

A thrackle on n vertices has fewer than $51n/38$ edges

TEJASVI SINGH TOMAR

ABSTRACT. A thrackle is a drawing of a graph in the plane in which every two edges meet exactly once, either at a common endpoint or at a proper crossing. Conway conjectured that a thrackle with n vertices has at most n edges. We prove that a thrackle with $n \geq 4$ vertices has at most $\frac{51}{38}(n-1)$ edges, improving the bound $\frac{18}{13}(n-1)$ obtained by Tóth and, independently, by Scoppetto. The main new ingredient, suggested by Scoppetto, is that the dumbbell $\text{DB}(6, 8, -1)$, a 6-cycle and an 8-cycle sharing one edge, is not thracklean, which we show by a computer search. It follows, by an enumeration, that no 6-cycle of a triangle-free thracklean graph shares an edge with an 8-cycle. With a second enumeration, which shows that a face of size 10 shares at most seven edges with faces of size 6, Xu's discharging argument then gives $51/38$. The search tests the planarity of graphs that encode the orders of the crossings along the edges, following Fulek and Pach, and prunes with the lists of all such orders for subgraphs; for $\text{DB}(6, 8, -1)$ it makes about $2.1 \cdot 10^{11}$ planarity tests. We repeated the computations with a second implementation, written independently of the first.

1. INTRODUCTION

A *drawing* of a graph in the plane represents the vertices by distinct points and the edges by simple arcs between the points of their endpoints, so that no arc passes through a vertex other than its endpoints, two arcs have finitely many common points, and two arcs never touch without crossing. A *thrackle* is a drawing in which every two edges meet exactly once: two edges with a common endpoint meet only there, and two edges without a common endpoint cross exactly once, properly. A graph is *thracklean* if it has a thrackle drawing. Let $t(n)$ be the largest number of edges of a thracklean graph with n vertices. Conway's thrackle conjecture states that $t(n) \leq n$ for every n . It would be sharp, since every cycle of length at least five is thracklean [14], [7, Section 1].

Lovász, Pach and Szegedy proved $t(n) \leq 2n - 3$ for $n \geq 2$ [10], and Cairns and Nikolayevsky proved $t(n) \leq \frac{3}{2}(n-1)$ [3]. Fulek and Pach introduced a computational method for deciding whether a small graph is thracklean and proved $t(n) \leq \frac{167}{117}n < 1.428n$ [7]. Goddyn and Xu proved $t(n) \leq 1.4n$ [9], Fulek and Pach proved $t(n) \leq 1.3984n$ for $n > 3$ [8], and Xu's discharging argument gives $t(n) \leq 1.393(n-1)$ for $n \geq 4$ [15], [11, Section 3.1]. The constant $\frac{18}{13}$ appeared in [8, Remark 2], conditional on a computer verification of the conjecture for $n \leq 11$ that had been stated without proof in [4]. In 2026 the bound $\frac{18}{13}(n-1)$ was obtained by Tóth in a bachelor thesis [13] and, independently and for $n \geq 4$, by Scoppetto [11, Theorem 1.1]. According to its abstract, [13] derives the bound by discharging from new non-thracklean graphs found by an algorithm. Scoppetto shows with an adapted reconstruction of the program of Fulek and Pach that the dumbbell $\text{DB}(6, 8, -2)$ is not thracklean, deduces that every 8-face of the embedding used by Xu has at most four edges in common with 6-faces, and adapts Xu's discharging. We prove the following.

Theorem 1.1. *Every thracklean graph with $n \geq 4$ vertices has at most $\frac{51}{38}(n-1)$ edges.*

2020 *Mathematics Subject Classification.* 05C10, 05C35; 05C62, 68R10.

Key words and phrases. Thrackle, Conway's thrackle conjecture, topological graph, discharging, planarity testing, computer-assisted proof.

Since $t(1) = 0$, $t(2) = 1$ and $t(3) = 3$, Theorem 1.1 gives $t(n) < \frac{51}{38}n$ for every $n \geq 1$. In the numbering of the repository [12], the constant $C_{78} = \sup_{n \geq 1} t(n)/n$ satisfies $1 \leq C_{78} \leq \frac{51}{38} = 1.3421\dots$, and Conway's conjecture asserts $C_{78} = 1$.

For integers $c', c'' \geq 3$ and $0 < k < \min(c', c'')$, the dumbbell $\text{DB}(c', c'', -k)$ is the union of a c' -cycle and a c'' -cycle whose intersection is a path with k edges, and $\text{DB}(c', c'', 0)$ is the union of a c' -cycle and a c'' -cycle with exactly one common vertex [7]. The dumbbell $\text{DB}(c', c'', -k)$ has $c' + c'' - k - 1$ vertices and one more edge than vertices, so a thrackable dumbbell would disprove Conway's conjecture. Scoppetto pointed out [11, Section 7.2] that if $\text{DB}(6, 8, -1)$ is not thrackable, and if this forces every 8-face to have no edge in common with a 6-face, then the discharging would give $\frac{15}{11}$. We carry this out, and we go further by bounding the number of edges that a face of size 10 has in common with 6-faces. The new computational ingredient is the following.

