

A LOWER BOUND FOR THE SMALLEST LIMIT POINT OF THE ABSOLUTE MAHLER MEASURE OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

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ABSTRACT. Let $\Omega(\alpha) = M(\alpha)^{1/\deg \alpha}$ be the absolute Mahler measure of an algebraic number α , so that $\log \Omega(\alpha)$ is its absolute logarithmic Weil height. By work of Smyth, every number in $[1.7273\dots, \infty)$ is a limit point of the values of Ω on totally positive algebraic integers, and Flammang proved that $\Omega(\alpha) \geq 1.722069$ for every totally positive algebraic integer α outside an explicit finite set. We prove that all but finitely many totally positive algebraic numbers α satisfy $\Omega(\alpha) > 1.7234615$. Hence the smallest limit point of these values lies between 1.7234615 and 1.7274, and all but finitely many totally real algebraic numbers β satisfy $\log \Omega(\beta) > 0.2721673$. The proof combines Smyth's method of auxiliary functions with the logarithmic energy term that Orloski, Talebizadeh Sardari and Smith derived from the integrality of discriminants. The auxiliary function uses x and eighteen of Flammang's polynomials, and the reference measure has a piecewise linear density on 5473 rational nodes, so that its potential and energy have closed forms. Two separately written programs verify the resulting inequality on the positive half-line in ball arithmetic.

1. INTRODUCTION

Let α be an algebraic number of degree n , with minimal polynomial $P(x) = a \prod_{i=1}^n (x - \alpha_i)$ in $\mathbb{Z}[x]$, primitive and with leading coefficient $a \geq 1$. The Mahler measure and the absolute Mahler measure of α are

$$M(\alpha) = a \prod_{i=1}^n \max(1, |\alpha_i|), \quad \Omega(\alpha) = M(\alpha)^{1/n},$$

and $h(\alpha) = \log \Omega(\alpha)$ is the absolute logarithmic Weil height of α . We call α totally positive if all its conjugates α_i are real and positive. Let $\mathcal{L}_+$ be the set of the values $\Omega(\alpha)$, where α runs through the totally positive algebraic integers, and let ℓ_+ be the smallest limit point of $\mathcal{L}_+$.

Schinzel [17] proved that $\Omega(\alpha) \geq (1 + \sqrt{5})/2$ for every totally positive algebraic integer $\alpha \neq 0, 1$, with equality for the roots of $x^2 - 3x + 1$. Smyth [20, Theorems 1 and 2] proved that the values $\Omega(\beta)$, for totally real algebraic integers β , have the limit point $\ell = 1.31427\dots$ and are dense in (ℓ, ∞) . If $\beta \neq 0$ is a totally real algebraic integer, then β^2 is a totally positive algebraic integer and $\Omega(\beta^2) = \Omega(\beta)^2$. Hence every number in $[\ell^2, \infty)$ is a limit point of $\mathcal{L}_+$, and $\ell_+ \leq \ell^2 = 1.7273\dots$

Lower bounds for ℓ_+ come from the method of auxiliary functions. Suppose that

$$(1.1) \quad \log^+ x \geq m + \sum_j c_j \log |Q_j(x)| \quad \text{for all } x > 0,$$

where $\log^+ x = \log \max(1, x)$, the Q_j are finitely many nonzero integer polynomials and $c_j \geq 0$. If α is a totally positive algebraic integer with $Q_j(\alpha) \neq 0$ for all j , then each product $\prod_i Q_j(\alpha_i)$ is a nonzero integer, since it is a resultant, and averaging (1.1) over the conjugates of α gives

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$\log \Omega(\alpha) \geq m$. Smyth [21] found such an inequality by linear programming and checked it in interval arithmetic. He stated the result for totally real algebraic integers β , as $\Omega(\beta) > 1.31040$ when β^2 is not in an explicit finite set, and his inequality gives $\Omega(\alpha) > 1.31040^2 > 1.7171$ for all totally positive algebraic integers α outside that set [21, (6) and (9)]. Flammang raised the bound to 1.720566 [6], Mu and Wu to 1.721916 [11], and Flammang to 1.722069 [7, Theorem 1]. In each case the bound holds for all but finitely many totally positive algebraic integers, so it is a lower bound for ℓ_+ . In [7] the exceptions are the roots of seven listed polynomials, two of which, Q_6 and Q_9 in Table 1, are reducible (their factors reappear in Section 4.1). In Section 4.4 we check the weights printed in [7, Table 1] against her constant. Flammang conjectures that the next point of $\mathcal{L}_+$ is 1.722325, attained at the roots of $x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1$.

Theorem 1.1. *Let $m_0 = \frac{19152077587359311629}{35184372088832000000} = 0.5443347841\dots$. For every $\varepsilon > 0$, all but finitely many totally positive algebraic numbers α satisfy $\log \Omega(\alpha) \geq m_0 - \varepsilon$. In particular, all but finitely many totally positive algebraic numbers α satisfy $\Omega(\alpha) > 1.7234615$, and*

$$1.7234615 < e^{m_0} \leq \ell_+ \leq \ell^2 = 1.7273\dots$$

The first statement of Theorem 1.1 is equivalent to saying that the essential minimum $\sup\{H : \{\alpha \text{ totally positive} : \Omega(\alpha) \leq H\} \text{ is finite}\}$ is at least e^{m_0} . Since $M(\alpha)$ includes the leading coefficient, the theorem concerns all totally positive algebraic numbers, not only integers. It leaves open which algebraic integers have $\Omega(\alpha) < e^{m_0}$ and which points of $\mathcal{L}_+$ lie below e^{m_0} , such as the conjectured point 1.722325. Since $\beta \mapsto \beta^2$ is at most two-to-one and $h(\beta^2) = 2h(\beta)$, we obtain the following.

