

AN UPPER BOUND OF $2\sqrt{2}$ FOR THE THREE-DIMENSIONAL SOFA PROBLEM IN A SNAKE-SHAPED CORRIDOR

TEJASVI SINGH TOMAR

ABSTRACT. Georgiev, Gómez-Serrano, Tao and Wagner ask for the largest volume of a connected body that can be moved through a three-dimensional corridor of unit-square cross-section with two turns that are far apart, one in the x - y plane and one in the y - z plane (their Problem 6.64). Tang proved upper bounds for this problem, the best of which, 3.4083, rests on a branch-and-bound computation in double precision. We prove, without a computer, that the volume is at most $2\sqrt{2} = 2.8284\dots$ whenever the straight section between the turns is longer than $2 + \sqrt{2}$. The proof combines Tang's first-crossing argument with two observations. First, two tubes of unit-square cross-section whose axes make an angle θ meet in volume at most $1/\sin \theta$, and this bound is sharp. Second, unless the volume is less than $\sqrt{2}$, the axis of the body's entry tube, the moving copy of the infinite tube that contains the entry arm, makes the angle $\pi/4$ with the middle arm at some time when the body still lies in the first two arms. This is the three-dimensional analogue of Hammersley's bound $2\sqrt{2}$ for the planar moving sofa problem, and the information used at this single time cannot yield a better bound. For shorter straight sections the same argument gives explicit bounds that, together with the main bound, improve Tang's bounds on the largest volume for every length greater than $\sqrt{2}$ outside the interval $(2.2, 2.26)$.

1. INTRODUCTION

The moving sofa problem of Moser [6] asks for the largest area of a connected planar region that can be moved around a right-angled corner in a hallway of unit width. Hammersley [4] proved that the area is at most $2\sqrt{2}$. Gerver [2] constructed a region of area $2.2195\dots$, Kallus and Romik [5] proved the upper bound 2.37 with the help of a computer, and Baek [1] has given a proof that Gerver's region is optimal. Romik [8] extended Gerver's method and found, in closed form, a candidate for the variant in which the region must turn both ways.

O'Rourke [7] posed a three-dimensional version. A corridor of unit-square cross-section has an entry arm parallel to the x -axis, a middle arm parallel to the y -axis and an exit arm parallel to the z -axis. The first turn lies in the x - y plane and the second in the y - z plane. Georgiev, Gómez-Serrano, Tao and Wagner [3, Problem 6.64] ask for the largest volume $C_{6.64}$ of a connected bounded set that can pass continuously through such a corridor when the two turns are far apart. The code that accompanies [3] uses a straight section of length 4 between the turns [9, Definition 2.1]. The authors of [3] report a body of volume at least 1.81, and Tang [10] reports a body of volume at least 1.8558404, certified with the same code. Tang [9] also proved upper bounds: for a straight section of length M , the bounds 7.8691 for $M \geq 3.53$ and 4.2862 for $M \geq 2.61$ are proved by hand, and the best bound, 3.4083 for $M > 2.2$, rests on a branch-and-bound computation in double precision.

We define the corridor K_M and passages in Section 2, and we write $V^*(K_M)$ for the supremum of the volumes of compact connected sets that pass through K_M .

2020 *Mathematics Subject Classification.* 51M16, 49Q10; 52A38.

Key words and phrases. Moving sofa problem, snake-shaped corridor, rigid motion, intersection of tubes, volume bound.

Theorem 1.1. *For every $M > 2 + \sqrt{2}$, we have $V^*(K_M) \leq 2\sqrt{2}$.*

The bound $2\sqrt{2} = 2.8284\dots$ does not depend on M . It therefore holds for each reading of “far apart” with $M > 2 + \sqrt{2}$, including $M = 4$, and in the notation of [3] it gives $C_{6.64} \leq 2\sqrt{2}$. By Remark 2.3, it also holds for bounded connected sets and for stronger notions of passage.

The method. We follow Tang [9] in the first steps. The body starts in the entry arm, so at each time it lies in a moving copy of the infinite tube $\mathbb{R} \times [0, 1]^2$, which we call its *entry tube*. Fix $u > \sqrt{2}$ and consider the first time at which the top of the body reaches the height $1 + u$ in the middle arm. At that time either the body lies in the middle arm above height 1, or, by connectedness, the body has points in the unit windows at heights 1 and $1 + u$. In the second case the axis of the entry tube is steep: the cosine of its angle with the y -axis is at least $\mu^*(u) = (u^2 - 2)/(u^2 + 2)$ [9, Lemma 5.1]. We take $u = 2 + \sqrt{2}$, so that $\mu^*(u) = 1/\sqrt{2} = \cos(\pi/4)$.

To this argument our proof adds two steps. First, we prove that two tubes of unit-square cross-section whose axes make an angle θ meet in volume at most $1/\sin \theta$ (Lemma 3.3). This bound is sharp, and it is the constant conjectured in [9, Remark 8.1]. In the first case above, it shows that the axis is again steep unless the volume is less than $\sqrt{2}$. Second, we do not bound the volume at that first time. Since the axis of the entry tube starts parallel to the x -axis, it makes the angle $\pi/4$ with the y -axis at some time up to that first time. The body then lies in the union of the entry arm and the middle arm, and Lemma 3.3 bounds the parts of the body in the two arms by $\sqrt{2}$ each.