Theorem 1.2. *The dumbbell $\text{DB}(6, 8, -1)$ is not thrackable.*

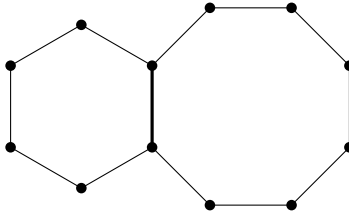


FIGURE 1. The dumbbell $\text{DB}(6, 8, -1)$: a 6-cycle and an 8-cycle sharing one edge (thick). It has 12 vertices and 13 edges.

We prove Theorem 1.2 with the method of Fulek and Pach [7]: a graph G is thrackable if and only if, for some choice of the orders in which the edges cross each other, an associated graph G' is planar (Lemma 2.4). Our search builds these orders edge by edge, tests the planarity of the partial graphs, and prunes with the complete lists of admissible orders for the 8-cycle and for the 8-cycle with a pendant edge, an idea of Scoppetto [11]. The dumbbell has 13 edges, and the search for it splits into 694,211 independent subproblems after a reduction by its symmetry. Section 5 describes the search and the computations; the search for $\text{DB}(6, 8, -1)$ took about 2,160 thread-hours.

With the same program we also check that $\text{DB}(6, 8, -2)$ and $\text{DB}(6, 8, -3)$ are not thrackable, the first being Scoppetto's result [11, Lemma 4.2], and that every union of a 6-cycle and an 8-cycle sharing an edge, other than $\text{DB}(6, 8, -1)$, contains a triangle or is not thrackable (Lemma 3.2). So in a triangle-free thrackable graph no 6-cycle shares an edge with an 8-cycle (Corollary 3.3). We also verify that two distinct 6-cycles of a triangle-free thrackable graph have no common vertex (Lemma 3.4), a statement used in [8, 11] for which only the dumbbell cases had been checked. We repeated these computations with a second implementation of the search, written without reference to the first and with a different planarity test (Section 5.4).

Section 4 then proves Theorem 1.1 along the lines of Xu [15] and Scoppetto [11]. A 2-connected triangle-free thrackable graph has an embedding in the plane or the projective plane in which every face has even size (Lemma 2.3), hence, by Lemma 2.1, size at least six; since no 6-face shares an edge with an 8-face or with another 6-face, the faces adjacent to a 6-face have size at least 10. Moreover, a face of size 10 has at most seven edges in common with 6-faces (Lemmas 4.2 and 4.3); the second lemma rests on a further enumeration. A discharging argument gives the bound for triangle-free graphs, Conway's doubling reduces graphs with a triangle to bipartite ones, and disconnected graphs are handled separately. Section 6 explains where the constant $\frac{51}{38}$ comes from and what a further improvement along these lines would need.

2. PRELIMINARIES

All graphs are finite and simple. For a graph G we write n for the number of vertices and m for the number of edges, and C_k for the cycle of length k . A subgraph of a thrackable graph is thrackable, since deleting edges and vertices from a thrackle leaves a thrackle. We use the following known facts.

Lemma 2.1 ([8, Lemma 4]). *The cycle C_4 is not thrackable.*

In fact, C_4 cannot even be drawn so that every two edges without a common endpoint cross an odd number of times and adjacent edges meet only at their common endpoint [8, Lemma 4 and Section 5].

Let G be drawn as a thrackle, and let C be an odd cycle of G . Conway's doubling of C replaces every vertex v of C by two nearby vertices v_1, v_2 , every edge vv' of C by the two edges $v_1v'_2$ and $v_2v'_1$, and attaches each other edge at v to v_1 or to v_2 , according to the side of C on which it leaves v , so that every edge attached to v_1 crosses every edge attached to v_2 exactly once near v ; all other edges are kept [7, Section 2]. The cycle C of length k becomes a cycle of length $2k$, so the number of vertices and the number of edges both grow by k .

Lemma 2.2 (Conway; [7, Lemma 5]; see also [3, Lemma 2]). *Let G be drawn as a thrackle, and let C be an odd cycle of G . Then the graph obtained by Conway's doubling of C is bipartite, and it can be drawn as a thrackle.*

An embedding of a graph in the projective plane is a *parity embedding* if every odd cycle is one-sided and every even cycle is two-sided. In a parity embedding every face has even size, where the size of a face is the length of its facial walk [8, Section 2].

Lemma 2.3 ([8, Corollary 1 and Proposition 1], based on [10, 4]). *Let G be a 2-connected triangle-free thrackable graph. If G is bipartite, it has an embedding in the plane; otherwise it has a parity embedding in the projective plane. In either embedding every face has even size, and the facial walk of every face of size at most 8 is a cycle.*

Both embeddings are cellular: in the plane since G is connected, and in the projective plane since G contains an odd cycle C , which is one-sided, so that the complement of C is an open disc, and the faces of the connected graph $G \supseteq C$ in this disc are open discs. So every face has one facial walk, and every edge is traversed twice by the facial walks, once on each side. In the plane case the last statement of Lemma 2.3 holds for all faces, by the 2-connectivity of G .