Corollary 1.2. *For every $\varepsilon > 0$, all but finitely many totally real algebraic numbers β satisfy $h(\beta) \geq m_0/2 - \varepsilon$. In particular, all but finitely many totally real algebraic numbers β satisfy $h(\beta) > 0.2721673$ and $\Omega(\beta) > 1.3128067$.*

For totally real algebraic integers, the theorem of Flammang [7] gives $h(\beta) \geq \frac{1}{2} \log 1.722069 = 0.271763\dots$ with finitely many exceptions, and Smyth's construction shows that the lower limit of h is at most $\log \ell = 0.27328\dots$. For totally real algebraic numbers $\beta \neq 0, \pm 1$, Schinzel's inequality $h(\beta) \geq \frac{1}{2} \log \frac{1+\sqrt{5}}{2} = 0.2406059\dots$ holds without integrality [8, Theorem 1], [23, Section 6.1]. Remark 2.4 explains why the auxiliary functions of [7] also apply to algebraic numbers that are not integers.

We add to (1.1) a term that comes from discriminants. For a probability measure ν on $\mathbb{C}$ with compact support, let $U_\nu(z) = \int \log |z - y| d\nu(y)$ be its logarithmic potential and $I(\nu) = \int U_\nu d\nu$ its logarithmic energy. The average of $2U_\nu - I(\nu)$ over the conjugates of α is bounded below by the energy of a slightly smoothed copy of the uniform measure on the conjugates, up to a small error. That energy is bounded below because the discriminant of the minimal polynomial is a nonzero integer. So it suffices to prove

$$\log^+ x \geq m + \sum_j c_j \log |Q_j(x)| + \lambda(2U_\nu(x) - I(\nu)) \quad \text{for all } x > 0$$

for some $\lambda \geq 0$ and some ν that gives mass at most Kr to every disc of radius $r \leq 1$, for a constant K . A measure on $\mathbb{R}$ with a bounded density has this property. The case $\lambda = 0$ is (1.1). Lemma 2.2 below proves this, with an explicit error term for each degree. The leading coefficient of α enters through $M(\alpha)$, the resultants and the discriminant, and it is absorbed because $\sum_j c_j \deg Q_j + 2\lambda \leq 1$, a condition that the auxiliary inequality itself forces.

Orloski, Talebizadeh Sardari and Smith [12] introduced this energy condition for the Schur–Siegel–Smyth trace problem, the lower limit of $\text{tr}(\alpha)/\deg(\alpha)$ over totally positive algebraic integers, where the left side is x instead of $\log^+ x$. For that problem, Smyth's method [22] and its refinements gave a sequence of improved lower bounds. Serre showed, in a letter published

as an appendix to [1], that inequalities with x on the left and only polynomial terms on the right cannot hold with a constant larger than $1.89830\dots$, and Smith [19] later proved that the trace constant is less than 1.89831 . Discriminants entered the trace problem already in Siegel's work [18]. Orloski and Talebizadeh Sardari [13] proved that every weak* limit μ of the uniform measures on the conjugates of a sequence of distinct algebraic integers, all of whose conjugates lie in a fixed compact set, satisfies $\int \log |Q(x_1, \dots, x_n)| d\mu(x_1) \cdots d\mu(x_n) \geq 0$ for every nonzero $Q \in \mathbb{Z}[x_1, \dots, x_n]$ [12, Section 1.1]. For $Q = x_1 - x_2$ this says that the energy of μ is nonnegative. For the Mahler measure of totally positive algebraic integers, Rhin [15] proved a version of Schinzel's inequality that improves as the discriminant grows [23, Section 9.2].

Our auxiliary function uses the polynomial x and the eighteen reciprocal polynomials $x^d U_j(x + 1/x - 2)$ built from the polynomials U_j of degree d in Flammang's table [7, Table 1] with $j = 1, \dots, 8, 10, 11, 13, 14, 16, 18, 20, 21, 24, 25$. The reference measure ν has a continuous piecewise linear density on 5473 rational nodes in $[0, 11.2]$. Its potential and energy have closed forms (Lemma 3.1), and two programs verify the auxiliary inequality on $(0, \infty)$ in ball arithmetic [10] (Section 3). Without the energy term, the best auxiliary function with the same nineteen polynomials gives only about 1.72159, and with x and all twenty-seven polynomials of [7, Table 1] about 1.72207 (Section 4).

Section 2 proves the lower bound from an auxiliary function with an energy term. Section 3 describes the certificate and its verification and proves Theorem 1.1. Section 4 explains how the certificate was found and collects numerical observations.

2. AUXILIARY FUNCTIONS WITH AN ENERGY TERM

For finite positive Borel measures θ, η on $\mathbb{C}$ with compact support we write

$$U_\eta(z) = \int \log |z - y| d\eta(y), \quad I(\theta, \eta) = \int U_\eta d\theta, \quad I(\theta) = I(\theta, \theta).$$

Since $\log |z - y|$ is bounded above on compact sets, these integrals exist in $[-\infty, \infty)$, and $I(\theta, \eta) = I(\eta, \theta)$ by Tonelli's theorem. We write $D(y, r) = \{z \in \mathbb{C} : |z - y| < r\}$ for the open disc and $\omega_{y,r}$ for the normalized arc length measure on the circle $|z - y| = r$. By Jensen's formula,

$$(2.1) \quad U_{\omega_{y,r}}(z) = \max(\log |z - y|, \log r) \quad (z \in \mathbb{C}).$$

We use the following classical fact, see [16, Lemma I.1.8], where the energy is defined with the opposite sign. We include the short proof.

Lemma 2.1. *Let θ and η be probability measures on $\mathbb{C}$ with compact support such that $I(\theta)$, $I(\eta)$ and $I(\theta, \eta)$ are finite. Then $I(\theta) \leq 2I(\theta, \eta) - I(\eta)$.*

Proof. Let $\sigma = \theta - \eta$ and let R be such that both supports lie in $D(0, R)$. For z, w in the supports, $|\log |z - w|| = 2\log^+ |z - w| - \log |z - w| \leq 2\log^+(2R) - \log |z - w|$, so $\iint |\log |z - w|| d|\sigma|(z) d|\sigma|(w) < \infty$, because $|\sigma| \leq \theta + \eta$ and all mixed energies are finite. For $t > 0$ we have $-2\log t = \int_0^\infty (e^{-ut^2} - e^{-u}) \frac{du}{u}$, and the integrand has constant sign, so its absolute value integrates to $2|\log t|$. A measure with an atom has energy $-\infty$, so θ and η have no atoms, and the diagonal $\{z = w\}$ is null for $|\sigma| \times |\sigma|$. By Fubini's theorem and $\sigma(\mathbb{C}) = 0$,

$$-2 \iint \log |z - w| d\sigma(z) d\sigma(w) = \int_0^\infty \left(\iint e^{-u|z-w|^2} d\sigma(z) d\sigma(w) \right) \frac{du}{u} \geq 0,$$

since the Gaussian kernel $e^{-u|z-w|^2}$ is positive definite on $\mathbb{C} = \mathbb{R}^2$ (its Fourier transform is positive). The left side is $-2(I(\theta) - 2I(\theta, \eta) + I(\eta))$ by bilinearity. $\square$