This is Hammersley’s argument in three dimensions: in the plane, after a rotation by $\pi/4$ the sofa lies in a strip of width one that makes the angle $\pi/4$ with both arms of the hallway, and this strip meets each arm in area at most $\sqrt{2}$ [5, Section 2]. Section 6 shows that, in three dimensions as in the plane, $2\sqrt{2}$ is the exact value of the relaxation that the proof uses. A smaller bound needs more information, such as the positions of the body at several times, which Kallus and Romik combine in the plane [5].

Section 5 applies the same argument to shorter straight sections. Corollary 5.1 bounds $V^*(K_M)$ for $\sqrt{2} < M \leq 2 + \sqrt{2}$. Together with Theorem 1.1, it improves the bounds of [9] on $V^*(K_M)$ for every $M > \sqrt{2}$ outside the interval $(2.2, 2.26)$. Section 6 discusses the limits of the method, notes that the proof uses only the first turn, and reports numerical observations on relaxations with several times.

2. THE CORRIDOR AND PASSAGES

We write e_x, e_y, e_z for the standard basis of $\mathbb{R}^3$ and vol for Lebesgue measure. For $c \in \mathbb{R}$ we write $\{y \leq c\}$ for the half-space $\{p \in \mathbb{R}^3 : p \cdot e_y \leq c\}$, and similarly for other inequalities.

Definition 2.1. Let $M > 0$. The *corridor* is $K_M = S_X \cup S_Y \cup S_Z$, where

$$S_X = (-\infty, 0] \times [0, 1] \times [0, 1], \quad S_Y = [0, 1] \times [0, 1 + M] \times [0, 1], \quad S_Z = [0, 1] \times [1 + M, 2 + M] \times [0, \infty).$$

The cube $[0, 1]^3$ is the first corner, where the corridor turns from the x -direction into the y -direction. The cube $[0, 1] \times [1 + M, 2 + M] \times [0, 1]$ is the second corner, where it turns into the z -direction. Between the corners lies a straight section of length M . These are the corridors of [9, Definition 2.1]. Other choices of the directions of the turns give images of K_M under reflections, which preserve volumes and passages.

For $c \geq 0$ we put

$$S_Y(c) = [0, 1] \times [0, c] \times [0, 1], \quad W_c = [0, 1] \times \{c\} \times [0, 1].$$

We also use the infinite tubes $X = \mathbb{R} \times [0, 1] \times [0, 1]$ and $Y = [0, 1] \times \mathbb{R} \times [0, 1]$. Since $S_X \subseteq \{y \leq 1\}$, $S_Y \cap \{y \leq c\} = S_Y(c)$ and $S_Z \subseteq \{y \geq 1 + M\}$, we have

$$(2.1) \quad K_M \cap \{y \leq c\} = S_X \cup S_Y(c) \subseteq X \cup Y \quad (1 \leq c < 1 + M).$$

In particular, $K_M \cap \{y \leq 1\} = (-\infty, 1] \times [0, 1] \times [0, 1] \subseteq X$.

A *rigid motion* is a map $g(p) = Rp + v$ with $R \in SO(3)$ and $v \in \mathbb{R}^3$. The rigid motions form the group $SE(3)$, which we view as a subset of $\mathbb{R}^{3 \times 3} \times \mathbb{R}^3$.

Definition 2.2. A *body* is a nonempty compact connected set $B \subset \mathbb{R}^3$. A *passage* of B through K_M is a continuous map $t \mapsto g_t$ from $[0, 1]$ to $SE(3)$ such that the sets $B_t = g_t(B)$ satisfy

- (i) $B_t \subseteq K_M$ for all $t \in [0, 1]$,
- (ii) $B_0 \subseteq S_X$,
- (iii) $B_1 \subseteq S_Z$.

We write $V^*(K_M)$ for the supremum of $\text{vol}(B)$ over all bodies B that have a passage through K_M .

This is the weak traversal of [9, Definition 2.2]. There a body must also have positive volume, which does not change $V^*(K_M)$.

Remark 2.3 (Other readings). The proof of Theorem 1.1 uses the conditions (ii) and (iii) only in the weaker forms

- (ii') $B_0 \subseteq K_M \cap \{y \leq 1\}$,
- (iii') B_1 meets $\{y \geq 1 + M\}$.

The theorem therefore holds for each notion of passage that implies (i), (ii') and (iii'). Stronger conditions on the end positions, such as $B_1 \subseteq S_Z \cap \{z \geq 1\}$ or the fixed start and end positions in the code of [3] (see [9, Remark 2.3]), are of this kind. If a bounded connected set E passes, then its closure is a body that passes with the same motion, because K_M , S_X and S_Z are closed. Its volume is at least the outer measure of E . Similarly, if E passes through the interior of K_M , with the interiors of S_X and S_Z as start and end regions, then its closure passes through K_M .

