SMALE'S SIXTH PROBLEM FOR FIVE BODIES IN THE PLANE, WITH A PROOF FORMALIZED IN LEAN

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Abstract. We prove that for every choice of five positive masses the planar five-body problem has finitely many central configurations up to similarity. This settles Smale's sixth problem for five bodies in the plane. Albouy and Kaloshin proved it for masses outside a closed algebraic subset of codimension two. We follow their method. Infinitely many central configurations would give an algebraic curve of complex solutions. At the places at infinity of the curve the valuations of the mutual distances must balance. We determine the possible valuation vectors at all positive masses, together with the mass conditions under which each can occur. We treat limits in which four bodies form a cluster of zero potential energy by expanding the potential, which is constant along the curve, at a place. This excludes, at every positive mass, one of the two diagrams in which such clusters occur, and part of the other. The same expansion excludes one further diagram at every positive mass. What remains is an exact analysis, by computer algebra, of the sets of mass conditions that can hold together at positive masses. Exact rational certificates show that the balance fails under each of them, except on a few one-parameter families of masses, which we treat separately. On the rays through $(1, 4, 4, 4, 4)$ and $(1, 1, 4, 4, 4)$ the certificates do not apply. On the first ray they cannot apply: there the complex equations have a continuum of solutions, as Albouy and Kaloshin observed. On these rays we count the real central configurations: there are 306 and 294 up to orientation-preserving similarity. The certificates, the programs that check them and a proof of the theorem in the Lean proof assistant are available.

1. Introduction

A configuration $q = (q_1, \dots, q_n)$ of n distinct points in the plane, carrying positive masses $m_1, \dots, m_n$, is a *central configuration* if there are $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}^2$ such that

$$\lambda\, m_i(q_i - c) = \sum_{j \neq i} \frac{m_i m_j\,(q_i - q_j)}{|q_i - q_j|^3} \qquad (i = 1, \dots, n). \tag{1.1}$$

The right side is minus the Newtonian force on the body i: the force on each body points towards the centre of mass c and is proportional to its distance from it. Central configurations are the shapes of the homographic solutions of the n-body problem, and in particular the shapes of its relative equilibria. If q is a central configuration, so is any configuration similar to it. Smale's sixth problem [15] (Problem 9 of [1]) asks whether, for every choice of positive masses, the number of central configurations up to similarity is finite.

For $n = 3$ there are five, the two equilateral triangles of Lagrange and the three collinear configurations of Euler. Moulton [9] counted the collinear ones for all n. Hampton and Moeckel [4] proved finiteness for four bodies and all positive masses. Albouy and Kaloshin [2] proved finiteness for five bodies in the plane when the masses lie outside a closed algebraic subset $\mathcal{A} \subset (\mathbb{R}_{>0})^5$ of codimension two. Their argument produces singular sequences that approach one of the first eight limit diagrams in their list of sixteen. Each of these eight forces a polynomial

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condition on the masses. The set $\mathcal{A}$ is where two independent conditions hold at once. Jensen and Leykin [5] gave a tropical proof of finiteness for generic masses. Moczurad and Zgliczyński [12] counted the central configurations exactly, with a computer-assisted proof, at several mass vectors of the exceptional set. We remove the exceptional set.

Theorem 1.1. *For every $m \in (\mathbb{R}_{>0})^5$ the planar five-body problem with masses m has finitely many central configurations up to similarity.*

We use the results of [2] listed in §2.6, in particular their classification of the limit diagrams (their Fig. 11) and the mass conditions of their §8.

Part of the exceptional set of [2] cannot be avoided by an argument about the complexified equations alone. Roberts [14] found a continuum of central configurations for the masses $(1, 1, 1, 1, -\frac{1}{4})$. Albouy and Kaloshin observe that changing the signs of m_5 and of the four distances r_{k5} turns it into a continuum of solutions, with masses $(4, 4, 4, 4, 1)$, of the polynomial system they work with [2, pp. 543–544]. These solutions are not central configurations, because the distances r_{k5} are negative square roots, but an argument that sees only the complexified system cannot tell them apart. At $(1, 4, 4, 4, 4)$ the complex solution set is therefore infinite. Finiteness of the real central configurations there must use the signs of the distances. On the rays through $(1, 4, 4, 4, 4)$ and $(1, 1, 4, 4, 4)$ and their images under permutations of the bodies we count the real central configurations directly: for every positive multiple of a permutation of $(1, 4, 4, 4, 4)$ there are exactly 306 such configurations up to orientation-preserving similarity, and for $(1, 1, 4, 4, 4)$ exactly 294 (Theorems 10.1 and 10.2). We call these rays the *special rays*: a mass vector lies on a special ray if and only if it is a positive multiple of a permutation of $(1, 4, 4, 4, 4)$ or of $(1, 1, 4, 4, 4)$.

The method. Our argument has six parts.

Curves and balancing. If there are infinitely many central configurations, the complex solution set contains an irreducible curve on which the potential U is a nonzero constant and some mutual distance is not constant. At each place at infinity of this curve the inverse distances $p_{kl} = r_{kl}^{-1}$ have valuations. The vectors $\rho(P) = (-\operatorname{val}_P p_{kl})_{k<l}$ sum to zero over the places (§2). This is the starting point of [2].

Types and mass conditions. At a singular place, Albouy and Kaloshin show that the configuration approaches one of sixteen diagrams and that most diagrams force polynomial conditions on the masses. We make this sharp at all positive masses. For each diagram and labelling of the bodies we give the set of vectors $\rho(P)$ that can occur, as an explicit polyhedral set (a *piece*) that is present only when certain mass conditions hold (Theorem 3.12). Besides results of [2], Theorem 3.12 contains our sharper type statements for several diagrams and a lemma on diagram 7 (§3.5) that is not in [2].

Lexicographic separation. If the pieces present at m admit a staged family of linear functionals that are nonnegative on all of them and positive on each in turn, the vectors $\rho(P)$ cannot sum to zero and m has no such curve (§4). On the sixteen strata of the sorted, scaled masses we analyse by exact computer algebra the sets of mass conditions that can hold together (§8). We certify such a family for each of these sets, with rational functionals checked in exact arithmetic, except for the sets at two mass points, where we use a smaller hull of diagram 5, and on a few one-parameter families of masses, which we treat separately (§§6, 9 and 10.4).

Clusters of zero energy. Using all the singular places of a curve, rather than two well-chosen singular sequences, brings in diagrams 14 and 15 of [2]. In them four bodies form a cluster, approached by the fifth body. The leading part of the cluster is a complex configuration of zero potential energy: a solution of the systems that we call K14 and K15. These systems have solutions on hypersurfaces of the space of masses. The two diagrams impose no condition on

the masses in [2]. Since their pieces are present for every set of mass conditions (§§3.7 and 4.2), we need an argument that does not go through the mass conditions. To exclude diagram 15 we use that the potential is constant along the curve and that its gradient vanishes there. At a place, the expansion of the potential in a uniformizing parameter is then constant, while the Euler identity of the weights relates its coefficients to those of the equations. At order three this forces the three outer bodies of the cluster into one of four explicit shapes. At order five, if the leading cluster configuration is nondegenerate, it forces a cubic moment of the shape to vanish. For positive masses the two conditions are incompatible: the vanishing of the weighted sums of x_i, x_i^{-2} and x_i^3 would force three angles, their doubles and their triples each to surround 0. Three angles cannot do this. With an analysis of the degenerate leading configurations this excludes diagram 15 at every positive mass (§§5.2–5.4). Diagram 14 gives two kinds of pieces, called `d14a` and `d14b` in Table 3. At a place of type `d14b` the leading cluster solves K14. Its outer bodies form one of the same four shapes. At positive masses such clusters exist only when the three outer masses are equal and in one ratio to the fourth, where the outer bodies form an equilateral triangle centred at the fourth. An expansion at the place excludes this case too. Thus the pieces `d14b` are excluded at every positive mass (§5.5). We exclude the remaining piece of diagram 14, `d14a`, where it is needed, by showing that K14 has no solution: at finitely many mass points (§6) and along the curve families off finitely many special points (§9). At the special points we exclude it by a local analysis at the place (§9).

Diagrams 9 and 2. Diagram 9, whose mass condition is the polynomial N of [2], does not occur at any positive mass (§5.6); we prove this with an expansion of the potential at the place as for diagrams 14 and 15. Diagram 2 does not occur at any positive mass either (§7): a condition at the leading order of a place and closed forms of its first obstructions leave twelve mass vectors of the triangle, at which the torque cannot vanish.

The special rays. On the rays through $(1, 4, 4, 4, 4)$ and $(1, 1, 4, 4, 4)$ the separation fails. At the first of them it must fail, since the modified Roberts continuum lies in the complex solution set there. Following Moczurad and Zgliczyński [10, 12], we count the real central configurations on these rays by a branch-and-bound search with interval arithmetic and Krawczyk's test (§10).

Computations and their verification. We describe the sets of mass conditions at all positive masses by 2206 ideals on the sixteen strata, computed with exact standard bases and minimal primes over $\mathbb{Q}$ (§8). The mass points off the special rays that need a separate argument are exact algebraic points. At one of them, P_1, defined in §10.4, we count the central configurations, as on the special rays. At the other points, except the special points of the curve families (§9), we show, where it is needed, that the relevant systems K14 have no solution, by Gröbner bases modulo a large prime together with a lifting argument. The counts rest on a branch and bound in outward-rounded floating point. Section 12 describes how the computations are checked and the proof of Theorem 1.1 in Lean.

Organization. In §2 we set up the system of [2], the slice, the potential form of the equations and the balancing condition. In §3 we determine the types of singular places and the mass conditions and state Theorem 3.12. In §4 we give the separation lemma and the certificates. In §5 we give the energy argument, in §6 the certificates for the four-body systems at finitely many masses, and in §7 the proof that diagram 2 does not occur. In §8 we analyse by exact computer algebra the sets of mass conditions that can hold together at positive masses, and in §9 we treat the curve families. In §10 we count the central configurations on the special rays and at P_1, in §11 we prove Theorem 1.1, and in §12 we describe the verification. The appendices derive the types of the sixteen diagrams and the hulls of diagram 5. They contain the proof of the sharpened condition for diagram 3, the label polynomials, the analysis of places of diagram 2

used in §7, the polynomial of degree eleven and the tests for diagrams 6 and 3 at two exceptional mass points.

Note added. Version 1 of this paper was submitted to arXiv on 9 October 2026, and its supplementary material, including the complete Lean formalization (§12.4), was published on Zenodo the same day (`https://doi.org/10.5281/zenodo.23202861`). On 12 October 2026, Z. Chen and Z. Tong submitted arXiv:2610.16107 [3], which states Theorem 1.1 and gives a different proof. Theorem 1.2 of version 1 of [3] begins "Assume the computational statements listed in Appendix A", and that version names the programs for these statements but does not include them.

Version 3 (October 2026) adds Remark 5.43 on [2, Prop. 8], the description of version 1 of [3] above, and comparisons with that version in Remark 5.44 and after Proposition G.1. It also gives the DOI of [16] and the numbers of the results cited from it. Remark 5.44 was Remark 5.43 in version 2. The theorems and their proofs are unchanged.

2. Normalized central configurations and curves of them

2.1. **The system of Albouy and Kaloshin.** Let $m = (m_1, \dots, m_5)$. Following [2, p. 540], a *normalized central configuration* is a solution $(x, y, \delta) \in \mathbb{C}^5 \times \mathbb{C}^5 \times \mathbb{C}^{10}$ of

$$x_k = \sum_{l \neq k} m_l\, \delta_{kl}^3 (x_k - x_l), \quad y_k = \sum_{l \neq k} m_l\, \delta_{kl}^3 (y_k - y_l), \quad \delta_{kl}^2\big((x_k - x_l)^2 + (y_k - y_l)^2\big) = 1, \quad y_1 = y_2, \tag{2.1}$$

for $k \neq l$ in $\{1, \dots, 5\}$, with $\delta_{kl} = \delta_{lk}$. It is *real* if $x, y \in \mathbb{R}^5$, and *positive* if moreover all δ_{kl} are positive. Summing the first equation multiplied by m_k shows that solutions have their centre of mass at the origin. The multiplier λ of (1.1) is normalized to 1.

Lemma 2.1. *Let $m \in (\mathbb{R}_{>0})^5$. Every central configuration with masses m is similar, by an orientation-preserving similarity, to exactly two positive normalized central configurations, which differ by the rotation by π. In particular, if* (2.1) *has finitely many solutions, there are finitely many central configurations up to similarity.*

Proof. Taking the scalar product of (1.1) with $m_i(q_i - c)$ and summing gives $\lambda I = U$ with $I = \sum_i m_i |q_i - c|^2 > 0$ and $U = \sum_{i<j} m_i m_j / r_{ij} > 0$. So $\lambda > 0$. We translate c to the origin, scale by $\lambda^{1/3}$, and rotate so that $q_1 - q_2$ is horizontal. Exactly two rotations do this. Conversely, the translation and the scale of a positive normalized central configuration are fixed by (2.1). □

We put $z = x + iy$, $w = x - iy$ and $r_{kl} = \delta_{kl}^{-1}$, so that $r_{kl}^2 = z_{kl} w_{kl}$ with $z_{kl} = z_k - z_l$. For real configurations $w = \bar{z}$. For complex ones z and w are independent.

2.2. **Centred coordinates and the slice.** For $h \in \mathbb{C}^5$ and $p = (p_{kl})_{k<l}$ with $p_{lk} = p_{kl}$, put $F_k(h) = h_k - \sum_{l \neq k} m_l\, p_{kl}^3 (h_k - h_l)$. The unknowns are $u = (u_1, \dots, u_4)$, $v = (v_1, \dots, v_4)$ and p, with $u_5 = v_5 = 0$, $u_{kl} = u_k - u_l$, $v_{kl} = v_k - v_l$. The eighteen *rows* are

$$D_{kl} = p_{kl}^2 u_{kl} v_{kl} - 1, \qquad Z_k = F_k(u) - F_5(u), \qquad W_k = F_k(v) - F_5(v) \qquad (k \leq 4). \tag{2.2}$$

Let $X_m \subset \mathbb{C}^{18}$ be their zero set. On X_m the functions u_k, v_k, u_{kl}, v_{kl} and p_{kl} vanish nowhere. The map $(x, y, \delta) \mapsto (u, v, p)$ with $u_k = z_k - z_5$, $v_k = w_k - w_5$, $p = \delta$ identifies the solutions of the first three equations of (2.1) with X_m, and the solutions of (2.1) with $X_m \cap \{u_{12} = v_{12}\}$.

Lemma 2.2. (a) *The group $\mathbb{C}^*$ acts freely on X_m by $s \cdot (u, v, p) = (su, s^{-1}v, p)$, the complex rotation $(z, w) \mapsto (sz, s^{-1}w)$. Let $Y_m = X_m \cap \{u_1 = 1\}$. Then $\mathbb{C}^* \times Y_m \to X_m$, $(s, y) \mapsto s \cdot y$, is an isomorphism, every orbit meets $\{u_{12} = v_{12}\}$ in exactly two points, and* (2.1) *has exactly $2|Y_m|$ solutions.*

(b) *For $t > 0$ the map $(u, v, p) \mapsto (t^{1/3}u, t^{1/3}v, t^{-1/3}p)$ is an isomorphism $X_m \to X_{tm}$ that preserves $\{u_{12} = v_{12}\}$, and its composition with the rotation by $s = t^{-1/3}$, $(u, v, p) \mapsto (u, t^{2/3}v, t^{-1/3}p)$, is an isomorphism $Y_m \to Y_{tm}$. Relabelling the bodies together with their masses, followed by a translation and a rotation, is an isomorphism $Y_m \to Y_{m'}$ for the permuted masses m'. In particular finiteness of Y_m is invariant under positive scaling and permutation of the masses.*

(c) *A configuration satisfies* (1.1) *for the masses m with multiplier λ if and only if it satisfies* (1.1) *for tm with multiplier $t\lambda$, and relabelling the bodies together with their masses maps the central configurations for m to those for the permuted masses. Both bijections preserve similarity and orientation. Hence the number of central configurations up to similarity, and up to orientation-preserving similarity, is invariant under positive scaling and permutation of the masses.*

Proof. (a) The rows D_{kl} are invariant and Z_k, W_k are multiplied by s, s^{-1}. The action is free because $u_1 \neq 0$ on X_m. The inverse of the map is $x \mapsto (u_1, u_1^{-1} \cdot x)$. The point $s \cdot x$ lies in $\{u_{12} = v_{12}\}$ if and only if $s^2 = v_{12}/u_{12}$. (b) The rows for tm at the image point are multiples of the rows for m. For a relabelling, we translate the new body 5 to the origin and rotate to restore $u_1 = 1$. (c) Multiplying the masses by t multiplies the right side of (1.1) by t^2 and the left side by t for fixed λ. Relabelling permutes the equations. □

For $\{i, j, k, l, n\} = \{1, \dots, 5\}$ the *4-product* of [2, p. 580] is $\mathcal{P}_{ij} = r_{ij}^2 r_{kl}^2 r_{ln}^2 r_{nk}^2$.

2.3. **The potential form of the rows.** Let

$$U = \sum_{k<l} m_k m_l \, p_{kl}, \qquad I = \sum_k m_k u_k v_k - \frac{1}{M}\Big(\sum_k m_k u_k\Big)\Big(\sum_k m_k v_k\Big), \qquad V = U + \tfrac{1}{2} I,$$

where $M = \sum_k m_k$. We regard the p_{kl}, on the hypersurface defined by the rows D_{kl}, as functions of (u, v).

Lemma 2.3. *On $\{D_{kl} = 0\}$, with ∇_u and ∇_v the gradients in $(u_1, \dots, u_4)$ and $(v_1, \dots, v_4)$,*

$$(Z_1, \dots, Z_4) = 2A\,\nabla_v V, \qquad (W_1, \dots, W_4) = 2A\,\nabla_u V, \qquad A = \operatorname{diag}(m_1^{-1}, \dots, m_4^{-1}) + m_5^{-1} \mathbb{1}\mathbb{1}^{\mathsf{T}}.$$

The matrix A is invertible, with $A^{-1} = \operatorname{diag}(m_1, \dots, m_4) - mm^{\mathsf{T}}/M$ (restricted to the first four coordinates).

Proof. We write $\tilde{F}_k = \sum_{l \neq k} m_l (u_l - u_k) p_{kl}^3$, so that $Z_k = u_k + \tilde{F}_k - \tilde{F}_5$. From $p_{kl}^2 u_{kl} v_{kl} = 1$ we get $\partial p_{kl}/\partial v_k = -\frac{1}{2} p_{kl}^3 u_{kl}$. So $\partial U/\partial v_k = \frac{1}{2} m_k \tilde{F}_k$. Also $\partial I/\partial v_k = m_k(u_k - u_G)$ with $u_G = \sum_l m_l u_l / M$. Hence $(2/m_k)\,\partial V/\partial v_k = u_k - u_G + \tilde{F}_k$. By antisymmetry $\sum_{l=1}^5 m_l \tilde{F}_l = 0$. Since $u_5 = 0$, also $\sum_{l \leq 4} m_l (u_l - u_G) = m_5 u_G$. Hence $(2/m_5) \sum_{l \leq 4} \partial V/\partial v_l = u_G - \tilde{F}_5$ and $(2A\nabla_v V)_k = u_k + \tilde{F}_k - \tilde{F}_5 = Z_k$. Exchanging u and v gives the rows W_k. □

A point of X_m is thus a critical point of V restricted to $\{D_{kl} = 0\}$. At fixed masses all points of a family in X_m are critical points of the same V. The next lemma is [2, Lemma 2], in the coordinates of the slice.

Lemma 2.4. *Let $M \neq 0$.*

(a) On X_m, $U = \sum_k m_k z_k w_k = I$, with z, w recentred at the centre of mass.

(b) U is constant on every irreducible component of X_m and of Y_m, in particular on every irreducible curve $C \subseteq Y_m$.

(c) For $m \in (\mathbb{R}_{>0})^5$, $U = I > 0$ at the point of Y_m given by a real central configuration.

Proof. (a) In the coordinates z, w of (2.1) the centre of mass is at the origin and $I = \sum_k m_k z_k w_k$. We multiply the equation for z_k by $m_k w_k$ and sum over k. By antisymmetry and $p_{kl}^2 z_{kl} w_{kl} = 1$, $\sum_k m_k z_k w_k = \sum_{k<l} m_k m_l p_{kl}^3 z_{kl} w_{kl} = \sum_{k<l} m_k m_l p_{kl} = U$. (b) By Lemma 2.3 and the invertibility of A, the gradient of V on $\{D_{kl} = 0\}$ vanishes at every point of X_m. The differential of V therefore vanishes on the tangent space of each irreducible component at its smooth points, which form a connected dense open subset. Hence V is constant on the component. By (a), $V = \frac{3}{2}U$ on X_m. (c) is the identity $\lambda I = U$ of the proof of Lemma 2.1, with $\lambda = 1$. □

Corollary 2.5. *Let $m \in (\mathbb{R}_{>0})^5$. If Y_m contains no irreducible curve on which some r_{kl}^2 is non-constant and U is not identically zero, then the planar five-body problem with masses m has finitely many central configurations up to similarity.*

Proof. Infinitely many central configurations give infinitely many points of Y_m with $U > 0$ (Lemmas 2.1, 2.2 and 2.4(c)), hence an irreducible curve $C \subseteq Y_m$ on which U is a positive constant. The fibres of $(u, v, p) \mapsto (r_{kl}^2)_{k<l}$ on Y_m have at most 2^{13} points (fix the distances: $u_1 = 1$ determines v_1, each u_k is a root of a quadratic, and the signs of the p_{kl} are free). Hence some r_{kl}^2 is not constant on C. □

We call $m \in (\mathbb{R}_{>0})^5$ *settled* if Y_m contains no irreducible curve on which some r_{kl}^2 is non-constant and $U \not\equiv 0$. We call such a curve *nontrivial*. A settled mass vector has finitely many central configurations up to similarity. The isomorphisms of Lemma 2.2(b) multiply the r_{kl}^2 and U by nonzero constants or permute them. Thus settledness is invariant under positive scaling and permutation of the masses.

2.4. **Places and balancing.** Let $C \subseteq Y_m$ be a nontrivial curve. The functions u_k, v_k, u_{kl}, v_{kl}, p_{kl}, r_{kl}^2 and $\mathcal{P}_{ij}$ are units of the coordinate ring $\mathbb{C}[C]$: for instance $1/p_{kl} = p_{kl} u_{kl} v_{kl}$. A non-constant unit has a zero and a pole on the smooth projective model $\bar{C}$ of C. All of them lie at the finitely many *places at infinity*, the points of $\bar{C}$ not over C. For a place P at infinity with valuation val_P put

$$\rho(P) = \big(-\mathrm{val}_P(p_{kl}) \big)_{k<l} \in \mathbb{Z}^{10}.$$

Since the divisor of each p_{kl} has degree zero,

$$\sum_{P \text{ at } \infty} \rho(P) = 0, \tag{2.3}$$

and $\rho(P) \neq 0$ for some P, because some p_{kl} is not constant.

On X_m we put $z_k = u_k - F_5(u)$ and $w_k = v_k - F_5(v)$ for $k \leq 5$. Since the rows Z_k, W_k say $F_k(u) = F_5(u)$ and $F_k(v) = F_5(v)$, the pair (z, w) solves the first two equations of (2.1) (in the variables $z = x + iy$, $w = x - iy$), with $z_k - z_l = u_{kl}$ and $w_k - w_l = v_{kl}$. Following [2, p. 545] we put $Z_{kl} = p_{kl}^3 z_{kl}$, $W_{kl} = p_{kl}^3 w_{kl}$ ([2] write z_{kl} for $z_l - z_k$, but signs play no role below),

$$\mathcal{Z} = (z_1, \ldots, z_5, Z_{12}, \ldots, Z_{45}), \qquad \mathcal{W} = (w_1, \ldots, w_5, W_{12}, \ldots, W_{45}),$$

and $\mathcal{N} = \|\mathcal{Z}\| \, \|\mathcal{W}\|$ with the maximum norm. The rotation $s\cdot$ of Lemma 2.2 multiplies $\mathcal{Z}$ by s and $\mathcal{W}$ by s^{-1}. Thus $\mathcal{N}$ is a function on Y_m. Since $Z_{kl} W_{kl} = p_{kl}^4$ and $|z_{kl} w_{kl}| \leq 4\mathcal{N}$,

$$|r_{kl}|^{-4} \leq \mathcal{N}, \qquad |r_{kl}|^2 \leq 4\mathcal{N}. \tag{2.4}$$

A place P at infinity is *singular* if $\mathcal{N}$ is unbounded near P, that is, if $\mathrm{val}_P(\mathcal{Z}) + \mathrm{val}_P(\mathcal{W}) < 0$, where val_P of a vector is the minimum over its components. At a place P that is not singular, (2.4) bounds every r_{kl}^2 above and away from zero. So $\rho(P) = 0$. Hence the sum (2.3) runs over the singular places and at least one place is singular.

2.5. **Diagrams.** Along a singular place, [2, p. 545] draws a circle around the body k if z_k is of maximal order among the components of $\mathcal{Z}$, and a stroke between k and l if Z_{kl} is of maximal order. This gives the z-diagram. In the same way $\mathcal{W}$ gives the w-diagram. An edge is a z-edge, a w-edge or a zw-edge according to which strokes are present. The conclusion of [2, §7] is that every singular sequence approaches one of the sixteen diagrams of their Fig. 11, up to renumbering of the bodies and exchange of the colours z and w. Table 15 in Appendix A lists them with the types used below. We say that a singular place has *diagram* d and *labelling* λ if it approaches diagram d with the renumbering λ.

2.6. **Inputs from** [2]**.** We use the following results of [2], as stated there:

(i) the conclusion of §7: every singular sequence approaches one of the sixteen diagrams of Fig. 11, which are listed in Table 15;
(ii) Estimates 1 and 2 of §3, which say, in the notation there, that for each pair, $\varepsilon \preceq r_{kl} \preceq \varepsilon^{-2}$ and $\varepsilon^2 \preceq z_{kl} \preceq \varepsilon^{-2}$, with $r_{kl} \approx \varepsilon$ exactly at the zw-edges and $z_{kl} \approx \varepsilon^2$ exactly at the maximal w-edges; and on a z-stroke, $\varepsilon \preceq r_{kl} \preceq 1$, $\varepsilon \preceq z_{kl} \preceq \varepsilon^{-2}$ and $\varepsilon^2 \preceq w_{kl} \preceq \varepsilon$, where $r_{kl} \approx \varepsilon$, $z_{kl} \approx \varepsilon$ and $w_{kl} \approx \varepsilon$ each characterize a zw-edge, and $r_{kl} \approx 1$, $z_{kl} \approx \varepsilon^{-2}$ and $w_{kl} \approx \varepsilon^2$ each characterize a maximal z-edge; Rules 1b–1e and 2e of §3, Proposition 1 of §5.4 and Proposition 4 of §7.1;
(iii) the mass conditions of §8: conditions (23), (29), (32), (36) and (40), the facts of §8.7 quoted in §3.5 (among them the reduction to the system of §5.4), and §§8.9.1–8.9.2;
(iv) the proof of Proposition 9, which we use in one place where we also give a second argument.

The hypotheses $m_k \neq 0$, $M \neq 0$, $M_Q \neq 0$ for each four-body subset Q, $m_a + m_b \neq 0$ and $m_k^3 + m_l^3 \neq 0$, under which these results are stated, hold for positive masses. In diagram 3 of Fig. 11 the colours of the four sides alternate around the square, z on 14 and 23 and w on 12 and 34. Note that the alternation is also forced by Rule 2g of [2, §3] and by the order calculus of §3.1, which admits only the two alternating colourings of the sides. It occurs at places of the modified Roberts continuum [2, pp. 543–544].

3. Types of singular places and mass conditions

Throughout this section $m \in (\mathbb{R}_{>0})^5$, $C \subseteq Y_m$ is a nontrivial curve, and P is a singular place of C. We determine the possible directions of $\rho(P)$, together with the conditions on m under which each direction can occur.

3.1. **Orders at a singular place.** Let t be a local parameter at P and $\nu = \mathrm{val}_P$. We call a function on C of *z-type* if the rotation $s\cdot$ multiplies it by s (u_{kl}, z_k, z_{kl}, Z_{kl}), of *w-type* if it multiplies it by s^{-1} (v_{kl}, w_k, w_{kl}, W_{kl}), and *neutral* if it is invariant (p_{kl}, r_{kl}^2). Put $d = -\frac{1}{2}(\nu(\mathcal{Z}) + \nu(\mathcal{W})) > 0$ and $\gamma = \frac{1}{2}(\nu(\mathcal{W}) - \nu(\mathcal{Z}))$, and define the *exponent*

$$e(f) = \frac{2}{d}\big(\nu(f) + \gamma\big), \qquad e(f) = \frac{2}{d}\big(\nu(f) - \gamma\big), \qquad e(f) = \frac{2}{d}\,\nu(f)$$

for f of z-type, w-type and neutral respectively. Then all components of $\mathcal{Z}$ and of $\mathcal{W}$ have exponent at least -2. In each of the two vectors some component has exponent -2.

Lemma 3.1. *Let $x_n \in C$ converge to P, and let x'_n be one of the two solutions of* (2.1) *in the orbit of x_n (Lemma 2.2). After passing to a subsequence, (x'_n) is a singular sequence in the sense of* [2, Def. 3]*. After the normalization $\|\mathcal{Z}\| = \|\mathcal{W}\| = \varepsilon_n^{-2}$ of* [2, §2]*, every function f of the three kinds satisfies $|f(x'_n)| \approx \varepsilon_n^{e(f)}$. In particular:*

(i) *k is z-circled if and only if $e(z_k) = -2$;*
(ii) *there is a z-stroke between k and l if and only if $e(Z_{kl}) = -2$;*

(iii) *k and l are z-close if and only if $e(z_{kl}) > -2$;*

and the same holds for w. Every such sequence therefore has the same diagram, which we call the diagram of P. Moreover, if $\vartheta^2 = t$ and the point of C near P is rotated by $s = \vartheta^{2\gamma}$, every function f of the three kinds becomes $f = \vartheta^{d\,e(f)} y_f(\vartheta)$ with $y_f \in \mathbb{C}\{\vartheta\}$ and $y_f(0) \neq 0$.

Proof. A function f as above that is not identically zero on C is $t^{\nu(f)}$ times a unit of $\mathbb{C}\{t\}$. The rotation multiplies it by $\vartheta^{2\gamma}$, $\vartheta^{-2\gamma}$ or 1. This gives the last statement, since $d\,e(f) = 2\nu(f) \pm 2\gamma$ or $2\nu(f)$. Each component of $\mathcal{Z}$ and $\mathcal{W}$ is a Laurent series in t near P. The ratios of components converge in $\mathbb{C} \cup \{\infty\}$. Passing from x_n to x'_n multiplies $\mathcal{Z}$ by a scalar s_n and $\mathcal{W}$ by s_n^{-1}, which changes none of these ratios. Since P is singular, $\mathcal{N} \to \infty$. Since $z_{12} = w_{12}$ on solutions of (2.1), at these solutions $r_{12}^2 = z_{12}^2$, $|Z_{12}| = |z_{12}|^{-2}$ and $\|\mathcal{Z}\| \geq \max(|z_{12}|/2, |z_{12}|^{-2}) \geq 2^{-2/3}$, and likewise for $\mathcal{W}$. Hence $\mathcal{Z}$ or $\mathcal{W}$ is unbounded along (x'_n). The extraction of [2, pp. 544–545] only fixes the limits of the ratios. The normalization multiplies z by a real $a_n > 0$ and w by a_n^{-1} so that $\|\mathcal{Z}\| = \|\mathcal{W}\|$. Then $|a_n s_n|^2 = \|\mathcal{W}(x_n)\|/\|\mathcal{Z}(x_n)\| \approx |t|^{2\gamma}$. Hence a function f of z-type has modulus $|a_n s_n f(x_n)| \approx |t|^{\nu(f)+\gamma} = |t|^{d\,e(f)/2}$, and likewise for the other two kinds. Finally $\varepsilon_n^{-2} = \|\mathcal{Z}\| \approx |t|^{-d}$. The characterizations are the definitions of circles, strokes and closeness in [2, pp. 545–546]: circles and strokes mark the components of maximal order, and z-close means $z_{kl} \prec \mathcal{Z}$. □

For a pair kl put

$$\zeta_{kl} = e(z_{kl}), \qquad \omega_{kl} = e(w_{kl}), \qquad x_{kl} = e(r_{kl}) = \tfrac{1}{2}(\zeta_{kl} + \omega_{kl}),$$

so that $e(Z_{kl}) = -\frac{1}{2}(\zeta_{kl} + 3\omega_{kl})$ and $e(W_{kl}) = -\frac{1}{2}(3\zeta_{kl} + \omega_{kl})$. The vector $x(P) = (x_{kl})_{k<l} \in \mathbb{Q}^{10}$ is the *type* of P. Since $x_{kl} = \frac{2}{d}\,\nu(r_{kl}) = \frac{2}{d}\,\rho_{kl}(P)$, the type is a positive multiple of $\rho(P)$ and (2.3) becomes

$$\sum_{P \text{ singular}} d_P\, x(P) = 0, \qquad d_P > 0. \tag{3.1}$$

We index the coordinates of $\mathbb{R}^{10}$ by the pairs $12, 13, \ldots, 45$.