Lemma 2.2 (Auxiliary function with an energy term). *Let $Q_1, \dots, Q_J \in \mathbb{Z}[x]$ be nonzero, $c_1, \dots, c_J > 0$, $\lambda \geq 0$ and $m \in \mathbb{R}$. Let ν be a probability measure on $\mathbb{C}$ with compact support,*

and let $K > 0$ be such that $\nu(D(y, r)) \leq Kr$ for all $y \in \mathbb{C}$ and $0 < r \leq 1$. Let Z be the set of positive real roots of the polynomials Q_j . Assume that

$$(2.2) \quad F(x) = \log^+ x - m - \sum_{j=1}^J c_j \log |Q_j(x)| - \lambda(2U_\nu(x) - I(\nu)) \geq 0 \quad \text{for all } x \in (0, \infty) \setminus Z.$$

Then $\sum_j c_j \deg Q_j + 2\lambda \leq 1$. Moreover, every totally positive algebraic number α of degree $n \geq 1/(2K)$ that is not a root of one of the polynomials Q_j satisfies

$$(2.3) \quad \log \Omega(\alpha) \geq m - \frac{\lambda(1 + \log(2Kn))}{n}.$$

Proof. For $y \in \mathbb{C}$ and $0 < r \leq 1$ we have $\max(\log |x - y|, \log r) - \log |x - y| = \int_0^r \mathbf{1}_{\{|x-y| < u\}} \frac{du}{u}$. Integrating over x with respect to ν gives

$$(2.4) \quad \int \max(\log |x - y|, \log r) d\nu(x) = U_\nu(y) + \int_0^r \nu(D(y, u)) \frac{du}{u} \leq U_\nu(y) + Kr.$$

With $r = 1$ this gives $U_\nu(y) \geq \int \log^+ |x - y| d\nu(x) - K \geq -K$. Since ν has compact support, U_ν is also bounded above on compact sets, so U_ν is finite everywhere and $I(\nu)$ is finite. As $x \rightarrow +\infty$ we have $U_\nu(x) = \log x + o(1)$ and $\log |Q_j(x)| = \deg(Q_j) \log x + O(1)$, so

$$F(x) = \left(1 - \sum_j c_j \deg Q_j - 2\lambda\right) \log x + O(1),$$

and (2.2) forces $\sum_j c_j \deg Q_j + 2\lambda \leq 1$.

Now let α be as in the statement, with minimal polynomial $P(x) = a \prod_{i=1}^n (x - \alpha_i)$. All α_i are positive, so

$$(2.5) \quad n \log \Omega(\alpha) = \log a + \sum_{i=1}^n \log^+ \alpha_i.$$

First, let $1 \leq j \leq J$. Since $Q_j(\alpha) \neq 0$ and P is irreducible, Q_j vanishes at no α_i . In particular no α_i lies in Z , so every $F(\alpha_i)$ is a real number. The resultant $\text{Res}(P, Q_j) = a^{\deg Q_j} \prod_i Q_j(\alpha_i)$ is a nonzero integer, hence

$$(2.6) \quad \sum_{i=1}^n \log |Q_j(\alpha_i)| \geq -\deg(Q_j) \log a.$$

Since P is irreducible, its discriminant $\text{disc}(P) = a^{2n-2} \prod_{i < k} (\alpha_i - \alpha_k)^2$ is also a nonzero integer, and

$$(2.7) \quad \sum_{i \neq k} \log |\alpha_i - \alpha_k| \geq -(2n-2) \log a.$$

To smooth the conjugates over small circles, let $0 < r \leq 1$ and $\tau_r = \frac{1}{n} \sum_{i=1}^n \omega_{\alpha_i, r}$. By (2.1), $I(\omega_{y, r}) = \log r$, and for $i \neq k$

$$I(\omega_{\alpha_i, r}, \omega_{\alpha_k, r}) = \int \max(\log |z - \alpha_k|, \log r) d\omega_{\alpha_i, r}(z) \geq \int \log |z - \alpha_k| d\omega_{\alpha_i, r}(z) \geq \log |\alpha_i - \alpha_k|,$$

where the last step is (2.1) again. With (2.7) we get

$$(2.8) \quad I(\tau_r) \geq \frac{\log r}{n} + \frac{1}{n^2} \sum_{i \neq k} \log |\alpha_i - \alpha_k| \geq \frac{\log r}{n} - \frac{(2n-2) \log a}{n^2}.$$

Next we compare τ_r with ν . The potential of τ_r is at least $\log r$ by (2.1), and the potential of ν is at least $-K$. Both potentials are bounded above on compact sets, so $I(\tau_r)$, $I(\nu)$ and $I(\tau_r, \nu)$ are finite. Lemma 2.1 gives $I(\tau_r) \leq 2I(\tau_r, \nu) - I(\nu)$. By Fubini's theorem, (2.1) and (2.4),

$$I(\tau_r, \nu) = \frac{1}{n} \sum_{i=1}^n \int \max(\log |x - \alpha_i|, \log r) d\nu(x) \leq \frac{1}{n} \sum_{i=1}^n U_\nu(\alpha_i) + Kr.$$

Together with (2.8),

$$(2.9) \quad \frac{1}{n} \sum_{i=1}^n (2U_\nu(\alpha_i) - I(\nu)) \geq I(\tau_r) - 2Kr \geq \frac{\log r}{n} - \frac{(2n-2)\log a}{n^2} - 2Kr.$$