Condition (iii') only asks the body to reach the far end of the middle arm, so the second turn plays no role. Consider the corridor $(-\infty, 1] \times [0, 1]^2 \cup [0, 1] \times [0, \infty) \times [0, 1]$ with one turn, and a body that moves inside it from S_X into $[0, 1] \times [0, \infty) \times [0, 1]$. Take $M > 2 + \sqrt{2}$ at least the largest height that the body reaches during the motion, and then push the body in the direction e_y until its top reaches the height $1 + M$. The part of the corridor with one turn in $\{y \leq 1 + M\}$ lies in K_M , so the extended motion satisfies (i), (ii') and (iii'). Hence the bound $2\sqrt{2}$ also holds for the corridor with one turn.

3. TWO TUBES

We first bound the volume of the intersection of two tubes with unit-square cross-sections.

Definition 3.1. A *tube* is a set

$$T = \{\rho + \lambda e + a f_1 + b f_2 : \lambda \in \mathbb{R}, |a| \leq \frac{1}{2}, |b| \leq \frac{1}{2}\},$$

where (e, f_1, f_2) is an orthonormal basis of $\mathbb{R}^3$ and $\rho \in \mathbb{R}^3$. The line $\rho + \mathbb{R}e$ is the *axis* of T , the vector e is an *axis direction*, and the unit square $Q = \{a f_1 + b f_2 : |a|, |b| \leq \frac{1}{2}\}$ in the plane $e^\perp$ is the *cross-section*. The *angle* between two tubes with axis directions e and e' is the angle $\theta \in [0, \pi/2]$ between their axes, so that $\cos \theta = |e \cdot e'|$.

Each $p \in \mathbb{R}^3$ can be written uniquely as $p = \rho + \lambda e + \xi$ with $\lambda \in \mathbb{R}$ and $\xi \in e^\perp$, and $p \in T$ if and only if $\xi \in Q$. The point ρ lies on the axis, and Q is centred at 0. Rigid motions map tubes to tubes, and X and Y are tubes with axis directions e_x and e_y .

Lemma 3.2. *Let Q be a unit square centred at 0 in a plane, and let m, n be orthonormal vectors in this plane. For $b \in \mathbb{R}$ let $\ell(b)$ be the length of the interval $\{a \in \mathbb{R} : am + bn \in Q\}$. Then $\int_{\mathbb{R}} \ell(b)^2 db \leq 1$, with equality if and only if m is parallel to a side of Q .*

Proof. Let $\gamma(x) = \text{area}(Q \cap (Q + x))$ for x in the plane. For $\xi = am + bn$, the set of $r \in \mathbb{R}$ with $\xi - rm \in Q$ is an interval of length $\ell(b)$. By Tonelli's theorem,

$$\int_{\mathbb{R}} \gamma(rm) dr = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_Q(\xi) \mathbf{1}_Q(\xi - rm) d\xi dr = \int_{\mathbb{R}} \mathbf{1}_Q(\xi) \ell(\xi \cdot n) d\xi = \int_{\mathbb{R}} \ell(b)^2 db.$$

Let d_1, d_2 be unit vectors along the sides of Q , and let $\kappa \geq \sigma \geq 0$ be the numbers $|m \cdot d_1|$ and $|m \cdot d_2|$ in decreasing order. Then $\kappa^2 + \sigma^2 = 1$, so $\kappa \geq 1/\sqrt{2}$. Since $\gamma(x) = (1 - |x \cdot d_1|)_+(1 - |x \cdot d_2|)_+$, and since $1 - r\sigma \geq 1 - \sigma/\kappa \geq 0$ for $0 \leq r \leq 1/\kappa$, we get

$$\int_{\mathbb{R}} \gamma(rm) dr = 2 \int_0^{1/\kappa} (1 - r\kappa)(1 - r\sigma) dr = \frac{1}{\kappa} - \frac{\sigma}{3\kappa^2}.$$

This is at most 1 if and only if $3\kappa(1 - \kappa) \leq \sigma$. If $\sigma = 0$, then $\kappa = 1$ and equality holds. If $\sigma > 0$, then $1 - \kappa = \sigma^2/(1 + \kappa)$, and the inequality becomes $3\kappa\sigma \leq 1 + \kappa$. It holds strictly, because $3\kappa\sigma \leq \frac{3}{2}(\kappa^2 + \sigma^2) = \frac{3}{2} < 1 + \frac{1}{\sqrt{2}} \leq 1 + \kappa$. $\square$

Lemma 3.3. *Let T and T' be tubes at angle $\theta \in (0, \pi/2]$. Then*

$$\text{vol}(T \cap T') \leq \frac{1}{\sin \theta}.$$

Equality holds if and only if both tubes have a pair of faces orthogonal to the unit normal n of the plane spanned by their axis directions, and T and T' have the same orthogonal projection to the line $\mathbb{R}n$.

Proof. Write $T = \{\rho + \lambda e + \xi : \lambda \in \mathbb{R}, \xi \in Q\}$ and $T' = \{\rho' + \lambda e' + \xi : \lambda \in \mathbb{R}, \xi \in Q'\}$ as in Definition 3.1. Let $n = e \times e'/|e \times e'|$, where $|e \times e'| = \sin \theta > 0$, and put $m = n \times e$ and $m' = n \times e'$. Then (e, m, n) and (e', m', n) are orthonormal bases, and $m \cdot m' = e \cdot e'$. Let ℓ and ℓ' be the functions of Lemma 3.2 for Q with (m, n) and for Q' with (m', n) .

