

The Steiner ratio is at least 0.8564

TEJASVI SINGH TOMAR

ABSTRACT. The Steiner ratio is the infimum, over finite sets of points in the Euclidean plane, of the length of a Steiner minimal tree divided by the length of a minimum spanning tree. Gilbert and Pollak conjectured that it equals $\sqrt{3}/2 = 0.8660\dots$. We prove that it is at least $2141/2500 = 0.8564$, improving the bound 0.8559 of Ke, Huang, Shu, He, Gai and Wang. Their method reduces the induction step of Du and Hwang to finitely many inequalities on boxes in the space of local configurations of a Steiner minimal tree. We use it with complete proofs of the geometric lemmas it needs, with two new lemmas that use the minimality of the tree among all networks, and with explicit networks for reconnections of four points. Our certificate consists of 555,102,905 boxes. A checker written independently of the program that generated them verifies, in ball arithmetic at the value $2141/2500$, the inequality of every box that meets the domain of its family, and a second program verifies that the boxes of each family tile its parameter space. We also show that, near one configuration, every splitting of the form used by Ke et al., priced with their lemmas, has positive cost once the ratio is at least 0.856. A complete certificate of this form for any value of at least 0.856, such as the conditional value 0.860 listed in Tao's repository of optimization constants, therefore needs lemmas beyond the published ones.

1. INTRODUCTION

For a finite set V of points in the Euclidean plane, let $L_S(V)$ be the length of a Steiner minimal tree for V , that is, of a shortest network that connects the points of V and may use further vertices, and let $L_M(V)$ be the length of a minimum spanning tree of V . The *Steiner ratio* is

$$\rho_S = \inf \left\{ \frac{L_S(V)}{L_M(V)} : V \subset \mathbb{R}^2, 2 \leq |V| < \infty \right\}.$$

Gilbert and Pollak conjectured that $\rho_S = \sqrt{3}/2 = 0.8660\dots$, the value for the vertices of an equilateral triangle [11]. The lower bounds $\frac{1}{2}$, attributed to Moore in [11], $1/\sqrt{3} = 0.577\dots$ [10], 0.743 [2], 0.8 [3] and 0.824 [1] followed. The conjecture is known for sets of at most four points [15, 8], five points [7] and six points [16]. Du and Hwang announced a proof of the conjecture [4, 5]; Ivanov and Tuzhilin explained why the argument does not prove it [12]. In 2026, Ke, Huang, Shu, He, Gai and Wang gave a computer-assisted proof of $\rho_S \geq 0.8559$ [14].

Theorem 1.1. *For every finite set V of points in the plane, $L_S(V) \geq \frac{2141}{2500} L_M(V)$. Hence $\rho_S \geq 2141/2500 = 0.8564$.*

The proof follows the method of [14], which rests on the induction of Du and Hwang [3]. In a Steiner minimal tree one removes a few edges near a leaf, connects the separated terminals to the others by segments, and reconnects the rest by a new network; if the length removed exceeds ρ times the length of the connecting segments plus the length added, the inequality for fewer points gives it for all points (Lemma 2.2). Near a suitable leaf, every full Steiner minimal tree with at least five terminals contains one of two local configurations, which are determined up

2020 *Mathematics Subject Classification.* 05C05; 90C27, 65G40.

Key words and phrases. Steiner ratio, Gilbert–Pollak conjecture, Steiner minimal tree, minimum spanning tree, computer-assisted proof, interval arithmetic.

to similarity by six or eight edge lengths (Figure 1). For each configuration, [14] fixes a finite list of such *splittings* and bounds the cost of each by an explicit function of the edge lengths. The parameter space is covered by boxes, and each box carries one splitting whose cost bound is non-positive at the vertices of the box, which suffices because along every line parallel to a coordinate axis the bounds first do not increase and then do not decrease. Some splittings connect to a terminal that lies outside the configuration, at the end of a chain of Steiner points; geometric lemmas of [14] and [3] confine this terminal to a polygon under conditions on the edge lengths.

We use this method, the splitting tables of [14], and a modified version of its generator of boxes, which is published under the MIT license. The proof differs from that of [14] in four respects.

- (1) *Proofs.* We prove the reduction to the two configurations (Section 2) and every bound for the hidden terminals that the certificate uses (Section 3), relying only on standard facts about Steiner minimal trees and on the known cases of at most four points. The bounds follow from an exchange inequality and a turning-number argument (Lemmas 3.1 and 3.2). Two of the published proofs have gaps, described in Remark 3.9; the statements we use hold.
- (2) *Two new bounds.* The published lemmas bound one of the hidden terminals only when $f \leq d$, in the notation of Figure 1, and the other only under linear conditions. Lemmas 3.11 and 3.12 extend both. Their proofs use that the Steiner tree is shorter than every network on the same terminals, not only than the networks obtained by exchanging one edge.
- (3) *Four-point reconnections.* When a splitting reconnects four points, we bound the length of the new network by that of an explicit network with two Steiner points, so no condition for the existence of a full Steiner tree is needed (Section 4).
- (4) *Checks.* The certificate consists of 555,102,905 boxes in four families. A checker written independently of the generator verifies, in ball arithmetic at $\rho = 2141/2500$, every box that meets the domain of its family, and a second program verifies that the boxes of each family tile the orthant $[0, \infty)^n$ of its coordinates (Section 5). Run on the first 2,000,000 records of each family with $\rho = 2142/2500$ or $\rho = 4283/5000$, the checker rejects between 29,122 and 129,167 of them, and it also rejects altered records.

The repository of optimization constants maintained by Tao [18] also lists the bound 0.860, marked as conditional, from [17]. That work re-checks the records of the certificate of [14], and it checks every record of partitions regenerated at $\rho = 43/50$ with the generator and the lemmas of [14]. On coverage it says: “We verify that every emitted region certifies its inequality; that the emitted regions cover each case domain is a property of the generator’s recursive subdivision, exactly as in the published work” [17, Section 7]; and it states its bound as conditional on what it calls the lemma layer of [14], which includes the geometric lemmas, the case decomposition, the monotonicity of the lengths involved and, for regions checked at their vertices, a unimodality property of the checked functions. Proposition 6.1 and the remarks after it show that at the configurations of type `d.regular` with parameters (w_*, f) , where w_* is a fixed point and f is slightly larger than d , every splitting of the form used in [14] whose cost is bounded with the lemmas of [14] has positive cost when $\rho \geq 0.856$. So a complete certificate of this form for any value at least 0.856 needs further lemmas, such as Lemmas 3.11 and 3.12. The partitions of [17] were generated with the generator, the splitting tables and the lemmas of [14]; at the commit 709673a8 pinned there, these files are identical to those at the commit we use. In the family `d.regular` with $f \geq d$, which [14] checks on boxes in (b, c, d, s, e) with $f = d$, none of their records that pass at the values 0.856 and 0.860 reported there can have a box containing the point w_*

of Proposition 6.1. The proofs of the lemmas of [14] given here therefore do not make the value 0.860 a theorem.

Section 2 sets up the induction and the configurations, Section 3 proves the bounds for the hidden terminals, Section 4 describes the functions that are checked on the boxes, Section 5 describes the certificate and completes the proof of Theorem 1.1, and Section 6 proves Proposition 6.1.

2. SPLITTINGS AND LOCAL CONFIGURATIONS

2.1. Steiner minimal trees. We use the following facts [11]. Every finite set V with $|V| \geq 2$ has a Steiner minimal tree (SMT). It is a tree whose vertices are the points of V , called *terminals*, and possibly further points, called *Steiner points*; its edges are segments, and two edges meet only at a common endpoint. Every Steiner point has degree three, and its three edges make angles of 120° . There are at most $|V| - 2$ Steiner points. An SMT with exactly $|V| - 2$ Steiner points is *full*; then every terminal is a leaf. Conversely, if every terminal is a leaf, counting degrees shows that there are $|V| - 2$ Steiner points. We call a connected finite union of segments a *network*; no network that contains V is shorter than an SMT for V .

2.2. Splittings. For a tree S and a set S^- of its edges, $S \setminus S^-$ denotes the union of the edges of S that are not in S^- . For a finite set N of segments, $L(N)$ is the sum of their lengths; a network formed by these segments has length at most $L(N)$.

Definition 2.1. Let S be an SMT for V . A *splitting* of S is a tuple (V^*, S^-, S^+, t^*) such that

- (a) V^* is a nonempty proper subset of V ;
- (b) S^- is a set of edges of S ;
- (c) S^+ is a finite set of segments such that $V \setminus V^*$ lies in one connected component of $(S \setminus S^-) \cup S^+$;
- (d) t^* is a finite set of segments with endpoints in V such that every point of V^* is joined to a point of $V \setminus V^*$ by a path in t^* .

Its *cost* at ρ is $\rho L(t^*) + L(S^+) - L(S^-)$.