Finally, we average (2.2) over the conjugates α_i and use (2.6) and (2.9):

$$\frac{1}{n} \sum_{i=1}^n \log^+ \alpha_i \geq m - \frac{\log a}{n} \left(\sum_j c_j \deg Q_j + 2\lambda \frac{n-1}{n} \right) - \lambda \left(2Kr + \frac{\log(1/r)}{n} \right).$$

By (2.5),

$$\log \Omega(\alpha) \geq m + \frac{\log a}{n} \left(1 - \sum_j c_j \deg Q_j - 2\lambda \frac{n-1}{n} \right) - \lambda \left(2Kr + \frac{\log(1/r)}{n} \right).$$

The second term is nonnegative, because $\log a \geq 0$ and $\sum_j c_j \deg Q_j + 2\lambda \leq 1$. The choice $r = 1/(2Kn)$, admissible since $n \geq 1/(2K)$, gives (2.3). $\square$

Corollary 2.3. *Under the assumptions of Lemma 2.2, for every $\varepsilon > 0$ only finitely many totally positive algebraic numbers α satisfy $\log \Omega(\alpha) \leq m - \varepsilon$.*

Proof. Choose $n_0 \geq 1/(2K)$ such that $\lambda(1 + \log(2Kn))/n < \varepsilon$ for all $n \geq n_0$. If α is totally positive and $\log \Omega(\alpha) \leq m - \varepsilon$, then either α is one of the finitely many roots of the polynomials Q_j , or α has degree less than n_0 by (2.3). In the second case $h(\alpha) \leq m$, and by Northcott's theorem [2, Theorem 1.6.8] there are only finitely many such α . $\square$

Proof of Corollary 1.2 from Theorem 1.1. Let $\varepsilon > 0$, and let $\beta \neq 0$ be a totally real algebraic number with $h(\beta) < m_0/2 - \varepsilon$. Then β^2 is totally positive and $h(\beta^2) = 2h(\beta) < m_0 - 2\varepsilon$. By Theorem 1.1 only finitely many numbers occur as β^2 , and each of them has at most two square roots. With $\beta = 0$ there is one more exception. $\square$

Remark 2.4. For $\lambda = 0$, Lemma 2.2 is the classical argument of [21], extended to algebraic numbers that need not be integers. Its conclusion is then $\log \Omega(\alpha) \geq m$ for every totally positive algebraic number α that is not a root of one of the Q_j , because the measure ν plays no role. For instance, the uniform measure on $[0, 1]$ satisfies the hypothesis with $K = 2$, and then every $n \geq 1$ is admissible. Flammang [7, Section 2.1] uses auxiliary functions $\log^+ x - c_0 \log x - \sum_j c_j \log |Q_j(x)|$ with reciprocal Q_j and $2c_0 + \sum_j c_j \deg Q_j = 1$, which are invariant under $x \mapsto 1/x$. Lemma 2.2 with $\lambda = 0$ applies to them, so bounds obtained with such functions, which have no energy term, also hold for all totally positive algebraic numbers outside a finite set.

3. THE CERTIFICATE

3.1. The polynomials. The certificate uses the 19 polynomials of Table 1. For $1 \leq k \leq 18$, the polynomial Q_k is the reciprocal polynomial $Q_k(x) = x^{2d}U(x + 1/x - 2)$ of degree $2d$, where U is the polynomial of degree d listed in [7, Table 1] under the index j given in the table. The weights c_k are dyadic rationals. The polynomial $Q_1 = (x - 1)^2$ vanishes at the corner $x = 1$ of $\log^+ x$, so the auxiliary function tends to $+\infty$ there. The polynomials Q_6 , Q_9 and Q_{15} are

products of a polynomial and its reciprocal, and Q_{15} is the only Q_k whose leading coefficient is not 1. All roots of the Q_k are real and nonnegative, except for two pairs of complex roots of Q_{16} .

TABLE 1. The polynomials and their weights, rounded to four significant digits. For $1 \leq k \leq 18$, $Q_k(x) = x^{\deg U} U(x + 1/x - 2)$, where U is the polynomial U_j of [7, Table 1], given by its coefficients from the leading one down to the constant term.

k	j	U_j , or Q_0	$\deg Q_k$	c_k
0	–	$Q_0 = x$	1	$1.932 \cdot 10^{-1}$
1	1	1, 0	2	$1.558 \cdot 10^{-1}$
2	2	1, –1	2	$3.860 \cdot 10^{-2}$
3	3	1, –2	2	$4.143 \cdot 10^{-7}$
4	4	1, –3, 1	4	$1.295 \cdot 10^{-2}$
5	5	1, –4, 1	4	$2.372 \cdot 10^{-4}$
6	6	1, –5, 6, –1	6	$1.686 \cdot 10^{-3}$
7	7	1, –6, 5, –1	6	$5.191 \cdot 10^{-4}$
8	8	1, –7, 13, –7, 1	8	$4.143 \cdot 10^{-3}$
9	10	1, –7, 14, –8, 1	8	$7.517 \cdot 10^{-4}$
10	11	1, –8, 14, –7, 1	8	$9.505 \cdot 10^{-5}$
11	13	1, –13, 61, –131, 136, –66, 14, –1	14	$1.237 \cdot 10^{-5}$
12	14	1, –13, 61, –132, 138, –67, 14, –1	14	$1.946 \cdot 10^{-6}$
13	16	1, –13, 62, –137, 147, –73, 15, –1	14	$8.026 \cdot 10^{-7}$
14	18	1, –15, 83, –220, 303, –220, 83, –15, 1	16	$8.049 \cdot 10^{-4}$
15	20	2, –26, 128, –308, 391, –265, 94, –16, 1	16	$3.204 \cdot 10^{-7}$
16	21	1, –21, 182, –853, 2386, –4151, 4545, –3109, 1298, –315, 40, –2	22	$4.052 \cdot 10^{-7}$
17	24	1, –25, 266, –1582, 5817, –13835, 21682, –22449, 15233, –6667, 1833, –302, 27, –1	26	$3.650 \cdot 10^{-7}$
18	25	1, –31, 413, –3141, 15261, –50187, 115410, –189036, 222621, –189036, 115410, –50187, 15261, –3141, 413, –31, 1	32	$5.251 \cdot 10^{-8}$

The roots of the Q_k , which Lemma 2.2 excludes, include totally positive algebraic integers with $\Omega(\alpha) < e^{m_0}$, for instance 1, the roots of $x^2 - 3x + 1$, and the roots of the sextic Q_7 of Flammang’s conjecture, where $\Omega = 1.72232 \dots$. They are among the finitely many exceptions that Theorem 1.1 allows.