We slice by the planes $P_g = \{p : p \cdot n = g\}$, $g \in \mathbb{R}$. A point $p \in P_g$ has the form $\rho + \lambda e + am + bn$ with $b = g - \rho \cdot n$, so $p \in T$ if and only if $(p - \rho) \cdot m$ lies in an interval of length $\ell(g - \rho \cdot n)$. Similarly, $p \in T'$ if and only if $(p - \rho') \cdot m'$ lies in an interval of length $\ell'(g - \rho' \cdot n)$. Since the unit vectors m and m' lie in the plane $n^\perp$ and $|m \cdot m'| = \cos \theta$, the map $p \mapsto (p \cdot m, p \cdot m')$ from P_g to $\mathbb{R}^2$ is affine with Jacobian $\sin \theta$, and

$$\text{area}(T \cap T' \cap P_g) = \frac{\ell(g - \rho \cdot n) \ell'(g - \rho' \cdot n)}{\sin \theta}.$$

We integrate over g and apply the Cauchy–Schwarz inequality and Lemma 3.2:

$$\text{vol}(T \cap T') = \frac{1}{\sin \theta} \int_{\mathbb{R}} \ell(g - \rho \cdot n) \ell'(g - \rho' \cdot n) dg \leq \frac{1}{\sin \theta} \left(\int_{\mathbb{R}} \ell^2 \right)^{1/2} \left(\int_{\mathbb{R}} \ell'^2 \right)^{1/2} \leq \frac{1}{\sin \theta}.$$

If equality holds, then $\int \ell^2 = \int \ell'^2 = 1$, since both integrals are at most 1 and the product of their square roots is 1. Lemma 3.2 then shows that m is parallel to a side of Q . Then n is parallel to the other side, so T has a pair of faces orthogonal to n , and ℓ is the indicator function of $[-\frac{1}{2}, \frac{1}{2}]$. The same holds for T' , and equality in the Cauchy–Schwarz inequality forces $\rho \cdot n = \rho' \cdot n$. Conversely, under these conditions the integral equals $\int_{\mathbb{R}} \mathbf{1}_{[-1/2, 1/2]}(g - \rho \cdot n) dg = 1$. $\square$

Remark 3.4. Tang [9, Lemma 7.1] proves the bound $\sqrt{2}/\sin \theta$. Equality holds in Lemma 3.3, for example, for X and the tube $\{p : |p \cdot (-\sin \theta, \cos \theta, 0)| \leq \frac{1}{2}, 0 \leq p \cdot e_z \leq 1\}$. Their intersection is the product of a parallelogram of area $1/\sin \theta$ with $[0, 1]$.

4. PROOF OF THE MAIN THEOREM

In this section we fix $M > 0$, a body B , and a continuous map $t \mapsto g_t$ from $[0, 1]$ to $SE(3)$ that satisfies (i), (ii') and (iii'). We write $g_t(p) = R_t p + v_t$ and $B_t = g_t(B)$. Replacing B by B_0 and g_t by $g_t g_0^{-1}$ does not change the sets B_t , so we may assume that g_0 is the identity. Lemmas 4.1–4.3 are due to Tang [9], and we include short proofs.

4.1. The entry tube and the height.

Lemma 4.1 ([9, Lemmas 3.2 and 3.3]). (a) For each $t \in [0, 1]$, the set B_t lies in the tube $T_t = g_t(X)$, whose axis direction is $w_t = R_t e_x$. The function $\mu(t) = |w_t \cdot e_y|$ is continuous on $[0, 1]$, and $\mu(0) = 0$.

(b) The function $h(t) = \max\{p \cdot e_y : p \in B_t\}$ is continuous on $[0, 1]$, and $h(0) \leq 1$ and $h(1) \geq 1 + M$.

Proof. (a) By (ii'), $B = B_0 \subseteq K_M \cap \{y \leq 1\} \subseteq X$. Hence $B_t = g_t(B) \subseteq g_t(X)$. The rotation part R_t of g_t maps the axis direction e_x of X to w_t . The map $t \mapsto w_t$ is continuous, and $w_0 = e_x$ because R_0 is the identity.

(b) For $p \in B$ and $t, t' \in [0, 1]$ we have $|g_t(p) - g_{t'}(p)| \leq \|R_t - R_{t'}\| \max_{q \in B} |q| + |v_t - v_{t'}|$, with the operator norm. The right side does not depend on p and tends to 0 as $t' \rightarrow t$, so $|h(t) - h(t')|$ tends to 0. Finally, $h(0) \leq 1$ by (ii') and $h(1) \geq 1 + M$ by (iii'). $\square$

4.2. The first crossing.

Lemma 4.2 ([9, Lemmas 3.4 and 3.5]). Let $1 < c < 1 + M$, and let $\tau = \min\{t \in [0, 1] : h(t) \geq c\}$.

(a) The minimum exists, and $B_t \subseteq S_X \cup S_Y(c)$ for all $t \in [0, \tau]$.

(b) The set B_τ contains a point of W_c .

(c) If B_τ meets $\{y \leq 1\}$, then B_τ contains a point of W_1 .