Proposition 1 of [8] is stated for 2-connected thrackles, but its proof (in the arXiv version of [8]) uses that G is triangle-free; since that proof is brief, we include one for the projective case. There every odd cycle is one-sided and every even cycle two-sided, so a closed walk has odd length if and only if it is one-sided, that is, nonzero in the homology of the projective plane with coefficients in $\mathbb{Z}_2$ (decompose its edges, counted modulo 2, into cycles). Let W be the facial walk of a face D of size at most 8. Suppose first that W traverses an edge e twice. A simple closed curve γ that runs through D and meets G only once, crossing e , does not separate the projective plane, because e is not a bridge of G ; so γ is one-sided, and $W = ePeQ$ traverses e twice in the same direction. The closed walks eP and eQ cross γ once, so they have odd length, which is at least 3 since G has no loops; as their lengths add up to $|W| \leq 8$, one of them has length 3 and is a triangle. If W traverses no edge twice but repeats a vertex, it splits into two closed trails of total length at most 8, one of which has length 3 or 4 and is a triangle or a 4-cycle. Both cases are excluded, so W is a cycle.

For an edge e of a graph G , let $N(e)$ be the set of edges of G that have no endpoint in common with e . Orient every edge of G . A *crossing system* $\Pi = (\pi_e)_{e \in E(G)}$ consists of a linear order π_e of $N(e)$ for every edge e ; in a thrackle, π_e is the order in which the edges of $N(e)$ cross e when

e is traversed in its direction. Given Π , the *augmented graph* $G'(G, \Pi)$ is obtained as follows [7, Section 4]. Each edge e becomes a path from its tail to its head through one new vertex X_{ef} for each $f \in N(e)$, in the order π_e , with one further new vertex between any two consecutive crossing vertices; the vertex $X_{ef} = X_{fe}$ lies on the paths of both e and f . Finally, for every crossing vertex $X = X_{ef}$, with neighbours a_1, a_2 on the path of e and b_1, b_2 on the path of f , the 4-cycle $a_1b_1a_2b_2$ is added.

Lemma 2.4 ([7, Section 4]). *A graph G is thrackable if and only if $G'(G, \Pi)$ is planar for some crossing system Π .*

Sketch. A thrackle gives Π and a plane drawing of $G'(G, \Pi)$: the crossings become vertices, and the 4-cycles can be drawn along the arcs, close to the crossings. Conversely, in an embedding of $G'(G, \Pi)$ in the plane the 4-cycle around X_{ef} together with the four edges at X_{ef} form a wheel, which is 3-connected and therefore has a unique embedding up to reflection; so the two edges of the path of e at X_{ef} are separated by those of f . Deleting the 4-cycles and turning the vertices X_{ef} into crossings gives a drawing in which every two edges without a common endpoint cross exactly once, at a proper crossing, and adjacent edges meet only at their common endpoint. $\square$

3. THE COMPUTATIONAL RESULTS

Theorem 3.1. *The dumbbells $\text{DB}(6, 8, -1)$, $\text{DB}(6, 8, -2)$ and $\text{DB}(6, 8, -3)$ are not thrackable.*

For $\text{DB}(6, 8, -2)$ this is [11, Lemma 4.2]. For $\text{DB}(6, 8, -3)$, which is also $\text{DB}(8, 8, -5)$ and has 10 vertices, [11, Lemma 3.13] relies on the statement in [7] that Conway's conjecture holds for bipartite graphs with at most 10 vertices; since the computations in [7, Section 4] concern the dumbbells $\text{DB}(6, 6, \ell)$, we checked $\text{DB}(6, 8, -3)$ directly. The proof of Theorem 3.1 is the computation described in Section 5.

Lemma 3.2. *Let C be a 6-cycle and F an 8-cycle of a triangle-free thrackable graph, and suppose that C and F have at least one common edge. Then $C \cup F$ is $\text{DB}(6, 8, -1)$.*

Proof. Write $F = v_0v_1 \dots v_7$ and $C = w_0w_1 \dots w_5$, where $w_0w_1 = v_0v_1$ is a common edge; after reflecting F we may assume $w_0 = v_0$ and $w_1 = v_1$. Each of $w_2, \dots, w_5$ is either a vertex outside F or one of $v_2, \dots, v_7$, distinct ones for distinct w_i . We enumerated all these choices by computer and sorted the graphs $C \cup F$ into isomorphism classes. There are 343 classes. Of these, 266 contain a triangle; of the others, 60 contain a 4-cycle; of the remaining 17, four contain two distinct 6-cycles with a common vertex (among them $\text{DB}(6, 8, -4)$, which is also $\text{DB}(6, 6, -2)$), three are the dumbbells $\text{DB}(6, 8, -k)$ for $k = 1, 2, 3$, and 10 are of none of these kinds. Graphs with a triangle are excluded by assumption, and graphs with a 4-cycle by Lemma 2.1. The dumbbells $\text{DB}(6, 8, -2)$ and $\text{DB}(6, 8, -3)$ are not thrackable by Theorem 3.1. For the 4 + 10 remaining classes the search of Section 5 found no planar augmented graph, in at most 3 seconds each, so they are not thrackable. Hence $C \cup F = \text{DB}(6, 8, -1)$. $\square$