3.2. The reference measure. Let $0 = e_0 < e_1 < \dots < e_N = 11.2$ be the $N+1 = 5473$ nodes of the certificate, which are rational numbers. Let G be the continuous function on $\mathbb{R}$ that is linear on each interval $[e_k, e_{k+1}]$, takes rational values $G(e_k) = G_k \geq 0$ at the nodes, with $G_0 = G_N = 0$, and vanishes outside $[0, 11.2]$. Let ρ be the measure $G(x) dx$, let $s = \rho(\mathbb{R}) = 0.05288541 \dots$, and put

$$\nu = \rho/s, \quad \lambda = s.$$

Then $\lambda(2U_\nu - I(\nu)) = 2U_\rho - I(\rho)/s$, and the function (2.2) becomes

$$(3.1) \quad F(x) = \log^+ x - m_0 - \sum_{k=0}^{18} c_k \log |Q_k(x)| - 2U_\rho(x) + \frac{I(\rho)}{s}.$$

The function G vanishes outside $[0.08525, 11.2]$ and on 401 of the intervals $[e_k, e_{k+1}]$ inside it, which form nine gaps, each containing one root of a Q_k . A disc of radius r meets $\mathbb{R}$ in an interval

of length at most $2r$, so ν satisfies the hypothesis of Lemma 2.2 with $K = 2 \max G/s$. Computed exactly from the node values, $\max G/s$ is less than 2.44, so we may take $K = 4.88$, a value that enters only the error term of Remark 3.4.

For $t \in \mathbb{R}$ let $H(t) = \frac{t^2}{2} \log |t| - \frac{3}{4}t^2$ and $L(t) = \frac{t^4}{24} \log |t| - \frac{25}{288}t^4$, with $H(0) = L(0) = 0$. Let C_i be the jump of the slope of G at e_i , that is, $C_0 = g_0$, $C_N = -g_{N-1}$ and $C_i = g_i - g_{i-1}$ for $0 < i < N$, where $g_k = (G_{k+1} - G_k)/(e_{k+1} - e_k)$. Then $G'' = \sum_i C_i \delta_{e_i}$ in the sense of distributions, and $\sum_i C_i = \sum_i C_i e_i = 0$ because G vanishes outside $[e_0, e_N]$.

Lemma 3.1. *For $x \in \mathbb{R}$,*

$$U_\rho(x) = \sum_{i=0}^N C_i H(e_i - x), \quad I(\rho) = \sum_{i,k=0}^N C_i C_k L(e_i - e_k).$$

Proof. A direct computation gives $H''(t) = \log |t|$ and $L'' = H$ for $t \neq 0$. Both functions are even, H is continuously differentiable and L is three times continuously differentiable. Since G is Lipschitz with compact support, two integrations by parts give

$$U_\rho(x) = \int H''(y - x) G(y) dy = \int H(y - x) dG'(y) = \sum_i C_i H(e_i - x),$$

with no boundary terms. Here the second integral is a Stieltjes integral with respect to the step function G' , which jumps by C_i at e_i . In the same way $\int H(e_i - x) G(x) dx = \sum_k C_k L(e_k - e_i)$, and integrating the first formula against G gives the second. $\square$

Since H is continuously differentiable with $H'' = \log |t|$, the potential U_ρ is continuously differentiable on $\mathbb{R}$, and $U_\rho''(x) = \sum_i C_i \log |x - e_i|$ for x different from the nodes. With m_0 , c_k , e_i and G_i exact, all terms of (3.1) at a rational point x are elementary functions of rational numbers.

3.3. Verification. Two programs verify the following statement.

Proposition 3.2. *For the data of the certificate, $\sum_k c_k \deg Q_k + 2s < 1$, and the function (3.1) satisfies $F(x) \geq 0$ for every $x > 0$ that is not a root of one of the polynomials Q_k .*

Both programs use Arb ball arithmetic [10] through python-flint [14]. In ball arithmetic every computed quantity is an interval, given by a midpoint and a radius, that is guaranteed to contain the exact value. Both programs read the data as exact rationals, check that the polynomials are distinct with nonzero leading coefficients, that x is one of them with a positive weight, that the weights and the G_k are nonnegative, that the nodes increase, that $G_0 = G_N = 0$ and that $s > 0$, and compute $\sum_k c_k \deg Q_k + 2s$ exactly. Its value is $0.80677179 \dots < 1$. All nineteen weights are positive, as Lemma 2.2 requires. Both programs also rely on the certified root isolation of FLINT [5], which encloses the complex roots of an integer polynomial in disjoint balls. The second program also relies on FLINT returning the real roots with imaginary part zero, while the first checks its brackets for the real roots by Sturm counts and exact signs. Apart from this, and from FLINT's factorization over $\mathbb{Z}$ in the second program, acceptance depends only on exact comparisons of rationals and on the positivity of lower ends of balls, so heuristic choices, such as the widths of windows and the points of subdivision, affect the running time but not the validity of an accepted bound.

The first program works with 200-bit balls. It computes the slope jumps C_i , checks $\sum_i C_i = \sum_i C_i e_i = 0$ exactly, and evaluates U_ρ and $I(\rho)$ by Lemma 3.1. It treats the two tails separately and covers $[x_{lo}, x_{hi}]$ by root windows and chord intervals.

- On $(0, x_{lo}]$, with $x_{lo} = 1/1000$, it uses $\log^+ x = 0$, $-c_0 \log x \geq -c_0 \log x_{lo}$, an upper bound for $|Q_k|$ on $[0, x_{lo}]$ for $k \geq 1$, and $U_\rho(x) \leq U_\rho(0)$, which holds because U_ρ decreases to the

left of the support of G . The resulting lower bound for F is larger than 0.790. The program also checks with a Sturm sequence that no Q_k with $k \geq 1$ vanishes on $[0, x_{\text{lo}}]$.