Proof. (a) By Lemma 4.1(b), the set $\{t : h(t) \geq c\}$ is closed and contains 1, so τ exists. Since $h(0) \leq 1 < c$, we have $\tau > 0$ and $h(t) < c$ for $t < \tau$, and $h(\tau) = c$ by continuity. Hence $B_t \subseteq K_M \cap \{y \leq c\}$ for $t \leq \tau$, and (2.1) gives (a).

(b) Since B_τ is compact, some $q \in B_\tau$ has $q \cdot e_y = h(\tau) = c > 1$. By (a) and $S_X \subseteq \{y \leq 1\}$, we have $q \in S_Y(c)$, so $q \in W_c$.

(c) The sets $A_1 = B_\tau \cap \{y \leq 1\}$ and $A_2 = B_\tau \cap \{y > 1\}$ are disjoint and nonempty, their union is B_τ , and A_1 is closed. Since B_τ is connected, A_2 is not closed, and some point $p \in B_\tau \setminus A_2$ is a limit of points of A_2 . Then $p \cdot e_y = 1$. By (a), the points of A_2 lie in $S_Y(c)$, so their x - and z -coordinates lie in $[0, 1]$, and so do those of p . Thus $p \in W_1$. $\square$

4.3. The tilt.

Lemma 4.3 ([9, Lemma 5.1]). Let $u > \sqrt{2}$ and $\mu^*(u) = (u^2 - 2)/(u^2 + 2)$. If a tube with axis direction w contains a point $p \in W_1$ and a point $q \in W_{1+u}$, then $|w \cdot e_y| \geq \mu^*(u)$.

Proof. Let $\mu = |w \cdot e_y|$ and $\nu = \sqrt{1 - \mu^2}$. Write $d = q - p = (d_x, u, d_z)$ and $r = (d_x^2 + d_z^2)^{1/2}$. Then $r \leq \sqrt{2}$, because p and q have x - and z -coordinates in $[0, 1]$. Write the tube as $\{\rho + \lambda w + \xi : \lambda \in \mathbb{R}, \xi \in Q\}$. The component $d - (d \cdot w)w$ of d orthogonal to w is the difference of two points of Q , and Q has diameter $\sqrt{2}$. Hence $|d|^2 - (d \cdot w)^2 \leq 2$. By the Cauchy–Schwarz inequality, $|d \cdot w| \leq u\mu + |d_x w_x + d_z w_z| \leq u\mu + r\nu$. Since $|d|^2 = u^2 + r^2$, we get

$$u^2 + r^2 - 2 \leq (u\mu + r\nu)^2.$$

Since $\mu^2 + \nu^2 = 1$, this is equivalent to $(u\nu - r\mu)^2 \leq 2$, and therefore $u\nu \leq \sqrt{2} + r\mu \leq \sqrt{2}(1 + \mu)$. As $\nu = \sqrt{1 - \mu}\sqrt{1 + \mu}$ and $1 + \mu > 0$, we get $u\sqrt{1 - \mu} \leq \sqrt{2}\sqrt{1 + \mu}$. Squaring gives $u^2(1 - \mu) \leq 2(1 + \mu)$, that is, $\mu \geq \mu^*(u)$. $\square$

Remark 4.4. Lemma 4.3 is sharp. Let $p = (0, 1, 0)$, $q = (1, 1+u, 1)$ and $w = (\nu/\sqrt{2}, \mu^*(u), \nu/\sqrt{2})$ with $\nu = \sqrt{1 - \mu^*(u)^2}$. A short computation shows that $|q - p|^2 - ((q - p) \cdot w)^2 = 2$, that is, the component of $q - p$ orthogonal to w has length $\sqrt{2}$. Therefore p and q lie in a tube with axis direction w whose cross-section has this component as a diagonal.

4.4. The volume bound.

Proposition 4.5. *Let $\sqrt{2} < u < M$ and $0 < s \leq \mu^*(u)$. Then*

$$\text{vol}(B) \leq \frac{1}{s} + \frac{1}{\sqrt{1 - s^2}}.$$

Proof. Put $c = 1 + u$, so that $1 < c < 1 + M$, and let τ be as in Lemma 4.2. Since $\mu^*(u) < 1$, we have $s < 1$. We first show that $\text{vol}(B) < 1/\sqrt{1 - s^2}$ or $\mu(\tau) \geq s$.

Suppose that $B_\tau \subseteq \{y > 1\}$. By Lemma 4.2(a), and since S_X lies in $\{y \leq 1\}$, we have $B_\tau \subseteq S_Y(c) \subseteq Y$. Also $B_\tau \subseteq T_\tau$ by Lemma 4.1(a). If $\mu(\tau) < s$, then the angle θ between T_τ and Y satisfies $\cos \theta = \mu(\tau) < s$, and Lemma 3.3 gives

$$\text{vol}(B) = \text{vol}(B_\tau) \leq \text{vol}(T_\tau \cap Y) \leq \frac{1}{\sin \theta} < \frac{1}{\sqrt{1 - s^2}}.$$

Suppose now that B_τ meets $\{y \leq 1\}$. By Lemma 4.2(b) and (c), the tube T_τ contains points of W_1 and W_{1+u} , and Lemma 4.3 gives $\mu(\tau) \geq \mu^*(u) \geq s$.