This is weaker than [14, Definition 13], which also asks that no point of V^* is incident to $S \setminus S^-$, that $(S \setminus S^-) \cup S^+$ is a Steiner tree, and that no path in t^* joins two points of $V \setminus V^*$; Lemma 2.2 does not need these conditions. The following lemma is stated in [14, Lemma 14], where it is attributed to [3, Lemma 3]. We include a proof for the weaker definition.

Lemma 2.2. *Let $\rho > 0$ and $n \geq 2$, and suppose that $L_S(W) \geq \rho L_M(W)$ for every set W with $2 \leq |W| < n$. Let S be an SMT for a set V of n points. If S has a splitting of cost at most 0 at ρ , then $L_S(V) \geq \rho L_M(V)$.*

Proof. Let (V^*, S^-, S^+, t^*) be the splitting and $V' = V \setminus V^*$, so $1 \leq |V'| < n$. By (c), a connected component of $(S \setminus S^-) \cup S^+$ is a network containing V' , so $L_S(V') \leq L(S) - L(S^-) + L(S^+)$. If $|V'| \geq 2$, the hypothesis gives $\rho L_M(V') \leq L_S(V')$; if $|V'| = 1$, both sides are 0. The edges of a minimum spanning tree of V' together with the segments of t^* form a connected graph on V by (d), whose edges are segments between points of V ; so $L_M(V) \leq L_M(V') + L(t^*)$. Hence

$$\rho L_M(V) \leq L_S(V') + \rho L(t^*) \leq L(S) - (L(S^-) - L(S^+) - \rho L(t^*)) \leq L(S) = L_S(V). \quad \square$$

2.3. Reduction to full trees.

Lemma 2.3. *Let $\rho > 0$ and $n \geq 3$, and suppose that $L_S(W) \geq \rho L_M(W)$ for every set W with $2 \leq |W| < n$. If a set V of n points has an SMT S in which some terminal has degree at least two, then $L_S(V) \geq \rho L_M(V)$.*

Proof. Let x be such a terminal, let $S_1, \dots, S_k$ ($k \geq 2$) be the closures of the components of $S \setminus \{x\}$, and let $V_i = V \cap S_i$. Each S_i contains x and a leaf of S , which is a terminal, so $|V_i| \geq 2$; and V_i misses the leaves of S in the other branches, so $|V_i| < n$. Each S_i is an SMT for V_i , since a shorter network for V_i could replace it. The minimum spanning trees of the V_i all contain x , so their union is a connected graph on V , and $L_M(V) \leq \sum_i L_M(V_i) \leq \sum_i L(S_i)/\rho = L_S(V)/\rho$. $\square$

2.4. Local configurations.

Lemma 2.4. *Let S be a full SMT for a set of $n \geq 5$ terminals. Then S has Steiner points X, Y, Z and terminals A, B such that the neighbours of X are A, B, Y ; the neighbours of Y are X, Z and a vertex D that is either a terminal or a Steiner point whose two other neighbours U, V are terminals; and Z has two neighbours R, Q besides Y . If $n = 5$, then D, R and Q are terminals.*

Proof. The Steiner points span a subtree with $n - 2 \geq 3$ vertices, which contains a path with two edges. A longest path in this subtree ends at two of its leaves, each of which is adjacent to two terminals; so S has a path between two leaves with at least four edges. Let $v_0, v_1, \dots, v_k$ be a longest such path, so $k \geq 4$, and put $A = v_0, X = v_1, Y = v_2, Z = v_3$. The third neighbour B of X is a terminal: otherwise it would have a further neighbour, and a path from a leaf behind B through $B, X, Y, \dots, v_k$ would have at least $k + 1$ edges. Let D be the third neighbour of Y . If a neighbour of D other than Y were a Steiner point, a leaf behind it would give a path with at least $k + 1$ edges ending at v_k ; so D is a terminal or a Steiner point whose other two neighbours are terminals. Since $k \geq 4$, $Z = v_3$ is not an end of the path, so it is a Steiner point; let R, Q be its other neighbours. If $n = 5$, the only Steiner points are X, Y, Z , so D, R, Q are terminals. $\square$

Following [14], we call the configuration *d_regular* if D is a terminal and *d_steiner* otherwise. We write T_R and T_Q for the closures of the components of $S \setminus [Z, R]$ and $S \setminus [Z, Q]$ that contain R and Q ; every terminal other than A, B and D (or U, V) lies in $T_R \cup T_Q$.

We identify the plane with $\mathbb{C}$ and put $\omega = e^{i\pi/3}$, so that ω and $\bar{\omega}$ are the unit vectors at angles 60° and -60° .

Lemma 2.5. *In Lemma 2.4, after possibly exchanging the names A, B , the names R, Q and the names U, V , and after a similarity of the plane, possibly reversing orientation, we have*

$$(2.1) \quad \begin{aligned} A = 0, \quad X = 1, \quad B = 1 + b\omega, \quad Y = 1 + s\bar{\omega}, \quad D = Y + d, \quad Z = Y - c\omega, \\ R = Z - e, \quad Q = Z + f\bar{\omega}, \quad \text{and in } d_steiner \quad U = D + u\omega, \quad V = D + v\bar{\omega}, \end{aligned}$$

*with positive numbers b, s, c, d, e, f (and u, v), the lengths of the edges XB, XY, YZ, YD, ZR, ZQ (and DU, DV) divided by $|AX|$. In *d_steiner* one may moreover assume that $\max(u, v) \leq \max(b, 1)$.*

Proof. Traversing $X \rightarrow Y$, one of the edges YD, YZ turns left by 60° and the other turns right. Reflecting the plane if necessary, YD turns left. Exactly one of the two leaves at X is such that $A \rightarrow X \rightarrow Y$ turns right; we call it A and the other one B , which is allowed since both are terminals adjacent to X . A translation, a rotation and a scaling by $1/|AX|$ give $A = 0$ and $X = 1$. Since the edges at each Steiner point make angles of 120° , the edge XY has direction -60° and XB direction 60° ; YD has direction 0° and YZ direction -120° . At Z we call R the neighbour in direction 180° and Q the one in direction -60° , and in *d_steiner* we call U and V the neighbours of D in directions 60° and -60° . This gives (2.1).

In *d_steiner*, suppose $\max(u, v) > \max(b, 1)$. The vertex D with its leaves U, V and the vertex X with its leaves A, B are both attached to Y , so exchanging their roles gives another labelling that satisfies Lemma 2.4 with D a Steiner point. After normalizing it as above, the

new unit length is $|DU|$ or $|DV|$, the new $\max(b, 1)$ is the old $\max(u, v)$ divided by it, and the new $\max(u, v)$ is the old $\max(b, 1)$ divided by it. So the new parameters satisfy $\max(u, v) < \max(b, 1)$. $\square$

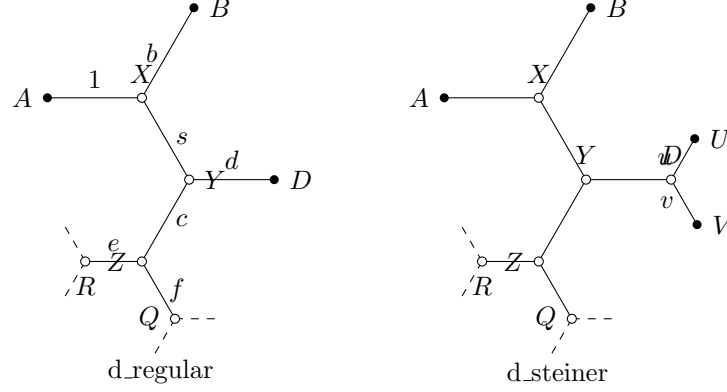


FIGURE 1. The two local configurations of Lemma 2.4 in the coordinates (2.1). Filled dots are terminals, open dots are Steiner points or, at R and Q , vertices that may be of either kind; the dashed edges continue into T_R and T_Q .

2.5. The subcases. Every configuration has $f \leq d$ or $f \geq d$. The certificate treats four families: $d_regular$ with $f \leq d$ and with $f \geq d$, and $d_steiner$ with $f \leq d$ and with $f \geq d$. In the parameter space of a family, the coordinates are (b, c, d, s, e, f) for $d_regular$ and (b, c, d, s, u, v, e, f) for $d_steiner$; for $d_steiner$ with $f \geq d$ we use (b, c, d, s, u, v, e) and the value $f = d$, see Lemma 4.3.

2.6. Small sets. For $|V| = 2$ the ratio is 1. For three points with sides $a \leq b' \leq c'$, let γ be the angle opposite c' , so $\gamma \geq 60^\circ$ and $L_M = a + b'$. If $\gamma \geq 120^\circ$, the SMT is the minimum spanning tree. Otherwise

$$L_S^2 = \frac{1}{2}(a^2 + b'^2 + c'^2) + 2\sqrt{3} \text{ area} = a^2 + b'^2 + 2ab' \sin(\gamma - 30^\circ) \geq a^2 + ab' + b'^2 \geq \frac{3}{4}(a + b')^2.$$

For four points the conjecture was proved by Pollak [15]; a shorter proof is in [8]. So $L_S(V) \geq \frac{\sqrt{3}}{2} L_M(V) > \frac{2141}{2500} L_M(V)$ whenever $2 \leq |V| \leq 4$.