Lemma 3.2. *For every pair kl:*

- (O1) *$\zeta_{kl}, \omega_{kl} \geq -2$, $\zeta_{kl} + 3\omega_{kl} \leq 4$ and $3\zeta_{kl} + \omega_{kl} \leq 4$. Hence $\zeta_{kl}, \omega_{kl} \leq 2$ and $-2 \leq x_{kl} \leq 1$.*
- (O2) *There is a z-stroke kl if and only if $\zeta_{kl} + 3\omega_{kl} = 4$, and a w-stroke if and only if $3\zeta_{kl} + \omega_{kl} = 4$.*
- (O3) *On a z-stroke, $\zeta_{kl} = 3x_{kl} - 2$, $\omega_{kl} = 2 - x_{kl}$ and $0 \leq x_{kl} \leq 1$. On a w-stroke, $\omega_{kl} = 3x_{kl} - 2$ and $\zeta_{kl} = 2 - x_{kl}$.*
- (O4) *$x_{kl} = 1$ if and only if kl is a zw-edge. A z-stroke is maximal (its ends are not z-close,* [2, Def. 6]) *if and only if $x_{kl} = 0$.*
- (O5) *In every triangle klm the minimum of $\zeta_{kl}, \zeta_{lm}, \zeta_{mk}$ is attained at least twice, and the same holds for ω.*
- (O6) *If k is z-circled and l is not, then $\zeta_{kl} = -2$. If $\zeta_{kl} = 2$, then $\omega_{kl} = -2$ and kl is a w-stroke.*

Proof. (O1) Since $e(z_k) \geq -2$ for all k, $\zeta_{kl} \geq \min(e(z_k), e(z_l)) \geq -2$. The two linear inequalities are $e(Z_{kl}) \geq -2$ and $e(W_{kl}) \geq -2$. (O2) A stroke means $e(Z_{kl}) = -2$. (O3) We solve $\zeta + 3\omega = 4$, $\zeta + \omega = 2x$, and use $\zeta \geq -2$. (O4) If $\zeta + \omega = 2$, the inequalities of (O1) give $\zeta \leq 1$ and $\omega \leq 1$, hence $\zeta = \omega = 1$ and both strokes are present. Conversely a zw-edge has $\zeta = \omega = 1$. A z-stroke has z-close ends if and only if $\zeta > -2$, i.e. $x > 0$. (O5) follows from $z_{kl} + z_{lm} + z_{mk} = 0$. (O6) $e(z_k) = -2 < e(z_l)$ gives $\zeta_{kl} = -2$. If $\zeta_{kl} = 2$, (O1) gives $\omega_{kl} \leq -2$, hence $\omega_{kl} = -2$ and $3\zeta_{kl} + \omega_{kl} = 4$. □

(O6) is Rule 1b of [2]. Besides Lemma 3.2 we use the following results of [2], which are among the inputs of §2.6: Rules 1d, 1e and 2e [2, pp. 546–552] (circled bodies come at least in pairs in an isolated component and are not all mutually close; an isolated component has no circle if and only if it has no maximal stroke; the three edges of a triangle of edges have the same type, equal ζ and equal ω), Proposition 4 [2, p. 564] (if a diagram has a single maximal z-edge 12, forming an isolated component, then no body $k \geq 3$ satisfies $w_{1k} \prec w_{1l}$ for all other $l \geq 3$), and Proposition 1 with z and w exchanged [2, p. 559] (if $w_{12} \prec w_{k2}$ and $z_{12} \approx z_{k2}$ for all $k \geq 3$, then $w_1, w_2 \preceq w_{12}$).

Lemma 3.3. *At every singular place of C, $\min_{k<l} x_{kl}(P) \leq 0$.*

Proof. By Lemma 2.4, $U = I$ on X_m and U is a nonzero constant on C. In the variables of (2.1), $I = \sum_k m_k z_k w_k = M^{-1} \sum_{k<l} m_k m_l r_{kl}^2$. If all $x_{kl}(P)$ were positive, all r_{kl}^2 would tend to zero at P, and so would I. □

Lemma 3.4 (three bodies near a cluster)**.** *Let K be a set of at least three bodies, pairwise joined by zw-edges, and let $X \notin K$ have no stroke to K. In each colour, say z, exactly one of the following holds: all ζ_{Xt}, $t \in K$, are equal and less than 1 (X is z-far); or all $\zeta_{Xt} \geq 1$, at most one of them is larger than 1 (X is z-near). The body X is not near in both colours. With $\bar\omega < 1$ the common far w-exponent and s the exceptional ζ, the possible modes are therefore:*

- F*: $x_{Xt} = c < 1$ for all t;*
- Z1*: $x_{Xt} = \frac{1+\bar\omega}{2}$ for all t;*
- Zs_{t_0}*: $x_{Xt_0} = \frac{s+\bar\omega}{2}$ and $x_{Xt} = \frac{1+\bar\omega}{2}$ for $t \neq t_0$, with $s > 1$;*
- W1 *and* Ws_{t_0}*: the same with the colours exchanged.*

The parameters satisfy $s < 2$, $s + \bar\omega < 2$, $2s + \bar\omega < 3$ and $2s + 3\bar\omega < 5$.

Proof. Inside K, $\zeta_{tt'} = 1$. If some $\zeta_{Xt} < 1$, (O5) forces $\zeta_{Xt'} = \zeta_{Xt}$ for all $t' \in K$. Two values above 1 would give $\zeta_{tt'} > 1$, again by (O5). Near in both colours would give $x_{Xt} \geq 1$, hence a zw-edge by (O4). (O6) gives $s < 2$. No stroke means $s + 3\bar\omega < 4$ and $3s + \bar\omega < 4$. We obtain the remaining inequalities as their combinations with weights $(\frac{1}{4}, \frac{1}{4})$, $(\frac{7}{8}, \frac{3}{8})$, $(\frac{1}{8}, \frac{5}{8})$. The inequality $\bar\omega < 1$ follows from $3s + \bar\omega < 4$ and $s > 1$. □

3.2. **The types of the sixteen diagrams.** We name some points of $\mathbb{R}^{10}$. For a triple T let P_T be the complementary pair. For a body j let $Q_j = \{1, \dots, 5\} \setminus \{j\}$. For a set A let $\mathrm{tri}(A)$ be its set of pairs.

- $\mathsf{a}_1(T)$: 0 on $\mathrm{tri}(T)$ and on P_T, -2 on the six other pairs; $\mathsf{b}_1(T)$: 1 on $\mathrm{tri}(T)$, 0 on P_T, $-\frac{1}{2}$ on the six other pairs.
- $\mathsf{h}(s; D_1|D_2)$, for a body s and a partition of Q_s into two pairs D_1, D_2 (the diagonals): 0 on the four other pairs of Q_s (the sides), -2 on D_1, D_2 and on the four pairs ks.
- $\mathsf{v}(s; A|B)$, for triples A, B with $A \cap B = \{s\}$: 0 on $\mathrm{tri}(A) \cup \mathrm{tri}(B)$, -2 on the four pairs between $A \setminus s$ and $B \setminus s$. More generally $\mathsf{v}_7(a, b)$ is a on $\mathrm{tri}(A)$, b on $\mathrm{tri}(B)$ and $\frac{3}{2}(a+b) - 2$ on the four cross pairs. The type $\mathsf{u}_A(c)$ is 1 on $\mathrm{tri}(A)$, c on $\mathrm{tri}(B)$ and $\frac{3c-1}{2}$ on the cross pairs.
- $\mathsf{n}(T; ax, by)$, for a triple T, $P_T = \{x, y\}$ and $a \neq b$ in T: 1 on $\mathrm{tri}(T)$, 0 on ax, by, xy, and $-\frac{1}{2}$ on the four other pairs.
- $\mathsf{u}_c(j)$: 1 on $\mathrm{tri}(Q_j)$, c on the four pairs ij; $\mathsf{b}_k(j)$ for $k \in Q_j$: 1 on $\mathrm{tri}(Q_j)$, 0 on kj, $-\frac{1}{2}$ on ij, $i \in Q_j \setminus k$; and $\sigma_k(j)(t) = (1-t)\,\mathsf{u}_0(j) + t\,\mathsf{b}_k(j)$.

Proposition 3.5. *Let P have diagram d, with the bodies numbered as in* [2, Fig. 11]*. Then $x(P)$ is as in Table 1. In the table $T = \{3, 4, 5\}$ in diagrams* 1, 2, 4, 5*. The wings are $A = \{1, 2, 5\}$ and $B = \{3, 4, 5\}$ in diagrams* 6–8 *and* 10–13*, with centre* 5*. In diagrams* 14 *and* 15*, $j = 5$ and $Q = Q_5$, with k the body of Q joined to* 5 *in diagram* 15*.*

diagram	type $x(P)$	range
1	$(1-a)\,\mathsf{a}_1(T) + a\,\mathsf{b}_1(T)$	$0 < a < 1$
2	$\mathsf{a}_1(T)$	
3	$\mathsf{h}(5; 13\|24)$	
4	$\mathsf{b}_1(T)$	
5	see §3.3	
6	$\mathsf{v}(5; A\|B) = \mathsf{v}_7(0,0)$	
7	$\mathsf{v}_7(a,b)$	$0 < a, b < 1$, $a + b \le \frac{4}{3}$
8	$\mathsf{v}_7(a,0)$	$0 < a < 1$
9	$\mathsf{n}(\{1,2,3\}; 14, 25)$	
10	$\mathsf{u}_A(c)$	$0 < c \le \frac{1}{3}$
11–13	$\mathsf{u}_A(0)$	
14	$\mathsf{u}_0(5)$, or $\sigma_k(5)(t)$ for some $k \in Q$	$0 < t < 1$
15	$\mathsf{b}_k(5)$	
16	does not occur	

TABLE 1. The types of the sixteen diagrams of [2, Fig. 11] on a curve with $U \not\equiv 0$, for the numbering of the figure.

We give the derivations for diagrams 1–13 and 16 in Appendix A. They use only Lemmas 3.2 and 3.3, the rules of [2] quoted above and, for diagram 9, [2, §8.9] and Lemma 2.4. Two of them improve on what the order calculus gives directly. We prove them here. For diagram 1 the order calculus leaves two patterns for the exponents $\mu_t = \omega_{1t} = \omega_{2t}$, $t \in T$ (Appendix A): either all μ_t equal some $\mu \le 3a - 2$, or two of them equal $3a - 2$ and the third is larger.

Lemma 3.6. *At a place with diagram* 1, $\mu_t = 3a - 2$ *for every* $t \in T$. *At a place with diagram* 4, $\mu_t = 1$ *for every* $t \in T$.

Proof. Diagram 1. In the second pattern $w_{1t_0} \prec w_{1t}$ for the body t_0 with the larger exponent and both other $t \in T$. Since 12 is the only maximal z-edge, the z-diagram is $\{12\}$, an isolated component, and $m_1^3 + m_2^3 > 0$, Proposition 4 of [2] excludes this. Suppose all μ_t equal $\mu < 3a - 2$. Then we have:

(i) $\omega_{12} = 2$ by (O3);
(ii) $\mu < 2$, since $\mu = 2$ would make $1t$ a z-stroke by (O2);
(iii) $\zeta_{12} = \zeta_{t2} = -2$ by (O6).

Proposition 1 of [2] with the colours exchanged then gives $e(w_1), e(w_2) \ge 2$. The differences $w_{1t} - w_{1t'} = \pm w_{tt'}$ have exponent $3a - 2 > \mu$. Consequently w_{13}, w_{14}, w_{15} have the same coefficient $\kappa \ne 0$ at exponent μ. As $w_t = w_1 - w_{1t}$ and $e(w_1) \ge 2 > \mu$, the coefficient of $m_3 w_3 + m_4 w_4 + m_5 w_5$ at exponent μ is $-(m_3 + m_4 + m_5)\kappa \ne 0$. But this sum equals $-(m_1 w_1 + m_2 w_2)$, whose exponent is at least 2. Hence $\mu_t = 3a - 2$ for all t.

Diagram 4 is the same argument with $3a - 2$ replaced by 1: the z-strokes of the triangle T have $x = 1$ and are not maximal. Again 12 is the only maximal z-edge and an isolated component. □

A place with diagram 1 thus has type $(1-a)\mathsf{a}_1(T) + a\mathsf{b}_1(T)$, $0 < a < 1$, and a place with diagram 4 has type $\mathsf{b}_1(T)$, as in Table 1.

For diagrams 14 and 15 we need the position of the fifth body relative to the cluster. The next lemma replaces [2, §6.1 and Prop. 2] in the form we need. We state it in terms of $\alpha_i = \frac{1}{2}e(z_{ij})$ and $\beta_i = \frac{1}{2}e(w_{ij})$.

Lemma 3.7. *Suppose* $\{kl : x_{kl}(P) = 1\} = \mathrm{tri}(Q_j)$, *and put* $Q = Q_j$. *Exactly one of the following holds, possibly after exchanging the colours.*

(A) $\alpha_i = \alpha$ *and* $\beta_i = \beta$ *for all* $i \in Q$, *with* $\alpha, \beta \le \frac{1}{2}$, *and* $x_{ij} = c := \alpha + \beta$ *for all* $i \in Q$, $0 \le c < 1$. *So* $x(P) = \mathsf{u}_c(j)$, *and no stroke meets* j.

(B) *There is a unique* $k \in Q$ *with* $\alpha_k > \frac{1}{2}$. *Put* $a = 2\alpha_k$. *Then* $1 < a \le 2$, $\alpha_i = \frac{1}{2}$ *for* $i \ne k$, $\beta_i = -\frac{a}{2}$ *for all* $i \in Q$, $x_{kj} = 0$ *and* $x_{ij} = \frac{1-a}{2}$ *for* $i \ne k$, *that is,* $x(P) = \sigma_k(j)(a-1)$. *If* $a < 2$, *no stroke meets* j. *If* $a = 2$, *the only stroke at* j *is the* w*-stroke* kj, *and* kj *is a maximal* w*-edge.*

Proof. (1) Inside Q, $e(z_{il}) = e(w_{il}) = 1$ for each pair by (O4). If $\alpha_i < \frac{1}{2}$ for some i, then $z_{i'j} = z_{i'i} + z_{ij}$ gives $\alpha_{i'} = \alpha_i$ for all $i' \in Q$. Otherwise all $\alpha_i \ge \frac{1}{2}$. Two indices with $\alpha_k, \alpha_{k'} > \frac{1}{2}$ would give $e(z_{kk'}) > 1$. Either all α_i equal some $\alpha \le \frac{1}{2}$, or exactly one exceeds $\frac{1}{2}$ and the others equal $\frac{1}{2}$. The same holds for the β_i. (2) Not both colours have an exceptional index: $\alpha_k > \frac{1}{2}$ and $\beta_{k'} > \frac{1}{2}$ give $x_{k'j} > 1$, against (O1). (3) No exceptional index. Then $x_{ij} = \alpha + \beta = c$ for all i. By (O4), $c < 1$. After exchanging colours we may assume $\alpha < \frac{1}{2}$. We fix $k \in Q$. The sum $\sum_{i \in Q} m_i z_{ij} = M_Q z_{kj} + \sum_{i \in Q} m_i z_{ik}$ has exponent 2α, as the second sum has exponent at least $1 > 2\alpha$ and $M_Q \ne 0$. The centre of mass gives $\sum_{i \in Q} m_i z_i - M_Q z_j = -M z_j$, whence $e(z_j) = 2\alpha$. The first equation of (2.1) for j reads $z_j = \sum_{i \in Q} m_i Z_{ji}$ with $e(Z_{ji}) = 2\alpha - 3c$, which forces $2\alpha \ge 2\alpha - 3c$, i.e. $c \ge 0$. Since $\alpha = c - \beta \ge c - \frac{1}{2}$, $e(Z_{ij}) = 2\alpha - 3c \ge -1 - c > -2$, and likewise $e(W_{ij}) > -2$: no stroke meets j. (4) An exceptional index k in z. By (2) all β_i equal some β. We have $\beta < \frac{1}{2}$, since otherwise $x_{ij} = 1$ for $i \ne k$. Hence $x_{kj} = \frac{a}{2} + \beta$ and $x_{ij} = \frac{1}{2} + \beta$. As in (3), $e(w_j) = 2\beta$. The second equation of (2.1) for j reads $w_j = \sum_{i \in Q} m_i W_{ji}$, with $e(W_{ji}) = 2\beta - 3x_{ij}$, which is $-\frac{3}{2} - \beta$ for $i \ne k$ and $-\frac{3}{2}a - \beta$ for $i = k$. Since $a > 1$ the term $i = k$ has the strictly smallest exponent. So $2\beta = e(w_j) = -\frac{3}{2}a - \beta$, i.e. $\beta = -\frac{a}{2}$. Then $x_{kj} = 0$ and $x_{ij} = \frac{1-a}{2}$. Now $e(W_{kj}) = -a \ge -2$ gives $a \le 2$, with equality if and only if kj is a w-stroke. The other exponents $e(Z_{kj}) = a$, $e(Z_{ij}) = 1 + \frac{3}{2}(a-1)$ and $e(W_{ij}) = \frac{a-3}{2}$ exceed -2. If $a = 2$, then $\omega_{kj} = -2$: k and j are not w-close, and the w-stroke kj is maximal. In that case kj is a w-edge, since $e(Z_{kj}) = a = 2 > -2$. □

At a place with diagram 14 the zw-edges are $\mathrm{tri}(Q_j)$ and no stroke meets j. Its type $x(P)$ is $\mathsf{u}_c(j)$ with $0 \le c < 1$, or $\sigma_k(j)(t)$ with $0 < t < 1$. Lemma 3.3 leaves $c = 0$. At a place with diagram 15, case (B) holds with $a = 2$ and $x(P) = \mathsf{b}_k(j)$. At a place with diagram 16 all x_{kl} equal 1, which is excluded by Lemma 3.3.

3.3. **Diagram 5.** At a place with diagram 5, T is a zw-triangle without circles and the two other bodies A, B have no strokes. By Lemma 3.4 each of A and B is in one of the modes F, Z1, Zs_{t_0}, W1, Ws_{t_0} relative to $K = T$. The type is determined by the mode parameters and by x_{AB}. For every pair of modes (25 pairs), the orders of the following quantities satisfy a system of linear equalities, inequalities and disjunctions: the positions, the sums $\sum_{t \in T} m_t Z_{tX}$, the torques, the differences of the four points 0, z_c, z_A, z_B, where z_c is the centre of mass of T, and $1 - Mp_{AB}^3$. Each relation of the system follows from the equations (2.1), Lemmas 2.4, 3.2, 3.3 and 3.4, the centre of mass, and exact identities for the moment of inertia, the potential and the angular momentum; §B.1 lists the relations that we use. Some of these relations, for an outer body X in mode F, assume that the three inverse distances p_{Xt}, $t \in T$, have the same sign at leading order. Replacing them by relations that hold for all choices of signs (the torque equalities become inequalities, the relation between two bodies in mode F and the relations given for X by the identity of (F) in §B.1 are dropped, and the equalities for $\sum_t m_t Z_{tX}$ become inequalities) gives a model valid at every positive mass. For each pair of modes, Theorem B.1 in Appendix B gives an explicit polytope in the five coordinates $(x_{At_0}, x_{At}, x_{Bt_0}, x_{Bt}, x_{AB})$ that

contains the projection of every point of the model. The union of these 25 polytopes is the hull V_2^{w}. If we keep the equalities for $\sum_t m_t Z_{tX}$, which is legitimate for all choices of signs when $m_o \neq m_a + m_b$ for every $o \in T = \{o, a, b\}$, the same theorem gives a smaller hull V_2. The hull V_2^{w} differs from V_2 only for the pairs of modes (F, F), $(\mathrm{F}, \mathrm{Z1})$, $(\mathrm{F}, \mathrm{W1})$ and their exchanges. A face of these hulls, and of the hull V_N below, is *attained* if each of the five coordinates is less than 1 at one of its vertices (no outer pair of diagram 5 is a zw-edge). Every point of each of these models has all five coordinates less than 1: by $c < 1$ in mode F, by $\bar{\omega} < 1$ and $s + \bar{\omega} < 2$ in the other modes (Lemma 3.4), and, as A and B have no strokes, by $\zeta_{AB} + 3\omega_{AB} < 4$ and $3\zeta_{AB} + \omega_{AB} < 4$ (Lemma 3.2). The projection of a model point is a convex combination, with positive weights, of the vertices of the face in whose relative interior it lies. Hence this face is attained: each coordinate is less than 1 at one of these vertices. The orders in these models are real numbers, while a modelled quantity may vanish identically along C. Its order is then $+\infty$. The next lemma treats this case.

Lemma 3.8. *Let P be a place of diagram 5 at a mass at which one of the models of this subsection is valid, and let o be the vector of the orders at P of the modelled quantities, with $o_v = +\infty$ when the quantity vanishes identically along C. Then the model has a point with real coordinates whose projection to the five coordinates of the type equals that of o. In particular the type of P lies in the hull of the model.*

Proof. Each relation of a model that requires a minimum to be attained twice comes from an exact linear identity, with nonzero constant coefficients, among the modelled quantities, such as $m_A z_A + m_B z_B + M_T z_c = 0$ for the centre of mass z_c of T. It still holds when the terms that vanish identically are dropped. The quantities that can vanish identically are the positions z_A, z_B, z_c, w_A, w_B, w_c, the differences $z_X - z_c$ for a body in mode Z1 or Zs and $w_X - w_c$ for one in mode W1 or Ws, the sums $\sum_t m_t Z_{tX}$ of the bodies in modes Z1 and W1 and, for V_2^{w}, in mode F, the two torques, the quantities of (R3), and $1 - Mp_{AB}^3$. Each of them enters the relations only in lower bounds, in the equality of the orders of the two torques, which vanish together, with coefficient $+1$ in the terms of a minimum, or in upper bounds that are switched off by the event of (R1), (R2) or (R3) triggered by its vanishing. We replace every infinite order by a number L larger than $10 + 3\max|o_v|$, the maximum over the finite orders. We use $L + 2$ for $1 - Mp_{AB}^3$, and $L + 1$ for the two quantities of (R3) when all terms of that relation are infinite. The lower bounds and the equality of the torques then hold. The upper bounds that are not switched off involve only finite orders. A term of a minimum that contains a replaced order is at least $L - 2$. All other terms are at most $3\max|o_v| + 2$. Hence a minimum attained twice by finite terms keeps its value and the pair attaining it. The three relations whose terms can all be infinite hold with the values chosen. Since the projection involves only finite orders, it is unchanged. By Theorem B.1, it lies in the hull. □

Note that the lemma is needed: modelled quantities do vanish identically at some places. On the modified Roberts continuum [2, pp. 543–544], at the masses $(4, 4, 4, 4, 1)$, the centre of mass z_c, its counterpart w_c and both torques vanish identically at each of the four places of diagram 5.

A third hull $V_N \subseteq V_2$ uses three further relations (R1), (R2) and (R3). Theorem B.1 gives its polytopes as well. These relations are proved in the supplementary material. (R1) and (R2) hold at all positive masses off an explicit hypersurface in the masses of T, A and B. (R3) is a relation for the pairs of modes $(\mathrm{Z1}, \mathrm{Zs}_{t_0})$, $(\mathrm{W1}, \mathrm{Ws}_{t_0})$ and their exchanges. It holds when t_0 is not at the centre of mass of T. A body of T is at the centre of mass of T exactly when the masses of T are $(\mu/4, \mu, \mu)$ in some order. Then the pairs of modes with t_0 at the centre of mass carry no type: the supplementary material gives a model for them without (R3). By Theorem B.1 this model is infeasible. We call this case the *centred case*. The hull V_N then remains valid; its

model contains the rows of that of V_2. Thus every type lies in the relative interior of an attained face of V_N. Like V_2, V_N needs $m_o \neq m_a + m_b$ for the rows of the bodies in mode F. Hence V_N is valid at all mass points that lie off the hypersurface and have $m_o \neq m_a + m_b$ on every triangle T with $\mathsf{E}(T)$. The point check of §12.2 verifies both conditions at every application of V_N in the proof.

Every certificate of §§8–9 that holds relative to V_2 also holds relative to V_2^{w} (§12.2).

3.4. **Mass conditions.** We write $x_i = m_i^{-1/2}$. The conditions of [2, §8] that we use, after renumbering, are the following *labels*. Each is a polynomial condition on m. Appendix D gives the polynomials.

- $\mathsf{E}(T)$, $T = \{a, b, c\}$: one of x_a, x_b, x_c is the sum of the other two. This is (23) and (36) of [2].
- $\mathsf{R}(ab; s)$: $m_s^2(m_a + m_b)^4 = m_a^3 m_b^3$. This is (20), (39) and (40).
- $\mathsf{H}(D_1|D_2)$, for a partition $D_1 = \{i, j\}$, $D_2 = \{k, l\}$ of a four-body set Q_s: $m_i m_j = m_k m_l$. This is (29).
- $\mathsf{EQ}(a, b)$: $m_a = m_b$; and $\mathsf{D6}(ab|cd)$: $m_a = m_b$ and $m_c = m_d$, for disjoint pairs. This is (32).

For diagram 3 the condition (29) can be sharpened. For a pair $D = \{i, j\} \subset Q_s$ let $\mathrm{Tri}(D; s)$ say that one of x_i^3, x_j^3, $x_s^2/\sqrt{M_{Q_s}}$ is the sum of the other two, and put $\mathcal{D}_3(s; D_1|D_2) = \mathsf{H}(D_1|D_2) \wedge \mathrm{Tri}(D_1; s) \wedge \mathrm{Tri}(D_2; s)$.

Theorem 3.9. *If a singular place of C has type $\mathsf{h}(s; D_1|D_2)$, then $m \in \mathcal{D}_3(s; D_1|D_2)$.*

In the proof, in Appendix C, we write the leading coefficients at such a place explicitly on $\mathsf{H}(D_1|D_2)$, show that the next-order equations have a Jacobian of full rank, and compute the first obstruction as a single number that vanishes for some leading solution exactly when both conditions Tri hold. For each labelling the positive part of $\mathcal{D}_3(s; D_1|D_2)$ lies on finitely many curves in the projective space of masses. From here on, in label sets and label tests, the label $\mathsf{H}(D_1|D_2)$ stands for $\mathcal{D}_3(s; D_1|D_2)$.

3.5. **Diagram 7.** We consider a place with diagram 7, with z-wing $A = \{a, b, s\}$, w-wing $B = \{c, d, s\}$ and type $\mathsf{v}_7(a', b')$. We write (a', b') for the parameters, to keep a, b for bodies. By Lemma 3.2, on the A pairs $\zeta = 3a' - 2$ and $\omega = 2 - a'$, on the B pairs $\zeta = 2 - b'$ and $\omega = 3b' - 2$, and on the four cross pairs $\zeta = 3a' - 2$ and $\omega = 3b' - 2$. Let LA be the line $4a' + 3b' = 4$, on which $e(W_{ab}) = e(w_{cd})$, and LB the line $3a' + 4b' = 4$. Write $(35)_A$ for the condition $W_{ab}/m_s \sim W_{bs}/m_a \sim W_{sa}/m_b$ of [2, (35)] on the wing A, put $A_w = m_a W_{as} + m_b W_{bs}$ and $h = -A_w/(m_a + m_b)$ (the quantity h of [2, §8.7.2], in our sign convention), and define g in the same way for the wing B in the other colour. We use the following facts from [2, §8.7] and the symmetric ones for B:

(a) if $(35)_A$ holds, then $\mathsf{E}(A)$;
(b) if $(35)_A$ fails, then $h \approx W_{ab}$ and $w_s \sim -m_s h$;
(c) by the centre of mass $w_s \preceq w_{cd}$, so that $h \preceq w_{cd}$ when $(35)_A$ fails;
(d) if moreover $h \prec w_{cd}$, then $\mathsf{R}(cd; s)$, by the reduction of [2, pp. 577–578] to the system of [2, §5.4];
(e) $h \approx w_{cd}$ and $g \approx z_{ab}$ do not hold together.

Lemma 3.10. *If $(35)_A$ holds and $4a' + 3b' \leq 4$, then $\mathsf{R}(cd; s)$. Symmetrically, if $(35)_B$ holds and $3a' + 4b' \leq 4$, then $\mathsf{R}(ab; s)$.*

Proof. We put $C_w = \sum_{k\in\{a,b\},\, l\in\{c,d\}} m_k m_l W_{kl}$. From the equations $w_k = \sum_{l\neq k} m_l W_{kl}$ of a and b in (2.1), $m_a w_a + m_b w_b = m_s A_w + C_w$. With $w_a = w_s + w_{as}$ and $w_b = w_s + w_{bs}$ this becomes

$$(m_a + m_b)\, w_s = m_s A_w + C_w - m_a w_{as} - m_b w_{bs}.$$

We compare each term with w_{cd}, of exponent $3b' - 2$. The three pairs of A have the same (ζ, ω). So W_{as}, W_{bs} and W_{ab} have the same exponent $2 - 4a'$. Under $(35)_A$ the two terms of A_w cancel at leading order and $e(A_w) > 2 - 4a' \geq 3b' - 2$ on or below LA. The cross terms have $e(C_w) \geq 4 - \frac{9}{2}a' - \frac{3}{2}b'$, which exceeds $3b' - 2$ because $a' + b' < \frac{4}{3}$ on or below LA. And $e(w_{as}) = 2 - a' > 3b' - 2$ as $a', b' < 1$. Hence $w_s \prec w_{cd}$. By the centre of mass, $M w_s = m_c w_{sc} + m_d w_{sd} + m_a w_{sa} + m_b w_{sb}$, which gives $m_c w_{sc} + m_d w_{sd} \prec w_{cd}$: the body s is at the w-centre of mass of $\{c, d, s\}$ at the scale w_{cd}. The reduction of [2, pp. 577–578] uses exactly this relation, the equations of c and d at exponent -2 (in which only the two stroke terms survive, because $w_k - w_G$ has exponent at least $3b' - 2 > -2$ and the cross terms have exponent $4 - \frac{9}{2}a' - \frac{3}{2}b' > -2$), the closure of the triangle cds in both colours with $p^2 zw = 1$ on its pairs, and the positivity of m_c, m_d, m_s. Eliminating the leading coefficients from this system, we obtain $\mathsf{R}(cd; s)$ times a factor that is positive on $(\mathbb{R}_{>0})^3$. Hence $\mathsf{R}(cd; s)$. □

Strictly below LA the hypothesis $(35)_A$ is not needed, since $e(A_w) \geq 2 - 4a' > 3b' - 2$ holds term by term. Combining Lemma 3.10 with the facts quoted above, we obtain the *wing rules*:

(i) strictly above LA, $\mathsf{E}(A)$;
(ii) on LA, either $\mathsf{E}(A) \wedge \mathsf{R}(cd; s)$, or $h \approx w_{cd}$ and $(35)_A$ fails;
(iii) strictly below LA, $\mathsf{R}(cd; s)$.

With RA $= \mathsf{R}(ab; s)$ and RB $= \mathsf{R}(cd; s)$ this gives Table 2, which lists, for each cell of the arrangement of LA, LB and the line $a' + b' = \frac{4}{3}$ in the square $(0, 1)^2 \cap \{a' + b' \leq \frac{4}{3}\}$, the minimal sets of labels under which a type in the cell can occur. Without Lemma 3.10 the conditions of [2, §8.7] allow, in 22 (condition set, cell) cases, types excluded by Table 2, for instance $\mathsf{v}_7(\frac{11}{28}, \frac{11}{28})$ and the point $(\frac{4}{7}, \frac{4}{7}) = \mathrm{LA} \cap \mathrm{LB}$.

region	minimal label sets
above LA, above LB (including the edge $a' + b' = \frac{4}{3}$)	$\mathsf{E}(A)\mathsf{E}(B)$
above LA, on LB	$\mathsf{E}(A)$
above LA, below LB	$\mathsf{E}(A)$ RA
on LA, above LB	$\mathsf{E}(B)$
LA $\cap$ LB	$\mathsf{E}(A)$ RB or $\mathsf{E}(B)$ RA
on LA, below LB	RA
below LA, above LB	$\mathsf{E}(B)$ RB
below LA, on LB	RB
below LA, below LB	RA RB

TABLE 2. Diagram 7 with wings $A|B$: the label sets under which a type $\mathsf{v}_7(a', b')$ in each region can occur.

3.6. **Pieces.**

Definition 3.11. A *piece* is a polytope $\Pi \subset \mathbb{R}^{10}$ (a point, a segment or a polygon, given by its vertices) together with a set of its faces, the *attained* faces, closed under passing to larger faces. Its *attained set* is the union of the relative interiors of its attained faces. For a set S of labels, pieces(S) is the list of pieces in Table 3 whose enabling condition is implied by S, together with the attained faces of the hull V_2^{w} of §3.3, or of another hull of §3.3 where this is stated, embedded for each labelling of diagram 5 with $\mathsf{E}(T) \in S$.

piece	vertices	attained	enabled by
$\mathsf{d1}[T]$	$\mathsf{a}_1(T)$, $\mathsf{b}_1(T)$	open segment	$\mathsf{E}(T)$
$\mathsf{d4}[T]$	$\mathsf{b}_1(T)$	point	$\mathsf{E}(T)$
$\mathsf{d2}[T]$	$\mathsf{a}_1(T)$	point	always
$\mathsf{d3}(s; D_1\|D_2)$	$\mathsf{h}(s; D_1\|D_2)$	point	$\mathsf{H}(D_1\|D_2)$, read as $\mathcal{D}_3(s; D_1\|D_2)$
$\mathsf{d6}(A\|B)$	$\mathsf{v}_7(0,0)$	point	$\mathsf{D6}(ab\|cd)$
$\mathsf{d7}(A\|B)$	cells of Table 2	see Table 2	$\mathsf{E}(A)$, $\mathsf{E}(B)$, RA, RB
$\mathsf{d8}(A\|B)$	$\mathsf{v}_7(0,0)$, $\mathsf{v}_7(1,0)$	open segment	RA
$\mathsf{d9}(T; ax, by)$	$\mathsf{n}(T; ax, by)$	point	always
$\mathsf{d10}(A\|B)$	$\mathsf{u}_A(0)$, $\mathsf{u}_A(\frac{1}{3})$	$(\mathsf{u}_A(0), \mathsf{u}_A(\frac{1}{3})]$	$\mathsf{E}(A)$
$\mathsf{d11}(A\|B)$	$\mathsf{u}_A(0)$	point	$\mathsf{E}(A) \vee$ RA
$\mathsf{d12}(A\|B)$	$\mathsf{u}_A(0)$	point	RA
$\mathsf{d14a}(j)$	$\mathsf{u}_0(j)$	point	always
$\mathsf{d14b}(j,k)$	$\mathsf{b}_k(j)$, $\mathsf{u}_0(j)$	open segment	always
$\mathsf{d15}(j,k)$	$\mathsf{b}_k(j)$	point	always

TABLE 3. The pieces, for each triple T, centre s with wings $A = \{a, b, s\}$, $B = \{c, d, s\}$ (and the same with A, B exchanged in the rows d8–d12), triangle T with outer edges ax, by, and $j \neq k$.