It remains to treat the case $\mu(\tau) \geq s$. The function μ is continuous and $\mu(0) = 0 < s$, so $\mu(t) = s$ for some $t \in (0, \tau]$. By Lemma 4.2(a) and (2.1), $B_t \subseteq X \cup Y$, and $B_t \subseteq T_t$ by Lemma 4.1(a). Let θ_x and θ_y be the angles between T_t and the tubes X and Y . Then $\cos \theta_y = |w_t \cdot e_y| = s$ and $\cos^2 \theta_x = (w_t \cdot e_x)^2 \leq 1 - (w_t \cdot e_y)^2 = 1 - s^2$. So $\sin \theta_y = \sqrt{1 - s^2} > 0$ and $\sin \theta_x \geq s > 0$, and Lemma 3.3 gives

$$\text{vol}(B) = \text{vol}(B_t) \leq \text{vol}(T_t \cap X) + \text{vol}(T_t \cap Y) \leq \frac{1}{s} + \frac{1}{\sqrt{1 - s^2}}.$$

In both cases, $\text{vol}(B) \leq \max\{1/\sqrt{1 - s^2}, 1/s + 1/\sqrt{1 - s^2}\} = 1/s + 1/\sqrt{1 - s^2}$. $\square$

Proof of Theorem 1.1. Let $M > 2 + \sqrt{2}$, let B be a body with a passage through K_M , and let $u = 2 + \sqrt{2}$. Then $\sqrt{2} < u < M$, and

$$\mu^*(u) = \frac{u^2 - 2}{u^2 + 2} = \frac{4 + 4\sqrt{2}}{8 + 4\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Since a passage satisfies (i), (ii') and (iii'), Proposition 4.5 with $s = 1/\sqrt{2}$ gives $\text{vol}(B) \leq \sqrt{2} + \sqrt{2} = 2\sqrt{2}$. $\square$

5. SHORTER STRAIGHT SECTIONS

We now apply Proposition 4.5 to straight sections of length at most $2 + \sqrt{2}$. For $u > \sqrt{2}$ put

$$\Gamma(u) = \frac{u^2 + 2}{u^2 - 2} + \frac{u^2 + 2}{2\sqrt{2}u}.$$

Corollary 5.1. *For every $M \in (\sqrt{2}, 2 + \sqrt{2}]$, we have $V^*(K_M) \leq \Gamma(M)$.*

Proof. Let B be a body with a passage through K_M , and let $\sqrt{2} < u < M$. We apply Proposition 4.5 with $s = \mu^*(u)$. Since $1/s = (u^2 + 2)/(u^2 - 2)$ and $1 - s^2 = 8u^2/(u^2 + 2)^2$, it gives $\text{vol}(B) \leq \Gamma(u)$. We let u tend to M and use the continuity of Γ . $\square$

Remark 5.2. (a) Since $\Gamma(2 + \sqrt{2}) = 2\sqrt{2}$, Theorem 1.1 also holds for $M = 2 + \sqrt{2}$.

- (b) The derivative $\Gamma'(u) = -8u/(u^2 - 2)^2 + (u^2 - 2)/(2\sqrt{2}u^2)$ is negative if and only if $(u^2 - 2)^3 < 16\sqrt{2}u^3$, that is, $u^2 - 2\sqrt{2}u - 2 < 0$, that is, $u < 2 + \sqrt{2}$. Thus Γ decreases on $(\sqrt{2}, 2 + \sqrt{2}]$. Moreover, μ^* increases and $\mu^*(2 + \sqrt{2}) = 1/\sqrt{2}$, so Proposition 4.5 allows only $s < 1/\sqrt{2}$ when $M \leq 2 + \sqrt{2}$, and $1/s + 1/\sqrt{1 - s^2}$ decreases in s on $(0, 1/\sqrt{2}]$. Hence Corollary 5.1 is the best bound that Proposition 4.5 gives for these M .
- (c) We have $\Gamma(2.3) < 3.337$, $\Gamma(2.61) < 3.025$ and $\Gamma(3) < 2.868$. Since $\Gamma(2.26) < 3.4083$ and Γ decreases, Corollary 5.1 improves the bound 3.4083 of [9, Theorem 6.2] for $2.26 \leq M \leq 2 + \sqrt{2}$. For $2 < M \leq 2.2$ we have $\Gamma(M) < \Gamma(2) = 3 + 3\sqrt{2}/4 < 4.061$, which is below the bound 4.0706 that [9, Table 2] gives for $M > 2$. In Tang's analytic bound [9, (5)], the first term $\sqrt{2}(u^2 + 2)/(u^2 - 2) + (u^2 + 2)/(2u)$ of the maximum equals $\sqrt{2}\Gamma(u)$. The factor $\sqrt{2}$ comes from the chord estimates of [9, Lemma 5.3], which exceed the bounds of Lemma 3.3 for X and Y by this factor.
- (d) For $\sqrt{2} < M \leq 2$, [9, Remarks 4.4 and 5.6] give the bounds $f_T(M)$ and $\inf_{u < M} \Psi(u)$, where $f_T(u) = u + (1 + \sqrt{2})\sqrt{u^2 + 2}/(u - \sqrt{2})$ and Ψ is the maximum in [9, (5)]. Both exceed $\Gamma(M)$. Indeed, $\Psi \geq \sqrt{2}\Gamma$ by (c), and Γ decreases. Moreover, $f_T(u) > \Gamma(u)$ for $u > \sqrt{2}$: the inequality $(1 + \sqrt{2})(u + \sqrt{2}) > \sqrt{u^2 + 2}$ gives $(1 + \sqrt{2})\sqrt{u^2 + 2}/(u - \sqrt{2}) > (u^2 + 2)/(u^2 - 2)$, and $(2\sqrt{2} - 1)u^2 > 2$ gives $u > (u^2 + 2)/(2\sqrt{2}u)$. Together with (c) and Theorem 1.1, this shows that our bounds on $V^*(K_M)$ are below those of [9] for every $M > \sqrt{2}$ outside the interval $(2.2, 2.26)$.