Corollary 3.3. *In a triangle-free thrackable graph, no 6-cycle has an edge in common with an 8-cycle.*

Proof. By Lemma 3.2 the union of such cycles would be $\text{DB}(6, 8, -1)$, a subgraph of a thrackable graph, contradicting Theorem 1.2. $\square$

Fulek and Pach [8, Section 3] state that two distinct 6-cycles of a thrackle have no common vertex, citing [7], where Lemma 6 shows that the dumbbells $\text{DB}(6, 6, -k)$, $0 \leq k \leq 3$, are not thrackable (the case $k = 3$ is due to Lovász, Pach and Szegedy [10, Theorem 5.1]). Since two 6-cycles can also meet in more than one path or vertex, we checked these cases as in Lemma 3.2.

Lemma 3.4 (see [8, Section 3]). *Two distinct 6-cycles of a triangle-free thrackable graph have no common vertex.*

Proof. Let $Q = v_0 \dots v_5$ and $Q' = w_0 \dots w_5$ be distinct 6-cycles with the common vertex $w_0 = v_0$. Each of $w_1, \dots, w_5$ is a vertex outside Q or one of $v_1, \dots, v_5$, distinct ones for distinct w_i . Up to isomorphism these choices give 84 graphs $Q \cup Q'$. Of these, 56 contain a triangle; of the others, 20 contain a 4-cycle; of the remaining eight, four are the dumbbells $\text{DB}(6, 6, -k)$ for $0 \leq k \leq 3$, and four are of none of these kinds. The dumbbells are not thrackable by [7, Lemma 6] and [10, Theorem 5.1], and our program confirms this. For the remaining four graphs our program finds no planar augmented graph. With Lemma 2.1 this proves the claim. $\square$

4. PROOF OF THEOREM 1.1

Throughout this section $c = \frac{51}{38}$ and $r = 2c = \frac{51}{19}$. Lemma 4.4 uses the charges of [11, proof of Theorem 1.1, Case 1], which follow Xu [15], and Lemmas 4.5 and 4.6 and the proof of Theorem 1.1 follow Cases 1, 2 and 3 of the same proof. Let G be a 2-connected triangle-free thrackable graph, embedded as in Lemma 2.3. A k -face is a face of size k . Every face has even size. A face of size 2 or 4 would have a cycle of length 2 or 4 as its facial walk, which is impossible since G is simple and by Lemma 2.1. So every face has size at least 6, and the facial walks of the 6-faces and 8-faces are cycles. For a face D , a 6-edge of D is a traversal of an edge by the facial walk of D such that the other side of this edge belongs to a 6-face. An edge with a 6-face on one side is traversed only once by the facial walk of the face on its other side, so the 6-edges of D correspond to the edges that D has in common with 6-faces. We consider the traversals of a facial walk in their cyclic order, and call a maximal sequence of consecutive 6-edges a *run*.

Lemma 4.1. *Let $G \neq C_6$ be a 2-connected triangle-free thrackable graph, embedded as in Lemma 2.3.*

- (a) *Distinct 6-faces have distinct boundary cycles.*
- (b) *Every edge of a 6-face D has a face of size at least 10 on its other side.*

Proof. (a) Suppose that distinct faces D, D' have the same cycle Q as facial walk. At every vertex v of Q both D and D' occupy an angle between the two edges of Q at v . If v had a third edge, at most one of the angles at v would be bounded by these two edges. So every vertex of Q has degree 2, and $G = Q$; for a 6-face this means $G = C_6$, which is excluded.

(b) The facial walk of D is a cycle Q and traverses each edge e of Q once, so the other side of e belongs to a face $D' \neq D$. If D' were a 6-face, its boundary cycle would be a 6-cycle different from Q , by (a), with the common edge e , contradicting Lemma 3.4. If D' were an 8-face, its boundary would be an 8-cycle with the edge e in common with Q , contradicting Corollary 3.3. Since D' has even size at least 6, it has size at least 10. $\square$

Lemma 4.2. *Let $G \neq C_6$ be a 2-connected triangle-free thrackable graph, embedded as in Lemma 2.3. Then every 10-face has at most eight 6-edges.*

Proof. Let D be a 10-face, and consider the ten traversals of its facial walk in their cyclic order. Two consecutive traversals share a vertex, at which the walk passes from one to the next. If both are 6-edges, with 6-faces D_1 and D_2 on their other sides, then the boundary cycles of D_1 and D_2 have this vertex in common, so they coincide by Lemma 3.4, and $D_1 = D_2$ by Lemma 4.1(a). Hence the 6-edges of a run all have the same 6-face D^* on their other side. An edge traversed twice by the facial walk of D has D on both sides and gives no 6-edge. So the 6-edges of such a run are traversals of distinct edges of the boundary cycle of D^* , and there are at most six of them. In particular not all ten traversals are 6-edges. If the 6-edges form $b \geq 2$ runs, these are

cyclically separated by at least b other traversals, so there are at most $10 - b \leq 8$ of them; if $b \leq 1$, there are at most six. $\square$