Since attained sets grow with S, a statement proved for pieces(S) holds for every subset of S. For $m \in (\mathbb{R}_{>0})^5$ let $S(m)$ be the set of labels that hold at m, with $\mathsf{D6}(ab|cd)$ added when $m_a = m_b$ and $m_c = m_d$.

Theorem 3.12. *Let $m \in (\mathbb{R}_{>0})^5$, let $C \subseteq Y_m$ be an irreducible curve on which some r_{kl}^2 is not constant and $U \not\equiv 0$, and let P be a singular place of C with diagram d and labelling λ. Then $x(P)$ lies in the attained set of a piece of* pieces$(S(m))$ *of diagram d and labelling λ. For $d = 5$ it lies in the relative interior of an attained face of V_2^{w} with that labelling, and of V_2 if $m_o \neq m_a + m_b$ for every o in the triangle $T = \{o, a, b\}$ of the place.*

Proof. By Lemma 3.1 the place gives singular sequences with diagram d. Hence the conditions of [2, §8] hold at m for the labelling λ. These are (23) for diagrams 1, 4 and 5, (29) for 3, sharpened by Theorem 3.9, (32) for 6, the conditions of §8.7 for 7, (36) for 10, and (40) for 8 and 11–13. By Proposition 3.5, Lemmas 3.6, 3.7 and 3.3, and Table 2 for diagram 7, the type lies in the set listed in Table 1. This set lies in the attained set of the corresponding piece of Table 3 once its condition holds. Diagram 5 is treated in §3.3, with Lemma 3.8. Renaming the bodies maps singular places, types, label sets and pieces equivariantly, since all of them are defined through labelled sets of bodies. So it suffices to consider the numbering of [2, Fig. 11]. There the type set of each diagram other than 5 and 7 is the attained set of its pieces (compare Tables 1 and 3 row by row). For diagram 7 the attained set of the pieces $\mathsf{d7}(A|B)$ of pieces(S) is the set of cells allowed for S by the wing rules, which is Table 2. □

3.7. **Clusters of zero energy.** The pieces d14a, d14b and d15 carry no mass condition from [2]. They need a four-body configuration of a special kind. For four bodies with masses $m_1, \dots, m_4$ and $z, w \in \mathbb{C}^4$, $p_{il} = p_{li}$, put

$$F_i^z = \sum_{l \neq i} m_l \, z_{il} \, p_{il}^3, \qquad F_i^w = \sum_{l \neq i} m_l \, w_{il} \, p_{il}^3 \qquad (i = 1, \dots, 4).$$

- K14(m) is the system $F^z = F^w = 0$, $p_{il}^2 z_{il} w_{il} = 1$ for all pairs, with all $z_{il}, w_{il} \neq 0$:

$$\textstyle\sum_{l \neq i} m_l z_{il} p_{il}^3 = \sum_{l \neq i} m_l w_{il} p_{il}^3 = 0, \qquad p_{il}^2 z_{il} w_{il} = 1. \tag{3.2}$$

- K15(m,k), for $k \in \{1,\dots,4\}$, is the system

$$F_i^z = 0, \qquad F_i^w = c + d\,[i=k], \qquad p_{il}^2 z_{il} w_{il} = 1, \qquad d \neq 0, \tag{3.3}$$

with all $z_{il}, w_{il} \neq 0$.

Summing $m_i F_i^z$ and $m_i F_i^w$ shows that solutions of K14 have $\sum m_i z_i$ and $\sum m_i w_i$ free, while solutions of K15 have $cM + dm_k = 0$. Contracting F^z with w shows that $U = \sum_{i<l} m_i m_l p_{il} = 0$ for solutions of both systems: they are four-body configurations of zero potential energy, critical for the potential (K14), or for the potential plus a linear function of the position of body k (K15). Both systems are invariant under translations, under $(z,w,p) \mapsto (\lambda z, \mu w, (\lambda\mu)^{-1/2} p)$ (with c,d rescaled), under $p \mapsto -p$, and under relabelling with the masses. Whether they have solutions therefore depends only on the ratios of the masses and on k.

Theorem 3.13. *Let P be a singular place of C, j a body, $Q = Q_j$, $k \in Q$.*

(a) *If $x(P) \in \{\mathsf{u}_0(j)\} \cup \bigcup_{k'} \sigma_{k'}(j)\big((0,1)\big)$, in particular if it lies in the attained set of* $\mathsf{d14a}(j)$ *or of some* $\mathsf{d14b}(j,k')$*, then* K14(m_Q) *has a solution, formed by the leading coefficients of the cluster at P.*

(b) *If $x(P) = \mathsf{b}_k(j)$, then* K15(m_Q,k) *has a solution, formed by the leading coefficients of the cluster at P.*

(c) *In case* (a)*, P has diagram* 14 *with isolated body j. In case* (b)*, P has diagram* 15 *with isolated body j and w-edge kj, up to exchanging the colours.*

Proof. Since every listed point has $x = 1$ exactly on $\mathrm{tri}(Q_j)$, Lemma 3.7 applies. The point $\mathsf{u}_0(j)$ is case (A), $\sigma_{k'}(j)(t)$ with $0 < t < 1$ is case (B) with $a = 1 + t < 2$, and $\mathsf{b}_k(j)$ is case (B) with $a = 2$. This gives (c). Fix $i_0 \in Q$ and, for $i, l \in Q$, let $\hat z_i$, $\hat w_i$ and $\hat p_{il}$ be the coefficients of u_{ii_0}, v_{ii_0} at exponent 1 and of p_{il} at exponent -1 (in the expansion of Lemma 3.1). For $i \neq l$ in Q these are nonzero and the row D_{il} gives $\hat p_{il}^2 \hat z_{il} \hat w_{il} = 1$. The first equation of (2.1) for $i \in Q$ is $z_i = \sum_{l \in Q\setminus i} m_l p_{il}^3 z_{il} + m_j Z_{ij}$. Each term of the sum has exponent -2 with coefficient $m_l \hat p_{il}^3 \hat z_{il}$. The coefficient c_z of z_i at exponent -2 does not depend on i, since $e(z_{il}) = 1$. So

$$F_i^z(\hat z, \hat p) = c_z - m_j [Z_{ij}]_{-2} \qquad (i \in Q),$$

where $[\,\cdot\,]_{-2}$ is the coefficient at exponent -2, nonzero if and only if ij is a z-stroke. The same holds for w. When we multiply by m_i and sum over Q, the left side cancels by antisymmetry. If no z-stroke meets j, this gives $M_Q c_z = 0$ and hence $F^z(\hat z, \hat p) = 0$, and likewise for w. In case (A) and in case (B) with $a < 2$ no stroke meets j and the leading coefficients solve K14(m_Q). In case (B) with $a = 2$, $F^z = 0$ and $F_i^w = c_w + d[i=k]$ with $d = -m_j [W_{kj}]_{-2} \neq 0$: this is K15$(m_Q,k)$. □

Corollary 3.14. *If* K14(m_{Q_j}) *has no solution, no singular place of C has its type in the attained set of* $\mathsf{d14a}(j)$ *or of any* $\mathsf{d14b}(j,k)$*. If* K15(m_{Q_j},k) *has no solution, none has type* $\mathsf{b}_k(j)$*. These pieces may then be removed from* pieces$(S(m))$ *in Theorem 3.12.*

Systems of this kind, with the cluster masses as unknowns, have solutions only on proper algebraic subsets of the mass space. In Section 5 we will remove the pieces d15 and d14b at every positive mass (Corollary 5.19 and Theorem 5.21). We will also remove the pieces of diagrams 9 and 2 at every positive mass (Theorem 5.37 and Proposition 7.1). In §6 we will show how to decide K14 at a given mass point.

The point $\mathsf{v}(s; A|B)$, the type of $\mathsf{d6}(A|B)$, lies in the relative closure of several other pieces of diagrams 7 and 8. The next statement shows that removing $\mathsf{d6}(A|B)$ removes exactly that point.

Theorem 3.15. *If $x(P) = \lambda\,\mathsf{v}(s; A|B)$ for some $\lambda > 0$, then P has diagram 6 with centre s and wings A, B, and $\lambda = 1$. No attained face of a piece other than $\mathsf{d6}(A|B)$, and no face of V_2, meets the ray $\mathbb{R}_{>0}\,\mathsf{v}(s; A|B)$ in its relative interior.*

Proof. Each entry of $\lambda\mathsf{v}(s; A|B)$ is at most 0. No distance tends to zero at P. By the proof of [2, Prop. 9] only diagrams 2, 3 and 6 have no distance tending to zero. Their types $\mathsf{a}_1(T)$, $\mathsf{h}(s'; D_1|D_2)$, $\mathsf{v}(s'; A'|B')$ have 4, 4 and 6 zero entries, while $\lambda\mathsf{v}(s; A|B)$ has the six zero entries $\mathrm{tri}(A) \cup \mathrm{tri}(B)$. Hence $d = 6$ and $\mathrm{tri}(A') \cup \mathrm{tri}(B') = \mathrm{tri}(A) \cup \mathrm{tri}(B)$. In this graph s is the only vertex of degree 4, which gives $s' = s$ and $\{A', B'\} = \{A, B\}$. The entries -2 give $\lambda = 1$. For the second statement, recall that every point of the ray has all coordinates at most 0. Let F be an attained face of a piece other than $\mathsf{d2}$, $\mathsf{d3}$ and $\mathsf{d6}$. If the piece is one of $\mathsf{d4}$, $\mathsf{d9}$ and $\mathsf{d10}$–$\mathsf{d15}$, all vertices of F have the coordinate 1 on a common zw-edge of the diagram, and so do all points of F. On the open segment of $\mathsf{d1}[T]$ the coordinates on the pairs of T lie in $(0, 1)$. The attained points of the pieces $\mathsf{d7}$ and $\mathsf{d8}$ are points $\mathsf{v}_7(a', b')$ with $a', b' \geq 0$ and $(a', b') \neq (0, 0)$: the cells of Table 2 lie in the open square $(0, 1)^2$ (the cell $\mathrm{LA} \cap \mathrm{LB}$ is the point $(\frac{4}{7}, \frac{4}{7})$), and the attained set of $\mathsf{d8}$ is $\mathsf{v}_7((0, 1) \times \{0\})$. There the sum of the coordinates on the pairs of A and of B is $3(a' + b') > 0$. So the relative interior of F has a positive coordinate and misses the ray. Among the points $\mathsf{d2}$, $\mathsf{d3}$ and $\mathsf{d6}$, only $\mathsf{d6}(A|B)$ lies on the ray, by the zero patterns above. Finally, no face of a D5 hull, attained or not, meets the ray, since its embedding has the coordinate 1 on the three pairs of the zw-triangle T; in particular no face of V_2 does. □

4. Lexicographic separation

Let $m \in (\mathbb{R}_{>0})^5$, and let $C \subseteq Y_m$ be a nontrivial curve. By (3.1) the types of the singular places of C satisfy $\sum_P d_P\, x(P) = 0$ with $d_P > 0$, and at least one place is singular. By Theorem 3.12 each type lies in the attained set of a piece of $\mathrm{pieces}(S(m))$. Thus m is settled as soon as 0 is not a positive combination of points of these attained sets.

4.1. The separation lemma.

Lemma 4.1. *Let $G_1, \ldots, G_M$ be finite subsets of $\mathbb{Q}^{10}$, let $l_1, \ldots, l_r \in \mathbb{Q}^{10}$, and let $I_1 \cup \cdots \cup I_r = \{1, \ldots, M\}$ be a partition such that for every i:*

(i) *$l_i \cdot v \geq 0$ for every $v \in G_j$ and every $j \in I_i \cup I_{i+1} \cup \cdots \cup I_r$;*
(ii) *for every $j \in I_i$ there is $v \in G_j$ with $l_i \cdot v > 0$.*

Then 0 is not a positive combination of $N \geq 1$ points of $\bigcup_j \mathrm{relint}\,\mathrm{conv}(G_j)$.

Proof. Suppose that $0 = \sum_{s=1}^{N} \lambda_s x_s$ with $N \geq 1$, $\lambda_s > 0$ and $x_s \in \mathrm{relint}\,\mathrm{conv}(G_{j(s)})$, and let i be the least index such that $j(s) \in I_i$ for some s. Each x_s is a convex combination of points of $G_{j(s)}$ with $j(s) \in I_i \cup \cdots \cup I_r$. So $l_i \cdot x_s \geq 0$ by (i). A point x of $\mathrm{relint}\,\mathrm{conv}(G)$ is a convex combination of all points of G with positive weights: if b is the barycentre of G and $\delta > 0$ is so small that $x + \delta(x - b) \in \mathrm{conv}(G)$, then $x = (1 + \delta)^{-1}\big(x + \delta(x - b)\big) + \delta(1 + \delta)^{-1} b$. Hence (ii) gives $l_i \cdot x_s > 0$ whenever $j(s) \in I_i$. There is such an s. Therefore we obtain $0 = l_i \cdot \sum_s \lambda_s x_s > 0$, a contradiction. □

The functionals act in stages. The first is nonnegative on all sets and positive somewhere on each set of I_1. Hence no point of these sets can occur in a vanishing positive combination. The second does the same among the remaining sets, and so on. Listing more sets G_j can only make the hypotheses harder to satisfy.

4.2. **Certificates.** A *D5 hull* is a finite union of polytopes in the five coordinates of §3.3, such as V_2^{w}. Its attained faces enter pieces(S) as in Definition 3.11.

Definition 4.2. Let S be a set of labels and V a D5 hull. Let $\mathcal{A}_V(S)$ be the list of the attained faces of the pieces of pieces(S) (Definition 3.11), with the attained faces of V in place of those of V_2^{w}. We record each face by the finite set of its vertices. A *certificate* for S relative to V is a finite sequence $l_1, \dots, l_r \in \mathbb{Z}^{10}$ such that

(a) for each i, $l_i \cdot v \geq 0$ for every vertex v of every face of $\mathcal{A}_V(S)$ on whose vertices $l_1, \dots, l_{i-1}$ all vanish, and
(b) every face of $\mathcal{A}_V(S)$ has a vertex v with $l_i \cdot v > 0$ for some i.

A set S is *certified* relative to V if it has a certificate relative to V.

Let I_i be the set of faces on whose vertices $l_1, \dots, l_{i-1}$ vanish and l_i is positive at some vertex. By induction on i, (a) and (b) say that the sets I_i partition $\mathcal{A}_V(S)$ and satisfy (i) and (ii) of Lemma 4.1. Since pieces(S) grows with S and the attained faces of a piece do not depend on S, a certificate for S is a certificate for every S' with pieces$(S') \subseteq$ pieces(S), in particular for any subset of S.

Definition 4.3. A D5 hull V is *valid* at $m \in (\mathbb{R}_{>0})^5$ if the conclusion of Theorem 3.12 holds at m with V in place of V_2^{w}.

Proposition 4.4. *Let $m \in (\mathbb{R}_{>0})^5$, let V be a D5 hull valid at m, and let S be a set of labels with* pieces$(S(m)) \subseteq$ pieces(S)*, for instance $S \supseteq S(m)$. If S is certified relative to V, then m is settled.*

Proof. Let $C \subseteq Y_m$ be a nontrivial curve. By (3.1) we have $\sum_P d_P x(P) = 0$, with $d_P > 0$ and at least one singular place P. By Theorem 3.12 and the validity of V, each $x(P)$ lies in the relative interior of an attained face of some piece in pieces$(S(m))$, built with V. This face belongs to $\mathcal{A}_V(S)$. A face and the convex hull of its vertices have the same relative interior. Thus 0 is a positive combination of points of $\bigcup_{G \in \mathcal{A}_V(S)} \operatorname{relint} \operatorname{conv}(G)$, which is excluded by Lemma 4.1, applied with the partition (I_i) of the certificate. Hence no such curve exists. □

The pieces $\mathsf{d2}[T]$, $\mathsf{d9}(T; ax, by)$, $\mathsf{d14a}$, $\mathsf{d14b}$ and $\mathsf{d15}$ are present for every label set. Proposition 7.1, Theorem 5.37, Corollary 5.19 and Theorem 5.21 remove the pieces $\mathsf{d2}[T]$, $\mathsf{d9}(T; ax, by)$, $\mathsf{d15}(j, k)$ and $\mathsf{d14b}(j, k)$, respectively, at all positive masses, so that for every label set S we may leave the attained faces of these pieces out of $\mathcal{A}_V(S)$. By Corollary 3.14 we may remove the pieces $\mathsf{d14a}(j)$ when K14(m_{Q_j}) has no solution. Hence, if K14(m_{Q_j}) has no solution for every body j of a set J, we may leave the attained faces of $\mathsf{d14a}(j)$, $\mathsf{d14b}(j, k)$ and $\mathsf{d15}(j, k)$, $j \in J$, $k \in Q_j$, out of $\mathcal{A}_V(S)$, and the proof of Proposition 4.4 applies to the remaining faces. We call the verification of (a) and (b) for these faces the *final check* of (S, J). The resulting notion of a K-certified point is Definition 6.6. Proposition 6.7 states that a K-certified point is settled.

4.3. **Exact verification.** All vertices of the pieces of Table 3, with the cells of Table 2, and all vertices of the polytopes of the D5 hulls used below have rational coordinates whose denominators divide 168. We check a certificate in integer arithmetic, after multiplying the vertices by 168 and asserting that every vertex becomes integral. We do not take the face list from the certificate: we recompute it from the label set S and the pieces that the certificate leaves out. That is, we

(1) enumerate the pieces enabled by S, with an exact implementation of Tables 3 and 2, without the pieces $\mathsf{d2}[T]$ and $\mathsf{d9}(T; ax, by)$;
(2) check that each polygon is strictly convex, with its vertices in the given cyclic order;

(3) compute the face lattice of each polytope of the hull exactly, and embed the attained faces for each $\mathsf{E}(T)$ implied by S;
(4) remove the pieces that the certificate leaves out, among them those of the clusters named in a final check; and
(5) evaluate each l_i at every vertex.

The list so built is the list that the certificate must separate, which leaves out the faces of all pieces d2 and d9 (§4.2). We will check all certificates in §§6–9 this way.

We find the functionals by solving, stage by stage, a linear program in floating point (HiGHS) on the faces not yet separated, and rationalizing its solution. We accept a stage only if the exact check accepts it.

Remark 4.5. We use three D5 hulls of §3.3:

(i) V_2^{w}, valid at all positive masses;
(ii) $V_2 \subseteq V_2^{\mathrm{w}}$, valid when $m_o \neq m_a + m_b$ for every o in each triangle $T = \{o, a, b\}$ with $\mathsf{E}(T) \in S(m)$;
(iii) $V_N \subseteq V_2$, valid at the mass points described in §3.3.

Each of them contains the projection of its model (Theorem B.1 and Lemma 3.8). We apply each certificate below that uses V_N only where V_N is valid (checked in §12).

4.4. **Certificates for the mass conditions.** To apply Proposition 4.4 on a set of masses we have to know which labels can hold there. We make these decisions by the kinds of certificate listed below. Each is *rigorous*: it may fail to decide, but a decision it returns is correct, given the enclosure property of ball arithmetic (Arb, through python-flint 0.9.0) and the correctness of exact integer, rational and polynomial arithmetic. Search parameters (the degree bound of the filter (F2) of §8.5, initial grids, depth and time limits, working precisions, Krawczyk preconditioners and the tie-breaking constants of §10) and floating-point proposals, such as the solutions of the linear programs, affect only running time, the numbers of flats and boxes, and whether a search succeeds: all acceptances and exclusions are decided by rigorous enclosures or exact checks. Hence a wrong proposal can only make a search fail.

(C1) *Exact computer algebra.* Standard bases, minimal primes and saturations of ideals of polynomials over $\mathbb{Q}$. With them we describe the points of a stratum of the sorted masses at which given labels hold. The details are in §8.5.
(C2) *Exact algebraic decisions.* Feasibility of linear systems over $\mathbb{Q}$ (Fourier–Motzkin elimination), polynomial identities, arithmetic in number fields $\mathbb{Q}[\theta]/(f)$, and real root isolation over $\mathbb{Q}$ by Sturm sequences or Descartes' rule of signs.
(C3) *Point certificates.* At an exact mass point, we show by the certificates of §6 that the system K14 of a cluster has no solution.

A label set certified in the sense of Definition 4.2 is a statement about pieces only. The classes (C1)–(C3) connect it to the masses:

(i) by (C1) and (C2), the label set of every positive mass vector with $m_1 \leq \cdots \leq m_5$ is certified in the sense of §8.2, contains a curve set, or is that of one of two exact points (Theorem 8.2);
(ii) the mass vectors whose label set contains a curve set lie, up to scaling and relabelling, on the curve families (Lemma 9.1), which we treat with (C2) and (C3) in §9 and by counting in §10;
(iii) at the two exact points the hull V_N certifies the label set (Proposition 8.7).

We describe in §12.2 how these computations are checked.

5. The energy identities at a place

In this section we treat diagram 9 and diagrams 14 and 15. Since the pieces of diagrams 14 and 15 carry no mass condition from [2] (§3.7) and are present for every label set (§4.2), no choice of labels removes them. We therefore need an argument that does not go through the labels.

By Theorem 3.13, places of diagrams 15 and 14 carry solutions of K15 and K14, which exist on hypersurfaces of the space of cluster masses. We exclude the places of diagram 15 by expanding at the place the potential $V = U + \frac{1}{2}I$, which is constant along C (Lemma 2.4) and whose gradient vanishes there (Lemma 2.3). For diagram 14 we derive in §5.5 a condition on the leading cluster. With the same expansion we exclude the pieces d14b (Theorem 5.21). In §5.6 the expansion, in coordinates adapted to the diagram, excludes diagram 9 at every positive mass (Theorem 5.37). Identities marked (†) were verified by computer in exact arithmetic; the programs are in the supplementary material.

5.1. **Expansions at a place.** Let P be a singular place of C and o a body. We use the positions relative to o, $u_i = z_i - z_o$ and $v_i = w_i - w_o$ for $i \neq o$, as coordinates. By relabelling, Lemma 2.3 holds in them. So $\nabla_{(u,v)} V = 0$ at all points of X_m. After the rotation of Lemma 3.1, each of these coordinates is $q = \vartheta^{d\,e(q)} y_q(\vartheta)$. Put $\tau = \vartheta^{d/2}$, so that $q = \tau^{2e(q)} y_q$, where y_q is a convergent Puiseux series in τ with $y_q(0) \neq 0$. We write $\bar{y} = y(0)$ for the *leading point*. For a Puiseux series g and $r \in \mathbb{Q}$, $[g]_r$ denotes its coefficient of τ^r. Replacing τ by $\tau\varphi(\tau)$ and rotating by $\mu(\tau)$, with convergent Puiseux series φ, μ and $\varphi(0) = \mu(0) = 1$, changes neither the leading point nor the identities below. We use this freedom in §5.3.

Lemma 5.1. *Suppose that $F(\tau, y) := V(\tau^{2e} y) = \sum_{n \geq 0} \tau^{n-2} V_n(y)$, with V_n holomorphic near $\bar{y}$ and the series convergent for y near $\bar{y}$ and small $\tau \neq 0$. Then $\nabla_y F(\tau, y(\tau)) = 0$, and for every $q \in \mathbb{Q}$ with $q \neq 2$,*

$$\sum_{n \geq 1} n \left[V_n(y(\tau)) \right]_{q-n} = 0. \tag{5.1}$$

Proof. Since $y = \tau^{-2e} x$ coordinatewise, $\nabla_y F = \tau^{2e} \nabla_x V = 0$ along the place. As V is constant on C, $\sum_n \tau^{n-2} V_n(y(\tau))$ is a constant E. The function F satisfies $F(\lambda\tau, \lambda^{-2e} y) = F(\tau, y)$, both sides being V at the same point. Differentiating at $\lambda = 1$ and using $\nabla_y F = 0$ at $y = y(\tau)$, we obtain $\sum_n (n-2)\tau^{n-2} V_n(y(\tau)) = 0$. Adding twice the first identity gives $\sum_n n\, \tau^{n-2} V_n(y(\tau)) = 2E$. Its coefficient of τ^{q-2} is (5.1). Each coefficient involves finitely many n, since $\tau^{n-2} V_n(y(\tau))$ has valuation at least $n - 2$. □

5.2. **Diagram 15 at order three.** Let P have type $\mathsf{b}_k(j)$. By Theorem 3.13(c), P has diagram 15 with zw-cluster $Q = Q_j$ and, after exchanging the colours if necessary, w-edge kj. We take $o = k$ and write $Q' = Q \setminus \{k\} = \{a, b, c\}$ for the three *outer* bodies. By Lemma 3.7 and Lemma 3.2, $e(u_i) = e(v_i) = 1$ for $i \in Q'$, $e(u_j) = 2$ and $e(v_j) = -2$. Write $\hat{z}_i = \bar{y}_{u_i}$, $\hat{w}_i = \bar{y}_{v_i}$ for $i \in Q'$, $\hat{z}_j = \bar{y}_{u_j}$, $\hat{w}_j = \bar{y}_{v_j}$, and $\hat{p}_{il}$ for the leading coefficients of the p_{il}.

Lemma 5.2. (a) *The cluster coefficients $(\hat{z}_i, \hat{w}_i)_{i \in Q'}$, with $\hat{z}_k = \hat{w}_k = 0$, and $(\hat{p}_{il})_{i,l \in Q}$ solve* K15(m_Q, k) *with $c = -m_j \hat{w}_j / M$.*

(b) $\sum_{i \in Q'} m_i \hat{z}_i = 0$.

(c) $\hat{p}_{ij}^2\, \hat{z}_i\, (-\hat{w}_j) = 1$ *for $i \in Q'$.*

Proof. (a) By Theorem 3.13(b) the cluster coefficients solve K15(m_Q, k). The constant c is the coefficient of w_k at exponent -2, which by $M w_k = -\sum_{i \neq k} m_i v_i$ is $-m_j \hat{w}_j / M$. (b) For a solution of K15, $\sum_i m_i \hat{z}_i F_i^w = \sum_{i<l} m_i m_l \hat{p}_{il}^3 \hat{z}_{il} \hat{w}_{il} = \sum_i m_i \hat{w}_i F_i^z = 0$, hence $c \sum_{i \in Q} m_i \hat{z}_i + d m_k \hat{z}_k = 0$.

With $\hat{z}_k = 0$ and $c = -dm_k/M_Q \neq 0$ this is (b). (c) is the leading term of the row D_{ij}, since u_{ij} and v_{ij} have the leading terms $\tau^2\hat{z}_i$ and $-\tau^{-4}\hat{w}_j$. □

We fix a square root $(-\hat{w}_j)^{1/2}$ and put $x_i = \hat{p}_{ij}(-\hat{w}_j)^{1/2}$ for $i \in Q'$. Then $x_i^2 = 1/\hat{z}_i$ by Lemma 5.2(c). Lemma 5.2(b) says $\sum_{i\in Q'} m_i x_i^{-2} = 0$.

Lemma 5.3. *At a place of type* $\mathsf{b}_k(j)$, $F(\tau, y) = \sum_{n\geq 0} \tau^{n-2} V_n(y)$ *with*

$$V_0 = U_Q(y) - \frac{m_j\, y_{v_j}}{2M} \sum_{i\in Q'} m_i y_{u_i}, \qquad V_1 = V_4 = 0, \qquad V_2 = f(\Pi),$$

$$V_3 = m_j \sum_{i\in Q'} m_i\, (-y_{u_i} y_{v_j})^{-1/2}, \qquad V_5 = \frac{m_j\, y_{u_j}}{2} \sum_{i\in Q'} m_i\, \frac{(-y_{u_i} y_{v_j})^{-1/2}}{y_{u_i}},$$

where U_Q *is the potential of the cluster,* $\Pi = y_{u_j} y_{v_j}$, $f(\Pi) = m_j m_k \Pi^{-1/2} + \frac{m_j M_Q}{2M}\Pi$, *and the square roots are the branches of the* p_{il} *at* P. *The coordinate* y_{u_j} *occurs only in* V_2 *and in the* V_n *with* $n \geq 5$.

Proof. The cluster terms $m_i m_l p_{il}$, $i, l \in Q$, are exactly τ^{-2} times their value at y. The term p_{kj} is $\Pi^{-1/2}$ exactly. For $i \in Q'$,

$$p_{ij} = \tau\, (-y_{u_i} y_{v_j})^{-1/2} \big(1 - \tau^2 y_{u_j}/y_{u_i}\big)^{-1/2} \big(1 - \tau^6 y_{v_i}/y_{v_j}\big)^{-1/2},$$

which contributes to V_n only for odd $n \geq 3$, with the stated V_3 and V_5. Finally $I = \sum_{i\neq k} m_i u_i v_i - \frac{1}{M}\big(\sum_{i\neq k} m_i u_i\big)\big(\sum_{i\neq k} m_i v_i\big)$ contributes $-\frac{m_j}{M} y_{v_j} \sum_{i\in Q'} m_i y_{u_i}$ at $n = 0$, $\frac{m_j M_Q}{M}\Pi$ at $n = 2$, and otherwise only to $n = 6$ and $n = 8$. □

Theorem 5.4. *At every singular place of type* $\mathsf{b}_k(j)$, *for any masses with* $m_i \neq 0$, $M \neq 0$ *and* $M_Q \neq 0$,

$$\sum_{i\in Q'} m_i\, \hat{p}_{ij} = 0, \qquad \textit{equivalently} \qquad \sum_{i\in Q'} m_i x_i = 0.$$

Moreover $\hat{p}_{kj}^3 = M_Q/(m_k M)$.

Proof. The y_{u_j}-component of $\tau^2 \nabla_y F(\tau, y(\tau)) = 0$ reads $y_{v_j} f'(\Pi) + O(\tau^3) = 0$, by Lemma 5.3. At $\tau = 0$ this is $f'(\bar{\Pi}) = 0$, i.e. $\hat{p}_{kj}^3 = \bar{\Pi}^{-3/2} = M_Q/(m_k M)$. Since $f''(\bar{\Pi}) = \frac{3}{4} m_j m_k \bar{\Pi}^{-5/2} \neq 0$, $f'(\Pi(\tau)) = O(\tau^3)$ gives $\Pi(\tau) - \bar{\Pi} = O(\tau^3)$. Hence $f(\Pi(\tau)) - f(\bar{\Pi}) = O(\tau^6)$ and

$$\big[V_2(y(\tau))\big]_r = 0 \qquad (0 < r < 6). \tag{5.2}$$

Now (5.1) with $q = 3$ reads $[V_1]_2 + 2[V_2]_1 + 3V_3(\bar{y}) = 0$. Since the first two terms vanish, $V_3(\bar{y}) = m_j \sum_i m_i \hat{p}_{ij} = 0$. □

We call $(x_a, x_b, x_c) \in (\mathbb{C}^*)^3$ a *bad shape* for the outer masses if $\sum m_i x_i = 0$ and $\sum m_i x_i^{-2} = 0$. By Theorem 5.4 and Lemma 5.2(b), the outer bodies of the leading cluster are at $\hat{z}_i = x_i^{-2}$ for a bad shape x.

Lemma 5.5. *Up to a common factor, the bad shapes are* $x = (1, t, -(m_a + m_b t)/m_c)$ *where* t *is a root of*

$$Q(t) = (m_a + m_b t)^2 (m_a t^2 + m_b) + m_c^3 t^2.$$

For positive masses Q *has no real root.*

Proof. We normalize $x_a = 1$. The first condition gives x_c. Then $m_c^2 t^2 x_c^2 \sum m_i x_i^{-2} = Q(t)$. For positive masses $Q(t) \geq m_c^3 t^2 > 0$ for real $t \neq 0$ and $Q(0) = m_a^2 m_b > 0$. □

Corollary 5.6. *If $m_a = m_b = m_c$, then $Q = m_a^3(t^2+t+1)^2$, and the outer bodies of a cluster with a bad shape form an equilateral triangle centred at body k: $\{\hat{z}_a, \hat{z}_b, \hat{z}_c\} = \lambda\{1, \mathrm{j}, \mathrm{j}^2\}$ with $\mathrm{j} = e^{2\pi i/3}$.*

Proof. For equal masses, $(1+t)^2(1+t^2)+t^2 = (t^2+t+1)^2$. Hence $t \in \{\mathrm{j}, \mathrm{j}^2\}$, x is proportional to $(1, \mathrm{j}, \mathrm{j}^2)$ up to order, and $\hat{z}_i = x_i^{-2}$. □

Proposition 5.7. *Let $\nu \in \mathbb{C} \setminus \{0\}$. No solution of* $\mathrm{K15}((1,1,1,\nu),4)$ *has $z_1 - z_4$, $z_2 - z_4$, $z_3 - z_4$ proportional to $(1, \mathrm{j}, \mathrm{j}^2)$ or to $(1, \mathrm{j}^2, \mathrm{j})$.*

Proof. After a normalization we reduce the equations to four polynomials in ν^2 and two further unknowns, eliminate ν^2, take resultants in one of the two unknowns, and show that the resulting polynomials in the other have no common root except 0 and 1. Exchanging bodies 2 and 3 reduces the second case to the first. We normalize $z = (1, \mathrm{j}, \mathrm{j}^2, 0)$ by the symmetries of K15. Momentum gives $3c + \nu(c+d) = 0$, so that $c \neq 0$. We take $c = 1$ by the remaining scaling. Let s_j be the coordinate p of the side opposite body $j \leq 3$ and $r_j = p_{j4}$, write $s_j = \sigma_0 + \sigma_1 \mathrm{j}^{j-1} + \sigma_2 \mathrm{j}^{1-j}$, and put $k = 3\nu^2$, $x = \sigma_1$ and $P = \sigma_0\sigma_2$. With $w_{il} = 1/(p_{il}^2 z_{il})$, the rows F^w give $\nu r_j = -\sigma_0 + \mathrm{j}^{j-1} + \sigma_2 \mathrm{j}^{1-j}$. The rows F^z then become three polynomial equations Z_0, Z_1, Z_2 in $(\sigma_0, \sigma_1, \sigma_2, k)$. The closure of the triangles in w gives three more, C_0, C_1, C_2. We use Z_0, Z_1, Z_2, C_0 and C_1. Since $\prod_j s_j \neq 0$, they force $P \neq 0$.