6. LIMITS OF THE METHOD

At the time t in the proof of Proposition 4.5, the argument uses only that the body lies in $T_t \cap (S_X \cup S_Y(c))$, where T_t is a tube whose axis direction w satisfies $|w \cdot e_y| = s$. We first show that this information cannot lead to a bound below $2\sqrt{2}$.

Proposition 6.1. *Let $0 < s < 1$ and $c \geq 1 + (1 + s)/\sqrt{1 - s^2}$. The maximum of $\text{vol}(T \cap (S_X \cup S_Y(c)))$ over all tubes T whose axis direction w satisfies $|w \cdot e_y| = s$ exists and equals $1/s + 1/\sqrt{1 - s^2}$. This value is at least $2\sqrt{2}$, with equality if and only if $s = 1/\sqrt{2}$.*

Proof. For each such tube, $\text{vol}(T \cap (S_X \cup S_Y(c))) \leq \text{vol}(T \cap X) + \text{vol}(T \cap Y) \leq 1/s + 1/\sqrt{1 - s^2}$, as in the proof of Proposition 4.5. For the converse, let $\alpha = \arcsin s$, $n_\alpha = (-\sin \alpha, \cos \alpha, 0)$ and

$$T = \{p \in \mathbb{R}^3 : 0 \leq (p - e_y) \cdot n_\alpha \leq 1, 0 \leq p \cdot e_z \leq 1\},$$

a tube with axis direction $w = (\cos \alpha, \sin \alpha, 0)$. For $p = (x, y, z)$ we have $(p - e_y) \cdot n_\alpha = (y - 1)\cos \alpha - x \sin \alpha$. If $p \in T \cap X$, then $0 \leq y \leq 1$ and $x \sin \alpha \leq (y - 1)\cos \alpha \leq 0$, so $p \in S_X$. If $p \in T \cap Y$, then $0 \leq x \leq 1$ and

$$1 \leq 1 + x \tan \alpha \leq y \leq 1 + \frac{1 + x \sin \alpha}{\cos \alpha} \leq 1 + \frac{1 + s}{\sqrt{1 - s^2}} \leq c,$$

so $p \in S_Y(c)$. Since $S_X \cup S_Y(c) \subseteq X \cup Y$, the set $T \cap (S_X \cup S_Y(c))$ is the union of the convex sets $T \cap X$ and $T \cap Y$. These meet only in the segment $\{(0, 1, z) : 0 \leq z \leq 1\}$, so their union is connected. The set $T \cap X$ is the product of $[0, 1]$ with a parallelogram bounded by two pairs of parallel lines at distance 1 whose normals e_y and n_α make the angle α , so its volume is $1/\sin \alpha = 1/s$. Similarly, $\text{vol}(T \cap Y) = 1/\cos \alpha = 1/\sqrt{1 - s^2}$. Hence $\text{vol}(T \cap (S_X \cup S_Y(c))) = 1/s + 1/\sqrt{1 - s^2}$. Finally, $s\sqrt{1 - s^2} \leq \frac{1}{2}$, so

$$\frac{1}{s} + \frac{1}{\sqrt{1 - s^2}} \geq \frac{2}{(s\sqrt{1 - s^2})^{1/2}} \geq 2\sqrt{2},$$

and both inequalities are equalities if and only if $s = 1/\sqrt{2}$. □

For $s = 1/\sqrt{2}$ the condition on c reads $c \geq 2 + \sqrt{2}$, and the proof of Theorem 1.1 uses $c = 3 + \sqrt{2}$, so $2\sqrt{2}$ is the exact value of the relaxation in that proof. More generally, if $s \leq \mu^*(u)$, then $(1+s)/\sqrt{1-s^2} = \sqrt{(1+s)/(1-s)} \leq \sqrt{(1+\mu^*(u))/(1-\mu^*(u))} = u/\sqrt{2} < u$, so Proposition 6.1 applies with $c = 1 + u$. Hence no choice of u and s in Proposition 4.5 leads to a bound below $2\sqrt{2}$.

Remark 6.2 (The plane). In the plane, Kallus and Romik note that the analogous relaxation for a single angle α of rotation has the value $\sec \alpha + \csc \alpha$ [5, (12)], and its value $2\sqrt{2}$ at $\alpha = \pi/4$ is Hammersley's bound. With $s = \sin \alpha$, Proposition 6.1 gives the same value in three dimensions, and the tube in its proof is the product of an extremal planar strip with $[0, 1]$. Kallus and Romik improve Hammersley's bound by combining the constraints of several angles at once [5].