3. HIDDEN TERMINALS

Throughout this section S is a full SMT in the position of Lemma 2.5, with at least five terminals. For a point p and a segment σ , $\text{dist}(p, \sigma)$ is the distance between them, and $B(p, r)$ is the open disk of radius r about p .

3.1. Chains. Put $A_0 = A$, $A_1 = X$, $A_2 = Y$, $A_3 = Z$, $A_4 = R$. If A_k ($k \geq 4$) is a Steiner point, let A_{k+1} be its neighbour reached by turning right by 60° . The walk never goes back, so it follows a path of the tree and ends at a leaf A_N , $N \geq 4$, which is a terminal; we call it E . We write $\sigma_k = [A_{k-1}, A_k]$ and $a_k = |\sigma_k|$, so $a_1 = 1$, $a_2 = s$, $a_3 = c$, $a_4 = e$, and σ_k has direction $\bar{\omega}^{k-1}$. Similarly the walk $D \rightarrow Y \rightarrow Z \rightarrow Q$ turns left by 60° at Y and at Z ; continued by left turns at Steiner points, it ends at a terminal F . In $d_steiner$ the walk $V \rightarrow D \rightarrow Y \rightarrow Z \rightarrow Q \rightarrow \dots$ also turns left at every vertex and ends at the same F .

The terminal E lies in T_R and F lies in T_Q . These two trees are disjoint and contain none of A, B, D, U, V . In particular $E \neq F$, and both are terminals of $V \setminus \{A, B, D, U, V\}$.

3.2. Exchange and trapping.

Lemma 3.1. *Let p, q be points of S and let τ be a segment contained in an edge of S and in the path of S from p to q . Then $|pq| \geq |\tau|$. In particular, for $0 \leq u \leq k-2$,*

$$(3.1) \quad \text{dist}(A_u, \sigma_k) \geq \max(a_{u+1}, \dots, a_{k-1}),$$

and $|A_u A_v| \geq a_t$ for $u < t \leq v$.

Proof. Removing the relative interior of τ splits S into two parts, one containing p and the other q ; no terminal is removed. Adding $[p, q]$ gives a network that contains all terminals and has length $L(S) - |\tau| + |pq|$, which is at least $L(S)$. For (3.1), the path from A_u to a point of σ_k contains $\sigma_{u+1}, \dots, \sigma_{k-1}$. $\square$

A point p is to the *right* of an oriented line ℓ through p_0 with unit direction δ if $\text{Im}((p - p_0)\bar{\delta}) < 0$. Below, $\lambda_\ell(p) = \text{Im}((p - p_0)\bar{\delta})$ is the signed distance to the left of ℓ .

Lemma 3.2 (trap principle). *Let $1 \leq m \leq N$, and let ℓ be the line through A_{m-1} and A_m , oriented by σ_m . Let P be a simple polygon in the closed half-plane to the right of ℓ whose boundary consists of pieces of the edges $\sigma_1, \dots, \sigma_m$, a segment $J = [A_m, z_0] \subset \ell$ with z_0 beyond A_m , and further segments called doors, such that A_m lies in the relative interior of the side of P that contains J . If no point of the tail $A_m A_{m+1} \dots A_N$ other than A_m lies on a door outside J , then the tail lies in P , so $E \in P$.*

Proof. If $m = N$, then $E = A_m \in P$. Let $m < N$. Near A_m the polygon coincides with the half-disk to the right of ℓ , and σ_{m+1} turns right by 60° , so the tail enters the interior of P . Suppose it leaves P , and let z be the first point of the tail after A_m that lies on ∂P . The tail meets the edges $\sigma_1, \dots, \sigma_m$ only at A_m , since S is a tree whose edges meet only at common endpoints, and it does not meet a door outside J ; so $z \in J \setminus \{A_m\}$. The tail from A_m to z followed by $[z, A_m]$ is a simple closed polygon in the closed right half-plane of ℓ . Its angle at A_m is 60° , so it is traversed counterclockwise and its turning angles add up to 360° . They are 120° at A_m , -60° at each vertex of the tail before z , and $180^\circ - \varphi$ at z for some $\varphi \in (0^\circ, 180^\circ)$, since the edge that arrives at z does not lie on ℓ (its earlier points would lie on ∂P). Their sum is less than 300° , a contradiction. $\square$

Lemma 3.3. *Let $0 \leq u < t \leq v < N$, let ℓ be the line through A_v and A_{v+1} , oriented by σ_{v+1} , and let H be the foot of the perpendicular from A_u to ℓ . Suppose that*

- (T1) H lies on the ray from A_v through A_{v+1} ;
- (T2) $|A_u H| < a_t$;
- (T3) $A_u, \dots, A_v$ lie in the closed half-plane to the right of ℓ , and $A_{u+1}, \dots, A_{v-1}$ in the open one;
- (T4) $[A_u, H]$ meets the edges $\sigma_{u+1}, \dots, \sigma_t$ only at A_u .

Then H lies beyond A_{v+1} , the polygon $P = A_u A_{u+1} \dots A_v H$ is simple, and $E \in P$.

Lemma 3.3 is a form of [14, Lemma 8] with the additional hypotheses (T3) and (T4); see Remark 3.9.

Proof. The door $[A_u, H]$ lies in $B(A_u, a_t)$ by (T2), and by (3.1) the edges σ_k with $k \geq t+1$ avoid this disk. In particular $H \notin \sigma_{v+1}$, so by (T1) H lies beyond A_{v+1} . By (T3) the edges $\sigma_{u+1}, \dots, \sigma_v$ meet ℓ only at A_v , and at A_u if $A_u \in \ell$, in which case $H = A_u$. The door is perpendicular to ℓ , so it meets $[A_v, H]$ only at H ; by (T4) and the disk it meets $\sigma_{u+1}, \dots, \sigma_v$ only at A_u . So P is simple, and it lies in the closed right half-plane of ℓ . Lemma 3.2 with $m = v+1$, $J = [A_{v+1}, H]$ and the door $[H, A_u]$ applies, since the tail consists of edges σ_k with $k \geq v+2 \geq t+2$. $\square$

3.3. Bounds for E from the published lemmas. The following lemmas are Lemmas 27, 29, 30, 33 and 36 of [14]; the first is due to Du and Hwang [3, Lemma 2]. Let ℓ_5 be the line through $A_4 = R$ in direction $\omega^2 = e^{2\pi i/3}$, which contains σ_5 if $N \geq 5$, and let $W = 1 - c - e$ be its intersection with the real axis; then $|RW| = s + c$. The pentagon $\Pi = WA_1A_2A_3A_4$ is convex, it is traversed clockwise, and it lies to the right of ℓ_5 oriented by ω^2 .

Lemma 3.4 ([3, Lemma 2], [14, Lemma 27]). *If $c \leq 1$, then $E \in \Pi$ and $|AE| \leq \max(|AY|, |AZ|)$.*

Proof. If $N = 4$, then $E = R$. Let $N \geq 5$. We first show $a_5 < s + c$, that is, A_5 lies strictly between R and W . If $c + e \leq 1$, then $W \in \sigma_1$, and $a_5 \geq s + c$ would put W on σ_5 , while $\text{dist}(\sigma_1, \sigma_5) \geq a_2 > 0$ by Lemma 3.1. If $c + e > 1$ and $c < 1$, then $|AW| = c + e - 1 < e = a_4$, so $W \in B(A, a_4)$, which σ_5 avoids by (3.1). If $c = 1$, the foot of the perpendicular from A to ℓ_5 is at distance $1 + s - e/2$ from R , and $0 < e \leq 1 + s$ because $|AR|^2 - e^2 = (1 + s)(1 + s - e)$ and $|AR| \geq e$ (Lemma 3.1); so the foot lies on $[R, W]$, at distance $\frac{\sqrt{3}}{2}e < e$ from A . By (3.1), σ_5 does not contain the foot, so it ends before reaching it, and $a_5 < s + c$.

Now apply Lemma 3.2 with $m = 5$, $P = \Pi$ and $J = [A_5, W]$. If $c + e \leq 1$, the boundary of Π consists of $[W, A_1] \subset \sigma_1$, σ_2 , σ_3 , σ_4 and $[A_4, W]$, and there is no door. If $c + e > 1$, the door is $[W, A]$. All its points other than W lie in $B(A, e)$, since $|AW| = c + e - 1 \leq e$, and the tail consists of edges σ_k with $k \geq 6$, which avoid $B(A, a_4)$ by (3.1); the point W lies on J . So $E \in \Pi$.

