) $p_2(u) \leq \omega_2(u) + \varepsilon_2$ for $u \in [0, \frac{1}{2}]$;
- (3) $p_3(u, v, t) \geq \omega_3(u, v, t) - \varepsilon_3$ for every admissible triple with $q(u, v, t) \geq 0$.

By Lemma 2.4, $\omega_2 = 0$ on $[-1, 0]$ and $\omega_3 = 0$ on admissible triples with $q \leq 0$. The right-hand side 0 on those pieces is therefore exact.

5. PROOF OF THEOREM 1.1

By Corollary 2.2 it suffices to show $T_3(C) \geq 8$ for every kissing configuration $C \subset S^3$, with equality only for D_4 . By [9] we have $|C| \leq 24$, and by [7] a configuration with $|C| = 24$ is a D_4 root system, for which $T_3 = 8$ by Lemma 2.5.

Let $|C| \leq 23$. Then C has at most 253 pairs and 1771 triples. Lemmas 2.4 and 4.1 give

$$T_3(C) \geq F(C) - 253\varepsilon_2 - 1771\varepsilon_3,$$

since the singleton term only improves, the pair and triple terms on the pieces with right-hand side 0 are exact, and each remaining term loses at most ε_2 or ε_3 . Theorem 3.2 then gives

$$T_3(C) \geq 2\pi^2 - \frac{1828197}{156250} - 253\varepsilon_2 - 1771\varepsilon_3 = 8.0369770014675 \dots > 8.$$

If $\text{vol}(V_0) = 8$, then $T_3(C) \leq 8$ by Corollary 2.2, so C is a D_4 root system and V_0 is the regular 24-cell. Conversely, if the contacts form $\sqrt{2}$ times a D_4 root system, their half-spaces cut out a regular 24-cell of circumradius $\sqrt{2}$. Every other centre y has $|y| \geq 2\sqrt{2}$, so H_y lies in $\{|x| > \sqrt{2}\}$ and misses the cell. Hence V_0 is this 24-cell and $\text{vol}(V_0) = 8$. $\square$

6. REMARKS

Remark 6.1 (The bound on the number of points). The values in this remark are floating-point solutions of the semidefinite programs, with duality gaps below 10^{-6} ; they are not certified. Without the constraint $|C| \leq 23$, the same relaxation at degrees (14, 16) gives only $T_3 \geq 7.9617$, and 7.9932 at degrees (16, 18). The optimal moment vector at (14, 16) has $|C| \approx 23.78$, and its numbers of pairs, triples and quadruples are close to those of a mixture of a 23-point and a 24-point configuration. Bounding $|C|$ excludes it. With the bound $|C| \leq 23.5$, which is equivalent for integer sizes, the relaxation gives 7.9807. Replacing T_3 by the exact truncated volume $T_3 + S_4$ of Corollary 2.2, with a fourth piece for quadruples, raises the value without the bound to 7.9953 and with the bound $|C| \leq 23.5$ to 8.0239. Whether the relaxation for $T_3 + S_4$ reaches 8 without a bound on $|C|$ at some degree is open.

Remark 6.2 (The packing constraint is needed). The bound concerns cells whose facets come from a kissing configuration. Without the angular constraint it fails. The polytope $\{x : \langle x, q_i \rangle \leq 1, i = 1, \dots, 24\}$ for the rational vectors q_i listed in the supplement contains the unit ball and has 24 facets, yet its volume is $7.9655697\dots < 8$. This was computed exactly, by two independent methods [11]. Its normals have minimal angle about 54.6° . A volume below 8 in this setting was observed numerically in the code accompanying [2]. So no argument about facet directions alone, in the manner of Fejes Tóth's bound for polyhedra with twelve faces containing a ball [6], can give the conjecture. Any proof must use the 60° separation.

Remark 6.3 (Beyond contacts). For a general cell, the centres at distance between 2 and $2\sqrt{2}$ cut smaller caps from B , and Proposition 2.1 still applies. The natural extension of Theorem 3.2 is a level-two relaxation in which each point carries its distance as a label. By Proposition 2.3, rounding labels down is valid, but near D_4 the bound grows only linearly in the displacement, so such a relaxation must keep the labels continuous there. We do not pursue this here.

DATA AVAILABILITY

The certificate, the checker, the interval-arithmetic proofs of Lemma 4.1, the construction scripts and the exact computation behind Remark 6.2 are in [11].

USE OF AI TOOLS

Generative AI tools were used extensively to write and run code, to carry out and check computations, and to draft text. The author checked the results and the text, and takes responsibility for the content.

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