Remark 6.3 (Several times). The observations in this remark are numerical and are not proved, and nothing in this paper depends on them. Suppose that $\text{vol}(B) > 1$. For $t \leq \tau$ the set B_t lies in the slab $\{0 \leq z \leq 1\}$, which meets a tube with axis direction $\pm e_z$ in volume 1. So $w_t \neq \pm e_z$ for $t \leq \tau$, and the angle in $[0, \pi]$ between e_x and the projection of w_t to the x - y plane varies continuously in t . The analogue of the method of Kallus and Romik uses times $t_1, \dots, t_k \leq \tau$ at which this angle takes prescribed values $\alpha_1, \dots, \alpha_k$. Since the sine of this angle is at least $\mu(t) = |w_t \cdot e_y|$, the proof of Proposition 4.5 with $s = \sin \alpha_j$ shows that such times exist when $\mu^*(u) \geq \sin \alpha_j$ and $\text{vol}(B) \geq 1/\cos \alpha_j$. At these times the projection of w_t makes the oriented angle $+\alpha_j$ or $-\alpha_j$ with e_x . The resulting relaxation takes the supremum, over the signs and over all rigid motions $\varphi_1, \dots, \varphi_k$ whose rotation parts map e_x to vectors with these oriented projected angles, of the largest volume of a connected component of $X \cap \bigcap_j \varphi_j^{-1}(S_X \cup S_Y(c))$.

We take $M = 4$. The angle 51° requires $\mu^*(u) \geq \sin 51^\circ$, that is, $c = 1 + u \geq 4.9936\dots$, and we compute with $c = 5$, which can only increase the relaxation. For the angles $+30^\circ$ and $+51^\circ$, the best local maximum found numerically is 2.5431, at configurations that are products of planar regions with $[0, 1]$. In the plane, the best values found for every other choice of signs and of a roll of the body by π about its axis are at most 1.923, attained with both angles positive and a roll. Also in the plane, with positive angles and no roll, the triples $(51^\circ, 38^\circ, 22^\circ)$, $(51^\circ, 35^\circ, 20^\circ)$ and $(51^\circ, 36^\circ, 18^\circ)$ give best local maxima of about 2.48. Local searches in three dimensions with the angles $+30^\circ$ and $+51^\circ$, over tilts of the axis out of the x - y plane with $|w_z| \leq 0.95$, rolls of the body about its axis and offsets, found no larger value, and the best configurations they found are planar. The main obstacle to a computer-assisted bound along these lines is the roll of the body about its axis between the times.

Remark 6.4 (The second turn). The proof of Theorem 1.1 does not use the second turn, so $2\sqrt{2}$ is also a bound for the corridor with one turn (Remark 2.3). The product of Gerver's sofa [2] with $[0, 1]$ passes through that corridor, and its volume is the area 2.2195... of Gerver's sofa. Consequently, no argument that uses only the first turn can give a bound below this area.

DATA AVAILABILITY

The proofs use no computer. Scripts that check Lemmas 3.2, 3.3 and 4.3 and Proposition 6.1 numerically, a script that checks the inequality of Lemma 3.2 and the constants of Section 5 in interval arithmetic, scripts that follow the proof of Theorem 1.1 on two explicit motions, and the exploratory code behind Remark 6.3 are archived at doi:10.5281/zenodo.23505338 and available at <https://github.com/tejasvi/snake-sofa-bound>.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

REFERENCES

- [1] J. Baek, *Optimality of Gerver's sofa*, arXiv:2411.19826, 2024.
- [2] J. L. Gerver, *On moving a sofa around a corner*, Geom. Dedicata **42** (1992), 267–283.
- [3] B. Georgiev, J. Gómez-Serrano, T. Tao and A. Z. Wagner, *Mathematical exploration and discovery at scale*, arXiv:2511.02864, 2025.
- [4] J. M. Hammersley, *On the enfeeblement of mathematical skills by “Modern Mathematics” and by similar soft intellectual trash in schools and universities*, Bull. Inst. Math. Appl. **4** (1968), 66–85.
- [5] Y. Kallus and D. Romik, *Improved upper bounds in the moving sofa problem*, Adv. Math. **340** (2018), 960–982.
- [6] L. Moser, *Problem 66-11: Moving furniture through a hallway*, SIAM Rev. **8** (1966), 381.
- [7] J. O'Rourke, *Sofa in a snaky 3D corridor*, MathOverflow question 246914, 2016, <https://mathoverflow.net/q/246914>.
- [8] D. Romik, *Differential equations and exact solutions in the moving sofa problem*, Exp. Math. **27** (2018), 316–330.
- [9] Y. Tang, *An upper bound for the three-dimensional moving sofa problem in a snaky corridor*, arXiv:2610.14050, 2026.
- [10] Y. Tang, *New certified lower bounds for the moving sofa problem in a three-dimensional snaky corridor*, arXiv:2610.14051, 2026.

ORCID: <https://orcid.org/0000-0003-4668-3639>