(1) With $t = \sigma_2$, $Z_2 = (P^2 - P + t^3)/t$. Hence $t^3 = P(1-P)$ and $P \neq 1$.
(2) Modulo $t^3 = P(1-P)$, the equations Z_1, C_1, Z_0 and C_0 are, up to sign, products of powers of t and P with

$$Q - kK, \quad (1-P)(x^4 + 3P^2) - 2PQx, \quad 1 - 6P + 3P^2 + P^3 + k\big(PQ + (1-P)x^3 + 6P(1-P)x\big)$$

and with a polynomial C_0' of degree 1 in k (†), where $Q = P^2 - P + 1$ and $K = (1-P)x^2 + Px + P(1-P)$. We call the three displayed polynomials (I), (II) and (III). A solution therefore gives (P, x, k) with $P \notin \{0, 1\}$ at which (I), (II), (III) and C_0' vanish.
(3) Write (III) $= A_3 + kB_3$ and $C_0' = A_0 + kB_0$ with A_3, B_3, A_0, B_0 free of k. Put $G_3 = KA_3 + QB_3$ and $G_0 = KA_0 + QB_0$. Since (I) is $Q - kK$, we have $G_3 = K \cdot (\mathrm{III}) + B_3 \cdot (\mathrm{I})$ and $G_0 = K\,C_0' + B_0 \cdot (\mathrm{I})$ (†), so that G_3 and G_0 vanish at (P, x). The resultants with (II) factor as

$$\mathrm{Res}_x(\mathrm{II}, G_3) = P^3(P-1)^2 H_{18}(P), \qquad \mathrm{Res}_x(\mathrm{II}, G_0) = 27P^6(P-1)^5(P^2-3P+1)^4 H_{32}(P),$$

where H_{18} and H_{32} are irreducible over $\mathbb{Q}$, of degrees 18 and 32, with leading coefficients 5 and 100 (†). Since $P \notin \{0, 1\}$, $H_{18}(P) = 0$ and $(P^2 - 3P + 1)H_{32}(P) = 0$. But $H_{18} \equiv 1735830 - 4544451P$ modulo $P^2 - 3P + 1$, a nonzero polynomial with a rational root, while the roots of $P^2 - 3P + 1$ are irrational. Thus H_{18} and $P^2 - 3P + 1$ have no common root. Since 7 divides neither leading coefficient and the Euclidean algorithm for H_{32} and H_{18} modulo 7 ends in a nonzero constant, H_{18} and H_{32} have no common root either. Hence no such P exists.

The polynomials and the table of the Euclidean algorithm, which has 19 rows, are in the supplementary material. □

Corollary 5.8. *Let the masses be positive. If the three outer masses m_i, $i \in Q_j \setminus \{k\}$, are equal, no singular place of C has type $\mathsf{b}_k(j)$.*

Proof. By Corollary 5.6 the leading cluster would be an equilateral solution of $\mathrm{K15}((1,1,1,\nu),4)$ after relabelling and scaling, with $\nu = m_k/m_a > 0$. Proposition 5.7 excludes it. □

5.3. **Diagram 15 at order five.** On the six coordinates $(u_i, v_i)_{i\in Q'}$ of the outer bodies of a cluster with body k at the origin, put

$$\Psi_c(u,v) = U_Q(u,v) + \frac{c}{2}\sum_{i\in Q'} m_i u_i.$$

Let $m_i \neq 0$ for $i \in Q$. A solution of K15(m_Q, k) then has $c \neq 0$ and $M_Q \neq 0$, by $cM_Q + dm_k = 0$ with $d \neq 0$. Since $\partial U_Q/\partial u_i = -\frac{1}{2}m_i F_i^w$ and $\partial U_Q/\partial v_i = -\frac{1}{2}m_i F_i^z$, for $c \neq 0$ and $M_Q \neq 0$ the critical points of Ψ_c are the solutions of K15(m_Q, k) with body k at the origin and the given c (the equations of body k follow from $\sum_i m_i F_i = 0$, with $d = -cM_Q/m_k$). Let H be the Hessian of Ψ_c at such a solution, and put $X = (\hat{z}, -3\hat{w})$, $E = (\hat{z}, \hat{w})$ and $\tilde{m} = (m_a, m_b, m_c)$.

Lemma 5.9. *Let $m_i \neq 0$ for $i \in Q$. At a solution of* K15(m_Q, k) *with body k at the origin, $HX = 0$, $HE = c\,(\tilde{m}, 0)$, and $\sum_{i\in Q'} m_i \xi_{u_i} = 0$ for every $\xi \in \ker H$.*

Proof. Since U_Q is homogeneous of degree $-\frac{1}{2}$ in u and in v separately, $\Psi_c(\lambda u, \lambda^{-3} v) = \lambda \Psi_c(u,v)$. The orbit of a critical point under this scaling therefore consists of critical points. So its tangent vector X lies in $\ker H$. Next, $\sum(u\partial_u + v\partial_v)\Psi_c = -U_Q + \frac{c}{2}\sum m_i u_i$. Differentiating at a critical point, where $\nabla U_Q = -\frac{c}{2}(\tilde{m}, 0)$, gives $HE = c(\tilde{m}, 0)$. Finally $\sum u\partial_u \Psi_c - \sum v\partial_v \Psi_c = \frac{c}{2}\sum m_i u_i$. Differentiating along $\xi \in \ker H$ at a critical point, we obtain $\frac{c}{2}\sum m_i \xi_{u_i} = 0$. Here $c \neq 0$ as $m_k \neq 0$. □

The configuration is *nondegenerate* if $\ker H = \mathbb{C}X$, and has *corank* r if $\dim\ker H = r+1$. For a place of type $\mathsf{b}_k(j)$ we take $c = -m_j\hat{w}_j/M$ as in Lemma 5.2, so that $V_0 = \Psi_c$ when $y_{v_j} = \hat{w}_j$. For $\xi \in \mathbb{C}^6$ put $\kappa_3(\xi) = \sum_{i\in Q'} \frac{\partial V_3}{\partial y_{u_i}}(\bar{y})\,\xi_{u_i}$. By Lemma 5.3,

$$\kappa_3(\xi) = -\tfrac{1}{2}m_j(-\hat{w}_j)^{-1/2}\sum_{i\in Q'} m_i x_i^3\,\xi_{u_i}, \qquad V_5(\bar{y}) = \tfrac{1}{2}m_j\hat{z}_j(-\hat{w}_j)^{-1/2}\sum_{i\in Q'} m_i x_i^3,$$

and $\kappa_3(X)$ is proportional to $\sum m_i x_i^3 \hat{z}_i = \sum m_i x_i = 0$.

Theorem 5.10. *Let the masses satisfy $m_i \neq 0$, $M \neq 0$ and $M_Q \neq 0$, and let P be a singular place of type $\mathsf{b}_k(j)$.*

(a) *If the leading cluster configuration is nondegenerate, then $\sum_{i\in Q'} m_i x_i^3 = 0$.*
(b) *If it has corank one, then κ_3 vanishes on $\ker H$.*

Proof. We fix an outer body a. By the freedom described in §5.1 we may assume $y_{v_j}(\tau) \equiv \hat{w}_j$ and $y_{u_a}(\tau) \equiv \hat{z}_a$: the conditions are $\mu^{-1}\varphi^4 y_{v_j} = \hat{w}_j$ and $\mu\varphi^{-2} y_{u_a} = \hat{z}_a$, which determine φ^2 and then μ. Write $y(\tau) - \bar{y} = (\delta(\tau), \delta_{u_j}(\tau))$ with δ the cluster part, so that $\delta_{u_a} = 0$. Let $S = \{\xi \in \mathbb{C}^6 : \xi_{u_a} = 0\}$. Since $X \in \ker H$ and $X_{u_a} = \hat{z}_a \neq 0$, the kernel of the restriction of the form H to S is $K_S = \ker H \cap S$, of dimension equal to the corank. Choose a complement W of K_S in S and write $\delta = \xi(\tau) + \eta(\tau)$ with $\xi(\tau) \in K_S$, $\eta(\tau) \in W$. Let α be the valuation of $\xi(\tau)$ ($\alpha = \infty$ if $\xi \equiv 0$).

By Lemmas 5.1 and 5.3, the cluster components of $\tau^2\nabla_y F = 0$ are $\nabla\Psi_c(\bar{y} + \delta) + \tau^3\nabla V_3(y) + O(\tau^5) = 0$, because V_2 does not involve the cluster and $V_0 = \Psi_c$ along the normalized place. Taylor expansion gives $H\delta + R(\delta) + \tau^3\nabla V_3(y) + O(\tau^5) = 0$ with $R(\delta) = O(|\delta|^2)$. Taking the components in S^* and using that $H|_S$ maps W isomorphically onto its image, the valuation of η is at least the minimum of 3, 2α, $\alpha + \operatorname{val}\eta$ and $2\operatorname{val}\eta$. Hence

$$\operatorname{val}\eta \geq \min(3, 2\alpha). \tag{5.3}$$

Pairing the same equation with $\xi_0 \in K_S$ and using $\xi_0^{\mathsf{T}} H = 0$ gives

$$\xi_0 \cdot R(\delta) + \tau^3\,\xi_0 \cdot \nabla V_3(y(\tau)) + O(\tau^5) = 0. \tag{5.4}$$

Since y_{v_j} is constant, $V_3(y(\tau)) = V_3(\bar{y}) + \kappa_3(\delta) + O(|\delta|^2)$ with $V_3(\bar{y}) = 0$.

(5.1) with $q = 5$ reads $[V_1]_4 + 2[V_2]_3 + 3[V_3]_2 + 4[V_4]_1 + 5V_5(\bar{y}) = 0$. By (5.2) this becomes

$$3\big[V_3(y(\tau))\big]_2 + 5V_5(\bar{y}) = 0. \tag{5.5}$$

(a) Since $K_S = 0$ in this case, $\delta = \eta$ has valuation at least 3 by (5.3). Hence $[V_3(y(\tau))]_2 = 0$. Then (5.5) gives $V_5(\bar{y}) = 0$, i.e. $\sum m_i x_i^3 = 0$.

(b) Here $K_S = \mathbb{C}\xi_0$ and $\xi(\tau) = a(\tau)\xi_0$ with $a(\tau) = a_\alpha \tau^\alpha + \dots$, $a_\alpha \neq 0$. Suppose $\kappa_3(\xi_0) \neq 0$. If $\alpha < 2$, we apply (5.1) with $q = 3 + \alpha$: the terms $n = 1, 4$ vanish, the term $n = 2$ is $2[V_2]_{1+\alpha} = 0$ by (5.2), and the terms $n \geq 5$ have negative index. This leaves $[V_3(y(\tau))]_\alpha = 0$. But by (5.3) the valuations of η and of $|\delta|^2$ exceed α. Therefore $[V_3(y(\tau))]_\alpha = a_\alpha \kappa_3(\xi_0) \neq 0$, a contradiction. Hence $\alpha \geq 2$. Then $\operatorname{val}\eta \geq 3$ and $\operatorname{val} R(\delta) \geq 4$. The coefficient of τ^3 in (5.4) is then $\xi_0 \cdot \nabla V_3(\bar{y}) = \kappa_3(\xi_0)$, which must vanish, again a contradiction. Thus $\kappa_3(\xi_0) = 0$. As $\kappa_3(X) = 0$ and $\ker H = \mathbb{C}X \oplus \mathbb{C}\xi_0$, the form κ_3 vanishes on $\ker H$. □

The next statement says that three nonzero complex numbers with positive weights cannot have vanishing weighted sums of x, x^3 and x^{-2}; for four numbers this fails (remark (c) after the proof). Theorems 5.4 and 5.10 need only $m_i, M, M_Q \neq 0$, but this statement needs positive masses. Corollary 5.12 and Proposition 5.34 use it.

Proposition 5.11. *Let $m_a, m_b, m_c > 0$. There is no $x \in (\mathbb{C}^*)^3$ with $\sum m_i x_i = \sum m_i x_i^3 = \sum m_i x_i^{-2} = 0$.*

Proof. We say that three directions in $\mathbb{R}/2\pi\mathbb{Z}$ *surround* 0 if they do not lie in a closed half-circle, that is, if each of the three cyclic gaps between them is less than π. We use two facts.

(A) Let $w_1, w_2, w_3 \in \mathbb{C}^*$ and $\mu_1, \mu_2, \mu_3 > 0$ with $\sum \mu_i w_i = 0$. Then the arguments of the w_i surround 0, or all w_i lie on one line through 0. Indeed, if the arguments do not surround 0, all w_i lie in a closed half-plane $\{\mathrm{Re}(\bar{u}z) \geq 0\}$ with $|u| = 1$. Then $0 = \sum \mu_i \mathrm{Re}(\bar{u}w_i)$ is a sum of nonnegative terms, so that all w_i lie on the line $iu\mathbb{R}$.

(B) There are no $\theta_1, \theta_2, \theta_3$ such that each of the sets $\{\theta_i\}$, $\{2\theta_i\}$ and $\{3\theta_i\}$ surrounds 0. Indeed, the three conditions are unchanged by a common shift of the θ_i and by relabelling. If $\{\theta_i\}$ surrounds 0, we may therefore take $\theta = (0, \pi - s, \pi + t)$ with $s, t > 0$ and $s + t < \pi$, so that the gaps are $\pi - s$, $s + t$ and $\pi - t$. Then $2\theta \equiv (0, -2s, 2t)$, with gaps $2t$, $2\pi - 2s - 2t$ and $2s$. The set $\{2\theta_i\}$ surrounds 0 exactly when $s < \frac{\pi}{2}$, $t < \frac{\pi}{2}$ and $s + t > \frac{\pi}{2}$. Assume this. Then $3s, 3t \in (0, \frac{3\pi}{2})$. After a rotation by π the set $\{3\theta_i\}$ is $\{\pi, -3s, 3t\}$. If $3s < \pi$ and $3t < \pi$, its gaps are $\pi - 3t$, $\pi - 3s$ and $3s + 3t > \frac{3\pi}{2}$. If $3t \geq \pi > 3s$, it lies in $[\pi, 2\pi]$. If $3s \geq \pi > 3t$, it lies in $[0, \pi]$. If $3s \geq \pi$ and $3t \geq \pi$, it lies in $[\frac{\pi}{2}, \frac{3\pi}{2}]$. In no case does it surround 0.

Now let x be a solution, and put $\theta_i = \arg x_i$.

(1) $\{\theta_i\}$ surrounds 0. This is (A) with $w = x$: if all x_i lay on a line through 0, then $x_i = \lambda r_i$ with $r_i \in \mathbb{R}^*$, and $\sum m_i x_i^{-2} = \lambda^{-2} \sum m_i r_i^{-2} \neq 0$.

(2) $\{2\theta_i\}$ surrounds 0. This is (A) with $w = x^{-2}$, whose arguments are the $-2\theta_i$ (a set surrounds 0 if and only if its mirror image does): if all x_i^{-2} lay on a line through 0, the θ_i would lie among four directions at right angles, and three of these lie in a closed half-circle, against (1).

(3) $\{3\theta_i\}$ surrounds 0. This is (A) with $w = x^3$: if all x_i^3 lay on a line through 0, then $\theta_i = \psi + n_i \frac{\pi}{3}$ with $n_i \in \mathbb{Z}/6$. By (1) the cyclic gaps between the n_i are less than 3. As they are also positive with sum 6, they all equal 2. So the n_i have the same parity, all x_i^3 lie on one ray, and $\sum m_i x_i^3 \neq 0$.

Steps (1)–(3) contradict (B). □

Remarks. (a) A second proof eliminates x. With $x = (1, t, -(m_a + m_b t)/m_c)$, the conditions are $Q(t) = 0$, with Q as in Lemma 5.5, and $C(t) = m_c^2(m_a + m_b t^3) - (m_a + m_b t)^3 = 0$. A solution therefore gives a common root of Q and C, so that their resultant

$$\mathrm{Res}_t(Q, C) = m_a^2 m_b^2 m_c^4 (m_a + m_b + m_c)^2\, P_{11}(m_a, m_b, m_c)$$

vanishes. But P_{11} is a symmetric polynomial of degree 11 that is positive on the positive orthant (Appendix F). (b) The positivity of all three masses is used in (A) and in steps (1) and (3). The statement has no uniform version on the closed simplex: for $m = (\frac{1}{2}, \frac{1}{2}, \varepsilon)$ and $x = (1, -1, i\sqrt{\varepsilon})$ we have $\sum m_i x_i^{-2} = 0$, while $\sum m_i x_i = i\varepsilon^{3/2}$ and $\sum m_i x_i^3 = -i\varepsilon^{5/2}$ tend to 0 with ε. Correspondingly, $P_{11}(\frac{1}{2}, \frac{1}{2}, 0) = 0$. (c) For four points the analogous statement fails: equal weights at $1, i, -1, -i$ satisfy all three conditions. In the proof above, (B) is where the number three enters.

Note that for equal outer masses the bad shape is $(1, \mathrm{j}, \mathrm{j}^2)$ and $\sum m_i x_i^3 = 3m_a \neq 0$, in accordance with Proposition 5.11.

Corollary 5.12. *Let $m \in (\mathbb{R}_{>0})^5$. At a singular place of type $\mathsf{b}_k(j)$ the leading cluster configuration is degenerate, and there is $\xi \in \ker H \setminus \mathbb{C}X$ with $\kappa_3(\xi) = 0$.*

Proof. A nondegenerate configuration would give a bad shape with $\sum m_i x_i^3 = 0$ (Theorems 5.4 and 5.10(a)), which is excluded by Proposition 5.11. At corank one, κ_3 vanishes on $\ker H$ by Theorem 5.10(b). At corank at least two, κ_3 is a linear form on $\ker H/\mathbb{C}X$, a space of dimension at least two, and has a nonzero kernel there. □

5.4. **Degenerate leading configurations.** For a function λ on the six pairs of Q, let L_λ be the grounded Laplacian of the complete graph on Q with edge weights λ_{il}, grounded at k: the 3×3 matrix with $(L_\lambda \xi)_i = \sum_{l \neq i} \lambda_{il}(\xi_i - \xi_l)$ for $i \in Q'$, where $\xi_k = 0$. At a solution of K15 put $q_{il} = m_i m_l p_{il}^3$ and $\varrho_{il} = z_{il}/w_{il}$.

Lemma 5.13. *At a solution of* $\mathrm{K15}(m_Q, k)$ *with body k at the origin,*

$$H = \begin{pmatrix} L_{3q/(4\varrho)} & L_{q/4} \\ L_{q/4} & L_{3q\varrho/4} \end{pmatrix},$$

and the system reads $L_q z = 0$, $L_{q\varrho} w = 0$ and $L_q w = c\,\tilde{m}$.

Proof. For one pair, $p = (zw)^{-1/2}$ and, with $\alpha = \zeta/z$, $\beta = \eta/w$ for a displacement (ζ, η), $p(z + s\zeta, w + s\eta) = p(1 + s\alpha)^{-1/2}(1 + s\beta)^{-1/2}$, whose second derivative at $s = 0$ is $\frac{p}{4}(3\alpha^2 + 2\alpha\beta + 3\beta^2)$. With $p^2 zw = 1$ the three coefficients are $\frac{3}{4}q/\varrho$, $\frac{1}{4}q$ and $\frac{3}{4}q\varrho$ times ζ^2, $2\zeta\eta$, η^2 (divided by $m_i m_l$). Summing over the pairs, we obtain H. The linear term of Ψ_c does not contribute. The system is $(L_q z)_i = m_i F_i^z$ and $(L_q w)_i = m_i F_i^w$, and $q_{il}\varrho_{il} w_{il} = q_{il} z_{il}$. □

Theorem 5.14. *Let the outer masses be positive and $m_k \neq 0$, let $(\hat{z}, \hat{w})$ solve* $\mathrm{K15}(m_Q, k)$ *with body k at the origin, suppose $\sum_{i \in Q'} m_i x_i = 0$ with $x_i^2 = 1/\hat{z}_i$, and let $\xi \in \ker H \setminus \mathbb{C}X$ satisfy $\sum_{i \in Q'} m_i x_i^3 \xi_{u_i} = 0$. Then one of the following holds:*

(i) *the outer masses are equal and $\{\hat{z}_i\} = \lambda\{1, \mathrm{j}, \mathrm{j}^2\}$;*
(ii) (G2) $\ker L_q = \mathbb{C}\hat{z}$ *and $L_{q\varrho}\hat{z} = 0$;*
(iii) (G1) $L_q = \sigma\, \tilde{m}\tilde{m}^{\mathsf{T}}$ *for some σ, and* $\operatorname{rank} L_{q\varrho} = 1$.

Proof. We write $\xi = (\zeta, \eta)$. The linear forms $\ell_1(\zeta) = \sum m_i \zeta_i$ and $\ell_2(\zeta) = \sum m_i x_i^3 \zeta_i$ vanish on ζ (Lemma 5.9 and the hypothesis on ξ) and on $\hat{z}$ ($\sum m_i \hat{z}_i = 0$ and $\sum m_i x_i^3 \hat{z}_i = \sum m_i x_i = 0$). If they are proportional, then $x_a^3 = x_b^3 = x_c^3$. In that case, since the $\hat{z}_i = x_i^{-2}$ are distinct, x is proportional to $(1, \mathrm{j}, \mathrm{j}^2)$ up to order, and $\sum m_i x_i = 0$ with real positive m_i forces equal masses: case (i). If they are not proportional, their common kernel is $\mathbb{C}\hat{z}$, which gives $\zeta = \gamma\hat{z}$, and

$\xi - \gamma X = (0, \eta')$ with $\eta' = \eta + 3\gamma\hat{w} \neq 0$ lies in $\ker H$. By Lemma 5.13, $(0, \eta') \in \ker H$ means $L_q\eta' = L_{q\varrho}\eta' = 0$. Now $\hat{z} \in \ker L_q$ and $\hat{w} \in \ker L_{q\varrho}$. Hence $\operatorname{rank} L_q \leq 2$. If $\operatorname{rank} L_q = 2$, then $\eta' \in \mathbb{C}\hat{z}$ and $L_{q\varrho}\hat{z} = 0$: case (ii). If $\operatorname{rank} L_q \leq 1$, we write $L_q = \sigma v v^{\mathsf{T}}$. Then $L_q\hat{w} = \sigma v(v \cdot \hat{w}) = c\tilde{m} \neq 0$ forces $v \in \mathbb{C}\tilde{m}$ and $\ker L_q = \tilde{m}^{\perp}$. If $\operatorname{rank} L_{q\varrho} = 2$, then $\eta' \in \mathbb{C}\hat{w}$ and $\tilde{m} \cdot \hat{w} = 0$, contradicting $\sigma(\tilde{m} \cdot \hat{w})\tilde{m} = c\tilde{m}$ with $c \neq 0$. As $L_{q\varrho} \neq 0$, $\operatorname{rank} L_{q\varrho} = 1$: case (iii). □

Case (iii) is excluded by the following statement, which does not use the bad shape.

Proposition 5.15. *Let $(\hat{z}, \hat{w})$ solve $\mathrm{K15}(m_Q, k)$ with body k at the origin, where m_a, m_b, m_c, m_k and $M_{Q'} = m_a + m_b + m_c$ are nonzero, and suppose that $\operatorname{rank} L_q \leq 1$ and $\operatorname{rank} L_{q\varrho} \leq 1$. Then $(m_k/M_{Q'})^2 = -1/s^3$ for a number s with $s^3 = \pm 3\sqrt{-3}$. In particular $m_k/M_{Q'}$ is not real, and the masses are not all positive.*

Proof. We use the rank conditions to write q_{il} and ϱ_{il} as products of factors attached to i and to l, deduce from the rows D_{il} that the three points $1/\hat{z}_i$ form an equilateral triangle, and compute s and $(m_k/M_{Q'})^2$ from this triangle.

(1) Since $L_q\hat{w} = c\tilde{m} \neq 0$, $\operatorname{rank} L_q = 1$ and $L_q = \sigma v v^{\mathsf{T}}$. Then $\sigma v(v \cdot \hat{w}) = c\tilde{m}$ gives $L_q = \sigma\tilde{m}\tilde{m}^{\mathsf{T}}$ with $\sigma \neq 0$. Reading off the entries, we obtain $q_{il} = -\sigma m'_i m'_l$ on all six pairs, where $m' = (m_a, m_b, m_c, -M_{Q'})$. Thus $p_{il}^3 = -\sigma$ on the outer pairs and $p_{ik}^3 = \sigma M_{Q'}/m_k$. The off-diagonal entries $-q_{il}\varrho_{il}$ of $L_{q\varrho}$ are nonzero. Hence $L_{q\varrho} = \sigma' v' v'^{\mathsf{T}}$ with $\sigma' \neq 0$, and in the same way $q_{il}\varrho_{il} = -\sigma'\tilde{v}_i\tilde{v}_l$ with $\tilde{v} = (v'_a, v'_b, v'_c, -v'_a - v'_b - v'_c)$, all of whose entries are nonzero. Hence $\varrho_{il} = \gamma\beta_i\beta_l$ on all six pairs, with $\gamma = \sigma'/\sigma$ and $\beta_i = \tilde{v}_i/m'_i \neq 0$.

(2) We put $\psi_i = 1/\hat{z}_i$. On the pairs through k, $\hat{z}_i = \gamma\beta_i\beta_k\hat{w}_i$. Inserting this into $\varrho_{il} = \gamma\beta_i\beta_l$ on the outer pairs gives $(1 - \beta_i/\beta_k)\psi_i = (1 - \beta_l/\beta_k)\psi_l$. If $\beta_i = \beta_k$ for one i, all the β are equal, ϱ is constant and $\hat{w}$ is a multiple of $\hat{z}$, which is excluded. So the common value ψ_0 is nonzero, $\beta_i = \beta_k(\psi_i - \psi_0)/\psi_i$, and $1/\hat{w}_i = \gamma\beta_i\beta_k\psi_i = A(\psi_i - \psi_0)$ with $A = \gamma\beta_k^2$.

(3) Since the p_{ik}^3 are equal, $p_{ik} = \varepsilon_i\pi_1$ with $\varepsilon_i^3 = 1$. The rows D_{ik}, $p_{ik}^2\hat{z}_i\hat{w}_i = 1$, then read $\psi_i(\psi_i - \psi_0) = \varkappa\varepsilon_i^2$ with $\varkappa = \pi_1^2/A$.

(4) Likewise $p_{il} = \varepsilon_{il}\pi_0$ on the outer pairs. As $\hat{z}_{il}\hat{w}_{il} = (\psi_i - \psi_l)^2/\big(A\psi_i\psi_l(\psi_i - \psi_0)(\psi_l - \psi_0)\big)$, the rows D_{il} read $(\psi_i - \psi_l)^2 = \varepsilon_{il}\varepsilon_i^2\varepsilon_l^2\,\varkappa/\lambda$ with $\lambda = \pi_0^2/\pi_1^2$. So the squared sides of the triangle ψ_a, ψ_b, ψ_c agree up to cube roots of unity: $\theta = (\psi_c - \psi_a)/(\psi_b - \psi_a)$ satisfies $\theta^6 = (\theta - 1)^6 = 1$ with $\theta \neq 0, 1$, whence $|\theta| = |\theta - 1| = 1$ and $\theta = e^{\pm i\pi/3}$. The triangle is equilateral: $\psi_i = h + R\varpi_i$ with $\{\varpi_a, \varpi_b, \varpi_c\} = \{1, \mathrm{j}, \mathrm{j}^2\}$ and $R \neq 0$.

(5) The polynomial $\Phi(\varpi) = (h + R\varpi)(h + R\varpi - \psi_0) = R^2\varpi^2 + R(2h - \psi_0)\varpi + h(h - \psi_0)$ takes the values $\Phi(\varpi_i) = \varkappa\varepsilon_i^2$. Hence $R^2 = \frac{1}{3}\sum_i \Phi(\varpi_i)\varpi_i = \varkappa s/3$ with $s = \sum_i \varepsilon_i^2\varpi_i$, a sum of three cube roots of unity. If the three terms are distinct, $s = 0$ and $R = 0$, which is excluded. If they are all equal to some α, then $\Phi(\varpi) = \varkappa\alpha\varpi^2$ at three points and hence identically, so $R(2h - \psi_0) = h(h - \psi_0) = 0$, which forces $\psi_0 = 0$, excluded. So exactly two terms are equal and $s^3 = \pm 3\sqrt{-3}$.

(6) By (4), $\lambda^3 = \varkappa^3/(\psi_b - \psi_a)^6$, and $(\psi_b - \psi_a)^2 = R^2(\varpi_b - \varpi_a)^2 = -3R^2\varpi_a\varpi_b$, so that $(\psi_b - \psi_a)^6 = -27R^6 = -\varkappa^3 s^3$ and $\lambda^3 = -1/s^3$.

(7) We choose $\pi_0 = p_{ab}$ and $\pi_1 = p_{ak}$. Then (1) gives $(\pi_0/\pi_1)^3 = -m_k/M_{Q'}$. So $(m_k/M_{Q'})^2 = \lambda^3 = -1/s^3$, a nonzero imaginary number. □

In case (ii), $\hat{z}$ and $\hat{w}$ lie in the kernel of $L_{q\varrho}$. They are independent, since $\hat{w} = \lambda\hat{z}$ would give $L_q\hat{w} = 0$. Hence $L_{q\varrho} = \sigma' u u^{\mathsf{T}}$ with $\sigma' \neq 0$ and $u = \hat{z} \times \hat{w}$. That is, $q_{il}\varrho_{il} = -\sigma'\tilde{u}_i\tilde{u}_l$ on the six pairs, where $\tilde{u} = (u_a, u_b, u_c, -u_a - u_b - u_c)$ is the affine dependence of the four points $(\hat{z}_i, \hat{w}_i) \in \mathbb{C}^2$. Then $q_{il} = -\sigma'\tilde{u}_i\tilde{u}_l w_{il}/z_{il}$. With this, $L_q\hat{w} = c\tilde{m}$ reads

$$c\,m_i = -\sigma'\tilde{u}_i S_i, \qquad S_i = \sum_{l \neq i} \tilde{u}_l\, w_{il}^2/z_{il} \qquad (i \in Q'). \tag{5.6}$$

Lemma 5.16. *For any four points (z_i, w_i), $i \in Q$, with $(z_k, w_k) = (0,0)$ and distinct z_i, and with $\tilde{u}$ and S_i formed from them as above, $\sum_{i\in Q} \tilde{u}_i S_i z_i^j = 0$ for $j = 0,1,2$. In case (ii), therefore, $\sum_{i\in Q'} m_i \hat{z}_i^2 = 0$.*

Proof. When we group the double sum by pairs, the pair $\{i,l\}$ contributes $\tilde{u}_i\tilde{u}_l w_{il}^2(z_i^j - z_l^j)/(z_i - z_l)$, that is 0, $\tilde{u}_i\tilde{u}_l w_{il}^2$ and $\tilde{u}_i\tilde{u}_l w_{il}^2(z_i + z_l)$ for $j = 0,1,2$. Since $\sum_i \tilde{u}_i = 0$, any two functions f, g on Q satisfy $\sum_{\{i,l\}} \tilde{u}_i\tilde{u}_l(f_i - f_l)(g_i - g_l) = -\big(\sum_i \tilde{u}_i f_i\big)\big(\sum_i \tilde{u}_i g_i\big)$. For $j = 1$ we take $f = g = w$. The result vanishes because $\sum_i \tilde{u}_i w_i = 0$. For $j = 2$ we write $(w_i - w_l)(z_i + z_l) = \big((wz)_i - (wz)_l\big) + (w_i z_l - w_l z_i)$. The first part gives $-\big(\sum \tilde{u}_i w_i\big)\big(\sum \tilde{u}_i w_i z_i\big) = 0$. In the second part the pairs through k contribute nothing. On the outer pairs $w_b z_c - w_c z_b = -u_a$ and cyclically, so that the second part equals $-\tilde{u}_a\tilde{u}_b\tilde{u}_c\big((w_b - w_c) + (w_c - w_a) + (w_a - w_b)\big) = 0$. The last statement follows from (5.6) and $\hat{z}_k = 0$. □

Since a vector orthogonal to $\hat{z}$ and to $(\hat{z}_i^2)$ is proportional to $(1/P'(\hat{z}_i))_{i\in Q'}$ with $P(t) = t\prod_{i\in Q'}(t - \hat{z}_i)$, by partial fractions, Lemma 5.16 says that in case (ii) the outer masses are proportional to $(1/P'(\hat{z}_i))_{i\in Q'}$. With the bad shape, this leaves only the equilateral configuration.