For the bound, $|AA_1| = 1 \leq |AA_2|$. Let $x_3 = 1 + (s - c)/2 > 0$ be the real part of A_3 . From $|AA_4|^2 = |AA_3|^2 - e(2x_3 - e)$ and $|AA_4| \geq e$ we get $e \leq |AA_3|^2/(2x_3)$, and $4x_3^2 - |AA_3|^2 = 3(1 + s)(1 - c) \geq 0$ gives $e \leq 2x_3$, hence $|AA_4| \leq |AA_3|$. Finally $|AW| \leq \max(1, e) \leq |AA_4|$, because $|AA_4| \geq a_1$ and $|AA_4| \geq a_4$. $\square$

Lemma 3.5 ([14, Lemmas 29 and 30]). *Suppose that $c + e \geq 1$ and that $e < 1 - c + \frac{2}{\sqrt{3}}s$ or $e < 1 + (\frac{2}{\sqrt{3}} - 1)c$. Let H be the foot of the perpendicular from A to ℓ_5 . Then E lies in the polygon $AA_1A_2A_3A_4H$ and $|AE| \leq \max(1, |AY|, |AZ|, |AR|, \frac{\sqrt{3}}{2}(c + e - 1))$.*

Proof. If $N = 4$, then $E = R$. Otherwise we apply Lemma 3.3 with $(u, t, v) = (0, 2, 4)$ under the first condition and $(0, 3, 4)$ under the second; $\ell = \ell_5$. The signed distances to the left of ℓ_5 are

$$\lambda(A_3) = -\frac{\sqrt{3}}{2}e, \quad \lambda(A_1) = \lambda(A_2) = -\frac{\sqrt{3}}{2}(c + e), \quad \lambda(A) = -\frac{\sqrt{3}}{2}(c + e - 1),$$

so (T3) holds because $c + e \geq 1$, and $|AH| = \frac{\sqrt{3}}{2}(c + e - 1)$; (T2) is the respective condition. The point H is at signed distance $\frac{1}{2}(1 + 2s + c - e)$ from R along ℓ_5 , and $1 + 2s + c - e$ exceeds $2c + (2 - \frac{2}{\sqrt{3}})s$ under the first condition and $2s + (2 - \frac{2}{\sqrt{3}})c$ under the second; this gives (T1).

For (T4), $H = -(c + e - 1)(\frac{3}{4} + \frac{\sqrt{3}}{4}i)$, so the door $[A, H]$ lies in the half-plane $\text{Re} \leq 0$, which meets $\sigma_1 \cup \sigma_2$ only at A ; and the lines of $[A, H]$ and σ_3 meet at the point $Y + (1 + 2s)\omega$, which is not on $\sigma_3 = \{Y - t\omega : 0 \leq t \leq c\}$. The bound is the maximum of $|A \cdot |$ over the vertices of the polygon. $\square$

Lemma 3.6 ([14, Lemma 33]). *Suppose that $e < \kappa c$, where $\kappa = \frac{2}{\sqrt{3}} - 1$. Let $H = R - \frac{c-e}{2}\bar{\omega}$ be the foot of the perpendicular from Y to ℓ_5 . Then E lies in the quadrilateral $YZRH$. Consequently $|pE| \leq \max(|pY|, |pZ|, |pR|, |pH|)$ for every point p , in particular for $p = D$ in d -regular and $p \in \{U, V\}$ in d -steiner, and $|AE| \leq \max(|AY|, |AZ|, |AR|)$.*

Proof. If $N = 4$, then $E = R$. Otherwise we apply Lemma 3.3 with $(u, t, v) = (2, 3, 4)$. The point H is at signed distance $\frac{c-e}{2} > 0$ from R along ℓ_5 , which gives (T1), and $|YH| = \frac{\sqrt{3}}{2}(c + e)$, so (T2) is the condition $e < \kappa c$. The signed distances of Y and Z to the left of ℓ_5 are $-\frac{\sqrt{3}}{2}(c + e)$ and $-\frac{\sqrt{3}}{2}e$, which gives (T3). The door $[Y, H]$ leaves Y in direction 210° and σ_3 leaves Y in

direction 240° , which gives (T4). For the bound on $|AE|$, the vertex H can be dropped, because $|AH|^2 - |AZ|^2 = (\frac{3}{4}(c+e)^2 - c^2) - \frac{c}{2} - \frac{3e}{2} - cs < 0$. $\square$

Lemma 3.7 ([14, Lemma 36]). *If $e < s + 1$, then $E \in \Pi$ and $|AE| \leq \max(|AY|, |AZ|, |AR|)$.*

Proof. If $N = 4$, then $E = R$; let $N \geq 5$. If $c + e \leq 1$, the proof of Lemma 3.4 for this case applies; it did not use $c \leq 1$. Let $c + e > 1$, so $W = 1 - c - e < 0$. First, $a_5 < s + c$: otherwise $W \in \sigma_5$, and Lemma 3.1 with $\tau = [R, W]$ gives $c + e - 1 = |AW| \geq s + c$, contrary to $e < s + 1$. If $N = 5$ we are done. Let $N \geq 6$ and let ℓ_6 be the line through A_5 and A_6 , oriented by σ_6 , which has direction ω . The ray from A_5 in direction ω leaves Π through $[W, A_1]$ or through σ_2 , because σ_3 is parallel to it and σ_4 lies below it; let M' be the exit point and $M = 1 + s - e - a_5$ the intersection of ℓ_6 with the real axis, so $|A_5M| = s + c - a_5$. The chord $[A_5, M']$ cuts from Π the convex polygon P' that contains A_2, A_3, A_4 ; the signed distances to the left of ℓ_6 are

$$\lambda(A_4) = -\frac{\sqrt{3}}{2}a_5, \quad \lambda(A_3) = \lambda(A_2) = -\frac{\sqrt{3}}{2}(a_5 + e), \quad \lambda(A_1) = \frac{\sqrt{3}}{2}(M - 1), \quad \lambda(A) = \frac{\sqrt{3}}{2}M,$$

and P' lies to the right of ℓ_6 . Its boundary consists of pieces of $\sigma_1, \dots, \sigma_5$, the segment $[A_5, M'] \subset \ell_6$ and, if $M < 0$, the door $[M, A]$. We have $a_6 < |A_5M'|$, since σ_6 does not meet σ_1 or σ_2 (Lemma 3.1) and, if $M < 0$, $|AM| = a_5 - (s + 1 - e) < a_5$, so the door lies in $B(A, a_5)$, which $\sigma_6, \sigma_7, \dots$ avoid by (3.1). Lemma 3.2 with $m = 6$ and $J = [A_6, M']$ gives $E \in P' \subset \Pi$.

For the bound, $|AA_1| = 1 \leq |AA_2|$, and $|AW|$ is dropped: if $c + e \leq 1$ then $|AW| \leq 1$, and otherwise $|AA_4|^2 - (c + e - 1)^2 = (s + c)(s + 1 - e) > 0$. $\square$

3.4. Bounds for F when $f \leq d$. The proofs of Lemmas 3.2–3.7 use only that $A_0, A_1, \dots$ are consecutive vertices of S , that A_1, A_2, A_3 are Steiner points, that the chain turns right by 60° at each Steiner point, and Lemma 3.1; they do not use that A_0 is a terminal. So they apply, after a similarity that maps its first edge to $[0, 1]$, to every chain of S with these properties. The map $z \mapsto (D - z)/d$ sends D, Y, Z, Q to $0, 1, 1 + \frac{c}{d}\bar{\omega}, 1 + \frac{c}{d}\bar{\omega} - \frac{f}{d}\omega$, and the chain from D to F to a chain that turns right at every vertex, with edge lengths $1, \frac{c}{d}, \frac{f}{d}, \dots$. In $d_steiner$ the map $z \mapsto \bar{\omega}(V - z)/v$ sends the chain $V, D, Y, Z, Q, \dots$ to a right-turning chain with edge lengths $1, \frac{d}{v}, \frac{c}{v}, \frac{f}{v}, \dots$. Both maps are similarities, so they map S to an SMT.