Lemma 4.3. *Let $G \neq C_6$ be a 2-connected triangle-free thrackable graph, embedded as in Lemma 2.3. Then every run of 6-edges of a 10-face has at most three 6-edges, and every 10-face has at most seven 6-edges.*

Proof. Let D be a 10-face with facial walk W . By the proof of Lemma 4.2, the 6-edges of a run all have the same 6-face on their other side; let Q be its boundary cycle. They are traversals of distinct edges of Q , so there are $s \leq 6$ of them. A facial walk returns along the edge it has just traversed only at a vertex of degree 1; since G is 2-connected, W never does, so the run traverses s consecutive edges of Q (all of Q if $s = 6$). The other side of each edge of Q is the 6-face, not D , so every traversal of an edge of Q by W is a 6-edge. Hence W traverses each edge of Q at most once, and by maximality the edges of W just before and just after the run are not edges of Q . Label $Q = 012345$ and $W = w_0w_1 \dots w_9$ so that $w_i = i$ for $0 \leq i \leq s$ (with $w_6 = 0$ if $s = 6$). We enumerated by computer all closed walks W in which each of $w_{s+1}, \dots, w_9$ is a vertex of Q , an earlier vertex of W or a new vertex, such that: W never returns along the edge it has just traversed, also at w_0 ; $w_s w_{s+1}$ and $w_9 w_0$ are not edges of Q ; W traverses every edge of Q at most once and every other edge at most twice; and the union of Q and the edges of W has neither triangles nor 4-cycles. With this labelling, there is no such walk for $s = 6$. For $s = 5$ there are three, and in each of them the union contains two distinct 6-cycles with a common vertex, which Lemma 3.4 excludes. For $s = 4$ there are four, and in each of them the union contains a 6-cycle and an 8-cycle with a common edge, which Corollary 3.3 excludes. So every run has at most three 6-edges. If the 6-edges of D form b runs, there are at most $3b$ of them and, as in the proof of Lemma 4.2, at most $10 - b$ when $b \geq 2$; so there are at most seven. $\square$

Lemma 4.4. *Let G be a 2-connected triangle-free thrackable graph other than C_6 . Then $m \leq c(n - 1)$, and $m \leq c(n - 2)$ if G is bipartite.*

Proof. Embed G as in Lemma 2.3, and let f be the number of faces. Put $k = 2$ if G is bipartite and $k = 1$ otherwise. Since the embedding is cellular, $n - m + f = k$ by Euler's formula.

Give every vertex v the charge $d(v) - r$, where $d(v)$ is its degree, and every face D of size $|D|$ the charge $\frac{r-2}{2}|D| - r$, which is $\frac{13|D|-102}{38}$. The sum of the vertex degrees and the sum of the face sizes are both $2m$, so the total charge is

$$2m - rn + (r - 2)m - rf = -r(n - m + f) = -kr.$$

Now let every face of size at least 10 send $\frac{2}{19}$ across each of its 6-edges. By Lemma 4.1(b), every 6-face receives $\frac{2}{19}$ across each of its six edges and ends with charge $\frac{78-102}{38} + \frac{12}{19} = 0$. An 8-face sends nothing and keeps $\frac{104-102}{38} > 0$. A 10-face sends at most $\frac{14}{19}$, by Lemma 4.3, and keeps at least $\frac{130-102}{38} - \frac{14}{19} = 0$. A face of size $d \geq 12$ sends at most $\frac{2d}{19}$ and keeps at least $\frac{13d-102-4d}{38} = \frac{9d-102}{38} > 0$. The total charge does not change, so

$$2m - rn = \sum_v (d(v) - r) \leq -kr,$$

that is, $m \leq \frac{r}{2}(n - k) = c(n - k)$. $\square$

Lemma 4.5. *Every triangle-free thrackable graph with $n \geq 1$ vertices has at most $c(n - 1)$ edges.*

Proof. We use induction on n . If G is disconnected, apply the induction hypothesis to its components $G_1, \dots, G_j$: then $m \leq c(n - j) \leq c(n - 1)$. If G is connected and has a cut vertex, then G is the union of two connected graphs G_1, G_2 with exactly one common vertex, both

smaller than G , and $m = m_1 + m_2 \leq c(n_1 - 1) + c(n_2 - 1) = c(n - 1)$. The graphs with one or two vertices satisfy the bound. If G is 2-connected, the bound follows from Lemma 4.4, or from $6 \leq 5c$ if $G = C_6$. $\square$

Lemma 4.6. *Every connected thrackleable graph with $n \geq 4$ vertices that contains a triangle has at most $\frac{51n-52}{38}$ edges.*