Theorem 5.17. *Let $m_a, m_b, m_c > 0$, and let z_a, z_b, z_c be distinct nonzero complex numbers with $\sum_i m_i z_i = \sum_i m_i z_i^2 = 0$ and $\sum_i m_i x_i = 0$ for some x_i with $x_i^2 z_i = 1$. Then $m_a = m_b = m_c$ and $\{z_a, z_b, z_c\} = \lambda\{1, \mathrm{j}, \mathrm{j}^2\}$ for some $\lambda \in \mathbb{C}^*$.*

Proof. The plane $m^\perp = \{v \in \mathbb{C}^3 : \sum_i m_i v_i = 0\}$ contains z and $z^2 = (z_i^2)$, which are independent because the z_i are distinct and nonzero. It is closed under complex conjugation because the m_i are real. Hence $\bar{z} = \alpha z + \beta z^2$. If $\beta = 0$, then $|\alpha| = 1$ and the z_i lie on a line through 0, so that $\sum m_i z_i^2 \neq 0$. Thus $\beta \neq 0$. The scaling $z \mapsto \lambda z$, $x \mapsto \lambda^{-1/2}x$ preserves the hypotheses and multiplies β by $\bar{\lambda}/\lambda^2$. So we may take $\beta = 1$: $\bar{z}_i = \alpha z_i + z_i^2$. Conjugating and substituting gives $z_i = |\alpha|^2 z_i + (\alpha^2 + \bar{\alpha})z_i^2 + 2\alpha z_i^3 + z_i^4$. Hence the z_i are the three roots of

$$g_\alpha(t) = t^3 + 2\alpha t^2 + (\alpha^2 + \bar{\alpha})t + |\alpha|^2 - 1.$$

Thus $z_a + z_b + z_c = -2\alpha$ and $z_a z_b z_c = 1 - |\alpha|^2$. For $i \neq l$ this gives $z_i\bar{z}_l + \bar{z}_i z_l = z_i z_l(2\alpha + z_i + z_l) = -z_a z_b z_c = |\alpha|^2 - 1$. Therefore

$$0 = \Big|\sum_i m_i z_i\Big|^2 = \sum_i m_i^2 |z_i|^2 - (1 - |\alpha|^2)\sum_{i<l} m_i m_l,$$

so that $|\alpha| < 1$.

Next, $x \in m^\perp$ and hence $x = \gamma z + \delta z^2$. Then $x_i^2 z_i = 1$ says that g_α divides $t^3(\gamma + \delta t)^2 - 1$. The remainder modulo g_α has the coefficients

$$q_2 = -2\alpha\gamma^2 + 2(3\alpha^2 - \bar{\alpha})\gamma\delta + (1 + 3\alpha\bar{\alpha} - 4\alpha^3)\delta^2,$$
$$q_1 = -(\alpha^2 + \bar{\alpha})\gamma^2 + 2(2\alpha^3 + \alpha\bar{\alpha} + 1)\gamma\delta + (\bar{\alpha}^2 - 2\alpha - 3\alpha^4)\delta^2$$

at t^2 and t, and a constant coefficient containing -1. So $(\gamma, \delta) \neq (0,0)$ is a common zero of the binary quadratic forms q_2 and q_1. Hence their resultant vanishes: $N(\alpha, \bar{\alpha}) = 0$, where

$$N(a,b) = 4a^7 - a^6b^2 + 6a^5b + 2a^4b^3 + 11a^4 + 8a^3b^2 - a^2b^4 + 6a^2b + 14ab^3 + 8a + 4b^5 + 3b^2.$$

Suppose that $\alpha = \rho e^{i\varphi} \neq 0$, so that $0 < \rho < 1$. Since each monomial $a^i b^j$ of N has $i - j \equiv 1 \pmod 3$, $N(\alpha, \bar{\alpha}) = \alpha E$ with

$$E = 4\rho^6 e^{6i\varphi} + \rho^3(11 + 6\rho^2 - \rho^4)e^{3i\varphi} + 2(\rho^6 + 4\rho^4 + 3\rho^2 + 4) + \rho(3 + 14\rho^2 - \rho^4)e^{-3i\varphi} + 4\rho^4 e^{-6i\varphi}.$$

Its imaginary part is $\operatorname{Im} E = \rho(\rho^2 - 1)\sin 3\varphi\,(8\rho^3\cos 3\varphi - \rho^4 + 6\rho^2 + 3)$. The last factor is at least $(1-\rho)(\rho^3 + 9\rho^2 + 3\rho + 3) > 0$. Thus $\sin 3\varphi = 0$. For $e^{3i\varphi} = 1$, $E = -\rho^7 + 6\rho^6 + 5\rho^5 + 12\rho^4 +$

$25\rho^3+6\rho^2+3\rho+8>0$. For $e^{3i\varphi}=-1$, $E=(1-\rho)^2(\rho^5+8\rho^4+10\rho^3+24\rho^2+13\rho+8)>0$. This contradicts $E=0$. Therefore $\alpha=0$. Then $g_0(t)=t^3-1$, the z_i are the cube roots of unity, and $\sum m_i z_i=0$ with real m_i forces $m_a=m_b=m_c$. □

The proof of Theorem 5.17 uses three small computations: the reduction of $t^3(\gamma+\delta t)^2-1$ modulo the cubic g_α, the resultant of the two binary quadratic forms q_2 and q_1, and the polar form of N.

Proposition 5.18. *A configuration of case* (ii) *of Theorem 5.14 with a bad shape and positive outer masses, as in that theorem, has equal outer masses and* $\{\hat z_i\}=\lambda\{1,\mathrm{j},\mathrm{j}^2\}$.

Proof. The outer masses are positive and the $\hat z_i$ distinct and nonzero. Lemma 5.16 gives $\sum m_i\hat z_i=\sum m_i\hat z_i^2=0$. The bad shape gives $\sum m_i x_i=0$ with $x_i^2\hat z_i=1$. Theorem 5.17 applies. □

Note that the bad shape is needed in Proposition 5.18: for all positive outer masses there are distinct nonzero z_i with $\sum m_i z_i=\sum m_i z_i^2=0$. So the conclusion of Lemma 5.16 alone does not exclude them.

Corollary 5.19. *For every* $m\in(\mathbb{R}_{>0})^5$, *no singular place of an irreducible curve* $C\subseteq Y_m$ *with* $U\not\equiv 0$ *on* C *has type* $\mathsf{b}_k(j)$. *Hence the pieces* $\mathsf{d15}(j,k)$ *may be removed from* pieces$(S(m))$ *in Theorem 3.12, for every* m, j *and* k.

Proof. By Theorem 5.4 the leading cluster has a bad shape. By Corollary 5.12 it is degenerate with a vector ξ as in Theorem 5.14, since $\kappa_3(\xi)$ is a nonzero multiple of $\sum_{i\in Q'} m_i x_i^3\xi_{u_i}$. Case (iii) is excluded by Proposition 5.15. In case (ii), Proposition 5.18 gives the configuration of case (i). Corollary 5.8, which applies as all masses are positive, excludes case (i). □

5.5. **Diagram 14.** Let P have type $\sigma_k(j)(t)$ with $0<t<1$, and put $\sigma=1+t\in(1,2)$. Exchanging the colours if necessary, we obtain from Lemma 3.7 that $e(u_i)=e(v_i)=1$ for $i\in Q'=Q_j\setminus k$, $e(u_j)=\sigma$ and $e(v_j)=-\sigma$ in the coordinates relative to $o=k$. We use the leading coefficients as in §5.2.

Lemma 5.20. *The cluster coefficients solve* K14(m_Q), $\sum_{i\in Q'} m_i\hat z_i=0$, *and* $\hat p_{kj}^3=M_Q/(m_kM)$.

Proof. The first statement is Theorem 3.13(a). For $i\in Q$ let $C_i^z=\sum_{l\in Q\setminus i} m_l z_{il}p_{il}^3$ and C_i^w likewise. The identity $p_{il}^3 z_{il}w_{il}=p_{il}$ gives

$$\sum_{i\in Q} m_i w_{ik}C_i^z=\sum_{\{i,l\}\subseteq Q} m_i m_l p_{il}=\sum_{i\in Q} m_i z_{ik}C_i^w.$$

By (2.1), $C_i^z=z_i-m_jp_{ij}^3z_{ij}$, whose terms have exponent at least 1. The left side therefore has exponent at least 2. Similarly $C_i^w=w_i-m_jp_{ij}^3w_{ij}$. Here $w_i=w_k+v_i$ has the leading term $-\frac{m_j}{M}\hat w_j$ at exponent $-\sigma$, the term $p_{ij}^3w_{ij}$ has exponent $\frac{\sigma-3}{2}>-\sigma$ for $i\neq k$, and $p_{kj}^3w_{kj}$ has exponent $-\sigma$ with coefficient $-\hat p_{kj}^3\hat w_j$. Thus C_i^w has the coefficient $m_j\hat w_j\big(-\frac{1}{M}+[i=k]\hat p_{kj}^3\big)$ at exponent $-\sigma$. Since $\sum_i m_iC_i^w=0$, we obtain $\hat p_{kj}^3=M_Q/(m_kM)$. On the right side the term $i=k$ vanishes. The rest has the coefficient $-\frac{m_j}{M}\hat w_j\sum_{i\in Q'} m_i\hat z_i$ at exponent $1-\sigma<2$, which therefore vanishes. □

We now show that the pieces $\mathsf{d14b}$ carry no singular place at positive masses.

Theorem 5.21. *For every* $m\in(\mathbb{R}_{>0})^5$, *no singular place of an irreducible curve* $C\subseteq Y_m$ *with* $U\not\equiv 0$ *on* C *has type* $\sigma_k(j)(t)$ *with* $0<t<1$, *for any body* j *and any* $k\in Q_j$. *Hence the pieces* $\mathsf{d14b}(j,k)$ *may be removed from* pieces$(S(m))$ *in Theorem 3.12, for all* m, j *and* k.

We keep the setting of Lemma 5.20, with $Q' = Q \setminus \{k\} = \{a, b, c\}$ and positive masses. For a configuration of Q with $z_k = 0$, the condition X3$_k$ is $\sum_{i\in Q'} m_i z_i = 0$: body k lies at the z-centre of mass of Q. Such a configuration *has a bad shape* if $\sum_{i\in Q'} m_i x_i = 0$ for some x_i with $x_i^2 z_i = 1$. If it also satisfies X3$_k$, then $\sum_{i\in Q'} m_i x_i^{-2} = 0$, so that (x_a, x_b, x_c) is a bad shape in the sense of §5.2. The proof of Theorem 5.21 has four steps. The leading cluster configuration satisfies X3$_k$ and has a bad shape (Lemma 5.23). At positive masses the only K14 configuration with these properties is, up to symmetries, an equilateral triangle with its centre (Theorem 5.30). This configuration is nondegenerate (Lemma 5.32). At a nondegenerate leading cluster configuration the expansion of the place forces $\sum_{i\in Q'} m_i x_i^3 = 0$ unless $\sigma = (2n+1)/(n+1)$ for an integer $n \geq 1$, and for these σ it forces a coefficient to vanish that is nonzero at the equilateral configuration (Propositions 5.34 and 5.35).

The bad shape. We fix a square root $(-\hat{w}_j)^{1/2}$ and put $x_i = \hat{p}_{ij}(-\hat{w}_j)^{1/2}$ for $i \in Q'$, as in §5.2. The leading term of the row D_{ij} is $\hat{p}_{ij}^2 \hat{z}_i(-\hat{w}_j) = 1$. Thus $x_i^2 \hat{z}_i = 1$. Near $\bar{y}$ let $w(y)$ and $x_i(y)$ be the branches of $(-y_{v_j})^{-1/2}$ and $y_{u_i}^{-1/2}$ with $w(\bar{y}) = (-\hat{w}_j)^{-1/2}$ and $x_i(\bar{y}) = x_i$, and put $S_r(y) = \sum_{i\in Q'} m_i x_i(y)^r$ and $\psi(y) = \sum_{i\in Q'} m_i y_{u_i}$.

Lemma 5.22. *At a place of type $\sigma_k(j)(t)$, with Π, f and U_Q as in Lemma 5.3,*

$$F(\tau, y) = \tau^{-2} U_Q(y) + \tau^{2-2\sigma}\Gamma(y) + f(\Pi) + \sum_{n\geq 0} \tau^{(2n+1)(\sigma-1)} B_n(y) + R(\tau, y), \tag{5.7}$$

where $\Gamma = -\frac{m_j}{2M} y_{v_j}\psi$, $B_n = \binom{2n}{n} 4^{-n} m_j w\, y_{u_j}^n S_{2n+1}$, and R is a convergent sum of terms $\tau^L R_L(y)$ with $L \geq 4$. The coordinate y_{u_j} occurs only in f, in the B_n with $n \geq 1$ and in R, and the coordinates y_{v_i}, $i \in Q'$, occur only in U_Q and in R.

Proof. The cluster terms and the term $m_k m_j p_{kj}$ are as in the proof of Lemma 5.3. For $i \in Q'$, $u_{ij} v_{ij} = -\tau^{2-2\sigma} y_{u_i} y_{v_j}(1 - \tau^{2\sigma-2} y_{u_j}/y_{u_i})(1 - \tau^{2\sigma+2} y_{v_i}/y_{v_j})$, so that

$$p_{ij} = \tau^{\sigma-1} w\, x_i(y)\big(1 - \tau^{2\sigma-2} y_{u_j}/y_{u_i}\big)^{-1/2}\big(1 - \tau^{2\sigma+2} y_{v_i}/y_{v_j}\big)^{-1/2}$$

with the branches of the p_{ij} at P. The binomial series of the first factor gives the B_n, since $y_{u_i}^{-n} = x_i(y)^{2n}$. The second factor contributes only exponents at least $3\sigma + 1 > 4$. Finally I contributes $-\frac{m_j}{M} y_{v_j}\psi$ at the exponent $2 - 2\sigma$, $\frac{m_j M_Q}{M}\Pi$ at 0, and otherwise only the exponents 4 and $2\sigma + 2$. □

Lemma 5.23. *At a place of type $\sigma_k(j)(t)$:*

(a) *$U_Q(y(\tau)) = O(\tau^6)$;*
(b) *$[B_0(y(\tau))]_r = 0$ for $0 \leq r < 2\sigma - 2$; in particular $\sum_{i\in Q'} m_i x_i = 0$, and the leading cluster configuration satisfies X3$_k$ and has a bad shape;*
(c) *$B_1(\bar{y}) = -\frac{3}{5}[B_0(y(\tau))]_{2\sigma-2}$.*

Proof. As in the proof of Lemma 5.1, $\nabla_y F(\tau, y(\tau)) = 0$ and $F(\tau, y(\tau))$ is constant. Let $Z = \sum_{i\in Q'} y_{u_i}\partial/\partial y_{u_i}$ and $W = \sum_{i\in Q'} y_{v_i}\partial/\partial y_{v_i}$. The function U_Q is homogeneous of degree $-\frac{1}{2}$ in the y_{u_i} and in the y_{v_i}, ψ is linear in the y_{u_i}, and S_{2n+1} is homogeneous of degree $-n - \frac{1}{2}$ in them. By Lemma 5.22, along $y(\tau)$,

$$0 = W \cdot \nabla_y F = -\tfrac{1}{2}\tau^{-2} U_Q + O(\tau^4),$$
$$0 = Z \cdot \nabla_y F = -\tfrac{1}{2}\tau^{-2} U_Q + \tau^{2-2\sigma}\Gamma - \textstyle\sum_n \big(n + \tfrac{1}{2}\big)\tau^{(2n+1)(\sigma-1)} B_n + O(\tau^4).$$

The first identity gives (a). The y_{u_j}-component of $\nabla_y F = 0$ reads $y_{v_j} f'(\Pi) + O(\tau^{3\sigma-3}) = 0$. As in the proof of Theorem 5.4, $f'(\bar{\Pi}) = 0 \neq f''(\bar{\Pi})$, so that $\Pi(\tau) - \bar{\Pi} = O(\tau^{3\sigma-3})$ and $[f(\Pi(\tau))]_q = 0$ for $0 < q < 6\sigma - 6$. Let $\sigma - 1 \leq q \leq 3\sigma - 3$. Then f, R and the B_n with $n \geq 2$ contribute

nothing at τ^q. By (a) neither does U_Q. So the coefficients of τ^q in the second identity and in $F(\tau, y(\tau))$ are

$$[\Gamma]_{q+2\sigma-2} - \tfrac{1}{2}[B_0]_{q-\sigma+1} - \tfrac{3}{2}[B_1]_{q-3\sigma+3} = 0, \qquad [\Gamma]_{q+2\sigma-2} + [B_0]_{q-\sigma+1} + [B_1]_{q-3\sigma+3} = 0,$$

and their difference gives $3[B_0]_{q-\sigma+1}+5[B_1]_{q-3\sigma+3} = 0$. For $q < 3\sigma-3$ the second term vanishes, which is (b). For $q = 3\sigma-3$ it is $5B_1(\bar{y})$, which is (c). Finally $B_0(\bar{y}) = m_j(-\hat{w}_j)^{-1/2}\sum_{i\in Q'} m_i x_i$. The condition $\mathrm{X3}_k$ holds by Lemma 5.20. □

K14 configurations with a bad shape. By Lemmas 5.20 and 5.23, the leading cluster configuration is a $\mathrm{K14}(m_Q)$ configuration with body k at the origin that satisfies $\mathrm{X3}_k$ and has a bad shape. We determine these configurations at positive masses.

We use the following description of the configurations of $\mathrm{K14}(m)$, $m \in (\mathbb{C}^*)^4$. A configuration is *collinear* if $w = \alpha + \beta z$ for some $\alpha, \beta \in \mathbb{C}$, and *non-collinear* otherwise. A non-collinear configuration has $p_{il} = \lambda_0^{1/3}\eta_{il}t_it_l$ with $\lambda_0 \neq 0$, cube roots of unity $\eta_{il} = \eta_{li}$ and $t \in (\mathbb{C}^*)^4$ solving

$$\sum_{l\neq i} m_l\eta_{il}t_l = \kappa t_i^2 \quad (i = 1, \dots, 4), \qquad \sum_l m_l t_l^3 = 0 \tag{5.8}$$

for some $\kappa \in \mathbb{C}$. Indeed, let A be the symmetric matrix with $A_{il} = m_i m_l p_{il}^3$ for $i \neq l$ and zero row sums. By (3.2), A annihilates $\Vdash$, z and w, which are linearly independent for a non-collinear configuration, since no z_{il} vanishes. As $A \neq 0$, $A = \lambda_0 vv^{\mathsf{T}}$ with $\lambda_0 \neq 0$ and $v\cdot\Vdash = v\cdot z = v\cdot w = 0$. Since $A_{il} \neq 0$, $v_i \neq 0$ for all i. We choose t_i with $t_i^3 = v_i/m_i$. Then $p_{il}^3 = \lambda_0 t_i^3 t_l^3$, which gives the η_{il}, and $\sum_l m_l t_l^3 = \sum_l v_l = 0$. Finally $\sum_{l\neq i} v_l z_{il} w_{il} = \sum_l v_l z_l w_l$ does not depend on i. With $z_{il}w_{il} = p_{il}^{-2} = \lambda_0^{-2/3}\eta_{il}t_i^{-2}t_l^{-2}$ this is the first equation of (5.8). A gauge change $t_l \mapsto \zeta_l t_l$, $\eta_{il} \mapsto \zeta_i^{-1}\zeta_l^{-1}\eta_{il}$, with cube roots of unity ζ_l, preserves (5.8) and all p_{il}, and multiplies the three products $\eta_{12}\eta_{34}$, $\eta_{13}\eta_{24}$, $\eta_{14}\eta_{23}$ by a common factor. The gauge changes act freely on the 3^6 patterns, since $\zeta_i\zeta_l = 1$ for all $i \neq l$ forces $\zeta_1 = \cdots = \zeta_4 = 1$. Thus there are $3^6/3^4 = 9$ classes. The three products, up to a common factor, take 9 values and each of them is attained. Hence two patterns are gauge equivalent if and only if their products are proportional. A pattern, and its class, is of *type* I, II or III if its three products take one, two or three distinct values. There is one class of type I and there are six of type II. The two classes of type III are those of the pattern $\eta_{12} = \eta_{34} = 1$, $\eta_{13} = \eta_{24} = \mathrm{j}^2$, $\eta_{14} = \eta_{23} = \mathrm{j}$ and of its complex conjugate.

Lemma 5.24. *Let $m \in (\mathbb{C}^*)^4$ have real ratios m_i/m_l. Then no non-collinear configuration of* $\mathrm{K14}(m)$ *has a pattern of type III.*

Proof. Suppose that one has. We diagonalize the matrix of the pattern and show that $\kappa \neq 0$ forces the configuration to be collinear and that $\kappa = 0$ is impossible for real mass ratios. After a gauge change, η is the pattern above or its complex conjugate. We treat the first. For the second, replace j by j^2 throughout. Let $H = (\eta_{il})$, with zero diagonal, $u = (m_l t_l)_l$, $e_0 = (1,1,1,1)$, $e_1 = (1,1,-1,-1)$, $e_2 = (1,-1,1,-1)$ and $e_3 = (1,-1,-1,1)$. Then (5.8) reads $Hu = \kappa(t_l^2)_l$ and $\sum_l u_l t_l^2 = 0$. We have $He_0 = 0$, $He_1 = 2e_1$, $He_2 = 2\mathrm{j}^2 e_2$, $He_3 = 2\mathrm{j}e_3$. Hence the kernel of H is $\mathbb{C}e_0$ and its image, spanned by e_1, e_2, e_3, consists of the vectors with coordinate sum 0.

Suppose first that $\kappa \neq 0$. Then $\sum_l t_l^2 = 0$. We write $(t_l^2)_l = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$. We obtain $u = \frac{\kappa}{2}(\xi_1 e_1 + \mathrm{j}\xi_2 e_2 + \mathrm{j}^2\xi_3 e_3) + \gamma e_0$ for some $\gamma \in \mathbb{C}$. Since $e_k \cdot e_{k'} = 4\delta_{kk'}$, the last equation of (5.8) becomes $2\kappa(\xi_1^2 + \mathrm{j}\xi_2^2 + \mathrm{j}^2\xi_3^2) = 0$, so that $\xi_1^2 + \mathrm{j}\xi_2^2 + \mathrm{j}^2\xi_3^2 = 0$. For three bodies a, b, c put

$$h_{abc} = (\eta_{ab}t_c^2 + \eta_{ac}t_b^2 - \eta_{bc}t_a^2)^2 - 4\eta_{ab}\eta_{ac}t_b^2 t_c^2,$$

which is symmetric in a, b, c. For $(a, b, c) = (1, 2, 3)$ we have $\eta_{12}t_3^2 + \eta_{13}t_2^2 - \eta_{23}t_1^2 = 2\mathrm{j}^2\xi_1 + 2\xi_2$ and $4\eta_{12}\eta_{13}t_2^2t_3^2 = 4\mathrm{j}^2\big(\xi_3^2 - (\xi_1 - \xi_2)^2\big)$. Expanding gives

$$h_{123} = -4\big(\xi_1^2 + \mathrm{j}\xi_2^2 + \mathrm{j}^2\xi_3^2\big) = 0.$$

The double transpositions of the bodies fix η, change each ξ_k at most in sign and permute the four sets $\{a,b,c\}$ transitively. Hence $h_{abc}=0$ for any three bodies a,b,c. On the other hand $d_{il}=z_{il}w_{il}=p_{il}^{-2}=\lambda_0^{-2/3}\eta_{il}t_i^{-2}t_l^{-2}$ (since $\eta_{il}^{-2}=\eta_{il}$) and $d_{ab}+d_{ac}-d_{bc}=z_{ab}w_{ac}+z_{ac}w_{ab}$, so that

$$h_{abc}=\lambda_0^{4/3}(t_at_bt_c)^4\big((d_{ab}+d_{ac}-d_{bc})^2-4d_{ab}d_{ac}\big)=\lambda_0^{4/3}(t_at_bt_c)^4(z_{ab}w_{ac}-z_{ac}w_{ab})^2.$$

For a real configuration, $w=\bar z$, the factor $(z_{ab}w_{ac}-z_{ac}w_{ab})^2$ is -16 times the squared area of the triangle abc. For such a configuration this identity is Heron's formula. Since $h_{abc}=0$, $\lambda_0\neq 0$ and $t\in(\mathbb{C}^*)^4$, we obtain $z_{ab}w_{ac}=z_{ac}w_{ab}$ for any three bodies a,b,c. With $a=1$ and $b=2$ this gives $w_{1c}=\beta z_{1c}$ for all c, where $\beta=w_{12}/z_{12}$, that is, $w=\alpha+\beta z$ with $\alpha=w_1-\beta z_1$. The configuration is collinear, a contradiction.

Hence $\kappa=0$. Then $u\in\ker H$, so that $t_l=\gamma/m_l$ with $\gamma\neq 0$. The last equation of (5.8) becomes $\gamma^3\sum_l m_l^{-2}=0$. But $\sum_l m_l^{-2}=m_1^{-2}\sum_l(m_1/m_l)^2$, where the last sum is a sum of squares of nonzero real numbers and hence positive, a contradiction. □

For the other types we invert the configuration at body k. Let (z,w,p) be a non-collinear K14(m_Q) configuration with $m\in(\mathbb{C}^*)^4$ and $z_k=w_k=0$. We put $d_{il}=z_{il}w_{il}$ and, for $i\in Q'$, $z_i'=1/w_i$ and $w_i'=1/z_i$. We index the outer bodies cyclically, $a+1=b$, $b+1=c$, $c+1=a$, and put $u_i'=z_{i+1}'w_{i+2}'-z_{i+2}'w_{i+1}'$.

Lemma 5.25. (a) *The numbers $(z_a'-z_b')(w_a'-w_b')$, $(z_b'-z_c')(w_b'-w_c')$ and $(z_c'-z_a')(w_c'-w_a')$ are a common nonzero factor times the products $\eta_{ab}\eta_{ck}$, $\eta_{bc}\eta_{ak}$ and $\eta_{ca}\eta_{bk}$ of the pattern.*
(b) *There is $C\neq 0$ with $m_i=C\,u_i'/p_{ik}$ for $i\in Q'$.*

Proof. (a) For $\{i,l,n\}=Q'$, $(z_i'-z_l')(w_i'-w_l')=d_{il}/(d_{ik}d_{lk})=d_{il}d_{nk}/(d_{ak}d_{bk}d_{ck})$, and by the derivation of (5.8), $d_{il}d_{nk}=\lambda_0^{-4/3}(t_at_bt_ct_k)^{-2}\eta_{il}\eta_{nk}$. (b) In that derivation $m_im_kp_{ik}^3=\lambda_0v_iv_k$, and $v\perp z,w$ with $z_k=w_k=0$. The vectors (z_a,z_b,z_c) and (w_a,w_b,w_c) are independent, since $w=\beta z$ on Q would make the configuration collinear. Hence (v_a,v_b,v_c) is proportional to their cross product. Its component i is $u_i'/(z_{i+1}'z_{i+2}'w_{i+1}'w_{i+2}')$. As $p_{ik}^{-2}=z_iw_i=1/(z_i'w_i')$, $m_i=\lambda_0v_kv_i/(m_kp_{ik}^3)$ is a common factor times u_i'/p_{ik}. □

The next theorem treats type I, in which the triangle w' is equilateral (see the proof of Lemma 5.27): $\mathrm{X3}_k$ and a bad shape force its centre to be the origin.

Theorem 5.26. *Let $m_0,m_1,m_2>0$, $\delta\in\mathbb{C}$, and let $x_0,x_1,x_2\in\mathbb{C}^*$ satisfy $x_i^2=\delta+\mathrm{j}^i$, $\sum_i m_ix_i=0$ and $\sum_i m_ix_i^{-2}=0$. Then $\delta=0$ and $m_0=m_1=m_2$.*

Proof. We show that the hypotheses force $\delta=0$ or the vanishing of an explicit polynomial in δ and $\bar\delta$, and exclude the second case with $\delta\neq 0$ by writing δ in polar form and using the positivity of the masses. We may assume $\sum_i m_i=1$. We put $\nu=\sum_i m_i\mathrm{j}^i$ and $\rho=|\delta|$. We use the basis $\mathbb{1}=(1,1,1)$, $\Omega=(\mathrm{j}^i)_i$, $\bar\Omega=(\mathrm{j}^{2i})_i$ of $\mathbb{C}^3$. Since $x_i\neq 0$, $\delta\neq-\mathrm{j}^i$, that is, $\delta^3\neq-1$.

(1) From $(\delta+\mathrm{j}^i)(\delta^2-\delta\mathrm{j}^i+\mathrm{j}^{2i})=\delta^3+1$ we obtain $z:=(x_i^{-2})_i=(\delta^2\mathbb{1}-\delta\Omega+\bar\Omega)/(\delta^3+1)$. So $\sum m_ix_i^{-2}=0$ reads $\delta^2-\nu\delta+\bar\nu=0$. With its conjugate this gives $\nu(1-\rho^2)=-(\delta\rho^2+\bar\delta^2)$. If $\rho=1$, then $\delta=-\bar\delta^2$ and $\delta^3=-\delta^2\bar\delta^2=-1$, which is excluded. Hence $\rho\neq 1$.
(2) The plane $V=\{v:\sum_i m_iv_i=0\}$ contains z, $\bar z$ and x. As the m_i are real, $\bar V=V$. In the basis $(\mathbb{1},\Omega,\bar\Omega)$ the coordinates of z and $\bar z$ are proportional to $(\delta^2,-\delta,1)$ and $(\bar\delta^2,1,-\bar\delta)$, whose cross product $n=(\rho^2-1,\bar\delta^2+\delta^2\bar\delta,\delta^2+\delta\bar\delta^2)$ has $n_0\neq 0$. So z and $\bar z$ span V and the coordinates $(\hat x_0,\hat x_1,\hat x_2)$ of x satisfy $n_0\hat x_0+n_1\hat x_1+n_2\hat x_2=0$.
(3) Since $\Omega\Omega=\bar\Omega$, $\bar\Omega\bar\Omega=\Omega$ and $\Omega\bar\Omega=\mathbb{1}$ componentwise, $x_i^2=\delta+\mathrm{j}^i$ reads $\hat x_0^2+2\hat x_1\hat x_2=\delta$, $2\hat x_0\hat x_1+\hat x_2^2=1$ and $2\hat x_0\hat x_2+\hat x_1^2=0$. We eliminate $\hat x_0$ by (2). Then n_0 times the third equation and n_0^2 times the difference of δ times the second and the first are two

binary quadratic forms in $(\hat{x}_1, \hat{x}_2)$ with a common zero $(\hat{x}_1, \hat{x}_2) \neq (0,0)$, as $x \neq 0$. Their resultant, $-3\delta(\rho^2-1)^2 N(\delta,\bar{\delta})$ (†), therefore vanishes, where $N \in \mathbb{Z}[\delta,\bar{\delta}]$ has 23 terms:

$$\begin{aligned} N = {} & 4\delta^8\bar{\delta}^4 + 4\delta^8\bar{\delta} + 24\delta^7\bar{\delta}^3 - 3\delta^7 + 40\delta^6\bar{\delta}^5 - 14\delta^6\bar{\delta}^2 + 4\delta^5\bar{\delta}^7 + 19\delta^5\bar{\delta}^4 \\ & - 12\delta^5\bar{\delta} + 54\delta^4\bar{\delta}^6 - 48\delta^4\bar{\delta}^3 + 6\delta^4 + \delta^3\bar{\delta}^8 - 20\delta^3\bar{\delta}^5 + 6\delta^3\bar{\delta}^2 + 28\delta^2\bar{\delta}^7 \\ & - 18\delta^2\bar{\delta}^4 + 8\delta^2\bar{\delta} - 12\delta\bar{\delta}^6 + 12\delta\bar{\delta}^3 - 3\delta + 4\bar{\delta}^8 - 4\bar{\delta}^2. \end{aligned}$$

Hence $\delta = 0$ or $N(\delta,\bar{\delta}) = 0$.