- On $[x_{\text{hi}}, \infty)$, with $x_{\text{hi}} = 881$, it uses $|Q_k(x)| \leq \|Q_k\|_1 x^{\deg Q_k}$, where $\|Q_k\|_1$ is the sum of the absolute values of the coefficients, and $U_\rho(x) \leq s \log x$. This gives $F(x) \geq \kappa \log x - b$ with $\kappa = 1 - \sum_k c_k \deg Q_k - 2s = 0.19322820\dots$ and $b = m_0 - I(\rho)/s + \sum_k c_k \log \|Q_k\|_1$, and $\kappa \log x_{\text{hi}} - b > 0.3866$.
- On $[x_{\text{lo}}, x_{\text{hi}}]$ it first encloses each root. For each Q_k it takes the squarefree part, counts its distinct roots in $(x_{\text{lo}}, x_{\text{hi}})$ with a Sturm sequence, and finds for each root an interval with rational endpoints on which the squarefree part changes sign. These intervals come from the certified complex root isolation of FLINT, they are checked by exact sign evaluation, and their number must equal the Sturm count. Around each such interval it looks for a window W on which a lower bound for F is positive, shrinking W if necessary, and fails if it finds none. The bound uses the values at the rational centre x_c of W , with $Q_k(x_c)$ exact, and derivative bounds on W : each $\log |Q_k|$ is replaced by the logarithm of $|Q_k(x_c)| + \frac{|W|}{2} \max_W |Q'_k|$, which bounds $\log |Q_k|$ from above on all of W , roots included, and U_ρ by $U_\rho(x_c) + \frac{|W|}{2} \max_W |U'_\rho|$, while $\log^+ x$ is replaced by its value at the left end of W . There are 199 windows after merging, and each merged window is checked again, split if necessary. The corner $x = 1$ of $\log^+ x$ lies in the window of the double root of Q_1 .
- The nodes, the window ends, x_{lo} , x_{hi} and 1 cut the rest of $[x_{\text{lo}}, x_{\text{hi}}]$ into 5,568 intervals. On such an interval $[p, q]$ it uses Lemma 3.3 below with $f = F$. It evaluates $F(p)$ and $F(q)$ with $Q_k(p)$, $Q_k(q)$ and the differences $e_i - p$, $e_i - q$ computed exactly, and it bounds $\int_p^q (F'')_+$ from above. The term $(\log^+)'' \leq 0$ is dropped. For the polynomial terms it uses $-(\log |Q_k|)''(x) = \sum_\zeta \text{Re}(x - \zeta)^{-2}$, the sum over the complex roots ζ of Q_k with multiplicity, enclosed by FLINT, on a ball that contains $[p, q]$. If such a ball meets an enclosure of a root, the interval is bisected. For $-2U''_\rho = -2 \sum_i C_i \log |x - e_i|$ it uses a ball bound for the nodes off $[p, q]$. A node $e_i \in \{p, q\}$ contributes a logarithmic singularity, and the program requires $q - p \leq 1$. If $C_i > 0$, the positive part of $-2C_i \log |x - e_i|$ has integral at most $2C_i \delta (1 + \log(1/\delta))$ over $[p, q]$, where $\delta = q - p$. If $C_i < 0$, the term is at most $2|C_i| \log \delta$ on $[p, q]$. If the resulting lower bound for F on $[p, q]$ is not positive, the interval is bisected. The program fails if the lower bound for F at an end of an interval is not positive, or after 50 successive bisections. The windows and the intervals cover $[x_{\text{lo}}, x_{\text{hi}}]$, which the program checks, and bisection keeps the cover.

Lemma 3.3. *Let f be continuously differentiable on $[p, q]$ with f' absolutely continuous, and let $\delta = q - p$. Then for $p \leq x \leq q$,*

$$f(x) \geq \min(f(p), f(q)) - \frac{\delta}{4} \int_p^q (f''(t))_+ dt.$$

Proof. Let Λ be the linear function with $\Lambda(p) = f(p)$ and $\Lambda(q) = f(q)$. Then $f(x) = \Lambda(x) - \int_p^q k(x, t) f''(t) dt$ with $k(x, t) = (\min(x, t) - p)(q - \max(x, t))/\delta$. Since $0 \leq k \leq \delta/4$, the integral is at most $\frac{\delta}{4} \int_p^q (f'')_+$, and $\Lambda(x) \geq \min(f(p), f(q))$. $\square$

On an interval $[p, q]$ of the last step, F satisfies the hypotheses of Lemma 3.3. Indeed, $[p, q]$ contains no root of a Q_k , and 1 and the nodes are not interior points, so $\log^+ x$ and the polynomial terms are smooth on $[p, q]$. The function $U'_\rho(x) = -\sum_i C_i H'(e_i - x)$ is absolutely continuous, since $H'(t) = t \log |t| - t$ is an indefinite integral of the locally integrable function $\log |t|$.

The first program made 28,408 interval checks. The smallest lower bound for F at an endpoint of an accepted interval was $2.884 \cdot 10^{-6}$. With 64 processes on an x86-64 server with 96 threads the check took 474 seconds.

The second program shares no code with the first, but the certificate format from which it was written also states the closed forms of Lemma 3.1 and the tail bounds of the first program. It works with 128-bit balls and computes $I(\rho)$ with 192 bits. It factors the Q_k over $\mathbb{Z}$ with FLINT and merges equal irreducible factors, adding their weights. It encloses the real roots in FLINT's isolating balls, takes the rational root 1 as an exact breakpoint, and puts each of the 198 irrational positive roots ζ in a dyadic window $[p, q]$, which for this certificate always has width 2^{-100} , while the root 0 of x lies in the left tail. On such a window it bounds the term $-w \log |x - \zeta|$ of the root from below by $-w \log(q - p)$, where w is the total weight with which the irreducible factor vanishing at ζ occurs in the Q_k , and all other terms by ball arithmetic. It evaluates U_ρ and U'_ρ from closed forms on each interval of the grid, not from Lemma 3.1, and computes $I(\rho)$ from closed forms for each pair of grid intervals, cross-checked against Lemma 3.1. On each interval between breakpoints it bounds F from below by a second-order Taylor expansion at the midpoint, with a rigorous bound for F'' and with the logarithmic singularities at end nodes treated separately. The term $-w \log |x - \zeta|$ of a root at an end of an interval, or in an adjacent window, is bounded from below by $-w \log d$, where d is the distance from the far end of the interval to the far end of the window, or to the root if it is rational. The program bisects an interval when the bound is not positive. It fails if the bound on a window is not positive, if the ball for F at the midpoint of an interval lies below 0, or after 40 successive bisections. It chooses its own cut points for the tails, here $1/32$ and $112/5$. Its enclosure of $I(\rho)$ agrees with that of the first program to more than 35 digits. With 32 processes on the same server it took 18 seconds. Table 2 lists the data and the results of both programs.