Lemma 3.8 ([14, Lemma 28 and Corollary 26]). *Suppose $f \leq d$. In $d_regular$, $|DF| \leq \max(|DZ|, |DQ|)$. In $d_steiner$, $|VF| \leq \max(|VY|, |VZ|, |VQ|)$.*

Proof. If Q is a terminal, then $F = Q$. Otherwise the first three vertices after the start of the image chain are Steiner points: the images of Y, Z, Q in $d_regular$ and of D, Y, Z in $d_steiner$. In $d_regular$, the hypothesis $c \leq 1$ of Lemma 3.4 for the image chain reads $f/d \leq 1$, and its bound, mapped back, is $\max(|DZ|, |DQ|)$. In $d_steiner$, the hypothesis $e < s + 1$ of Lemma 3.7 for the image chain reads $f/v < d/v + 1$, which holds since $f \leq d$, and its bound, mapped back, is $\max(|VY|, |VZ|, |VQ|)$. $\square$

Remark 3.9. (a) [14, Lemma 8] states, for $0 \leq u < t \leq v < \min(6, N)$, that if the foot H of the perpendicular from A_u to the line A_vA_{v+1} lies on the ray from A_v through A_{v+1} and $|A_uH| < a_t$, then E lies in the polygon $A_u \cdots A_vH$. Its proof does not show that $A_u \cdots A_vH$ is simple, and it excludes exits through the sides $[A_{v+1}, H]$ and $[H, A_u]$, which are not part of the chain, by appeal to the fact that the chain does not cross itself. For chains that satisfy every exchange inequality of Lemma 3.1 between their points, the polygon need not be simple: for $(u, t, v) = (0, 1, 4)$ and edge lengths $(a_1, \dots, a_6) = (1, 2, \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40})$, the hypotheses hold and the side $[A_4, H]$ crosses σ_1 . We do not know whether this happens in an SMT. We use Lemma 3.3 instead and check its hypotheses in each instance.

(b) In the proof of [14, Lemma 36], in the notation there, $N = A_5$ and M is the point where the ray from A_5 in the direction of σ_6 meets $[A, W]$, and the inequality $|AM| < |NM|$ is derived

from $e < s + 1$. This implication is not valid: for $(s, c, e, a_5) = (1, \frac{3}{2}, \frac{3}{2}, \frac{17}{10})$, a chain that satisfies every exchange inequality of Lemma 3.1 up to A_5 , we get $|AM| = \frac{6}{5} > \frac{4}{5} = |NM|$. The case in question does not occur, because $|AM| < a_5$ (proof of Lemma 3.7).

3.5. Two new bounds.

Lemma 3.10 (removal exchange). *Let z_0 be a Steiner point of S with neighbours x, y, y' , and let $S_x, S_y, S_{y'}$ be the closures of the components of S minus the three edges at z_0 that contain x, y, y' . Then $|x'K| \geq |z_0x| + |z_0y| + |z_0y'| - |yy'|$ for every point $x' \in S_x$ and every point $K \in S_y \cup S_{y'}$.*

Proof. Removing the three edges at z_0 and adding the segments $[y, y']$ and $[x', K]$ gives a network that contains all terminals, since z_0 is not a terminal. Its length is $L(S) - |z_0x| - |z_0y| - |z_0y'| + |yy'| + |x'K| \geq L(S)$. $\square$

Lemma 3.11. *Let $\Phi = c + e + f - |RQ| - \frac{\sqrt{3}}{2}(c + e)$, where $|RQ| = \sqrt{e^2 + ef + f^2}$. If $e < c$ and $\Phi > 0$, the conclusions of Lemma 3.6 hold: E lies in the quadrilateral $YZRH$, with H as there.*

Proof. If $N = 4$, then $E = R$; let $N \geq 5$. Lemma 3.10 with $z_0 = Z$, $x = x' = Y$, $\{y, y'\} = \{R, Q\}$ gives $|YK| \geq c + e + f - |RQ|$ for every point K of T_R , and T_R contains R, σ_5 and the whole tail. As in the proof of Lemma 3.6, H is at signed distance $\frac{c-e}{2} > 0$ from R along ℓ_5 , and $|YH| = \frac{\sqrt{3}}{2}(c+e)$. Since $\Phi > 0$, every point of the door $[Y, H]$ is at distance less than $c + e + f - |RQ|$ from Y , so no point of T_R lies on the door; in particular A_5 lies strictly between R and H . The door leaves Y in direction 210° and σ_3 in direction 240° , and every point of the door lies above the line ZR , at height at least $\frac{\sqrt{3}}{4}(c - e) > 0$; so $YZRH$ is a convex quadrilateral, and it lies to the right of ℓ_5 . Lemma 3.2 with $m = 5$, $J = [A_5, H]$ and the door $[H, Y]$ gives $E \in YZRH$. The bounds follow as in Lemma 3.6; $|AH| < |AR|$ because $|AR|^2 - |AH|^2 = \frac{1}{4}(c - e)(2 + 4s + c - e)$. $\square$

If $e < \kappa c$, then $e < c$ and $\Phi \geq c - \frac{\sqrt{3}}{2}(c + e) > 0$, since $|RQ| \leq e + f$. So Lemma 3.11 contains Lemma 3.6, and its condition defines a convex set, because Φ is concave. Moreover $\partial\Phi/\partial f = 1 - (\frac{e}{2} + f)/|RQ| \geq 0$.

Lemma 3.12. *Let the configuration be d -regular. Put $Q^* = Q + (2d + c - f)$ and*

$$\Psi = 1 + b + s + 3d + 2c + e - L_S(A, B, D) - L_S(D, Q, Q^*) - |QR|,$$

where $L_S(p, q, r)$ is the length of an SMT for $\{p, q, r\}$. If $\Psi > 0$, then $|DF| \leq \max(|DZ|, |DQ|)$.

Proof. If Q is a terminal, then $F = Q$. Otherwise the walk continues from Q to $Q_2 = Q + g$, $g > 0$. Let S' be the union of the edges $AX, BX, XY, YD, YZ, ZR, ZQ, QQ_2$, of length $1 + b + s + d + c + e + f + g$. Since A, B, D are leaves and X, Y, Z have all their edges in S' , the rest of S consists of trees attached to S' only at R, Q and Q_2 . Hence S' is a shortest network containing $\{A, B, D, R, Q, Q_2\}$: replacing it by a shorter one would give a shorter network containing all terminals.

Put $x_Q = \text{Re}(D - Q) = d + \frac{c-f}{2}$, so $Q^* = Q + 2x_Q$, and suppose $g \geq 2x_Q$. Both Q_2 and Q^* lie on the horizontal line through Q , so $|Q^*Q_2| = g - 2x_Q$ and $L_S(D, Q, Q_2) \leq L_S(D, Q, Q^*) + g - 2x_Q$. The network formed by SMTs for $\{A, B, D\}$ and $\{D, Q, Q_2\}$ and the segment $[Q, R]$ contains the six points, and its length is at most

$$L_S(A, B, D) + L_S(D, Q, Q^*) + g - 2d - c + f + |QR| = L(S') - \Psi < L(S'),$$

a contradiction. So $g < 2x_Q$; if $2x_Q \leq 0$, this shows that Q is a terminal.

If $f \leq d$, the bound is Lemma 3.8. Let $f > d$. Then $g < 2x_Q < c + d$. For the image of the chain $D, Y, Z, Q, Q_2, \dots$ under $z \mapsto (D - z)/d$, with edge lengths $1, \frac{c}{d}, \frac{f}{d}, \frac{g}{d}, \dots$, the hypothesis of

Lemma 3.7 reads $g < d + c$, and its bound, mapped back, gives $|DF| \leq \max(|DZ|, |DQ|, |DQ_2|)$. Finally $|DQ_2|^2 = |DQ|^2 - g(2x_Q - g) < |DQ|^2$, since $0 < g < 2x_Q$. $\square$

The function Ψ is concave in the parameters: the points are affine functions of them, and the length of an SMT for three points, $\min_x(|xp| + |xq| + |xr|)$, is a convex function of (p, q, r) . So $\Psi > 0$ at the vertices of a bounded box implies $\Psi > 0$ on the box. The condition fails for large f ; Lemma 3.12 is used only on bounded boxes.

4. VERIFICATION FUNCTIONS

4.1. Splitting tables. The local configuration is the tree $\mathcal{T}$ with the vertices A, B, X, Y, D, Z, R, Q (and U, V) and the edges of Figure 1. The tables of [14] list 158 splittings for d.regular and 1418 for d.steiner. Each consists of a set V^* of terminals among A, B, D (or A, B, U, V), a set S^- of edges of $\mathcal{T}$, a set of vertices of $\mathcal{T}$ outside V^* that the new network S^+ joins, and a set t^* of segments between terminals among A, B, D, U, V, E, F . We have checked by a program that each of them has the following properties, which make it a splitting in the sense of Definition 2.1 for every configuration: V^* is nonempty and contains only terminals of $\mathcal{T}$; after removing S^- , no point of V^* keeps an edge; S^+ avoids V^* ; the terminals outside V^* , together with R and Q , which stand for $T_R \ni E$ and $T_Q \ni F$, are connected in $(\mathcal{T} \setminus S^-) \cup S^+$; and t^* is a forest on terminals which, after contracting $V \setminus V^*$ to one vertex, is a tree.