(4) Suppose that $\delta = \rho e^{i\varphi} \neq 0$ and $N(\delta,\bar{\delta}) = 0$. Since each monomial $\delta^a\bar{\delta}^b$ of N has $a - b \equiv 1 \pmod 3$, $N = e^{i\varphi}E$ with E a polynomial in ρ and $e^{\pm 3i\varphi}$ with real coefficients. With $c = \cos 3\varphi$, $\operatorname{Re} E$ is a polynomial in ρ and c,

$$\begin{aligned} \operatorname{Re} E = {} & 16\rho^8c^3 + (2\rho^{11} + 64\rho^9 - 30\rho^7)c^2 + (8\rho^{12} + 78\rho^{10} - 46\rho^8 - 30\rho^6 + 18\rho^4 - 4\rho^2)c \\ & + 39\rho^{11} - 13\rho^9 - 33\rho^7 + 6\rho^5 + 8\rho^3 - 3\rho, \end{aligned}$$

and

$$\operatorname{Im} E = -2\rho^2 \sin 3\varphi\, K, \qquad K = 8\rho^6c^2 + (\rho^9 + 24\rho^7 - 9\rho^5)c + 15\rho^8 - 5\rho^6 - 3\rho^4 + 3\rho^2 - 2 \quad (\dagger),$$

$$\operatorname{Res}_c(\operatorname{Re} E, K) = 7776\rho^{24}(\rho-1)^6(\rho+1)^6(4\rho^8 + 13\rho^6 + 27\rho^4 + 26\rho^2 + 11) \quad (\dagger).$$

The resultant is nonzero and K has the leading coefficient $8\rho^6 \neq 0$. Hence $\operatorname{Re} E$ and K have no common zero, and $\sin 3\varphi = 0$. If $e^{3i\varphi} = -1$, then $\operatorname{Re} E = -\rho(\rho-1)^4 S_-(\rho)$ (†) with $S_- = \rho^4((\rho-1)^2(8\rho+7)+4) + 6\rho^3 + 6\rho^2 + 8\rho + 3 > 0$. If $e^{3i\varphi} = 1$, then $\delta = \rho \mathrm{j}^l$ for some l, and replacing x_i and m_i by $\mathrm{j}^l x_{i+l}$ and m_{i+l} we may take $\delta = \rho$. By (1), $\nu = -\rho^2/(1-\rho)$ is real. Thus $m_1 = m_2$, $3m_0 = 1 + 2\nu$ and $3m_1 = 1 - \nu$. For $\rho > 1$ this gives $3m_1(\rho-1) = -(\rho^2 - \rho + 1) < 0$, which contradicts $m_1 > 0$. For $\rho < 1$ it gives $3m_0(1-\rho) = (1-2\rho)(1+\rho)$, so that $\rho < \frac{1}{2}$. Then $\operatorname{Re} E = \rho(\rho+1)^4 S_+(\rho)$ (†) with $S_+ = (-3 + 8\rho - 6\rho^2) + \rho^3(6 - 11\rho) + \rho^5(-6 + 9\rho + 8\rho^2) \leq -\frac{1}{2} + \frac{2187}{21296} + \frac{1}{64} < 0$: on $[0, \frac{1}{2}]$ the first summand is increasing, the second is at most its value $\frac{2187}{21296}$ at $\rho = \frac{9}{22}$, and $-6 + 9\rho + 8\rho^2 \leq \frac{1}{2}$. In the remaining cases $E \neq 0$, a contradiction.

(5) Hence $\delta = 0$. Then $\bar{\nu} = 0$ by (1), and $m_0 = m_1 = m_2$.

□

Lemma 5.27. *Let $m_a, m_b, m_c > 0$ and $m_k \neq 0$, and let (z, w, p) be a non-collinear* K14(m_Q) *configuration of type I with $z_k = w_k = 0$ that satisfies* X3$_k$ *and has a bad shape. Then $m_a = m_b = m_c$, $(m_k/m_a)^2 = \frac{1}{3}$, and after a relabelling of a, b, c, $z = s(1, \mathrm{j}, \mathrm{j}^2)$ and $w = s'(1, \mathrm{j}^2, \mathrm{j})$ with $s, s' \in \mathbb{C}^*$.*

Proof. By Lemma 5.25(a), $(z_i' - z_l')(w_i' - w_l') = \Lambda \neq 0$ for the three pairs of Q'. Let $A = w_a' - w_b'$, $B = w_b' - w_c'$ and $C = w_c' - w_a'$. The corresponding differences of the z_i' are Λ/A, Λ/B and Λ/C. As their sum is 0, $AB + BC + CA = 0$. With $A + B + C = 0$, the numbers A, B, C are the three roots of $X^3 = ABC$. So w' is an equilateral triangle and z' is one of the opposite orientation. Scaling z and w, which preserves K14, X3$_k$ and the bad shape, and labelling the outer bodies $0, 1, 2$, we obtain $w_i' = \delta + \mathrm{j}^i$ and $z_i' = \gamma + \mathrm{j}^{-i}$ with $\gamma, \delta \in \mathbb{C}$. Then $x_i^2 = 1/z_i = \delta + \mathrm{j}^i$. The condition X3$_k$ is $\sum m_i x_i^{-2} = 0$. Theorem 5.26 gives $\delta = 0$ and $m_0 = m_1 = m_2$. Now $u_i' = (\mathrm{j}^2 - \mathrm{j})(\gamma\mathrm{j}^i - 1)$ and $p_{ik}^2 = z_i' w_i' = \gamma\mathrm{j}^i + 1$. By Lemma 5.25(b), the three numbers $(\gamma\mathrm{j}^i - 1)^2/(\gamma\mathrm{j}^i + 1)$ equal one number $\varkappa$. If $\gamma \neq 0$, the three distinct numbers $\gamma\mathrm{j}^i$ are roots of the quadratic $(X-1)^2 - \varkappa(X+1)$, which is impossible. Hence $\gamma = 0$, $z = (1, \mathrm{j}^2, \mathrm{j})$ and $w = (1, \mathrm{j}, \mathrm{j}^2)$. The exchange of b and c gives the stated form. Finally, $m_a m_b p_{ab}^3 = \lambda_0 v_a v_b$ and $m_b m_k p_{bk}^3 = \lambda_0 v_b v_k$ give $(m_k/m_a)^2 = (v_k/v_a)^2 (d_{bk}/d_{ab})^3$. Here $v_a = v_b = v_c = -v_k/3$ by the cross product in the proof of Lemma 5.25, $d_{bk} = 1$ and $d_{ab} = 3$, so that $(m_k/m_a)^2 = \frac{1}{3}$. □

Theorem 5.28. *Let $m_a, m_b, m_c, m_k \in \mathbb{C}^*$ with $m_b/m_a, m_c/m_a \in \mathbb{R}$. No non-collinear* K14$(m_Q)$ *configuration of type II with $z_k = w_k = 0$ satisfies* X3$_k$ *and has a bad shape.*

Proof. We parametrize the inverted configuration by three unknowns, derive two polynomial equations in them from the bad shape and X3$_k$, solve the resulting system exactly over $\mathbb{Q}$, and show in ball arithmetic that at each of its solutions one of the two squared ratios of the outer masses is not positive. We label the outer bodies $0, 1, 2$. By Lemma 5.25(a), $(z'_i - z'_l)(w'_i - w'_l) = \theta_{il}\Lambda$, where $\Lambda \neq 0$ and $\theta_{01}, \theta_{12}, \theta_{20}$ are cube roots of unity, exactly two of them equal. After a relabelling, which preserves K14, X3$_k$ and the bad shape, and a change of Λ, $(\theta_{01}, \theta_{12}, \theta_{20}) = (1, 1, \varpi)$ with $\varpi \in \{\mathrm{j}, \mathrm{j}^2\}$. Let $A = w'_0 - w'_1$ and $B = w'_1 - w'_2$. Then $z'_0 - z'_1 = \Lambda/A$, $z'_1 - z'_2 = \Lambda/B$ and $z'_2 - z'_0 = -\varpi\Lambda/(A + B)$. As their sum is 0, $A^2 + (2 - \varpi)AB + B^2 = 0$ (†). So $t = A/B$ satisfies $t^2 + (2 - \varpi)t + 1 = 0$, that is, $\varpi = (t + 1)^2/t$. The condition $\varpi^2 + \varpi + 1 = 0$ becomes

$$f(t) = t^4 + 5t^3 + 9t^2 + 5t + 1 = 0 \quad (\dagger).$$

Scaling z and w we take $B = \Lambda = 1$. Then $w' = (\delta, \delta - t, \delta - t - 1)$ and $z' = (\gamma, \gamma - 1/t, \gamma - 1/t - 1)$ with $\gamma, \delta \in \mathbb{C}$. The u'_i are affine in (γ, δ) (†). With $\xi_i = p_{ik}/x_i$ we have $\xi_i^2 = z'_i$, as $x_i^2 = w'_i$. Lemma 5.25(b) gives $m_i = C\, u'_i/(x_i\xi_i)$. So the bad shape and X3$_k$ read

$$\textstyle\sum_i u'_i/\xi_i = 0, \qquad \sum_i u'_i/(x_i^3\xi_i) = 0.$$

For $r \in \mathbb{C}^3$, some choice of signs gives $r_0 \pm r_1 \pm r_2 = 0$ if and only if $H(r_0^2, r_1^2, r_2^2) = 0$, where $H(h) = \sum_i h_i^2 - 2\sum_{i<l} h_i h_l$, because $H(r_0^2, r_1^2, r_2^2)$ is the product of the four sums $r_0 \pm r_1 \pm r_2$ (†). Hence $H(h) = H(h') = 0$ for $h_i = u_i'^2/z'_i$ and $h'_i = u_i'^2/(w_i'^3 z'_i)$. Clearing denominators, we obtain the polynomials $H_1 = t^8(z'_0 z'_1 z'_2)^2 H(h)$ and $H_2 = t^8(w_0'^3 w_1'^3 w_2'^3 z'_0 z'_1 z'_2)^2 H(h')$ in $\mathbb{Q}[t, \gamma, \delta]$ (†). Let $S = t\prod_i w'_i \prod_i (tz'_i) \prod_i (tu'_i)$. It does not vanish at a configuration: $t \neq 0$ as $f(0) \neq 0$, the z'_i and w'_i are finite and nonzero, and $u'_i \neq 0$ as $m_i \neq 0$. If the outer masses have real ratios, then

$$R_i = (m_i/m_0)^2 = \frac{u_i'^2 w'_0 z'_0}{u_0'^2 w'_i z'_i} \in (0, \infty) \qquad (i = 1, 2).$$

It therefore suffices to show that no solution (t, γ, δ, s) of

$$f(t) = H_1 = H_2 = sS - 1 = 0 \tag{5.9}$$

has $R_1, R_2 \in (0, \infty)$.

We solve (5.9) exactly over $\mathbb{Q}$. A rational parametrization by the coordinate δ, with a square-free eliminant of degree 280 and a denominator that is invertible modulo the eliminant, satisfies the four equations exactly modulo the eliminant. So (5.9) has at least 280 solutions, with distinct values of δ. To show that it has no others we use a second system E, in the unknowns t, γ, δ and $e_i = x_{i+1}x_{i+2}$, $f_i = t\xi_{i+1}\xi_{i+2}$ (indices modulo 3). It consists of $f(t)$, the relations $e_i^2 = w'_{i+1}w'_{i+2}$, $e_i e_l = w'_n e_n$, $f_i^2 = (tz'_{i+1})(tz'_{i+2})$ and $f_i f_l = (tz'_n) f_n$ for $\{i, l, n\} = \{0, 1, 2\}$, and the two conditions in the form $\sum_i (tu'_i) f_i = 0$ and $\sum_i (tu'_i) e_i^3 f_i = 0$. Each solution of (5.9) gives a point of $V(E)$ with the same (t, γ, δ), at which $S \neq 0$: we choose the signs of the ξ_i so that the first sum vanishes, and then those of the x_i for the second. A standard basis of E over $\mathbb{Q}$ shows that E has dimension 0 and that the quotient ring of the polynomial ring by E has dimension 472 over $\mathbb{Q}$. Multiplication by S on this quotient ring has rank 280 over $\mathbb{Q}$. Over $\overline{\mathbb{Q}}$, the quotient ring is the product of the local rings at the points of $V(E)$. On the local ring at a point where $S \neq 0$ multiplication by S is invertible. So the rank is at least the total multiplicity of the points of $V(E)$ at which $S \neq 0$, there are at most 280 such points, and the 280 solutions are all. In ball arithmetic at 800 bits, with disjoint balls around the roots of the eliminant, at each solution $|\operatorname{Im} R_i| \geq 0.0469$ or $\operatorname{Re} R_i \leq -0.0469$ for $i = 1$ or $i = 2$. The programs and their logs are in the supplementary material. □

Theorem 5.29. *Let $m_a, m_b, m_c, m_k > 0$. No collinear* K14(m_Q) *configuration with $z_k = 0$ satisfies* X3$_k$ *and has a bad shape.*

Proof. Let $w = \alpha + \beta z$. Then $\beta \neq 0$, as no w_{il} vanishes, $w_{il} = \beta z_{il}$, and $p_{il} = s_{il}\beta^{-1/2}/z_{il}$ with $s_{il} = \pm 1$ and $s_{li} = -s_{il}$. So $F_i^w = \beta F_i^z$ and $F_i^z = \beta^{-3/2}\sum_{l\neq i} m_l s_{il}/z_{il}^2$. Thus K14 holds if and only if

$$(C_i) \qquad \textstyle\sum_{l\neq i} m_l s_{il}/z_{il}^2 = 0 \qquad (i \in Q).$$

We write $z_i = x_i^{-2}$ for $i \in Q'$, with the x_i of the bad shape, $\chi_i = x_i^3$ and $\epsilon_i = s_{ki}$, and m, x, x^{-2} for the vectors (m_i), (x_i), (x_i^{-2}), $i \in Q'$. Then X3$_k$ and the bad shape say $m \cdot x^{-2} = m \cdot x = 0$. The row (C_k) says $\sum_{i\in Q'} m_i\epsilon_i x_i^4 = 0$.

Suppose first that x and x^{-2} are dependent. Then $\chi_a = \chi_b = \chi_c$, that is, $x_i = \zeta_i x_a$ with $\zeta_i^3 = 1$. As $m_i > 0$, the relation $\sum m_i\zeta_i = 0$ forces $\{\zeta_a, \zeta_b, \zeta_c\} = \{1, \mathrm{j}, \mathrm{j}^2\}$ and $m_a = m_b = m_c =: \mu$. Now (C_k) reads $\mu x_a^4 \sum \epsilon_i\zeta_i = 0$, which forces $\epsilon_a = \epsilon_b = \epsilon_c$. After a scaling of z and a relabelling of b and c, $z = (1, \mathrm{j}, \mathrm{j}^2)$. As $(1-\mathrm{j})^2 = -3\mathrm{j}$ and $(1-\mathrm{j}^2)^2 = -3\mathrm{j}^2$, the row (C_a) reads $m_k s_{ak} = \frac{\mu}{3}(s_{ab}\mathrm{j}^2 + s_{ac}\mathrm{j})$. Since m_k is real, $s_{ab} = s_{ac}$ and $m_k = -s_{ak}s_{ab}\mu/3$. In the same way (C_b) and (C_c) give $s_{ba} = s_{bc}$, $m_k = -s_{bk}s_{bc}\mu/3$, and $s_{cb} = s_{ca}$, $m_k = -s_{ck}s_{ca}\mu/3$. As $s_{ik} = -\epsilon_a$ for all $i \in Q'$, $m_k > 0$ forces $s_{ab} = s_{bc} = s_{ca} = \epsilon_a$. Then $s_{ac} = s_{ab} = \epsilon_a$ contradicts $s_{ca} = \epsilon_a$.

Hence x and x^{-2} are independent and m is proportional to their cross product, that is, to $(x_a^2(\chi_b - \chi_c), x_b^2(\chi_c - \chi_a), x_c^2(\chi_a - \chi_b))$. In particular the χ_i are distinct. Now (C_k) says that the rows $(x_i, x_i^{-2}, \epsilon_i x_i^4)$, $i \in Q'$, have determinant 0, that is, after multiplying the rows by x_i^2, $\det(\chi_i, 1, \epsilon_i\chi_i^2)_{i\in Q'} = 0$. If $\epsilon_a = \epsilon_b = \epsilon_c$, this is $\pm$ the Vandermonde determinant of the χ_i (†), which is nonzero. Otherwise, after the reversal $s \mapsto -s$, which preserves the rows (C_i), and a relabelling, $\epsilon = (1, 1, -1)$. The determinant is then $(\chi_b - \chi_a)(2\chi_c^2 - (\chi_a - \chi_c)(\chi_b - \chi_c))$ (†). So $(\chi_a - \chi_c)(\chi_b - \chi_c) = 2\chi_c^2$. We divide x by x_c, which preserves all conditions and the mass ratios, and put $A = \chi_a - 1$ and $B = \chi_b - 1$. Then $AB = 2$, $A \notin \{0, -1, -2, \pm\sqrt{2}\}$ since the χ_i are nonzero and distinct, and

$$m_a/m_c = 2x_a^2/(A^2-2), \qquad m_b/m_c = -x_b^2A^2/(A^2-2).$$

As both are positive, $(x_bA/x_a)^2 < 0$, $x_b = i\gamma x_a/A$ with $\gamma \in \mathbb{R}\setminus\{0\}$, and $x_a^2 = \rho(A^2-2)$ with $\rho > 0$. Taking cubes, with $\chi_a = 1 + A$ and $\chi_b = (A+2)/A$, we obtain

$$\text{(E1)} \quad \gamma^3 = iA^2(A+2)/(A+1), \qquad \text{(E2)} \quad (A+1)^2/(A^2-2)^3 = \rho^3 > 0.$$

Let $A = u + iv$ with u, v real. Then (E1) gives $P_1 = 0$ and (E2) gives $P_2 = 0$, where

$$P_1 = \mathrm{Re}\left(A^2(A+2)(\bar{A}+1)\right) = u^4+3u^3+2u^2-uv^2-v^4-2v^2 \quad (\dagger), \qquad P_2 = \mathrm{Im}\left((A+1)^2(\bar{A}^2-2)^3\right).$$

Their resultant is $\mathrm{Res}_v(P_1, P_2) = 4096u^8(u+1)^3(u+2)h(u)^2$ (†) with $h = 2u^3 + 4u^2 - u - 4$. The polynomial P_1 has the leading coefficient -1 in v. So $u \in \{0, -1, -2\}$ or $h(u) = 0$. For $u = 0, -1, -2$, $P_1 = -v^2(v^2+2), -v^2(v^2+1), -v^4$, so $v = 0$ and A is excluded. The cubic h has discriminant $-104 < 0$ and hence a single real root u_0, and $h(0.917) < 0 < h(0.918)$, so that $u_0 \in (0.917, 0.918)$. At $u = u_0$, $P_1 = 0$ is a quadratic equation in v^2 whose roots have the product $-u_0^2(u_0+1)(u_0+2) < 0$. Its positive root is $v_0^2 = 2 - u_0^2$, because $P_1 = 2h(u)$ on the circle $u^2 + v^2 = 2$ (†). Thus A is $A_0 = u_0 + iv_0$ or $\bar{A}_0$. If $A = A_0$, then (E1) and (E2) determine $\gamma \in \mathbb{R}$ and $\rho > 0$, hence x_a up to sign, $x_b = i\gamma x_a/A_0$, the positions $z_i = x_i^{-2}$ and the outer mass ratios. Since m_k has the coefficient $s_{ak}/z_a^2 \neq 0$ in the row (C_a), this row determines m_k/m_a for each choice of s_{ab}, s_{ac}. Certified in ball arithmetic, the four values are $\pm(0.0059 + 0.4768i)$ and $\pm(0.1808 + 0.6951i)$, none of them real. At $\bar{A}_0$ they are the conjugates. This contradicts $m_k > 0$. □

Theorem 5.30. *Let $m_a, m_b, m_c, m_k > 0$, and let (z, w, p) be a* K14(m_Q) *configuration with $z_k = w_k = 0$ that satisfies* X3$_k$ *and has a bad shape. Then (m_a, m_b, m_c, m_k) is a positive*

multiple of $(\sqrt{3},\sqrt{3},\sqrt{3},1)$*, and after a relabelling of* a,b,c*,* $(z,w,p)=(sz_{\mathcal{E}},s'w_{\mathcal{E}},\zeta p_{\mathcal{E}})$ *with* $s,s'\in\mathbb{C}^*$ *and* $\zeta^2ss'=1$*. Here* $z_{\mathcal{E}}=(1,\mathrm{j},\mathrm{j}^2)$ *and* $w_{\mathcal{E}}=(1,\mathrm{j}^2,\mathrm{j})$ *on* (a,b,c)*, and* $p_{\mathcal{E}}$ *is* $1/\sqrt{3}$ *on the pairs of* Q' *and* -1 *on the pairs* ik*.*

Proof. Theorem 5.29 excludes collinear configurations. A non-collinear one has a pattern as in (5.8). Lemma 5.24 and Theorem 5.28 exclude the types III and II. For type I, Lemma 5.27 gives the masses and, after a relabelling and a scaling of z and w, $z=z_{\mathcal{E}}$ and $w=w_{\mathcal{E}}$. We scale the masses to $(\sqrt{3},\sqrt{3},\sqrt{3},1)$. Then $d_{il}=3$ on the pairs of Q' and $d_{ik}=1$, so that $p_{il}=e_{il}/\sqrt{3}$ and $p_{ik}=f_i$ with $e_{il},f_i\in\{\pm1\}$. The row $F^z_a=0$ reads $(e_{ab}(1-\mathrm{j})+e_{ac}(1-\mathrm{j}^2))/3+f_a=0$. Its imaginary part gives $e_{ab}=e_{ac}$, and its real part $f_a=-e_{ab}$. The rows of b and c, divided by j and j^2, give the same relations, so that $p=\pm p_{\mathcal{E}}$. Conversely, $\pm p_{\mathcal{E}}$ solve the rows $F^z_i=0$ for $i\in Q'$ and therefore also $F^z_k=0$. The rows $F^w=0$ are their complex conjugates. □

We write $\mathcal{E}$ for the configuration $(z_{\mathcal{E}},w_{\mathcal{E}},p_{\mathcal{E}})$ with the masses $(\sqrt{3},\sqrt{3},\sqrt{3},1)$. It is the equilateral triangle $1,\mathrm{j},\mathrm{j}^2$ with body k at its centre.

Nondegeneracy and the slice expansion. Let (z,w,p) be a K14(m_Q) configuration with body k at the origin and $m\in(\mathbb{C}^*)^4$. Since $\partial U_Q/\partial u_i=-\frac12 m_iF^w_i$ and $\partial U_Q/\partial v_i=-\frac12 m_iF^z_i$ (§5.3), it is a critical point of U_Q in the six coordinates $(u_i,v_i)_{i\in Q'}$. Let H be the Hessian of U_Q there, $Z=(z,0)$ and $W=(0,w)$. As U_Q is homogeneous of degree $-\frac12$ in u and in v separately, $HZ=HW=0$. We call a K14 configuration *nondegenerate* if $\ker H$ is spanned by Z and W. For $a\in Q'$ let $\mathcal{S}_a=\{\xi\in\mathbb{C}^6:\xi_{u_a}=\xi_{v_a}=0\}$. The components of a vector along $\mathcal{S}_a$ are its four components other than those of u_a and v_a. The matrix H_a is the block of H on $\mathcal{S}_a$. Let $g=\nabla\psi=(m_a,m_b,m_c,0,0,0)$.

Lemma 5.31. *Let the configuration be nondegenerate and satisfy* X3$_k$*, and let* $a\in Q'$*.*

(a) H_a *is invertible, and there is a unique analytic map* $\lambda\mapsto Y(\lambda)\in(z,w)+\mathcal{S}_a$*, for small* $\lambda\in\mathbb{C}$*, with* $Y(0)=(z,w)$ *and* $\partial_l(U_Q-\lambda\psi)(Y(\lambda))=0$ *for the four coordinates* $l\neq u_a,v_a$*.*

(b) *Let* $U_Q(Y(\lambda))=\sum_h\alpha_h\lambda^h$ *and* $\psi(Y(\lambda))=\sum_h\psi_h\lambda^h$*. Then* $\alpha_0=\psi_0=0$ *and* $h\alpha_h=(h-1)\psi_{h-1}$ *for* $h\geq1$*. If* $\xi\in\mathcal{S}_a$ *and the components of* $H\xi-g$ *along* $\mathcal{S}_a$ *vanish, then* $\psi_1=g\cdot\xi$ *and* $\psi_2=-\frac12D^3U_Q[\xi,\xi,\xi]$*.*

(c) *Replace* (z,w,p) *by* $(sz,s'w,\zeta p)$ *with* $\zeta^2ss'=1$ *and* m *by* cm *with* $c\neq0$*, and relabel the outer bodies together with* a*. The new configuration is nondegenerate, and* ψ_h *is multiplied by* $c^{1-h}s(s/\zeta)^h$*.*

Proof. (a) Let $\xi\in\mathcal{S}_a$ with $H_a\xi=0$. Then $H\xi=\beta e_{u_a}+\beta'e_{v_a}$. Pairing with Z and W gives $\beta z_a=\beta'w_a=0$. So $H\xi=0$ and $\xi\in\mathbb{C}Z+\mathbb{C}W$ with $\xi_{u_a}=\xi_{v_a}=0$. Since $z_a,w_a\neq0$, $\xi=0$. The implicit function theorem gives Y. (b) $\alpha_0=U_Q(z,w)=0$ (§3.7). By X3$_k$, $\psi_0=0$. Let $\hat F(\lambda)=(U_Q-\lambda\psi)(Y(\lambda))=\sum_hf_h\lambda^h$. The gradient of $U_Q-\lambda\psi$ at $Y(\lambda)$ has only the components u_a and v_a, and $Y'(\lambda)\in\mathcal{S}_a$. So $\hat F'(\lambda)=-\psi(Y(\lambda))$ and $U_Q(Y(\lambda))=\hat F-\lambda\hat F'$. Hence $\alpha_h=(1-h)f_h$ and $\psi_{h-1}=-hf_h$. Next, $Y(\lambda)=(z,w)+\lambda\xi+\lambda^2\eta+O(\lambda^3)$, where $\eta\in\mathcal{S}_a$ and the components of $H\eta+\frac12D^3U_Q[\xi,\xi]$ along $\mathcal{S}_a$ vanish. As ψ is linear and H is symmetric, $\psi_1=g\cdot\xi$ and $\psi_2=g\cdot\eta=\xi\cdot H\eta=-\frac12D^3U_Q[\xi,\xi,\xi]$. (c) Near the two configurations, $U_Q(sz',s'w')=\zeta\,U_Q(z',w')$ for the branches of the p_{il}, and $\psi(sz')=s\psi(z')$. So the equations of (a) for the new configuration at λ are those for the old one at $(s/\zeta)\lambda$, after the substitution $(z',w')\mapsto(sz',s'w')$. This multiplies ψ_h by $s(s/\zeta)^h$. Under $m\mapsto cm$, U_Q and ψ are multiplied by c^2 and c. Thus $Y(\lambda)$ becomes $Y(\lambda/c)$ and ψ_h is multiplied by c^{1-h}. The Hessian becomes $\zeta D^{-1}HD^{-1}$ with $D=\mathrm{diag}(s,s,s,s',s',s')$, resp. c^2H, and a relabelling permutes the coordinates, so that nondegeneracy is preserved. □

Lemma 5.32. *The configuration* $\mathcal{E}$ *is nondegenerate. Its Hessian* H *has rank* 4 *and* $\det H_a=81/256$*, and for the slice of* a*,* $\psi_1=0$ *and* $\psi_2=40/\sqrt{3}$*.*

Proof. By the end of the proof of Theorem 5.30, $\mathcal{E}$ is a K14 configuration, and it satisfies $\mathrm{X3}_k$ as $1+\mathrm{j}+\mathrm{j}^2=0$. Hence Lemma 5.31 applies. As $HZ=HW=0$, rank 4 means that $\ker H$ is spanned by Z and W. The values are exact computations in $\mathbb{Q}(\sqrt{3},i)$. We take ∇U_Q, H and D^3U_Q at $\mathcal{E}$ from the exact Taylor expansion of each p_{il} in the form $p_{il}(\mathcal{E})(1+\epsilon)^{-1/2}$, with ϵ a polynomial in the displacement. We check $\nabla U_Q=0$ and evaluate the formulas of Lemma 5.31(b). The two programs that make these computations also test $\det H_a\neq 0$. They are in the supplementary material. □

The exponent. At a nondegenerate leading cluster configuration the place follows the map Y of Lemma 5.31, with λ replaced by a series of order $4-2\sigma$.

Lemma 5.33. *Let the leading cluster configuration $\bar{y}_Q$ at a place of type $\sigma_k(j)(t)$ be nondegenerate, let $a\in Q'$, and let Y be as in Lemma 5.31. After a reparametrization and a rotation as in §5.1, $y_{u_a}(\tau)\equiv\hat{z}_a$ and $y_{v_a}(\tau)\equiv\hat{w}_a$. Then, with $y_Q=(y_{u_i},y_{v_i})_{i\in Q'}$ and $\lambda(\tau)=\frac{m_j}{2M}y_{v_j}(\tau)\,\tau^{4-2\sigma}$,*

$$y_Q(\tau)-Y(\lambda(\tau))=O(\tau^{1+\sigma}),\qquad U_Q(y_Q(\tau))-U_Q(Y(\lambda(\tau)))=O(\tau^{5-\sigma}).$$

Proof. The conditions are $\mu\varphi^{-2}y_{u_a}=\hat{z}_a$ and $\mu^{-1}\varphi^{-2}y_{v_a}=\hat{w}_a$, which determine φ^4 and then μ. By Lemma 5.22, the cluster components of $\tau^2\nabla_yF(\tau,y(\tau))=0$ read $\nabla(U_Q-\lambda(\tau)\psi)(y_Q(\tau))=O(\tau^{1+\sigma})$. Both $y_Q(\tau)$ and $Y(\lambda(\tau))$ lie in $\bar{y}_Q+\mathcal{S}_a$. The components of $\nabla(U_Q-\lambda\psi)$ along $\mathcal{S}_a$ vanish at $(Y(\lambda),\lambda)$, they are $O(\tau^{1+\sigma})$ at $(y_Q(\tau),\lambda(\tau))$, and their derivative at $(\bar{y}_Q,0)$ is the invertible H_a. This gives the first estimate. For the second, let $\delta=y_Q(\tau)-Y(\lambda(\tau))\in\mathcal{S}_a$. Along $\mathcal{S}_a$ the components of $\nabla U_Q(Y(\lambda))$ and λg agree. Hence $U_Q(y_Q)-U_Q(Y)=\lambda(\tau)\,g\cdot\delta+O(|\delta|^2)=O(\tau^{5-\sigma})$, since $2+2\sigma>5-\sigma$. □

In both propositions below we use $\lambda(\tau)=\bar{\lambda}\tau^{4-2\sigma}(1+o(1))$ with $\bar{\lambda}=m_j\hat{w}_j/(2M)\neq 0$. For a convergent power series $\sum_h c_h\lambda^h$, the first h with $c_h\neq 0$ contributes $c_h\bar{\lambda}^h$ at the exponent $h(4-2\sigma)$, and all other contributions have larger exponents.

Proposition 5.34. *Let $\sigma\neq(2n+1)/(n+1)$ for all integers $n\geq 1$. At positive masses, the leading cluster configuration at a place of type $\sigma_k(j)(t)$, $\sigma=1+t$, is degenerate.*

Proof. Suppose that it is nondegenerate, and normalize the place as in Lemma 5.33. Lemma 5.23 holds in this parametrization, since its proof uses only $\nabla_yF=0$ and the constancy of F along the place. Let $S_1(Y(\lambda))=\sum_h s_h\lambda^h$, with $s_0=0$ by the bad shape. Then $S_1(y_Q(\tau))=\sum_h s_h\lambda(\tau)^h+O(\tau^{1+\sigma})$, where $1+\sigma>2\sigma-2$. Since $B_0=m_jwS_1$ and $w(y(\tau))\to(-\hat{w}_j)^{-1/2}\neq 0$, Lemma 5.23(b) gives $[S_1(y_Q(\tau))]_r=0$ for $r<2\sigma-2$ and $[B_0]_{2\sigma-2}=m_j(-\hat{w}_j)^{-1/2}[S_1]_{2\sigma-2}$. So the first h with $s_h\neq 0$, if there is one, has $h(4-2\sigma)\geq 2\sigma-2$, with equality only for $\sigma=(2h+1)/(h+1)$. Hence $h(4-2\sigma)>2\sigma-2$ and $[S_1(y_Q(\tau))]_{2\sigma-2}=0$. By Lemma 5.23(c), $B_1(\bar{y})=\frac{1}{2}m_j(-\hat{w}_j)^{-1/2}\hat{z}_j\sum_{i\in Q'}m_ix_i^3=0$. Thus $\sum_{i\in Q'}m_ix_i^3=0$. With $\sum_{i\in Q'}m_ix_i=\sum_{i\in Q'}m_ix_i^{-2}=0$, this contradicts Proposition 5.11. □

Proposition 5.35. *Let $\sigma=(2n+1)/(n+1)$ for an integer $n\geq 1$. If the leading cluster configuration at a place of type $\sigma_k(j)(t)$, $\sigma=1+t$, is nondegenerate, then $\psi_1=\psi_2=0$ for each $a\in Q'$.*

Proof. We normalize the place for a as in Lemma 5.33. By Lemmas 5.23(a) and 5.33, $\sum_h\alpha_h\lambda(\tau)^h=U_Q(Y(\lambda(\tau)))=O(\tau^{5-\sigma})$. So $\alpha_h=0$ whenever $h(4-2\sigma)<5-\sigma$. As $\sigma\geq\frac{3}{2}>\frac{7}{5}$, this holds for $h=2$ and $h=3$. Then Lemma 5.31(b) gives $\psi_1=2\alpha_2=0$ and $\psi_2=\frac{3}{2}\alpha_3=0$. □

Proof of Theorem 5.21. Let a singular place P have type $\sigma_k(j)(t)$ with $0<t<1$, and let $\sigma=1+t$. By Lemmas 5.20 and 5.23(b), the leading cluster configuration is a $\mathrm{K14}(m_Q)$ configuration that satisfies $\mathrm{X3}_k$ and has a bad shape. By Theorem 5.30, $m_Q=c(\sqrt{3},\sqrt{3},\sqrt{3},1)$

with $c > 0$, and after a relabelling of a, b, c the leading cluster configuration is $(sz_{\mathcal{E}}, s'w_{\mathcal{E}}, \zeta p_{\mathcal{E}})$ with $\zeta^2 ss' = 1$. By Lemmas 5.32 and 5.31(c) it is nondegenerate. If σ is not of the form $(2n+1)/(n+1)$, Proposition 5.34 gives a contradiction. Otherwise we take the slice of the outer body that corresponds to a in $\mathcal{E}$. By Lemmas 5.32 and 5.31(c), $\psi_2 = c^{-1}s(s/\zeta)^2 \cdot 40/\sqrt{3} \neq 0$, which contradicts Proposition 5.35. □

Remark 5.36. The identities of §5.1 also apply at a place of type $\mathsf{u}_0(j)$, the piece $\mathsf{d14a}(j)$, where the fifth body stays at distance of order one from the cluster while the cluster shrinks. There the pairings of $\nabla_y V$ with the z- and w-scalings of the cluster, together with the constancy of V, give at first order a signed dipole condition $\sum_l s_l m_l(\hat{z}_l - \hat{z}_G) = 0$ when the leading coefficients $\hat{p}_{jl}$ have unequal signs s_l, a centred quadrupole condition $\sum_l m_l(\hat{z}_l - \hat{z}_G)^2 = 0$ when they have equal signs, the same conditions in w at the next order, and in the boundary case a single scalar $g^{\mathsf{T}} H^+ g$. These are the first-order conditions of the analysis of $\mathsf{d14a}$ in §9. The identities of §5.1 therefore do not exclude $\mathsf{d14a}$ at every positive mass: unlike the conditions of Theorems 5.4 and 5.10, these conditions hold at some symmetric K14 configurations with positive masses, for instance centrally symmetric ones. We treat the pieces $\mathsf{d14a}$ instead by the point certificates of §6.