Proof. Draw G as a thrackle, and let H be the graph obtained by Conway's doubling of a triangle. By Lemma 2.2, H is bipartite and thrackleable, and it is connected since G is connected. It has $N = n+3 \geq 7$ vertices and $M = m+3$ edges, so $H \neq C_6$, and it contains the 6-cycle that replaces the triangle. For a block B of H with v_B vertices and e_B edges, put $g(B) = c(v_B - 1) - e_B$. Since H is connected, $\sum_B (v_B - 1) = N - 1$, so

$$c(N - 2) - M = \sum_B g(B) - c.$$

A block is a single edge, with $g(B) = c - 1 = \frac{13}{38}$, or it is 2-connected and bipartite. If it is C_6 , then $g(B) = 5c - 6 = \frac{27}{38}$. Otherwise $e_B \leq c(v_B - 2)$ by Lemma 4.4, and $g(B) \geq c$. If some block is 2-connected and not C_6 , then $c(N - 2) - M \geq 0$. Otherwise some block is C_6 , since H contains a cycle, and there is at least one further block, since $H \neq C_6$; so $c(N - 2) - M \geq \frac{27}{38} + \frac{13}{38} - c = -\frac{11}{38}$. In both cases $M \leq c(N - 2) + \frac{11}{38}$, that is, $m + 3 \leq \frac{51}{38}(n + 1) + \frac{11}{38}$, which is $m \leq \frac{51n-52}{38}$. $\square$

Proof of Theorem 1.1. Let G be a thrackleable graph with $n \geq 4$ vertices and components $G_1, \dots, G_j$, where G_i has n_i vertices and m_i edges. If G_i is triangle-free, then $m_i \leq c(n_i - 1)$ by Lemma 4.5. If G_i contains a triangle and $n_i \geq 4$, then $m_i < c(n_i - 1)$ by Lemma 4.6. If G_i contains a triangle and $n_i \leq 3$, then $G_i = C_3$ and $m_i = 3 = c(n_i - 1) + \frac{6}{19}$. Let t be the number of components equal to C_3 . If $j = 1$, then $t = 0$ since $n \geq 4$, and $m \leq c(n - 1)$. If $j \geq 2$, then

$$m \leq c(n - j) + \frac{6t}{19} \leq c(n - j) + \frac{6j}{19} \leq c(n - j) + c(j - 1) = c(n - 1),$$

since $12j \leq 51(j - 1)$ for $j \geq 2$. $\square$

5. THE SEARCH

5.1. The search tree. Let G be a graph with oriented edges $e_0, e_1, \dots, e_{m-1}$, listed in a fixed order. For $0 \leq j \leq m$ let G_j be the subgraph formed by $e_0, \dots, e_{j-1}$, and for a crossing system Π of G let $\Pi|_{G_j}$ be the crossing system of G_j obtained by deleting all edges outside G_j from the orders. If $G'(G, \Pi)$ is planar, then so is $G'(G_j, \Pi|_{G_j})$, which is obtained from it by deleting edges and vertices and contracting edges.

The program builds crossing systems edge by edge, following the backtracking scheme of [7, Appendix A]. When the edge e_j is added, it is traced from its tail: each step chooses the next crossing of e_j , that is, an edge f of G_j in $N(e_j)$ that e_j has not crossed yet, and a position in the current order π_f at which e_j is inserted. After each step the program tests the planarity of the partial augmented graph, in which the edges of G_j are complete, e_j is drawn up to its last crossing, and the 4-cycle at that crossing is present only on existing arms. This graph is obtained from the augmented graph of every completion by deleting edges and vertices and contracting edges, so if it is not planar, no completion is, and the branch is pruned. When e_j has crossed all edges of $N(e_j) \cap E(G_j)$, its last segment is added and the complete augmented graph $G'(G_{j+1}, \cdot)$ is tested. Every crossing system Π of G determines, for each j , the restriction $\Pi|_{G_{j+1}}$ and the order in which e_j crosses the edges of G_j , hence exactly one branch of the search tree, and this branch is pruned only at a nonplanar graph obtained from $G'(G, \Pi)$ by deleting and contracting. So G is thrackleable if and only if the search reaches a planar complete augmented graph of G . The filters of Section 5.2 never prune the branch of a system Π with $G'(G, \Pi)$ planar.

Planarity is tested by the left-right algorithm [5, 2]. We also implemented the algorithm of Demoucron, Malgrange and Pertuiset [6] and compared the two implementations on 5,000 random graphs with 5 to 220 vertices and on 2,605 augmented graphs sampled during the search for $\text{DB}(6, 8, -1)$, with 52 to 185 vertices, of which 210 are planar. They agree on all of them.

5.2. Filters. If Π is a crossing system of G with $G'(G, \Pi)$ planar and S is a subgraph of G , then $G'(S, \Pi|_S)$ is planar. So if the list of all crossing systems Σ of S with $G'(S, \Sigma)$ planar is known, the program can prune every partial system whose restriction to S , as far as it is determined, extends to no system in the list. We call such lists *filters*. They are computed by the same program, which can list all complete planar systems of a graph instead of stopping at the first. During the search the program keeps, for each filter, the set of listed systems that are still compatible with the partial system, as a bit set, and updates it with precomputed bit sets for relations of the form “on the edge e , the crossing with x comes before the crossing with y ”. A branch is pruned when one of these sets becomes empty. Scoppetto used filters of this kind, also stored as bit sets, for C_6 , C_8 and C_8 with one further edge of $\text{DB}(6, 8, -2)$ [11, Section 4].

The cycle C_8 has 2,544 complete planar crossing systems, and C_8 with a pendant edge has 60,196.