4.2. The bound for a splitting. Each record of the certificate names a box, a splitting τ and one of the nine lemma sets of [14], numbered $0, \dots, 8$. A lemma set supplies bounds for the edges of t^* at E and F and, for four-point reconnections, a pairing of the four points. For a parameter w we put

$$\tilde{F}_\tau(w) = \rho \sum_{\epsilon \in t^*} \ell_\epsilon(w) + N_\tau(w) - \sum_{\epsilon \in S^-} |\epsilon|,$$

where $|\epsilon|$ is 1 for the edge AX and the corresponding coordinate otherwise, and:

- $\ell_\epsilon(w)$ is the distance of the endpoints if both are among A, B, D, U, V . For an edge at E or F it is the bound of the lemma that applies: for AE , Lemma 3.4, 3.5, 3.11 or 3.7 according to the lemma set (0; 1 or 2; 5; 8); for DE, UE, VE (lemma set 5), Lemma 3.11; for DF and VF , Lemma 3.8 in the families with $f \leq d$, and Lemma 3.12 in d.regular with $f \geq d$. Each bound is a maximum of distances from the terminal to points that are affine functions of w (for lemma set 8 the checker also includes the linear term $e + c - 1$ of [14, Lemma 36], which only increases it), and comes with the condition of its lemma.
- $N_\tau(w)$ is 0 if S^+ joins at most one point, the distance if it joins two, and the length of an SMT for the three points if it joins three. If it joins four points p_1, p_2, p_3, p_4 , with the pairing $\{p_1, p_2\}, \{p_3, p_4\}$ given by the lemma set, then $N_\tau(w) = \min_{x, y \in \mathbb{C}} (|p_1x| + |p_2x| + |xy| + |p_3y| + |p_4y|)$.

Among the sets of four or more points joined in the tables, only $\{A, D, Q, R\}$ and $\{B, D, Q, R\}$ receive a pairing from a lemma set: $\{A, R\}, \{D, Q\}$ for lemma set 3, $\{A, D\}, \{Q, R\}$ for lemma sets 4 and 7, and $\{B, D\}, \{Q, R\}$ for lemma set 6. The checker rejects records whose splitting joins any other set of four or more points. In [14] these lemma sets carry conditions under which $N_\tau(w)$ is the length of a full Steiner tree with that pairing, given by a closed formula [6]; we do not use these conditions or formulas. Table 1 lists the correspondence.

Lemma 4.1. *Let B be a bounded box in the parameter space of a family, and let a record name B , a splitting τ and a lemma set. Suppose that at every vertex of B the conditions of the bounds used by τ , other than the condition $f \leq d$ of the family, hold and $\tilde{F}_\tau \leq 0$. Then the conditions*

lemma set	[14]	here	condition	used for
0	Lemmas 27, 28	Lemmas 3.4, 3.8	$c \leq 1; f \leq d$	$AE; DF, VF$
1, 2	Lemmas 29, 30	Lemma 3.5	linear	AE
5	Lemma 33	Lemma 3.11	$e < c, \Phi > 0$	AE, DE, UE, VE
8	Lemma 36	Lemma 3.7	$e < s + 1$	AE
3, 4, 6, 7	Lemmas 31, 32, 34, 35	pairings only	none	four-point S^+
all	—	Lemma 3.12	$\Psi > 0$	DF (d_regular, $f \geq d$)

TABLE 1. Lemma sets and the bounds they supply. In the families with $f \leq d$, every lemma set supplies the bound of Lemma 3.8 for DF or VF ; in d_steiner with $f \geq d$, no edge at F is used.

hold and $\tilde{F}_\tau \leq 0$ on all of B , and every configuration of the family whose parameter w lies in B has a splitting of cost at most $\tilde{F}_\tau(w) \leq 0$ at ρ .

Proof. The conditions are linear inequalities, or $\Phi > 0$ together with $e < c$, or $\Psi > 0$; each defines a convex set, so they hold on all of B . Each term of $\tilde{F}_\tau$ is convex in w : distances between affine functions of w and their maxima are convex, the SMT length for three points and N_τ are partial minima of convex functions of (w, x) and (w, x, y) , and the last sum is linear. So $\tilde{F}_\tau$ is at most its maximum over the vertices of B , which is at most 0. For a configuration with parameter $w \in B$, the bounds of Section 3 apply to the edges of t^* at E and F ; in the families with $f \leq d$ we use that $f \leq d$ for the configuration, and in d_regular with $f \geq d$ Lemma 3.12 needs no condition on f . Take for S^+ an SMT of its points, or for four points a network attaining $N_\tau(w)$. Then the splitting τ has cost at most $\tilde{F}_\tau(w)$. $\square$

4.3. Unbounded boxes. Boxes of the form $\prod_i [\text{lo}_i, \text{hi}_i]$ with some $\text{hi}_i = \infty$ occur; their lower ends are finite. For a coordinate t and a splitting τ , let ϵ_t be the edge of $\mathcal{T}$ whose length is t . Increasing t with the other coordinates fixed translates the part of the configuration beyond ϵ_t . Let n_T be the number of edges of t^* whose term depends on t : an edge between two of A, B, D, U, V depends on t if ϵ_t separates its endpoints, and a bound from Lemma 3.4 or 3.8 depends on t if ϵ_t separates the terminal from one of the points in its maximum. Let $d_+ = 1$ if ϵ_t separates two points joined by S^+ , and $d_- = 1$ if $\epsilon_t \in S^-$, and $d_+ = 0, d_- = 0$ otherwise. We call t *monotone* for τ if $n_T + d_+ = d_- = 0$, or if $n_T + d_+ \leq 1$ and $d_- = 1$. This is the rule of [14, Algorithm 2 and Proposition 37], with the dependence of the bounds on t determined from the points they use.

Lemma 4.2. *Let B be a box whose lower ends are finite, and let B_0 be its face on which every unbounded coordinate equals its lower end. Suppose that every unbounded coordinate t of B is monotone for τ , that the lemma set is 0, so that the edges of t^* at E and F use only Lemmas 3.4 and 3.8 and S^+ joins at most three points, and that c is bounded on B if t^* contains AE . If the hypotheses of Lemma 4.1 hold at the vertices of B_0 , then its conclusion holds on B .*

Proof. The condition $c \leq 1$ of Lemma 3.4 holds at the vertices of B_0 , which include $c = \text{hi}_c$, and it does not involve the unbounded coordinates; the condition $f \leq d$ of Lemma 3.8 is that of the family. So the bounds apply on B . Along a coordinate t , each distance term of $\tilde{F}_\tau$ changes at rate at most 1 in absolute value, and not at all if ϵ_t does not separate its two points; the same holds for a maximum of such terms. When t increases by h , some of the points joined by S^+ , which are at most three, are translated by a vector of length h and the others stay fixed. The moved set or its complement has at most one element, so, up to a translation of all the points, a single point moves by h ; the length of an SMT of the points then changes by at most h , since a segment can be added. If all or none of the points move, the length does not change. The last

sum changes at rate d_- . So $\tilde{F}_\tau$ is non-increasing in t , because $\rho n_T + d_+ - d_- \leq 0$ for $\rho < 1$. Hence $\tilde{F}_\tau(w) \leq \tilde{F}_\tau(w_0)$, where $w_0 \in B_0$ has the unbounded coordinates of w replaced by their lower ends, and $\tilde{F}_\tau(w_0) \leq 0$ by Lemma 4.1 applied to B_0 . The rest of the proof of Lemma 4.1 applies unchanged. $\square$

4.4. The families with $f \geq d$.

Lemma 4.3. *Let τ be a splitting without edges at F such that, if Q is one of the points joined by S^+ , the edge ZQ is in S^- . Let $f \geq f_0 \geq 0$. If the conditions of the bounds used by τ hold at the parameter (w', f_0) , they hold at (w', f) , and the cost of τ for the configuration with parameter (w', f) is at most $\tilde{F}_\tau(w', f_0)$.*

Proof. Write $Q_0 = Z + f_0\bar{\omega}$ for the position of Q at (w', f_0) . The terms of t^* do not involve f : the bounds for edges at E use only A, D, U, V, Y, Z, R, H , and of their conditions only $\Phi > 0$ involves f , and Φ is non-decreasing in f . The sum over S^- grows by $f - f_0$ if $ZQ \in S^-$ and is constant otherwise. If Q is joined by S^+ , the network for (w', f_0) together with the segment $[Q_0, Q]$, which is part of the removed edge ZQ , joins the same points with Q in place of Q_0 , and its length is larger by $f - f_0$. Otherwise S^+ does not depend on f . $\square$

In `d.steiner` with $f \geq d$, the certificate uses only splittings as in Lemma 4.3 and checks $\tilde{F}_\tau(w', d)$ on boxes in the coordinates $w' = (b, c, d, s, u, v, e)$; by Lemma 4.3 with $f_0 = d$, this covers every configuration with $f \geq d$. This reduction of the case $f \geq d$ to $f = d$ is that of [14, Appendix H.5]. Then Q depends on d through $f = d$, and an unbounded d is accepted only if both d and f are monotone for τ and the lemma set is 0. The bounds for the rates of change along d and along f add up, by the triangle inequality, so the proof of Lemma 4.2 applies along the direction in which d and f increase together. In `d.regular` with $f \geq d$, f is a coordinate of the boxes, which also contain points with $f < d$; Lemma 3.12 is used only on bounded boxes, and a box with $\text{hi}_f = \infty$ needs a splitting as in Lemma 4.3. For such a box, Lemma 4.3 with $f_0 = \text{lo}_f$ reduces it to its face $f = \text{lo}_f$, and if other coordinates are unbounded too, Lemma 4.2 is then applied to this face.