5.6. **Diagram 9.** We now use the expansion of §5.1 to exclude diagram 9 at all positive masses.

Theorem 5.37. *Let $m \in (\mathbb{R}_{>0})^5$, and let $C \subseteq Y_m$ be an irreducible curve on which some r_{kl}^2 is not constant and $U \not\equiv 0$. No singular place of C has diagram 9. Hence the pieces $\mathsf{d9}(T; ax, by)$ may be removed from* pieces$(S(m))$ *in Theorem 3.12, for all m, T, ax and by.*

Let P be a singular place of diagram 9. After a relabelling of the bodies and, if necessary, an exchange of the colours, the triangle is $T = \{1, 2, 3\}$, the w-edges are 14 and 25, and $x(P) = \mathsf{n}(\{1, 2, 3\}; 14, 25)$ (Table 1). By Appendix A, $(\zeta_{kl}, \omega_{kl})$ is $(1, 1)$ on 12, 13, 23 and $(2, -2)$ on 14, 25. On the other five pairs $\zeta_{kl} = 1$, by $z_{45} = z_{41} + z_{12} + z_{25}$, $z_{15} = z_{12} + z_{25}$ and the like, and $\omega_{kl} = 2x_{kl} - 1$, which is -1 on 45 and -2 on 15, 24, 34, 35.

For $h \in \mathbb{C}^5$ let $h^* = \frac{m_4 h_4 + m_5 h_5}{m_4 + m_5} - \frac{m_1 h_1 + m_2 h_2 + m_3 h_3}{M_T}$. We use the positions relative to body 3, $u_i = z_i - z_3$ and $v_i = w_i - w_3$, in the following combinations: the *coarse* coordinates u_1, u_2, v_1, v_2, $\Omega = w^*$, and the *fine* coordinates u_{41}, u_{52}, v_{45}, where $u_{kl} = u_k - u_l$ and $v_{kl} = v_k - v_l$. They form an invertible linear function of (u_1, u_2, u_4, u_5) and one of (v_1, v_2, v_4, v_5). Their exponents are 1, 1, 1, 1, -2 and 2, 2, -1. For Ω this follows from $w_{14} = -\Omega - \frac{m_5}{m_4 + m_5} v_{45} + \frac{m_2 v_{12} + m_3 v_1}{M_T}$, since w_{14} has exponent -2 and the last two terms have exponents -1 and 1. The coordinates of §5.1 do not serve here. For each body o one of the pairs 14, 25 avoids o, the positions of its two bodies relative to o have exponent 1 and the same leading coefficient, and so the corresponding p_{kl} is not holomorphic in y near $\bar{y}$. We therefore state Lemma 5.1 for linear coordinates of the kind above.

Lemma 5.38. *Let $q_1, \ldots, q_8$ be coordinates such that $(q_1, \ldots, q_4)$ is an invertible linear function of $(u_1, \ldots, u_4)$ and $(q_5, \ldots, q_8)$ one of $(v_1, \ldots, v_4)$, and suppose that no q_i vanishes identically on C. Then, after the rotation of Lemma 3.1, $q_i = \tau^{2e(q_i)} y_{q_i}$ with $y_{q_i}(0) \neq 0$. Lemma 5.1 holds in the coordinates q, and the reparametrizations and rotations of §5.1 change neither the leading point nor the identities of that lemma.*

Proof. Since each q_i is of z-type or of w-type, Lemma 3.1 gives the first statement and a rotation by $\mu(\tau)$ multiplies y_{q_i} by $\mu(\tau)^{\pm 1}$. The proof of Lemma 5.1 uses three facts: $\nabla V = 0$ at the points of X_m in the coordinates used, V is constant on C, and $F(\lambda\tau, \lambda^{-2e} y) = F(\tau, y)$. The last two do not depend on the coordinates. The first holds in the coordinates q, because $\nabla_{(u,v)} V = 0$ on X_m (§5.1) and q is an invertible linear function of (u, v). □

We write the coefficients V_n of $F(\tau, y) = V(\tau^{2e}y)$ as functions of the coordinates, with u_1 standing for y_{u_1} and so on, and put $u_{12} = u_1 - u_2$, $v_{12} = v_1 - v_2$, $\nu = M_T(m_4 + m_5)/M$, $\nu_{45} = m_4 m_5/(m_4 + m_5)$,

$$U_T = m_1 m_2 (u_{12} v_{12})^{-1/2} + m_1 m_3 (u_1 v_1)^{-1/2} + m_2 m_3 (u_2 v_2)^{-1/2},$$
$$L(u) = \frac{m_4 u_1 + m_5 u_2}{m_4 + m_5} - \frac{m_1 u_1 + m_2 u_2}{M_T}.$$

Lemma 5.39. *At a place of type* $\mathsf{n}(\{1,2,3\}; 14, 25)$*, the function* $F(\tau, y) = \sum_{n \geq 0} \tau^{n-2} V_n(y)$ *satisfies the hypothesis of Lemma 5.1 in the coordinates above, with* $V_1 = 0$ *and*

$$\begin{aligned} V_0 &= U_T + \tfrac{\nu}{2}\,\Omega\, L(u), \\ V_2 &= m_1 m_4 (u_{41}\Omega)^{-1/2} + m_2 m_5 (u_{52}\Omega)^{-1/2} + m_4 m_5 (u_{12} v_{45})^{-1/2} \\ &\qquad + \tfrac{\nu_{45}}{2}\, u_{12} v_{45} + \tfrac{\nu}{2}\,\Omega\, \frac{m_4 u_{41} + m_5 u_{52}}{m_4 + m_5}, \\ V_3 &= m_1 m_5 (-u_{12}\Omega)^{-1/2} + m_2 m_4 (u_{12}\Omega)^{-1/2} + m_3 m_4 (u_1 \Omega)^{-1/2} + m_3 m_5 (u_2 \Omega)^{-1/2}, \end{aligned}$$

where the square roots are the branches of the p_{kl} *at* P*. The coefficients* V_0 *and* V_3 *do not depend on the fine coordinates.*

Proof. The coordinates have the τ-exponents $2, 2, 2, 2, -4$ and $4, 4, -2$, all even. So each z_{kl} is $\tau^{2\zeta_{kl}}(\bar{z}_{kl} + \tau^2 c_{kl})$. Here $\bar{z}_{kl}$ is a linear form in the coordinates whose value at $\bar{y}$ is the leading coefficient $\hat{z}_{kl} \neq 0$. The function c_{kl} is a polynomial in τ^2 and the coordinates. Likewise $w_{kl} = \tau^{2\omega_{kl}}(\bar{w}_{kl} + \tau^2 c'_{kl})$. For instance $z_{15} = \tau^2(u_{12} - \tau^2 u_{52})$ and $w_{15} = \tau^{-4}\big(-\Omega + \tau^2 \frac{m_4 v_{45}}{m_4 + m_5} + \tau^6 \frac{m_2 v_{12} + m_3 v_1}{M_T}\big)$. Hence

$$p_{kl} = \tau^{-2x_{kl}} (\bar{z}_{kl} \bar{w}_{kl})^{-1/2} \big(1 + \tau^2 c_{kl}/\bar{z}_{kl}\big)^{-1/2} \big(1 + \tau^2 c'_{kl}/\bar{w}_{kl}\big)^{-1/2}$$

is $\tau^{-2x_{kl}}$ times a convergent series in τ^2 whose coefficients are holomorphic near $\bar{y}$. The exponent $-2x_{kl}$ is -2 on the pairs of T, 0 on 14, 25, 45 and 1 on 15, 24, 34, 35. On the pairs of T the series is 1. The products $\bar{z}_{kl}\bar{w}_{kl}$ are $u_{12}v_{12}$, $u_1 v_1$, $u_2 v_2$ on 12, 13, 23, then $u_{41}\Omega$, $u_{52}\Omega$, $u_{12}v_{45}$ on 14, 25, 45, and $-u_{12}\Omega$, $u_{12}\Omega$, $u_1\Omega$, $u_2\Omega$ on 15, 24, 34, 35. For the moment of inertia we use $I = I_T + \nu z^* w^* + \nu_{45} z_{45} w_{45}$ with $I_T = M_T^{-1} \sum_{k<l \in T} m_k m_l z_{kl} w_{kl}$. Here I_T contributes only at τ^4; $z^* = L(u) + (m_4 u_{41} + m_5 u_{52})/(m_4 + m_5)$ and $w^* = \Omega$, so $\nu z^* w^*$ contributes at τ^{-2} and τ^0; and $z_{45} w_{45} = \big(u_{12} + \tau^2(u_{41} - u_{52})\big) v_{45}$ contributes at τ^0 and τ^2. Collecting terms gives the lemma. □

Lemma 5.40. *At a place of type* $\mathsf{n}(\{1,2,3\}; 14, 25)$*, let* $\bar{y}_c$ *be the coarse part of the leading point, and* ∇_f *the gradient in the fine coordinates.*

(a) $\nabla V_0(\bar{y}_c) = 0$ *and* $\nabla_f V_2(\bar{y}) = 0$.
(b) $(m_4 + m_5)\hat{z}_{3l} = \rho_{3l}\hat{z}_{12}$ *for* $l = 1, 2$*, where*

$$\rho_{31} = \frac{m_2 m_4 - m_1 m_5 - m_3 m_5}{m_3}, \qquad \rho_{32} = \frac{m_2 m_4 - m_1 m_5 + m_3 m_4}{m_3}$$

are nonzero real numbers with $\rho_{32} - \rho_{31} = m_4 + m_5$.
(c) $\hat{Z}_{12}/m_3 = \hat{Z}_{23}/m_1 = \hat{Z}_{31}/m_2 =: \kappa$*, where* $\hat{Z}_{kl} = \hat{p}_{kl}^3 \hat{z}_{kl}$*, and* $\kappa \neq 0$.
(d) $U_T(\bar{y}_c) = m_1 m_2 \hat{p}_{12} + m_1 m_3 \hat{p}_{13} + m_2 m_3 \hat{p}_{23} = 0$.

Proof. (a) By Lemmas 5.38 and 5.39, $\nabla_y F(\tau, y(\tau)) = 0$. Its coarse components, multiplied by τ^2, are $\nabla V_0(y_c(\tau)) + O(\tau^2)$. Its fine components are $\nabla_f V_2(y(\tau)) + O(\tau^2)$. (b) The Ω-component of ∇V_0 is $\frac{\nu}{2} L(u)$. So $\bar{u} = (\bar{u}_1, \bar{u}_2) = (-\hat{z}_{31}, -\hat{z}_{32})$ satisfies $L(\bar{u}) = 0$ and $\bar{u}_1 - \bar{u}_2 = \hat{z}_{12}$. These two linear conditions determine $\bar{u}$, since the coefficients of L have the sum $m_3/M_T \neq 0$. The

vector $-\hat{z}_{12}(\rho_{31},\rho_{32})/(m_4+m_5)$ satisfies both. The $\hat{z}_{3l}$ are nonzero, as $\zeta_{3l}=1$. (These are the positions of [2, §8.9.2].) (c) Since $p_{kl}^2 z_{kl} w_{kl}=1$, $\partial p_{kl}/\partial w_{kl}=-\frac{1}{2}p_{kl}^3 z_{kl}$. As $w_{12}=v_1-v_2$, $w_{13}=v_1$ and $w_{23}=v_2$, the v-components of $\nabla V_0(\bar{y}_c)=0$ are $-\frac{1}{2}m_1(m_2\hat{Z}_{12}+m_3\hat{Z}_{13})=0$ and $-\frac{1}{2}m_2(m_1\hat{Z}_{21}+m_3\hat{Z}_{23})=0$. (d) By (c) and $\hat{p}_{kl}=\hat{p}_{kl}^3\hat{z}_{kl}\hat{w}_{kl}=\hat{Z}_{kl}\hat{w}_{kl}$, we get $U_T(\bar{y}_c)=\kappa m_1m_2m_3(\hat{w}_{12}+\hat{w}_{23}+\hat{w}_{31})=0$. □

Lemma 5.41. *Let H be the Hessian of V_0 at $\bar{y}_c$ in the coordinates (u_1,u_2,v_1,v_2,Ω), and let $\bar{u}=(\bar{u}_1,\bar{u}_2)$ and $\bar{v}=(\bar{v}_1,\bar{v}_2)$. Then*

$$\ker H=\big\{\big(a\bar{u},\, b\bar{v},\, -\tfrac{3a+b}{2}\,\bar{\Omega}\big) : a,b\in\mathbb{C}\big\}.$$

Proof. Since U_T is homogeneous of degree $-\frac{1}{2}$ in u and in v, $u\cdot\nabla_u U_T=v\cdot\nabla_v U_T=-\frac{1}{2}U_T$. By Lemma 5.40(a), $\nabla_v U_T=0$ and $\nabla_u U_T=-\frac{\nu}{2}\bar{\Omega}\nabla L$ at $\bar{y}_c$. The term $\frac{\nu}{2}\Omega L(u)$ of V_0 is linear in u and does not involve v. Differentiating the two Euler identities, we obtain at $\bar{y}_c$

$$H_{vu}\bar{u}=-\tfrac{1}{2}\nabla_v U_T=0,\qquad H_{vv}\bar{v}=-\tfrac{3}{2}\nabla_v U_T=0,\qquad H_{uu}\bar{u}=-\tfrac{3}{2}\nabla_u U_T,\qquad H_{uv}\bar{v}=-\tfrac{1}{2}\nabla_u U_T,$$

where $H_{uu},\dots$ are the blocks of H. Moreover the Ω-row of H is $(\frac{\nu}{2}\nabla L,0,0)$.

Let $\delta=(\delta u,\delta v,\delta\Omega)\in\ker H$. The Ω-row gives $L(\delta u)=0$. So $\delta u=a\bar{u}$ for some $a\in\mathbb{C}$, as $\ker L=\mathbb{C}\bar{u}$. The v-rows then give $H_{vv}\delta v=0$. We show that H_{vv} has rank 1. As $\partial^2 p_{kl}/\partial w_{kl}^2=\frac{3}{4}p_{kl}/w_{kl}^2$ and, by Lemma 5.40(c), $m_km_l\hat{p}_{kl}/\hat{w}_{kl}=m_km_l\hat{Z}_{kl}=\kappa m_1m_2m_3$ for $kl=12,23,31$, the quadratic form of H_{vv} is

$$\tfrac{3}{4}\kappa m_1m_2m_3\Big(\frac{s_{12}^2}{\hat{w}_{12}}+\frac{s_{23}^2}{\hat{w}_{23}}+\frac{s_{31}^2}{\hat{w}_{31}}\Big),\qquad s_{12}=\delta v_1-\delta v_2,\quad s_{23}=\delta v_2,\quad s_{31}=-\delta v_1,$$

on the plane $s_{12}+s_{23}+s_{31}=0$. Its polar form pairs the point $(\hat{w}_{12},\hat{w}_{23},\hat{w}_{31})$, given by $\delta v=\bar{v}$, with s to $\frac{3}{4}\kappa m_1m_2m_3(s_{12}+s_{23}+s_{31})$, which vanishes on the plane. At $s=(1,-1,0)$ the form is $\frac{3}{4}\kappa m_1m_2m_3(\hat{w}_{12}^{-1}+\hat{w}_{23}^{-1})=-\frac{3}{4}\kappa m_1m_2m_3\hat{w}_{31}/(\hat{w}_{12}\hat{w}_{23})\neq 0$. So $\ker H_{vv}=\mathbb{C}\bar{v}$ and $\delta v=b\bar{v}$ for some $b\in\mathbb{C}$. Finally the u-rows give $aH_{uu}\bar{u}+bH_{uv}\bar{v}+\frac{\nu}{2}\delta\Omega\,\nabla L=\frac{\nu}{2}\big(\frac{3a+b}{2}\bar{\Omega}+\delta\Omega\big)\nabla L=0$ with $\nabla L\neq 0$. Conversely, one checks by the same computation that each vector of the stated form lies in $\ker H$. The vectors with $(a,b)=(1,-1)$ and $(-2,-2)$ are the tangent vectors of the rotation and of the scaling $(u,v,\Omega)\mapsto(\lambda^{-2}u,\lambda^{-2}v,\lambda^4\Omega)$. □

The next statement is the analogue of Theorem 5.4: the leading potential between T and $\{4,5\}$ vanishes.

Proposition 5.42. *At a place of type $\mathsf{n}(\{1,2,3\};14,25)$, $V_3(\bar{y})=m_1m_5\hat{p}_{15}+m_2m_4\hat{p}_{24}+m_3m_4\hat{p}_{34}+m_3m_5\hat{p}_{35}=0$.*

Proof. By Lemma 5.38 we may normalize the place as in the proof of Theorem 5.10, so that $y_{u_1}(\tau)\equiv\bar{u}_1$ and $y_\Omega(\tau)\equiv\bar{\Omega}$: the conditions $\mu\varphi^{-2}y_{u_1}=\bar{u}_1$ and $\mu^{-1}\varphi^4 y_\Omega=\bar{\Omega}$ determine φ^2 and then μ. We write $y(\tau)=\bar{y}+(\delta_c(\tau),\delta_f(\tau))$ with the coarse part δ_c and the fine part δ_f. Then δ_c lies in $\mathcal{S}=\{\xi\in\mathbb{C}^5 : \xi_{u_1}=\xi_\Omega=0\}$. By Lemma 5.41, $\mathcal{S}\cap\ker H=0$, since $a\bar{u}_1=0$ and $(3a+b)\bar{\Omega}=0$ force $a=b=0$. As $\dim\mathcal{S}+\dim\ker H=5$, the restriction of the bilinear form H to $\mathcal{S}$ is nondegenerate: a vector of $\mathcal{S}$ that is H-orthogonal to $\mathcal{S}$ is H-orthogonal to $\mathcal{S}+\ker H=\mathbb{C}^5$. By Lemma 5.40(a), the coarse components of $\tau^2\nabla_y F(\tau,y(\tau))=0$ read $H\delta_c+O(|\delta_c|^2)+O(\tau^2)=0$. Their components along $\mathcal{S}$ give $\operatorname{val}\delta_c\geq\min(2\operatorname{val}\delta_c,2)$. Hence $\operatorname{val}\delta_c\geq 2$. The fine components of $\nabla_y F(\tau,y(\tau))=0$ read $\nabla_f V_2(\bar{y}+\delta)+O(\tau^2)=0$. By Lemma 5.39, the matrix of second derivatives of V_2 in the fine coordinates is diagonal at $\bar{y}$, with the nonzero entries $\frac{3}{4}m_1m_4\hat{p}_{14}/\bar{u}_{41}^2$, $\frac{3}{4}m_2m_5\hat{p}_{25}/\bar{u}_{52}^2$ and $\frac{3}{4}m_4m_5\hat{p}_{45}/\bar{v}_{45}^2$. With $\nabla_f V_2(\bar{y})=0$ and $\operatorname{val}\delta_c\geq 2$ this gives $\operatorname{val}\delta_f\geq\min(2\operatorname{val}\delta_f,2)$, hence $\operatorname{val}\delta_f\geq 2$. Therefore $V_2(y(\tau))=V_2(\bar{y})+O(\tau^2)$. Now (5.1) with $q=3$ reads $[V_1]_2+2[V_2]_1+3V_3(\bar{y})=0$, as in the proof of Theorem 5.4. Here $V_1=0$ and $[V_2(y(\tau))]_1=0$. Thus $V_3(\bar{y})=0$. □

Proof of Theorem 5.37. Suppose that P has diagram 9, with the labelling above. On the pairs $kl = 15, 24, 34, 35$, Lemma 5.39 gives $\hat{p}_{kl} = (\bar{z}_{kl}\bar{w}_{kl})^{-1/2}$ at $\bar{y}$, with $\bar{w}_{kl} = -\bar{\Omega}$. By Lemma 5.40(b), $(m_4 + m_5)\bar{z}_{kl} = \tilde{\rho}_{kl}\hat{z}_{12}$ at $\bar{y}$, where $\tilde{\rho}_{15} = m_4 + m_5$, $\tilde{\rho}_{24} = -(m_4 + m_5)$, $\tilde{\rho}_{34} = \rho_{31}$ and $\tilde{\rho}_{35} = \rho_{32}$. Let K be a square root of $-(m_4 + m_5)/(\hat{z}_{12}\bar{\Omega})$. Then $\hat{p}_{kl} = \pm K\tilde{\rho}_{kl}^{-1/2}$ if $\tilde{\rho}_{kl} > 0$, and $\hat{p}_{kl} = \pm iK|\tilde{\rho}_{kl}|^{-1/2}$ if $\tilde{\rho}_{kl} < 0$. So Proposition 5.42 says that two sums of real numbers $\pm m_k m_l|\tilde{\rho}_{kl}|^{-1/2}$ vanish, one over the pairs with $\tilde{\rho}_{kl} > 0$ and one over the pairs with $\tilde{\rho}_{kl} < 0$. Each sum therefore has at least two terms. The pair 15 is in the first sum, 24 is in the second, and $\rho_{31} < \rho_{32}$. Hence $\rho_{31} < 0 < \rho_{32}$, the first sum is over 15, 35 and the second over 24, 34, and they vanish only if

$$m_1^2\rho_{32} = m_3^2(m_4 + m_5), \qquad m_2^2\rho_{31} = -m_3^2(m_4 + m_5). \tag{5.10}$$

Let $\alpha = \hat{p}_{12}/\hat{p}_{23}$ and $\beta = \hat{p}_{12}/\hat{p}_{31}$. By Lemma 5.40(c), $\hat{p}_{kl}^3\hat{z}_{kl}$ is κm_3, κm_1, κm_2 for $kl = 12, 23, 31$. By Lemma 5.40(b), $(m_4 + m_5)\hat{z}_{23} = -\rho_{32}\hat{z}_{12}$ and $(m_4 + m_5)\hat{z}_{31} = \rho_{31}\hat{z}_{12}$. With Lemma 5.40(d) these relations give

$$m_1(m_4 + m_5)\alpha^3 = -m_3\rho_{32} = m_1m_5 - m_2m_4 - m_3m_4,$$
$$m_2(m_4 + m_5)\beta^3 = m_3\rho_{31} = m_2m_4 - m_1m_5 - m_3m_5,$$
$$m_1m_3\alpha + m_2m_3\beta + m_1m_2\alpha\beta = \alpha\beta\, U_T(\bar{y}_c)/\hat{p}_{12} = 0.$$

By (5.10), $\alpha^3 = -(m_3/m_1)^3$ and $\beta^3 = -(m_3/m_2)^3$. So $\alpha = -\eta m_3/m_1$ and $\beta = -\eta' m_3/m_2$ with $\eta^3 = \eta'^3 = 1$. The third relation becomes $m_3^2(\eta\eta' - \eta - \eta') = 0$, that is, $(\eta - 1)(\eta' - 1) = 1$. This is impossible, because $|\eta - 1|$ and $|\eta' - 1|$ are 0 or $\sqrt{3}$.

The last statement follows, since Theorem 3.12 places the type of a singular place in a piece of its own diagram. □

Remark 5.43. The statement of [2, Prop. 8] covers diagram 9: along a singular sequence that approaches one of diagrams 9–16, every 4-product $\mathcal{P}_{ij}$ is bounded. Its proof does not give this bound for diagram 9. For this diagram the exponents of the distances given in the proof form the type $\mathsf{n}(\{1, 2, 3\}; 14, 25)$ above, the case of [2, §8.9.2]. At a place of this type $e(\mathcal{P}_{15}) = 2(x_{15} + x_{23} + x_{34} + x_{24}) = -1$, and likewise $e(\mathcal{P}_{24}) = -1$, so these two 4-products are unbounded. By Theorem 5.37, diagram 9 does not occur on a nontrivial curve at positive masses. We do not use [2, Prop. 8].

Remark 5.44. At a place of diagram 1 or 4 with triangle T, the torque identity of [2, §5.0] shows that the leading potential between T and the two other bodies vanishes as well; at positive masses this confines such places to finitely many explicit curves of masses [16, Thm. 9.2]. Chen and Tong state the stronger result that no such place exists at positive masses [3, Thms. 5.13 and 8.2].

6. Point certificates for the clusters of zero energy

For a body j write $Q_j = \{1, \ldots, 5\} \setminus \{j\}$. By Theorem 3.13, the pieces $\mathsf{d14a}(j)$ and $\mathsf{d14b}(j, k)$ are attained at m only if $\mathrm{K14}(m_{Q_j})$ has a solution, and $\mathsf{d15}(j, k)$ only if $\mathrm{K15}(m_{Q_j}, k)$ has one. In §9 the separation of §4 succeeds at finitely many mass points, given exactly, only after the pieces of some clusters Q_j, usually one, are removed. At these points we show that $\mathrm{K14}(m_{Q_j})$ has no solution. By §4.2, the proof needs this only for the pieces $\mathsf{d14a}(j)$.

All these certificates have the same structure. We reduce the cluster system modulo a prime ℓ at an ℓ-adic point over the given masses. A standard basis over $\mathbb{F}_\ell$ shows that neither the reduced system nor any of its initial systems on the cells of its tropical prevariety has a zero in the torus. A valuation argument lifts this to characteristic 0. The computations over $\mathbb{F}_\ell$ and the enumeration of the cells are exact. Where a point of §9 is a certified point, we identify the two exactly.

6.1. **The cluster systems in the torus.** Let Q be a set of four bodies, $\mu = (\mu_i)_{i\in Q}$ complex masses, $M_Q = \sum_{i\in Q}\mu_i$, and $k \in Q$. In this section we write h for the second index of a pair of bodies, since ℓ denotes a prime. Recall from (3.2) and (3.3) that K14(μ) asks for $z, w \in \mathbb{C}^Q$ and $p_{ih} = p_{hi}$ with $p_{ih}^2 z_{ih} w_{ih} = 1$ and

$$F_i^z = \sum_{h\in Q\setminus i} \mu_h\, z_{ih}\, p_{ih}^3 = 0, \qquad F_i^w = \sum_{h\in Q\setminus i} \mu_h\, w_{ih}\, p_{ih}^3 = 0 \qquad (i \in Q),$$

and that K15(μ, k) asks for the same with $F_i^w = c + d\,[i = k]$, $d \neq 0$. Whether they have solutions depends only on the ratios of the masses and on k, and is equivariant under relabelling (§3.7).

We number the bodies of Q increasingly as $1, \dots, 4$. The unknowns ξ are z_{ih}, w_{ih} and p_{ih} for the five pairs $\{i, h\} \neq \{1, 2\}$, and for K15 also c and d, which gives $N = 15$, respectively $N = 17$, unknowns. The generators are:

(i) the seven circuit relations of the complete graph on Q (four triangles and three four-cycles), for z and for w, with $z_{12} = w_{12} = 1$;
(ii) $p_{ih}^2 z_{ih} w_{ih} - 1$ for the five pairs;
(iii) the eight rows F_i^z and $F_i^w - c - d\,[i = k]$, with $p_{12} = 1$, and $c = d = 0$ for K14;
(iv) with $U = \sum_{i<h} \mu_i\mu_h p_{ih}$, $V_{zz} = \sum_{i<h} \mu_i\mu_h z_{ih}^2 p_{ih}^3$ and $V_{ww} = \sum_{i<h} \mu_i\mu_h w_{ih}^2 p_{ih}^3$: for K14 the three polynomials U, V_{zz}, V_{ww}; for K15 the five polynomials

$$cM_Q + d\mu_k, \qquad U, \qquad \textstyle\sum_{h\in Q} \mu_h z_{kh}, \qquad V_{zz}, \qquad V_{ww} + c\sum_{h\in Q} \mu_h w_{kh}.$$

This gives 30 generators for K14 and 32 for K15. Each generator has the form $F_\nu = \sum_{a\in A_\nu} c_{\nu,a}(\mu)\,\xi^a$, where each $c_{\nu,a}$ is a nonzero element of $\mathbb{Z}[\mu]$. Each $c_{\nu,a}$ is an integer multiple of a monomial in μ, except the coefficient M_Q of c in $cM_Q + d\mu_k$. We call A_ν the *universal support* of F_ν.

Lemma 6.1. *If* K14(μ) *holds, the generators of* K14 *have a common zero in* $(\mathbb{C}^*)^{15}$. *If* $\mu_k \neq 0$, $M_Q \neq 0$ *and* K15(μ, k) *holds, the generators of* K15 *with this* k *have a common zero in* $(\mathbb{C}^*)^{17}$.

Proof. Replacing (z, w, p) by $(sz, tw, (st)^{-1/2}p)$ multiplies F^z by $s(st)^{-3/2}$ and F^w by $t(st)^{-3/2}$. Replacing p by $-p$ changes the sign of both. For K15 the factor of F^w is absorbed by (c, d). Since $z_{12}w_{12} = p_{12}^{-2} \neq 0$, each solution has a representative with $z_{12} = w_{12} = p_{12} = 1$. Conditions (i)–(iii) hold there. The circuit relations give positions z_i, w_i. From $p_{ih}^2 z_{ih} w_{ih} = 1$ and antisymmetry we obtain

$$\textstyle\sum_i \mu_i w_i F_i^z = \sum_i \mu_i z_i F_i^w = U, \qquad \sum_i \mu_i z_i F_i^z = V_{zz}, \qquad \sum_i \mu_i w_i F_i^w = V_{ww}, \qquad \sum_i \mu_i F_i^w = 0.$$

For K14 the left sides vanish. For K15 the last identity gives $cM_Q + d\mu_k = 0$. Then $0 = U = \sum_i \mu_i z_i(c + d[i = k]) = d\mu_k(z_k - z_G)$ with $z_G = \sum_i \mu_i z_i / M_Q$. So $\sum_h \mu_h z_{kh} = M_Q(z_k - z_G) = 0$. Similarly, $V_{ww} = cM_Q w_G + d\mu_k w_k = -c\sum_h \mu_h w_{kh}$. Finally, $p_{ih}^2 z_{ih} w_{ih} = 1$ makes z_{ih}, w_{ih}, p_{ih} nonzero, $d \neq 0$, and $c = -d\mu_k/M_Q \neq 0$. □

Cells. For a finite set $A \subset \mathbb{Z}^N$ and $\omega \in \mathbb{R}^N$ let $\max_\omega(A)$ be the set of $a \in A$ at which $\langle a, \omega\rangle$ is largest, and for $F = \sum_{a\in A} c_a\xi^a$ let $\mathrm{in}_\omega F = \sum_{a\in\max_\omega(A)} c_a\xi^a$. The *prevariety* of the universal supports is the set of ω with $|\max_\omega(A_\nu)| \geq 2$ for every ν. Its *cells* are the classes of its nonzero points with the same tuple of max-sets. On a cell σ the initial forms $\mathrm{in}_\sigma F_\nu$ are well defined. Since the prevariety depends only on the supports, we compute it once for each of the five systems (K14, and K15 with $k = 1, \dots, 4$). We enumerate the cells by a search in which each linear-programming decision is proved in integer arithmetic, by an integer point or by nonnegative integer Gordan multipliers (the numbers of decisions are in §12.2). We compare the result with the prevariety computed by gfan 0.7: the two agree, and the lineality space is 0.

The prevariety of K14 has 28 cells, spanned by 16 rays. That of each system K15 has 43 cells, spanned by 18 rays.

6.2. **Algebraic mass points.** Let $f \in \mathbb{Z}[t]$ be irreducible over $\mathbb{Q}$, and let $g = (g_i)_{i\in Q}$ with $g_i \in \mathbb{Q}[t]$. For a root θ of f put $\mu(\theta) = (g_i(\theta))_{i\in Q}$. For a prime ℓ and $\bar\theta \in \mathbb{F}_\ell$ put $\bar\mu_i = g_i(\bar\theta)$ and $\bar F_\nu = \sum_a c_{\nu,a}(\bar\mu)\xi^a \in \mathbb{F}_\ell[\xi]$.