5.3. The computations. For $\text{DB}(6, 8, -1)$ we label the vertices $0, \dots, 11$ so that the common edge is $e_0 = 01$, the 6-cycle consists of e_0 and the path $1, 2, 3, 4, 5, 0$ (edges $e_1, \dots, e_5$, with e_5 oriented from 0 to 5), and the 8-cycle consists of e_0 and the path $1, 6, 7, \dots, 11, 0$ (edges $e_6, \dots, e_{12}$). The filters are the 8-cycle $e_0, e_6, \dots, e_{12}$ (2,544 systems) and the two graphs obtained from it by adding the pendant edge e_1 or e_5 (60,196 systems each). We first listed the complete planar crossing systems of the subgraph formed by $e_0, \dots, e_9$ that pass the filters; there are 1,215,372 of them, and every complete planar system of $\text{DB}(6, 8, -1)$ restricts to one of them. The permutation $\sigma = (01)(25)(34)(611)(710)(89)$ of the vertices is an automorphism of the dumbbell. For a crossing system Π let $\sigma\Pi$ be the system with $\pi'_{\sigma(e)} = \sigma(\pi_e)$, reversed when σ reverses the orientation of e ; then $G'(G, \sigma\Pi) \cong G'(G, \Pi)$. The set $R = \{e_0, \dots, e_5, e_9\}$ is mapped to itself by σ (the set $\{e_0, \dots, e_9\}$ is not), so $(\sigma\Pi)|_R = \sigma(\Pi|_R)$. We kept a system only if its restriction to R , written as the sequence of the orders on $e \in R$ in increasing order of e , is lexicographically at most its image under σ . Since σ is an involution and the order is total, for every complete planar system Π of the dumbbell, Π or $\sigma\Pi$ restricts to a kept system. This leaves 694,211 systems, and the program continued the search from each of them, in 576 parts run in parallel. All parts ended without a planar complete augmented graph.

Table 1 summarizes the runs. For $\text{DB}(6, 8, -2)$ we used the analogous filters (C_8 and C_8 with the pendant edges e_2 or e_5 in the labelling of the program) and split the search at the subgraph of the first nine edges, without symmetry reduction. For $\text{DB}(6, 8, -3)$, the 14 classes of Lemma 3.2 and the four remaining graphs of Lemma 3.4 we used as filters the complete planar systems of all cycles of length 5 to 8 in the graph. The dumbbells $\text{DB}(6, 6, -k)$ were run without filters ($k = 3$) or with filters formed by C_6 , C_8 , and C_6 with pendant edges or paths. We enumerated the walks of Lemma 4.3 with two programs that share no code; both find no walk for $s = 6$, three for $s = 5$, four for $s = 4$ and 19 for $s = 3$, and both classify the unions in the same way.

5.4. Checks. Besides the comparison of the planarity tests, we checked the program on graphs with known answers. It finds 1, 0, 1, 8, 92 and 2,544 complete planar systems for the cycles $C_3, \dots, C_8$, and 60,196 for C_8 with a pendant edge; the numbers for C_6 , C_8 and C_8 with a pendant edge are those found by Scoppetto with his program [11, Section 4]. It finds that $\text{DB}(6, 6, -k)$ is not thrackable for $0 \leq k \leq 3$, in agreement with [10, Theorem 5.1] and [7, Lemma 6], and it reproduces Scoppetto’s result for $\text{DB}(6, 8, -2)$, which was obtained with a different program. We also wrote a second implementation of the search, from a description of the method and without

graph	split states	planarity tests	longest part	total time
DB(6, 8, −1)	694,211	207,380,593,775	15,317 s	7,765,559 s
DB(6, 8, −2)	26,550	3,520,550,932	1,507 s	128,254 s
DB(6, 8, −3)	–	6,024,873	34 s	34 s

TABLE 1. The runs of Theorem 3.1. The times are thread-seconds, without the listing of the split states (for DB(6, 8, −1): 83,578,015 planarity tests, 1,197 s). All runs report “not thrackable”.

reference to the code of the first. It uses a different planarity test (the edge addition method of Boyer and Myrvold [1]), its own edge orders and splittings, and no symmetry reduction. With it we repeated the checks above and the computations of Theorem 3.1 and Lemmas 3.2 and 3.4, with the same results; for DB(6, 8, −1) it decided all 32,626 states on the first nine edges of its edge order in about 1,130 CPU-hours. We also enumerated the walks of Lemma 4.3 with a third program, written in the same way, with the same results.