5. THE CERTIFICATE

5.1. Families and records. For each family, a binary tree of boxes is built. Its roots are the 2^n boxes in which each of the n coordinates ranges over $[0, 1]$ or $[1, \infty)$. A box is split into two along one coordinate, at the midpoint of a bounded side or at 2lo for a side $[\text{lo}, \infty)$. Each leaf is a record: the box, a splitting and a lemma set. Table 2 lists the families. A record whose box lies outside its family's domain is skipped: in the families with $f \leq d$ if $\text{lo}_f > \text{hi}_d$, in `d.regular` with $f \geq d$ if $\text{hi}_f < \text{lo}_d$, and in `d.steiner` if $\max(\text{lo}_u, \text{lo}_v) > \max(\text{hi}_b, 1)$, so that the box contains no point with $\max(u, v) \leq \max(b, 1)$ (Lemma 2.5).

5.2. Generation. The trees were generated by a program derived from the generator of [14] (its files `certificate/d_*/plot.f_*.cpp`). It processes the boxes depth first and, for each box, tries the splittings of the table with the lemma sets until it finds one whose bound is below -10^{-6} at every vertex in floating point; otherwise it splits the box along its widest side, where a side $[\text{lo}, \infty)$ counts as having width $1/\text{lo}$. We changed the order in which splittings are tried, the distribution of the work over threads and the size of the record identifiers, and we added Lemmas 3.11 and 3.12, the explicit four-point networks of Section 5.3, and the treatment of $f \geq d$ described in Section 4. The generator is not part of the proof: its output is checked by the two programs below, which share no code with it.

family	coordinates	n	records	checked	skipped	depth
d.regular, $f \leq d$	b, c, d, s, e, f	6	95,569,010	63,854,665	31,714,345	51
d.regular, $f \geq d$	b, c, d, s, e, f	6	10,376,823	9,426,537	950,286	56
d.steiner, $f \leq d$	b, c, d, s, u, v, e, f	8	436,460,311	341,304,562	95,155,749	61
d.steiner, $f \geq d$	b, c, d, s, u, v, e	7	12,696,761	12,690,883	5,878	75
total			555,102,905	427,276,647	127,826,258	

TABLE 2. The four families. “Checked” records were verified; “skipped” records lie outside the domain of their family ($292,651 + 5,878$ d.steiner records by the condition on u, v , the others by the condition on f); “depth” is the largest depth of a leaf in the tree.

5.3. Checking. The checker was written from the statements of [14] and from reading its verifier, whose record format it uses; it shares no code with the generator. It works in ball arithmetic with 128-bit precision (the Arb part of FLINT [9, 13]), with ρ computed as a ball from the exact rational $2141/2500$, and it accepts a record only if the following hold.

- (1) The splitting and lemma set exist and every lower end of the box is finite. If the box is unbounded, the lemma set is 0 and every unbounded coordinate is monotone for the splitting, as in Lemma 4.2; the one exception is an unbounded f in d.regular with $f \geq d$, which needs a splitting as in Lemma 4.3. If t^* contains AE , then c is bounded. A splitting with an edge at F is rejected on unbounded boxes in d.regular with $f \geq d$ and on all boxes in d.steiner with $f \geq d$; in the latter family f must be monotone for the splitting, which for a splitting without edges at F is the hypothesis of Lemma 4.3.
- (2) At every vertex with finite coordinates, and there is at least one, the conditions of the bounds used hold for all points of the balls, and the computed upper bound for $\tilde{F}_\tau$ is at most 0. In d.steiner with $f \geq d$ the vertex is evaluated with $f = d$.

The checker computes upper bounds: distances and their maxima in ball arithmetic; the SMT length for three points by the formula $(\frac{1}{2}(|pq|^2 + |qr|^2 + |rp|^2) + 2\sqrt{3} \text{area})^{1/2}$ when every angle of the triangle is certainly at most 120° , and otherwise by the sum of the two sides at an angle that is not certainly at most 120° ; and for four points, with the pairing $\{p_1, p_2\}, \{p_3, p_4\}$ of the lemma set, the length of the network $p_1x, p_2x, xy, p_3y, p_4y$, where x and y are double-precision points obtained by 300 alternating steps of the Weiszfeld iteration [19] started at the midpoints of the pairs. Any x, y give an upper bound for N_τ . The lower bound for Ψ uses these upper bounds for the two SMT lengths.

A second program checks that the records of each family form the leaves of a tree as described: it rebuilds every box from the roots, recomputes the coordinate along which each internal box is split by the rule of the generator, and checks that every leaf is a record whose box equals the computed box, that every record is reached exactly once, and that every internal node has two distinct new children. So the boxes of each family tile $[0, \infty)^n$.

5.4. Results. All records of the four families pass the checker, with the numbers of Table 2, and the second program reports that the boxes of each family tile $[0, \infty)^n$. The checker ran in parallel on chunks of consecutive records; the logs give the counts for every chunk, and they add up to the numbers in Table 2.

Control runs gave the expected failures. At $\rho = 2142/2500$ and at $\rho = 4283/5000$, the first 2,000,000 records of each family had between 29,122 and 129,167 failures per run. In a copy of 100,000 records of each family, changing the splitting of one record to one whose bound is positive at a vertex made exactly that record fail. Removing one record from the d.regular

family with $f \geq d$ made the second program fail. On small test certificates, the following records failed: records that use Lemma 3.12 on an unbounded box, or with $\Psi \leq 0$ at a vertex; a record that uses the closed formula of [14] for a four-point network away from $f = d$, which the checker refuses (this control was run with the checker's switch that replaces the explicit networks by the closed formulas; all runs on the certificate used the explicit networks); and a record on a box with unbounded f whose splitting joins Q by S^+ without removing the edge ZQ . A record that needs Lemma 3.11 failed under a version of the checker without it.

An independent audit of the checker found one defect in an earlier version: a record with an edge AE and lemma set 0 on a box with $\text{hi}_c = \infty$ was accepted, because the condition $c \leq 1$ was tested only at the finite vertices. Such records do not occur in the certificate, and the version used here rejects them (the condition on c in Lemma 4.2).

5.5. Proof of Theorem 1.1. Let $\rho = 2141/2500$. We prove $L_S(V) \geq \rho L_M(V)$ by induction on $n = |V|$. For $n \leq 4$ this was shown in Section 2. Let $n \geq 5$, assume the inequality for all sets with fewer points, and let S be an SMT for V . If some terminal has degree at least two, Lemma 2.3 applies. Otherwise S is full, and Lemmas 2.4 and 2.5 give a configuration with a parameter in the domain of one of the four families, with $\max(u, v) \leq \max(b, 1)$ in `d_steiner`; in `d_steiner` with $f \geq d$ we take the parameter w' without f . The boxes of that family tile $[0, \infty)^n$, so some record has a box containing the parameter, and this box is not skipped, since skipped boxes are disjoint from the domain. The record passed the checker, so by Lemmas 4.1, 4.2 and 4.3 the configuration has a splitting of cost at most 0 at ρ ; by Section 4.1 it is a splitting of S . Lemma 2.2 gives $L_S(V) \geq \rho L_M(V)$. $\square$

6. LIMITS OF THE PUBLISHED LEMMAS

In this section only the bounds of [14] are used, in the form stated there: for $|AE|$, those of Lemmas 3.4, 3.5, 3.6 and 3.7 (lemma sets 0, 1, 2, 5, 8), with the condition $e < \kappa c$ in Lemma 3.6; for $|DE|$, that of Lemma 3.6 (lemma set 5); for F , Lemma 3.8, which needs $f \leq d$; and for four-point reconnections the closed formulas of [14] under their conditions. For a splitting τ and a lemma set, the *cost bound* at a parameter w is $\rho \sum_{\epsilon \in t^*} \ell_\epsilon(w) + L^+(w) - \sum_{\epsilon \in S^-} |\epsilon|$, with ℓ_ϵ as in Section 4 and L^+ the length of an SMT of the points joined by S^+ , or the closed formula for four points. It is affine in ρ , and its *critical value* is the value of ρ at which it vanishes.

Proposition 6.1. *Consider the d -regular configuration with*

$$w_* = (b, c, d, s, e) = (1.3656, 1.10927, 0.365604, 1, 0.171605)$$

and $f = d$.