TABLE 2. The certificate and the two checks.

Bound m_0	0.5443347841... (the fraction is given in Theorem 1.1)	
$\sum_k c_k \deg Q_k; s$	0.70100096...; 0.05288541...	
$\sum_k c_k \deg Q_k + 2s$	0.80677179... < 1	
Nodes; $G = 0$ outside; K	5473 in $[0, 11.2]$; $[0.08525, 11.2]$; 4.88	
$I(\rho)$	$2.8781306519... \cdot 10^{-11}$	
	first program	second program
Left tail	$(0, 1/1000], F \geq 0.790$	$(0, 1/32], F \geq 0.1253$
Right tail	$[881, \infty), F \geq 0.3866$	$[112/5, \infty), F \geq 0.0564$
Root windows	199	198, and a breakpoint at 1
Initial intervals	5,568	5,708
Interval checks	28,408	8,424
Smallest bound	$2.884 \cdot 10^{-6}$ (lower bound for F at the ends of the intervals)	$1.57 \cdot 10^{-9}$ (lower bound on an interval)
Time	474 s, 64 processes	18 s, 32 processes

Proof of Theorem 1.1. By Proposition 3.2, the function (3.1) is nonnegative on $(0, \infty)$ outside the roots of the Q_k . It is the function (2.2) with $\nu = \rho/s$, $\lambda = s$ and $m = m_0$, and ν satisfies the hypothesis of Lemma 2.2 with $K = 4.88$. Corollary 2.3 gives the first statement. Since $e^{m_0} = 1.72346152...$, Corollary 2.3 with $\varepsilon = m_0 - \log 1.7234615 > 0$ shows that only finitely many totally positive algebraic numbers satisfy $\Omega(\alpha) \leq 1.7234615$. Finally, by Corollary 2.3, for every $\varepsilon > 0$ only finitely many values $\Omega(\alpha)$ with α a totally positive algebraic integer lie below $e^{m_0 - \varepsilon}$, so $\ell_+ \geq e^{m_0}$. $\square$

Remark 3.4. The proof is effective. With $\lambda = s < 0.0529$ and $K = 4.88$, inequality (2.3) gives

$$\log \Omega(\alpha) \geq m_0 - \frac{0.0529(3.28 + \log n)}{n}$$

for every totally positive algebraic number α of degree n that is not a root of one of the Q_k . For instance $\Omega(\alpha) > 1.72334$ for $n \geq 10^4$.

4. HOW THE CERTIFICATE WAS FOUND, AND NUMERICAL OBSERVATIONS

Except for the discussion of duality at the start of Section 4.1, the statements of this section are numerical observations, computed in floating point, and the proof does not use them.

4.1. Search. For a fixed finite set of polynomials, the best bound that Lemma 2.2 can give is limited by a problem on measures. Let θ be a probability measure on $(0, \infty)$ with compact support, with $I(\theta) \geq 0$ and $\int \log |Q_k| d\theta \geq 0$ for all k . The condition $I(\theta) \geq 0$ excludes atoms. If (2.2) holds, then integrating it against θ and using Lemma 2.1 gives

$$\int \log^+ x d\theta \geq m + \lambda(2I(\theta, \nu) - I(\nu)) \geq m + \lambda I(\theta) \geq m.$$

So every value m that Lemma 2.2 gives with these polynomials is at most the infimum of $\int \log^+ x d\theta$ over such measures. For heights defined by a continuous Green function on $\mathbb{C}$, Burgos Gil, Menares, Qu and Sombra [3, Theorem D] proved that the best bound obtainable from auxiliary functions with arbitrary finite sets of integer polynomials equals the corresponding infimum over measures, which is the essential minimum of the height. Their setting includes neither the restriction to totally positive numbers nor an energy term, and we do not know whether the analogous equality holds here. We computed this infimum approximately over measures with piecewise linear densities, as a convex program with a linear objective, linear constraints and the energy constraint, using an interior-point solver [9] through a modeling language [4]. We then fixed the reference measure ν to the computed optimal measure and solved the linear program for m , the weights $c_k \geq 0$ and $\lambda \geq 0$, with the degree condition and with (2.2) imposed on a fine grid of points that includes many points near the roots of the Q_k and several points inside each interval between consecutive nodes. The nodes are decimal fractions, and the weights c_k and the values G_k are the exact values of the double-precision solution, with G equal to λ times the density of ν . Finally, m_0 is the value 0.54433778 of the linear program minus $3 \cdot 10^{-6}$. This margin is why the smallest value of F at the ends of the intervals in Table 2 is close to $3 \cdot 10^{-6}$.

The nodes are the points of the grid of spacing 0.004 on $[0, 11.2]$, together with points of spacing 0.001 on $[0.085, 0.72]$ and of spacing 0.00025 on $[0.085, 0.11]$. Around the roots $r \in (0.05, 11.2)$ of the polynomials with weight above 10^{-6} in the linear program of the preceding round of the search, there are points of spacing 0.00025 within 0.002 of r if $r < 0.75$, and of spacing 0.0005 within 0.003 of r otherwise, and r itself, and there are points of spacing 0.00025 near nineteen further points at which the auxiliary function of that round is below $1.1 \cdot 10^{-4}$. All points are rounded to ten decimal places, and a point closer than 10^{-4} to the previous node is dropped.