6. REMARKS

In the discharging of Lemma 4.4, a face of size d with h six-edges needs $\frac{r-2}{2}d - r - h(1 - \frac{r}{3}) \geq 0$, where $1 - \frac{r}{3}$ is the amount sent across each 6-edge, that is, $c = \frac{r}{2} \geq \frac{3(d+h)}{3d-6+2h}$. For 8-faces with $h \leq 5$, the case of [15], this gives $\frac{39}{28} = 1.3928\dots$, and for $h \leq 4$, the case of [11], it gives $\frac{18}{13}$. For $h \leq 3$ it would give $\frac{11}{8}$, and for $h \leq 2$ it would give $\frac{15}{11}$, which is also the requirement of a 10-face with ten 6-edges; for $h \leq 1$ it would give $\frac{27}{20}$, so at $\frac{15}{11}$ the 8-faces no longer bind. For $\frac{15}{11}$, therefore, Corollary 3.3 gives more than is needed; for $\frac{51}{38}$, which is less than $\frac{27}{20}$, it is needed in full, since an 8-face with one 6-edge would end with charge $-\frac{1}{19}$. With at most eight 6-edges on a 10-face (Lemma 4.2) the requirement of the 10-faces would be $\frac{27}{20}$, which is also that of an 8-face with one 6-edge, and with at most seven (Lemma 4.3) it is $\frac{51}{38}$. At most six would give $\frac{4}{3}$.

For $s = 3$ the enumeration of Lemma 4.3 finds 19 walks. In five of them the union contains two distinct 6-cycles with a common vertex, and in five others a 6-cycle and an 8-cycle with a common edge. The remaining nine unions fall into five isomorphism classes: one is DB(5, 5, −2), two contain it, one contains DB(5, 6, −2), and one is DB(6, 10, −3). Our program finds that DB(5, 5, −2) and DB(5, 6, −2) are not thrackable (157 and 1,680 planarity tests, with the filters of all cycles of length 5 to 8); we have not run DB(6, 10, −3). If it is not thrackable, runs have at most two 6-edges, every 10-face has at most six, and the proof of Theorem 1.1 would give $\frac{4}{3}(n - 1)$. The charges cannot give less than $\frac{4}{3}$, which is the requirement of an 8-face without 6-edges.

DATA AVAILABILITY

The programs, the input graphs, the filter lists, the split states (the 694,211 states after the symmetry reduction; the full list of 1,215,372 states is regenerated by the listing run), the data of the comparison of the planarity tests, the logs of all runs and of the enumerations of Lemmas 3.2, 3.4 and 4.3 are archived at doi:10.5281/zenodo.23549051 and in the repository <https://github.com/tejasvi/thrackle-bound>. The second implementation of Section 5.4 and its logs are in the directory `independent/` of the repository. The search program is written in C, and the enumerations and the scripts in Python 3. The large runs used static builds of earlier revisions of the search program, which differ from the archived version only in command-line options and in a faster implementation of the filter step; these binaries, their sources and

the differences are archived, and each binary is reproduced bit for bit by its recorded build command.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

REFERENCES

- [1] J. M. Boyer and W. J. Myrvold, *On the cutting edge: simplified $O(n)$ planarity by edge addition*, J. Graph Algorithms Appl. **8** (2004), 241–273.
- [2] U. Brandes, *The left-right planarity test*, manuscript, 2009.
- [3] G. Cairns and Y. Nikolayevsky, *Bounds for generalized thrackles*, Discrete Comput. Geom. **23** (2000), 191–206.
- [4] G. Cairns and Y. Nikolayevsky, *Generalized thrackle drawings of non-bipartite graphs*, Discrete Comput. Geom. **41** (2009), 119–134.
- [5] H. de Fraysseix, P. Ossona de Mendez and P. Rosenstiehl, *Trémaux trees and planarity*, Internat. J. Found. Comput. Sci. **17** (2006), 1017–1029.
- [6] G. Demoucron, Y. Malgrange and R. Pertuiset, *Graphes planaires: reconnaissance et construction de représentations planaires topologiques*, Rev. Française Recherche Opérationnelle **8** (1964), 33–47.
- [7] R. Fulek and J. Pach, *A computational approach to Conway’s thrackle conjecture*, Comput. Geom. **44** (2011), 345–355; arXiv:1002.3904.
- [8] R. Fulek and J. Pach, *Thrackles: an improved upper bound*, Discrete Appl. Math. **259** (2019), 226–231; we cite the arXiv version, arXiv:1708.08037v1.
- [9] L. Goddyn and Y. Xu, *On the bounds of Conway’s thrackles*, Discrete Comput. Geom. **58** (2017), 410–416.
- [10] L. Lovász, J. Pach and M. Szegedy, *On Conway’s thrackle conjecture*, Discrete Comput. Geom. **18** (1997), 369–376.
- [11] T. Scoppetto, *Another brick in the wall: a computationally assisted proof of an improved upper bound for thrackles*, arXiv:2610.17508v1, 2026.
- [12] T. Tao et al., *Optimization constants in mathematics*, online repository, <https://teorth.github.io/optimizationproblems/>, 2026.
- [13] M. Tóth, *Thrackle drawings of graphs*, bachelor thesis, 2026; National Repository of Grey Literature (Prague), record nusl-782605, <https://www.nusl.cz/ntk/nusl-782605>.
- [14] D. R. Woodall, *Thrackles and deadlock*, in: Combinatorial Mathematics and its Applications (D. J. A. Welsh, ed.), Academic Press, London, 1971, 335–348.
- [15] Y. Xu, *A new upper bound for Conway’s thrackles*, Appl. Math. Comput. **389** (2021), 125573.

ORCID: <https://orcid.org/0000-0003-4668-3639>