- (a) *For each of the 158 splittings of the d -regular table of [14] that has no edge at F , and each lemma set whose conditions hold at w_* , the cost bound is positive at $\rho = 0.856$. The largest critical value is $0.8559867412\dots$; it is attained by the splitting with $V^* = \{A, B\}$, S^- the seven edges of $\mathcal{T}$, S^+ joining D, Q, R , and $t^* = \{AB, BD\}$.*
- (b) *The same holds for every combination of a lemma set, a nonempty $V^* \subseteq \{A, B, D\}$, a set S^- of edges of $\mathcal{T}$ containing those at V^* , and a set of at most four vertices of $\mathcal{T}$ outside V^* that, joined by S^+ , connects the terminals outside V^* together with R and Q , when t^* is a shortest forest joining every point of V^* to the other terminals among A, B, D, E , with edges at E priced by the bound of the lemma set, and the length of S^+ is replaced by the length of an SMT of its points; there are 42,303 such combinations. Their critical values are at most 0.8559867413 .*

At $w_1 = (1.34514, 1.07421, 0.347656, 1.00391, 0.167969)$, with $f = d$, both statements hold with 0.8572 in place of 0.856 ; the largest critical value in (a) is $0.8571349212\dots$, and the bound in (b) is 0.8571349213 .

Proof. The statements are finite computations at points with rational coordinates, done by three programs. The first evaluates in ball arithmetic, from a transcription of the verifier of [14], all $158 \cdot 9$ pairs of a splitting and a lemma set; at both points the conditions of 527 pairs hold, and each of these has positive cost bound at 0.856 (at w_*) or 0.8572 (at w_1). The result is the same if the conditions of lemma sets 3 and 7, which are not linear, are ignored. The second evaluates the formulas generated by the code of [14] in floating point and finds the same largest critical values. The third enumerates the 42,303 combinations of (b) in ball arithmetic with lower bounds for $L(S^+)$: the distance, the length of an SMT for three points, and, for four points $p_1, \dots, p_4$, the minimum over the three pairings of a lower bound for $\min_{x,y}(|xp_i| + |xp_j| + |xy| + |yp_k| + |yp_l|)$ obtained from vectors $u_1, \dots, u_5$ of norm at most 1 with $u_1 + u_2 + u_3 = 0$ and $u_4 + u_5 = u_3$, since then the sum is at least $-\langle u_1, p_i \rangle - \langle u_2, p_j \rangle - \langle u_4, p_k \rangle - \langle u_5, p_l \rangle$; the program also computes, for each pairing, the bound $|U_{ij}U_{kl}|$ given by points U_{ij}, U_{kl} that form equilateral triangles with p_i, p_j and with p_k, p_l , and the largest bound for three of the points, and keeps the largest of these valid bounds. The shortest t^* is computed as a minimum spanning tree on lower bounds of the edge lengths. Every tree that joins four points with at most two Steiner points is a network of one of the three pairings, possibly with coinciding vertices, so the bound applies to every S^+ . $\square$

In (b), S^+ is one network, as in the tables of [14]. Joining at most four vertices of $\mathcal{T}$ is no restriction for such a network: after removing S^- , the terminals outside V^* and the trees T_R, T_Q lie in at most four components, one vertex from each suffices, and the length of an SMT does not decrease when points are added.

The cost bounds in (b) are continuous in f , and there are finitely many of them, so (b) also holds at (w_*, f) for $f \in (d, d + \delta)$ and some $\delta > 0$; for these f no bound for F from [14] applies. Let $\rho \geq 0.856$, and consider a certificate that uses only the lemmas of [14] and splittings of the kind in Proposition 6.1(b), and a record of it that is valid on its whole box, where the box contains (w_*, f) for such an f . Since $f > d$ there, the splitting has no edge at F . The conditions of the bounds for edges at E (lemma sets 0, 1, 2, 5, 8) are linear, so if they hold at the vertices of the box, they hold at (w_*, f) . The function that the record checks is $\rho \sum_{t^*} \ell_\epsilon + N - \sum_{S^-} |\epsilon|$, where N is the length of an SMT of the points joined by S^+ if there are at most three, and otherwise N_τ for the pairing of the lemma set, or any other convex function of w that is at least the length of an SMT of the points. It is convex in w (proof of Lemma 4.1), and at (w_*, f) it is at least the cost bound of (b), which is positive. So it is positive at a vertex of the box, and the record fails. In the formulation of [14], which checks this family on boxes in (b, c, d, s, e) with $f = d$ and uses the closed formulas for four points, part (a) gives the same conclusion at w_* , for lemma sets 3 and 7 even without their conditions, which are not linear; the closed formulas are convex. Hence, with the lemmas of [14] alone and with any partition, neither kind of certificate reaches $\rho = 0.856$, and a complete certificate of either kind for any value at least 0.856 needs further lemmas. Lemmas 3.11 and 3.12 both apply at $(w_*, f = d)$, with $\Phi > 0.06$ and $\Psi > 0.26$.

DATA AVAILABILITY

The certificate files (the records and the trees of the four families, 94,104,688,256 bytes uncompressed), compressed with xz, and the SHA-256 checksums of the uncompressed files are archived at doi:10.5281/zenodo.23570502. The checker, the coverage program, the splitting tables, the generator, the logs of all runs, the control runs, the build commands and the programs of Section 6 are in the repository <https://github.com/tejasvi/steiner-ratio-bound>. The generator, the splitting tables and the formulas of the splittings are taken or derived from the code of [14], published under the MIT License, Copyright (c) 2025–2026 Yisi Ke and Yankai Shu; the license text accompanies these files.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

REFERENCES

- [1] F. R. K. Chung and R. L. Graham, *A new bound for Euclidean Steiner minimal trees*, Ann. New York Acad. Sci. **440** (1985), 328–346.
- [2] F. R. K. Chung and F. K. Hwang, *A lower bound for the Steiner tree problem*, SIAM J. Appl. Math. **34** (1978), 27–36.
- [3] D.-Z. Du and F. K. Hwang, *A new bound for the Steiner ratio*, Trans. Amer. Math. Soc. **278** (1983), 137–148.
- [4] D.-Z. Du and F. K. Hwang, *The Steiner ratio conjecture of Gilbert and Pollak is true*, Proc. Natl. Acad. Sci. USA **87** (1990), 9464–9466.
- [5] D.-Z. Du and F. K. Hwang, *A proof of the Gilbert–Pollak conjecture on the Steiner ratio*, Algorithmica **7** (1992), 121–135.
- [6] D.-Z. Du, F. K. Hwang, G. D. Song and G. Y. Ting, *Steiner minimal trees on sets of four points*, Discrete Comput. Geom. **2** (1987), 401–414.
- [7] D.-Z. Du, F. K. Hwang and E. Y. Yao, *The Steiner ratio conjecture is true for five points*, J. Combin. Theory Ser. A **38** (1985), 230–240.
- [8] D.-Z. Du, E. Y. Yao and F. K. Hwang, *A short proof of a result of Pollak on Steiner minimal trees*, J. Combin. Theory Ser. A **32** (1982), 396–400.
- [9] The FLINT team, *FLINT: Fast Library for Number Theory*, <https://flintlib.org>.
- [10] R. L. Graham and F. K. Hwang, *Remarks on Steiner minimal trees*, Bull. Inst. Math. Acad. Sinica **4** (1976), 177–182.
- [11] E. N. Gilbert and H. O. Pollak, *Steiner minimal trees*, SIAM J. Appl. Math. **16** (1968), 1–29.
- [12] A. O. Ivanov and A. A. Tuzhilin, *The Steiner ratio Gilbert–Pollak conjecture is still open*, Algorithmica **62** (2012), 630–632.
- [13] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), 1281–1292.
- [14] Y. Ke, T. Huang, Y. Shu, D. He, J. Gai and L. Wang, *Towards solving the Gilbert–Pollak conjecture via large language models*, in: Proceedings of the 43rd International Conference on Machine Learning, PMLR **306** (2026), 56442–56498; we cite the numbering of arXiv:2601.22365v3. Code: <https://github.com/keyisi2006/Steiner-Ratio>, commit 8f19604.
- [15] H. O. Pollak, *Some remarks on the Steiner problem*, J. Combin. Theory Ser. A **24** (1978), 278–295.
- [16] J. H. Rubinstein and D. A. Thomas, *The Steiner ratio conjecture for six points*, J. Combin. Theory Ser. A **58** (1991), 54–77.
- [17] J. Savva, *Steiner_Ratio: independent verification of the Gilbert–Pollak certificate layer and a certificate-layer bound of 0.860*, Zenodo, 2026, doi:10.5281/zenodo.22223485.
- [18] T. Tao et al., *Optimization constants in mathematics*, online repository, <https://teorth.github.io/optimizationproblems/>, 2026.
- [19] E. Weiszfeld, *Sur le point pour lequel la somme des distances de n points donnés est minimum*, Tôhoku Math. J. **43** (1937), 355–386.

ORCID: <https://orcid.org/0000-0003-4668-3639>