The polynomial set came from a search. We started from x and the twenty-seven polynomials of [7, Table 1]. For approximate optimal measures θ we computed $\int \log |Q| d\theta$ for the 246,207 monic irreducible polynomials Q of degree $d \leq 8$ with all roots in $[0, 10]$ and with trace at most $3d$ for $d \leq 5$ and at most $2.2d$ for $6 \leq d \leq 8$, by interpolating the potential of θ on a grid of step 10^{-4} . For the 2000 lowest values with $d = 8$, this screen differs from the values computed below by up to $9.2 \cdot 10^{-5}$. In the first screen, of the polynomials with $d \leq 5$ against an early approximate measure, the quartic $x^4 - 8x^3 + 14x^2 - 7x + 1$ and the cubic $x^3 - 6x^2 + 5x - 1$ gave negative values, and we added them to the set. They divide the reciprocal polynomials built from U_{10} and U_6 , which are Q_9 and Q_6 in Table 1. The linear program over the resulting thirty polynomials gave positive weight to the nineteen polynomials of Table 1 and to no others, so the quartic and the cubic do not occur in the certificate. For the reference measure ν of

the certificate we also computed $\int \log |Q| d\nu$ from Lemma 3.1 at the roots of Q . The thirty polynomials of the set give values of at least $-2.7 \cdot 10^{-7}$. Outside the set, the reciprocals of the cubic and the quartic give $-2.50 \cdot 10^{-5}$ and $-1.99 \cdot 10^{-5}$, the factor $x - 1$ of Q_1 gives $-1.3 \cdot 10^{-7}$, and every other screened polynomial gives more than $7.9 \cdot 10^{-3}$.

The linear program evaluated $\log |Q_k|$ by Horner's rule in double precision, which is inaccurate near $x = 1.95$ for the polynomials of degree 32 to 38 among the thirty. A linear program of the same kind with $\log |Q_k|$ computed from the roots, for the reference measure of the certificate and on a slightly different grid, has the value 0.544338. It gives positive weight to twenty polynomials, namely those of Table 1 except Q_{15} , together with the reciprocal polynomials built from U_{17} and U_{27} . Which polynomials receive very small weights depends on details of the grid. Adding the other six irreducible factors of the reducible reciprocal polynomials built from U_6 , U_{10} , U_{19} and U_{20} raises the value of such a linear program by only $2 \cdot 10^{-7}$.

4.2. The discretization gap. For the thirty polynomials of the search, the measure program over piecewise linear densities on the nodes of the certificate has the approximate value 0.5443536. So no certificate with these polynomials gives more than about $m = 0.5443536$, that is, $\Omega \approx 1.72349$. The difference to the linear program with the fixed reference measure is $0.5443536 - 0.54433778 \approx 1.6 \cdot 10^{-5}$. On the 3,809 nodes of the preceding round of the search it was at least $5.5 \cdot 10^{-5}$: there the linear program gives 0.5442990, and the measure program gives 0.5443546 without the cubic and the quartic, and hence at least as much with them.

4.3. The energy term. If the energy term is removed, the linear program over the nineteen polynomials of Table 1 has the value 0.543251, that is, $\Omega \approx 1.72159$. With x and all twenty-seven polynomials of [7, Table 1] it gives $\Omega \approx 1.72207$, in agreement with [7, Theorem 1]. So the energy term raises the bound obtainable from the polynomials of Table 1 by about $1.1 \cdot 10^{-3}$ in $\log \Omega$, and Theorem 1.1 improves on [7] by about $8.1 \cdot 10^{-4}$ in $\log \Omega$.

4.4. Flammang's table. With the weights printed in [7, Table 1] (HAL version) and $c_0 = \frac{1}{2}(1 - \sum_j c_j \deg Q_j)$, the minimum of her auxiliary function is 0.543273, that is, $\Omega \approx 1.72163$, short of her constant. With the weight of U_{23} read as 0.000001784 instead of the printed 0.00001784, the minimum is 0.543526, that is, $\Omega \approx 1.722068$, which agrees with her constant up to the rounding of the printed weights. For the polynomials of degree 32 to 38 in her table, these values require $\log |Q_j|$ computed from the roots or with more than double precision.

4.5. Tests of the programs. Table 3 lists the results of both programs on thirty-nine modified certificates. The certificates that both programs accept raise m_0 by less than the margin, or set the weight of Q_3 or of Q_{18} , which is only $4.1 \cdot 10^{-7}$ or $5.3 \cdot 10^{-8}$, to 0.

TABLE 3. Modified certificates and the results of the two programs.

Modification	First program	Second program
m_0 increased by 10^{-5} , $4 \cdot 10^{-6}$ or $3.1 \cdot 10^{-6}$	rejected	rejected
m_0 increased by $2 \cdot 10^{-6}$	accepted	accepted
G multiplied by 0.99, 0.9999, 1.0001 or 1.01	rejected	rejected
weight of x multiplied by 1.001	rejected	rejected
$x^2 - 3x + 1$ replaced by $x^2 - 3x + 2$	rejected	rejected
$\sum_k c_k \deg Q_k + 2s$ equal to 1 or to $1 - 10^{-30}$	rejected	rejected
an additional polynomial that violates the degree condition	rejected	rejected
a negative G_k , $G_N \neq 0$, or two nodes swapped (two of each)	rejected	rejected
weight of one of $Q_0, \dots, Q_{18}$ other than Q_3 and Q_{18} set to 0	rejected	rejected
weight of Q_3 or of Q_{18} set to 0	accepted	accepted
x_{10} inside the support of G (not used by the second program)	rejected	accepted

DATA AVAILABILITY

The certificate, the two verification programs with their output, a short program that computes $\max G/s$ exactly and checks the factorizations used by the second program, the programs that write the modified certificates of Section 4 (49 files, which contain the thirty-nine certificates of Table 3) with the output of both verification programs on them, the programs that produced the certificate, and the programs behind the other numbers of Section 4 with their output are archived at doi:10.5281/zenodo.23524858 and available at <https://github.com/tejasvi/totally-positive-mahler-measure>. The certificate is the file `tp_certificate.json`, with SHA-256

9f1e51157f30ce8ab436c8552d7d6a9b5aaaa5483449de00ec68716eac3864.

The two verification programs are the files `check_certificate.py` and `check_independent.py`, with SHA-256

de4434350b21bbe8e831ced9729a8592f6496f671a256d18285004a28b4cd4ec and
33bce343517e35399626348d0e07b690215576cd81bdc0d58b98b01f97e0f5ad.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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