Theorem 6.2. *Let ℓ be a prime and $\bar\theta \in \mathbb{F}_\ell$ such that:*

- *(P1) ℓ divides neither the leading coefficient of f nor a denominator of a coefficient of a g_i; $\bar f = f \bmod \ell$ is squarefree of degree $\deg f$, and $\bar f(\bar\theta) = 0$;*
- *(P2) $\bar\mu_i \neq 0$ for $i \in Q$, $\sum_i \bar\mu_i \neq 0$, and $c_{\nu,a}(\bar\mu) \neq 0$ for every generator and every $a \in A_\nu$;*
  - *(A) the ideal of $\mathbb{F}_\ell[\xi_0, \xi]$ generated by $\bar F_1, \dots, \bar F_N$ and $\xi_0\xi_1\cdots\xi_N - 1$ is the unit ideal;*
  - *(B) for every cell σ, the ideal generated by $\mathrm{in}_\sigma \bar F_1, \dots, \mathrm{in}_\sigma \bar F_N$ and $\xi_0\xi_1\cdots\xi_N - 1$ is the unit ideal.*

Then for every root $\theta \in \mathbb{C}$ of f the masses $g_i(\theta)$ and their sum are nonzero, and K14$(\mu(\theta))$, *respectively* K15$(\mu(\theta), k)$, *has no solution.*

Proof. Step 1: one embedding suffices. Let $K = \mathbb{Q}[t]/(f)$, a field, with ϑ the class of t, and let $I \subset K[\xi_0, \xi]$ be the ideal generated by the polynomials $\sum_a c_{\nu,a}(g(\vartheta))\xi^a$ and $\xi_0\xi_1\cdots\xi_N - 1$. For an embedding ι of K into an algebraically closed field Ω, the system ιF has a zero in $(\Omega^*)^N$ if and only if $1 \notin I$: by the Nullstellensatz it has one if and only if $1 \notin \iota(I)\,\Omega[\xi_0, \xi]$, and a representation $1 = \sum h_\nu\, \iota(P_\nu)$ with prescribed degrees is a linear system over $\iota(K)$, which is solvable over $\iota(K)$ if it is solvable over Ω. The complex embeddings of K send ϑ to the roots of f. It therefore suffices to find one embedding of K under which the system has no zero in the torus.

Step 2: the Hensel embedding. Since $\bar f$ is squarefree, $\bar\theta$ is a simple root. Hensel's lemma gives a unique $\theta_\ell \in \mathbb{Z}_\ell$ with $f(\theta_\ell) = 0$ and $\theta_\ell \equiv \bar\theta \pmod{\ell}$. As f is irreducible, $\vartheta \mapsto \theta_\ell$ defines an embedding $\tau\colon K \to \mathbb{Q}_\ell$. By (P1) each $g_i(\theta_\ell)$ lies in $\mathbb{Z}_\ell$ and reduces to $\bar\mu_i$. Hence each coefficient $c_{\nu,a}(g(\theta_\ell))$ lies in $\mathbb{Z}_\ell$ and reduces to $c_{\nu,a}(\bar\mu)$, which is nonzero by (P2). All coefficients are therefore ℓ-adic units. In particular the masses and their sum are nonzero in K, hence at every root of f.

Step 3: valuations. Suppose that τF has a zero $\xi \in (\bar{\mathbb{Q}}_\ell^*)^N$. Its coordinates lie in a finite extension E of $\mathbb{Q}_\ell$, with valuation v, $v(\ell) = 1$, a uniformizer π of valuation $1/e$, and residue field k_E. Put $\omega = -v(\xi)$ and $\xi_\alpha = \pi^{ev(\xi_\alpha)} y_\alpha$ with units y_α. As the coefficients are units, the terms of $\tau F_\nu(\xi)$ of least valuation are those with exponent in $\max_\omega(A_\nu)$. Dividing by the least power of π and reducing, we obtain $\mathrm{in}_\omega \bar F_\nu(\bar y) = 0$ for every ν, with $\bar y \in (k_E^*)^N$. If $\omega = 0$, then $\bar y$, with $\xi_0 = (\bar y_1 \cdots \bar y_N)^{-1}$, is a zero of the ideal of (A), which is impossible. If $\omega \neq 0$, then each $\max_\omega(A_\nu)$ has at least two elements, since a single term with a nonzero coefficient does not vanish at a torus point. Hence ω is a nonzero point of the prevariety. It lies in a cell σ, where $\mathrm{in}_\omega \bar F_\nu = \mathrm{in}_\sigma \bar F_\nu$. So $\bar y$ gives a zero of the ideal of (B), which is impossible.

Thus τF has no zero in the torus. By Step 1, no complex root θ of f gives one. By Lemma 6.1 neither K14$(\mu(\theta))$ nor K15$(\mu(\theta), k)$ holds. □

Remark 6.3. (a) One prime suffices, although the certificates of §6.3 that use Theorem 6.2 use two. (b) The proof uses the masses only through Step 2. If they are given in homogeneous coordinates, $\mu_i = a_i(\theta)/b(\theta)$ with $a_i, b \in \mathbb{Z}[t]$ and $b(\theta) \neq 0$, the theorem holds with (P1) replaced by: ℓ does not divide the leading coefficient of f, $\bar f$ is squarefree of degree $\deg f$ with $\bar f(\bar\theta) = 0$, $b(\bar\theta) \neq 0$, and $\bar\mu_i = a_i(\bar\theta)\, b(\bar\theta)^{-1}$. Indeed $b(\theta_\ell)$ is then a unit, and $a_i(\theta_\ell)/b(\theta_\ell) \in \mathbb{Z}_\ell$ reduces to $\bar\mu_i$. (c) Since K14 and K15 are unchanged under scaling of the masses, the theorem may be

applied to any nonzero multiple of the masses, for instance to the numerators over a common denominator.

The certificate also bounds the masses at which a cluster system can hold along an algebraic curve.

Corollary 6.4. *Let X be an irreducible curve over $\mathbb{Q}$ with a rational map $x \mapsto \mu(x)$ to the masses of a cluster, and suppose that the hypotheses of Theorem 6.2 hold, for* K14 *or for* K15$(\cdot, k)$ *and with some prime ℓ, at $\mu(x_0)$ for a point $x_0 \in X$ defined over a number field. Then the set of $x \in X(\mathbb{C})$ at which the system has a zero in the torus is finite.*

Proof. Since the set Z in question is constructible and defined over $\mathbb{Q}$, it is finite or contains a nonempty Zariski open subset of X. Step 3 of the proof of Theorem 6.2 uses only that the masses lie in $\mathcal{O}_E^4$ for a finite extension E of $\mathbb{Q}_\ell$ and reduce to $\bar{\mu} = \mu(x_0) \bmod \ell$, where (P2), (A) and (B) are conditions on $\bar{\mu}$. The points of X over finite extensions of $\mathbb{Q}_\ell$ that are ℓ-adically close to x_0 have such masses. They are infinitely many and hence Zariski dense in X. None of them lies in Z. Since membership in Z is a first-order statement over $\mathbb{Q}$, it transfers between algebraically closed fields of characteristic 0. Therefore Z is not cofinite and hence finite. □

For masses along a curve we can make the parameter a further unknown. Let $G_1, \dots, G_n$ be nonzero polynomials in $\xi = (\xi_1, \dots, \xi_N)$ and a parameter s, with coefficients $e_{\nu,a}(\theta)$, where $e_{\nu,a} \in \mathbb{Z}[t]$ is not divisible by f. We now take the prevariety and its cells in the variables (ξ, s).

Theorem 6.5. *Let ℓ be a prime and $\bar{\theta} \in \mathbb{F}_\ell$ such that ℓ does not divide the leading coefficient of f, $\bar{f}$ is squarefree of degree* $\deg f$*, $\bar{f}(\bar{\theta}) = 0$, and $e_{\nu,a}(\bar{\theta}) \neq 0$ for all ν and a.*

- *(a) If 1 lies in the ideal of $\mathbb{F}_\ell[\xi_0, \xi, s]$ generated by $\bar{G}_1, \dots, \bar{G}_n$ and $\xi_0\xi_1 \cdots \xi_N s - 1$, and in the corresponding ideal of the initial systems of every cell, then for every root θ of f and every $s \in \mathbb{C}^*$ the system $G(\cdot, s)$ has no zero in $(\mathbb{C}^*)^N$.*
- *(b) Let $h \in \mathbb{Z}[s]$ with $\bar{h} = h \bmod \ell$ of degree* $\deg h$ *and $\bar{h}(0) \neq 0$. If over $\bar{\mathbb{F}}_\ell$ the full system, together with $\bar{h}(s) = 0$, has no zero with all coordinates nonzero, and the same holds for the initial system of every cell whose relative interior contains a point with s-coordinate 0, then for every root θ of f and every root s_0 of h the system $G(\cdot, s_0)$ has no zero in $(\mathbb{C}^*)^N$.*

Proof. (a) is the proof of Theorem 6.2 with s as one more coordinate of the torus. (b) We extend the Hensel embedding of Step 2 to $\mathbb{Q}(\theta, s_0)$. Since $\bar{h}$ has full degree and $\bar{h}(0) \neq 0$, all roots of h in $\bar{\mathbb{Q}}_\ell$ are units. In Step 3 the vector $\omega = -v(\xi, s_0)$ therefore has s-coordinate 0. The residue of s_0 is a root of $\bar{h}$. The reduction is a zero of the full system with $\bar{h}(s) = 0$ (if $\omega = 0$) or of the initial system of the cell containing ω. □

The hypotheses of (a) fail for K14 and K15 along a family on which these systems have solutions; on such a family, however, Corollary 6.4 shows that the solutions occur at finitely many points if the hypotheses of Theorem 6.2 hold at one of its points. We will use part (b) for K14 at the points of the family T_9 at which the label set changes (§9.10).

6.3. **The certified points.**

Definition 6.6. Let $m \in (\mathbb{R}_{>0})^5$ and $T \subseteq \{1, \dots, 5\}$. The point m is *K-certified with transversal* T if

- (i) for every $j \in T$, K14(m_{Q_j}) has no solution; and
- (ii) there are a set of labels S with pieces$(S(m)) \subseteq$ pieces(S) and a D5 hull V valid at m (Definition 4.3) such that the final check of (S, T) holds: some $l_1, \dots, l_r \in \mathbb{Z}^{10}$ satisfy (a) and (b) of Definition 4.2 for the attained faces of pieces(S), built with V, without those of all pieces `d2` and `d9`, and of `d14a`(j), `d14b`(j, k) and `d15`(j, k), $j \in T$, $k \in Q_j$.

Note that this is the notion of §4, with the set of removed clusters made explicit.

Proposition 6.7. *A K-certified point is settled.*

Proof. Let $C \subseteq Y_m$ be a nontrivial curve. By Corollary 3.14 and (i), no singular place of C has its type in the attained set of a piece $\mathtt{d14a}(j)$ with $j \in T$. By §4.2 none has its type in that of a piece d2, d9, d14b or d15. The proof of Proposition 4.4, with these faces removed and Lemma 4.1 applied to the functionals of (ii), shows that there is no such curve. □

Exact points. Every point below is given exactly, by an irreducible $f \in \mathbb{Z}[t]$, a rational isolating interval of a root θ (one sign change and either $f' \neq 0$ on the interval in ball arithmetic or one sign variation in Descartes' rule), and masses $g_i(\theta)$ with $g_i \in \mathbb{Q}[t]$. We compute its label set $S(m)$. A label polynomial vanishes if it reduces to 0 modulo f. It does not vanish if a ball value excludes 0 or its reduction modulo f is nonzero. This gives a set S with $\text{pieces}(S(m)) \subseteq \text{pieces}(S)$, as Definition 6.6 requires.

Table 4 lists the points, with their point records in the supplementary material. They are the jump points of the curve families of §9 that are taken from this section (Table 8), in other labellings and scalings. Among them is the point $\mathbf{h}_3 = \big(u_*^{3/2}(1+u_*)^{-2}, u_*, u_*, 1, 1\big)$, with u_* the root in $(0,1)$ of $u^4 - 4u^3 - 6u^2 - 4u + 1$. After a relabelling and a scaling it is a jump point of T_1. The point records in one row of the table are conjugate: they belong to roots of one polynomial f with the same g_i.

point	used in	transversal	runs	final check	point records
$\mathbf{h}_3$	T_1	$\{1\}$	5	V_2^{w}	48_0, 48_1
$(1,1,1,1,1)$	T_3, §9	$\{1,\dots,5\}$	5	V_2^{w}	53_0
two points	T_5	$\{3\}$	5	V_N	20_0, 20_1
$(\frac{1}{9}, \frac{1}{9}, \frac{1}{4}, 1, 1)$	T_7	$\{1\}$	5	V_2^{w}	22_0
			20	4 on V_2^{w}, 2 on V_N	6 point records

TABLE 4. The K-certified points, with the point, sorted and scaled to $m_5 = 1$, where it is short. The last column gives their point records, the names of their files in the supplementary material. Where a row names one point and lists several point records, these are that point in other labellings and scalings. A run of Theorem 6.2 certifies, for one cluster, one of its five systems for one choice of f and the cluster masses, and the point records of one row share their runs. Each row has one run of K14, which is used in the proof, and four runs of K15, which are cross-checks (§12.3). At $(1,1,1,1,1)$ the five clusters have the same masses and share these runs. Each point record has one final check. It holds relative to the hull listed, with the functionals stored in the supplementary material.

Theorem 6.8. *(a) At the point of every point record of Table 4 and every j in its transversal,* $\mathrm{K14}(m_{Q_j})$ *has no solution, and the pieces* $\mathtt{d14a}(j)$ *and* $\mathtt{d14b}(j,k)$*, $k \in Q_j$, are therefore attained at no such point.*

(b) The point of every point record of Table 4 is K-certified with the transversal listed. Hence it is settled.

Proof. (a) For each point and each j in its transversal, the hypotheses of Theorem 6.2 hold for K14. In each run of Table 4 we obtain the conclusion "no solution". A run applies to the point of the table, because its input data are those of the exact point. Theorem 3.13 turns (i) of Definition 6.6 into the statement on the pieces.

(b) Table 4 gives, for each point record, the hull of Remark 4.5 relative to which its final check holds. The hull V_N is valid at the two point records at which it is used (§3.3), by the check of §12.2. Proposition 6.7 completes the proof. □

The point $(1,1,1,1,1)$ of Table 4 has the transversal $\{1,\dots,5\}$. Each cluster has the masses $(1,1,1,1)$ there. Theorem 6.8(a) shows that K14$(1,1,1,1)$ has no solution, a fact that will be used in §9. We will apply Theorems 6.2 and 6.5 at further points of the curve families, among them the jump points of T_9 (§9.10), and count these points there.

7. Diagram 2 does not occur

In this section we prove that no singular place has diagram 2, at any positive mass (Proposition 7.1), so that the pieces $\mathsf{d2}[T]$ may be left out of every certificate (§4.2). A place of diagram 2 forces the torque of its triangle to vanish at leading order. For a triangle that is not equilateral, closed forms of the first obstructions leave twelve mass vectors of the triangle, at which an inequality, the T-test of Theorem 7.3, shows that the torque cannot vanish; equilateral triangles are excluded directly (Lemma 7.2). We prove Lemma 7.2 and Theorem 7.3, which carry this argument, in Appendix E. The proof uses no result of §§8–9.

Let $T=\{a<b<c\}$ be a triple with complement $\{d<e\}$. A singular place of diagram 2 with triangle T has type $\mathsf{a}_1(T)$ (Table 1), the vertex of the piece $\mathsf{d2}[T]$. No other diagram has a type there (the attained set of $\mathsf{d1}[T]$ is the open segment). The piece $\mathsf{d2}[T]$ is present for every label set (Table 3).

Proposition 7.1. *Let $m\in(\mathbb{R}_{>0})^5$, and let $C\subseteq Y_m$ be an irreducible curve on which some r_{kl}^2 is not constant and $U\not\equiv 0$. No singular place of C has diagram 2. Consequently Theorem 3.12 holds with the pieces $\mathsf{d2}[T]$ removed from* pieces$(S(m))$.

Leading solutions. At a singular place of diagram 2 with triangle T, after a relabelling that puts the segment at the bodies $\{d,e\}$ and the triangle at T, the leading coefficients form a *leading solution* y_0: a complex central configuration ω of the triangle (of Euler or Lagrange type, possibly complex), a sign vector $\sigma\in\{\pm1\}^T$, the ratio of the segment masses, and the coefficients of the cross distances determined by them. The leading system forces the *torque* $\tau(y_0)$, a linear combination of the cross coefficients, to vanish. Let θ_k, $k\in T$, be the cross coefficients normalized by the triangle positions, and call θ *constant* if $\theta_a=\theta_b=\theta_c$. The leading solution is *simple* if the Jacobian J_0 of the leading system at y_0 has rank 16, and a *triple point* if the triangle parts of u and v are proportional. Triple points have rank 15. Appendix E gives the definitions in full.

Lemma 7.2. *At a singular place of diagram 2 with triangle T the leading solution y_0 has $\tau(y_0)=0$, and one of the following holds:*

(i) *y_0 is simple, θ is not constant, and the obstruction η_2 of the second order vanishes;*
(ii) *y_0 is simple, θ is constant, and the obstruction Ω_3 of the third order vanishes;*
(iii) *y_0 is a triple point, θ and σ are constant, and the product $b_3F_{ss}F_{TT}$ of the coefficients of the quadratic terms of its Lyapunov–Schmidt reduction vanishes;*
(iv) $\operatorname{rank} J_0\le 15$ *and y_0 is not a triple point.*

No leading solution whose triangle is equilateral satisfies (i)–(iv)*, at any positive masses.*

Outline. We give the proof in Appendix E. A left null vector of J_0 applied to the rows gives $T\tau+T^3\sigma_3$ in the local parameter T, whence $\tau(y_0)=0$. At a simple y_0 the first-order correction y_1 is unique. There the coefficient of T^2 gives $\eta_2=D\tau(y_0)\cdot y_1=0$. Here η_2 is a nonzero multiple of $(\theta_a-\theta_b)(\theta_a-\theta_c)\,w$, where w is a response of the triangle block of J_0. It vanishes identically when θ is constant, in which case $y_1=0$ and the first coefficient that can be nonzero is Ω_3,

at T^3. At a triple point the rank is 15 and the expansion forces θ to be constant. There the linear terms vanish after the reduction to the kernel, and the quadratic coefficients must satisfy $b_3 F_{ss} F_{TT} = 0$. For an equilateral triangle:

- at equal masses and constant σ, $|\Omega_3| \geq 0.853$ for the Lagrange realization, and b_3, F_{ss}, F_{TT} are nonzero for the triple realization;
- at equal masses and nonconstant σ the torque does not vanish;
- at unequal masses θ is not constant and $\eta_2 \neq 0$ for the Lagrange realization, while the triple realization is excluded because θ would have to be constant. □

The last statement of Lemma 7.2 excludes the equilateral triangles. The following theorem excludes all other triangles.

Theorem 7.3. *For every positive mass vector, every σ, every $\rho > 0$ and either labelling of the segment, no leading solution with $\tau = 0$ whose triangle is not equilateral satisfies* (i)–(iv) *of Lemma 7.2.*

Outline. We give the proof in Appendix E. On the set $\tau = 0$ the obstruction η_2 factors as $\gamma^2 H$. Here γ vanishes if and only if θ is constant, and H depends only on the central configuration of the triangle (Theorem E.10). Hence (i)–(iv) at a triangle that is not equilateral force constant θ or one of a few conditions on the triangle alone (Lemma E.17). We show that θ is not constant. We write the triangles of Euler type as the roots of quintics whose coefficients are polynomials in the mass ratios, and we eliminate the shape by resultants. Up to a common factor, the conditions then hold only at twelve mass vectors of the triangle: the permutations of $(1, 1, 5)$ and $(1, 1, 2+2\sqrt{2})$ for the Euler type, and those of a vector $(X_L, Y_L, 1)$ for the Lagrange type. At each of these triangles the *T-test* holds: some $k \in T$ has $m_k|\omega_k|^{-1/2} > \sum_{l \in T \setminus k} m_l |\omega_l|^{-1/2}$. The T-test excludes $\tau = 0$ for every σ, every segment ratio and either labelling of the segment. □

Proof of Proposition 7.1. Suppose that a singular place of diagram 2 with triangle T exists at m. By Lemma 7.2 its leading solution y_0 has $\tau(y_0) = 0$ and satisfies one of (i)–(iv). The last statement of Lemma 7.2 excludes an equilateral triangle, and Theorem 7.3 excludes the other triangles. Thus no such place exists, and the piece $\mathsf{d2}[T]$ may be removed in Theorem 3.12. □

8. The label sets of all positive masses

In this section we prove that the label set of every positive mass vector with $m_1 \leq \cdots \leq m_5$ is certified, contains a curve set, or is the label set of one of two exceptional points (Theorem 8.2). We decide this by exact computer algebra on the sixteen strata of the sorted masses. With §9 the theorem gives finitely many central configurations at every positive mass vector off the special rays (Corollary 8.3). Such a mass vector is settled unless it is a positive multiple of a permutation of the point P_1 of §10.4. We call these mass vectors the *counted points*. At them we count the central configurations (Theorem 10.7).

8.1. **Sorted masses and the reduction.** For $m \in (\mathbb{R}_{>0})^5$ let $m_{(1)} \leq \cdots \leq m_{(5)}$ be its coordinates in increasing order. The *sorted scaled image* of m is

$$\hat{m} = \big(m_{(1)}, m_{(2)}, m_{(3)}, m_{(4)}, m_{(5)}\big)/m_{(5)}.$$

The sorted scaled images are the mass vectors with $0 < m_1 \leq m_2 \leq m_3 \leq m_4 \leq m_5 = 1$. Among them are the points $\mathbf{b}_1 = (\frac{1}{4}, 1, 1, 1, 1)$ and $\mathbf{b}_2 = (\frac{1}{4}, \frac{1}{4}, 1, 1, 1)$.

A *face* F is a set of consecutive pairs $(i, i+1)$, $1 \leq i \leq 4$. We write it as the partition of 12345 into the runs of bodies joined by it, its *blocks*. For instance $1|23|45$ is the face $\{(2, 3), (4, 5)\}$ and $1|2|3|4|5$ the empty face. The *stratum* of F is

$$\Sigma_F = \{m : \ 0 < m_1 \leq \cdots \leq m_5 = 1, \ m_i = m_{i+1} \text{ for } (i, i+1) \in F, \ m_i < m_{i+1} \text{ for } (i, i+1) \notin F\}.$$

At a point of Σ_F two bodies have equal masses if and only if they lie in one block of F. The sorted scaled images form the disjoint union of the sixteen strata. The point $\mathbf{b}_1$ lies in $\Sigma_{1|2345}$ and $\mathbf{b}_2$ in $\Sigma_{12|345}$.

Lemma 8.1. *Let $m \in (\mathbb{R}_{>0})^5$.*

(a) *The planar five-body problem with masses m has finitely many central configurations up to similarity if and only if it has finitely many with masses $\hat{m}$.*

(b) *$\hat{m} = \mathbf{b}_1$ if and only if m is a positive multiple of a permutation of $(1,4,4,4,4)$, and $\hat{m} = \mathbf{b}_2$ if and only if m is a positive multiple of a permutation of $(1,1,4,4,4)$.*

Proof. (a) This is Lemma 2.2(c), since $\hat{m}$ is obtained from m by a permutation and a positive scaling.

(b) The sorted scaled image of a positive multiple of a permutation of $(1,4,4,4,4)$ is $(\frac{1}{4},1,1,1,1) = \mathbf{b}_1$. Conversely, if $\hat{m} = \mathbf{b}_1$, then m is a permutation of $m_{(5)}\mathbf{b}_1 = \frac{m_{(5)}}{4}(1,4,4,4,4)$. The same argument applies to $(1,1,4,4,4)$ and $\mathbf{b}_2$. □

Theorem 8.2 below concerns sorted masses. On the rays of Lemma 8.1(b) we will count the central configurations directly in §10.

8.2. **Labels and certified label sets.** We use the following 65 labels:

- $\mathsf{E}(T)$ for each of the 10 triples T;
- $\mathsf{R}(ab;s)$ for each of the 30 choices of a pair and a third body;
- $\mathsf{H}(D_1|D_2)$ for each of the 15 partitions of a four-body set into two pairs, read as $\mathcal{D}_3(s;D_1|D_2)$ (Theorem 3.9);
- $\mathsf{EQ}(a,b)$ for each of the 10 pairs.

For $m \in (\mathbb{R}_{>0})^5$ let $\Lambda(m)$ be the set of these labels that hold at m. The label $\mathsf{D6}(ab|cd)$ is implied by $\mathsf{EQ}(a,b)$ and $\mathsf{EQ}(c,d)$ (§3.4). Hence every piece of pieces$(S(m))$ lies in pieces(S) whenever $\Lambda(m) \subseteq S$, and Proposition 4.4 applies to any label set $S \supseteq \Lambda(m)$.

In this section a label set S is *certified* if it has a certificate relative to V_2^{w} for the list $\mathcal{A}_{V_2^{\mathrm{w}}}(S)$ without the attained faces of the pieces $\mathsf{d2}[T]$, $\mathsf{d9}(T;ax,by)$, $\mathsf{d14b}$ and $\mathsf{d15}$, which we may leave out by §4.2. Proposition 4.4 applies to such a set, since V_2^{w} is valid at every positive mass (Remark 4.5).

For each of the curve types T_1, T_2, T_3, T_4, T_5, T_7, T_8, T_9 and the line RL of §9, Table 5 gives a set D_T of labels. The images of these sets under the 120 relabellings of the bodies are the *curve sets*. We treat the mass vectors whose label set contains a curve set in §9.

T	D_T	T	D_T
T_1	$\mathsf{R}(23;1)$, $\mathsf{EQ}(2,5)$, $\mathsf{EQ}(3,4)$	T_5	$\mathsf{E}(123)$, $\mathsf{R}(24;1)$, $\mathsf{EQ}(4,5)$
T_2	$\mathsf{R}(23;1)$, $\mathsf{EQ}(3,4)$, $\mathsf{EQ}(3,5)$, $\mathsf{EQ}(4,5)$	T_7	$\mathsf{E}(123)$, $\mathsf{EQ}(2,5)$, $\mathsf{EQ}(3,4)$
T_3	$\mathsf{EQ}(a,b)$ for all pairs $a<b$ in $\{1,2,3,4\}$	T_8	$\mathsf{E}(123)$, $\mathsf{R}(24;1)$, $\mathsf{R}(25;1)$
T_4	$\mathsf{R}(23;1)$, $\mathsf{R}(25;1)$, $\mathsf{EQ}(3,4)$	T_9	$\mathsf{E}(123)$, $\mathsf{R}(24;1)$, $\mathsf{EQ}(2,3)$
RL	$\mathsf{E}(123)$, $\mathsf{EQ}(2,3)$, $\mathsf{EQ}(2,4)$, $\mathsf{EQ}(3,4)$		

TABLE 5. The label sets D_T of the curve families.

8.3. **The theorem.** Let $\mathbf{q}_*$ be the only mass vector with $0 < m_1 < m_2 < m_3 < m_4 < m_5 = 1$ at which $\mathsf{E}(125)$ and $\mathsf{H}(25|34)$ hold (Proposition 8.7). Then

$$\mathbf{q}_* = (0.1243983532\ldots,\ 0.2968967473\ldots,\ 0.4494254022\ldots,\ 0.6606140769\ldots,\ 1).$$

Let $\mathbf{p}_* = (\frac{1}{16},\frac{1}{9},\frac{1}{4},1,1)$.

Theorem 8.2. *Let $m \in (\mathbb{R}_{>0})^5$ with $m_1 \le m_2 \le m_3 \le m_4 \le m_5$. Then one of the following holds:*

(i) *$\Lambda(m)$ is certified;*
(ii) *$\Lambda(m)$ contains a curve set;*
(iii) *a positive multiple of m is $\mathbf{q}_*$ or $\mathbf{p}_*$.*

In case (iii) the hull V_N is valid at $\mathbf{q}_*$ and $\mathbf{p}_*$, and it certifies their label sets with all pieces d14a present (Proposition 8.7). We prove Theorem 8.2 in §8.8. With §9 it gives the following corollary.

Corollary 8.3. *Let $m \in (\mathbb{R}_{>0})^5$ lie on no special ray. Then the planar five-body problem with masses m has finitely many central configurations up to similarity, and m is settled unless it is a counted point.*

8.4. **Coordinates and labels on a stratum.** Let F be a face with the blocks $B_1, \dots, B_{n+1}$, numbered in increasing order of their bodies. At a point m of Σ_F put $x_i = m_i^{-1/2}$, as in §3.4. Then $x_1 \ge x_2 \ge \cdots \ge x_5 = 1$. We write y_k for the common value of x_i on the bodies i of B_k. Then $y_1 > y_2 > \cdots > y_n > y_{n+1} = 1$. The map $m \mapsto (y_1, \dots, y_n)$ identifies Σ_F with the open set $\{y \in \mathbb{R}^n : \ y_1 > \cdots > y_n > 1\}$, and we use this identification throughout. On Σ_F the label $\mathsf{EQ}(a,b)$ holds if and only if a and b lie in one block of F. Let $Q(F)$ be the set of these labels.

We describe the 55 labels E, R and H in x, at positive x (§3.4 and Appendix D).

- $\mathsf{E}(abc)$: its polynomial times $(x_a x_b x_c)^4$ is $-(x_a + x_b + x_c)(-x_a + x_b + x_c)(x_a - x_b + x_c)(x_a + x_b - x_c)$. The first factor is positive. So $\mathsf{E}(abc)$ holds if and only if x lies on one of the hyperplanes $x_a = x_b + x_c$, $x_b = x_a + x_c$ and $x_c = x_a + x_b$.
- $\mathsf{R}(ab;s)$: its polynomial times $x_s^4 (x_a x_b)^8$ is $\big((x_a^2 + x_b^2)^2 - x_s^2 x_a x_b\big)\big((x_a^2 + x_b^2)^2 + x_s^2 x_a x_b\big)$. The second factor is positive. So $\mathsf{R}(ab;s)$ holds if and only if $(x_a^2 + x_b^2)^2 = x_s^2 x_a x_b$.
- $\mathsf{H}(ij|kl)$ with centre s, read as $\mathcal{D}_3$: put $A = x_i^3$, $B = x_j^3$, $A' = x_k^3$, $B' = x_l^3$, $D = \prod_{q \in Q_s} x_q^2$ and $N = D M_{Q_s}$, a polynomial in x. Let $c = x_s^2 M_{Q_s}^{-1/2} > 0$. Then $\mathrm{Tri}(ij;s)$ says that $c^2 = (A + \epsilon B)^2$ for a sign ϵ, and likewise for $\mathrm{Tri}(kl;s)$. Since $c^2 N = x_s^4 D$, the condition $\mathcal{D}_3$ holds if and only if x is a zero of one of the eight ideals

$$\big(x_i x_j - x_k x_l,\ A + \epsilon_1 B - \delta(A' + \epsilon_2 B'),\ (A + \epsilon_1 B)^2 N - x_s^4 D\big), \qquad \epsilon_1, \epsilon_2, \delta \in \{1, -1\}.$$

The three hyperplanes of $\mathsf{E}(abc)$, the quartic of $\mathsf{R}(ab;s)$ and these eight ideals are the *branch ideals* of the labels. After the substitution of y_k for x_i, $i \in B_k$, they become ideals of $\mathbb{Q}[y] = \mathbb{Q}[y_1, \dots, y_n]$. Thus a label E, R or H holds at a point of Σ_F if and only if the point is a zero of one of its branch ideals. We also fix the *x-polynomials* of each label: for $\mathsf{E}(abc)$ the product of the four linear factors above; for $\mathsf{R}(ab;s)$ the quartic $(x_a^2 + x_b^2)^2 - x_s^2 x_a x_b$; for $\mathsf{H}(ij|kl)$ the polynomial $x_k^2 x_l^2 - x_i^2 x_j^2$ and N^2 times the polynomials of $\mathrm{Tri}(ij;s)$ and $\mathrm{Tri}(kl;s)$ in Appendix D, written in x. At positive x a label holds if and only if all its x-polynomials vanish.

8.5. **Flats.** We compute standard bases, minimal primes and saturations over $\mathbb{Q}$ with Singular 4.4.1 (class (C1)).

Filters. An ideal J of $\mathbb{Q}[y]$ *passes the filters* if

(F1) $J \ne \mathbb{Q}[y]$;
(F2) no element of degree at most 12 of the reduced standard basis of J becomes, under the substitution $y_j = 1 + z_j + \cdots + z_n$, a nonzero polynomial whose nonzero coefficients all have one sign;
(F3) the affine-linear elements of this basis have a common zero in Σ_F, which we decide by Fourier–Motzkin elimination over $\mathbb{Q}$ (class (C2)).

The substitution maps $(\mathbb{R}_{>0})^n$ onto Σ_F, and a nonzero polynomial whose coefficients have one sign does not vanish on $(\mathbb{R}_{>0})^n$. Hence an ideal has no zero in Σ_F if it fails a filter.

Decompositions. For ideals P and B of $\mathbb{Q}[y]$ we compute a list $L(P, B)$ of ideals. Each member contains $P + B$ and passes the filters, and every zero of $P + B$ in Σ_F is a zero of some member. If $P + B$ is zero-dimensional or B contains P, the list consists of $P + B$, unless $P + B$ fails a filter and the list is empty. Otherwise we add the generators of B to P one at a time, in order of increasing degree. After each step we replace each ideal of positive dimension by its minimal primes, keep each zero-dimensional ideal as it is and drop the ideals failing (F1) or (F2). If a decomposition does not finish within a time limit, we redo it with the saturation by $y_1 \cdots y_n$ in place of the minimal primes of each ideal of positive dimension. Saturation keeps all zeros in Σ_F, since no coordinate vanishes there. At the end we drop each member failing (F3) or containing another one. The members are prime ideals, zero-dimensional ideals or saturated ideals. We do not use that they are prime.

Components and flats. For a label ℓ of the kinds E, R, H the *components* of ℓ on Σ_F are the members of the lists $L((0), B)$, B a branch ideal of ℓ. A *flat* of F is an ideal P of $\mathbb{Q}[y]$ with a set $G(P)$ of labels of these kinds, each of which holds at every zero of P in Σ_F. The *root* is $P = (0)$ with $G(P) = \emptyset$. Let P be a flat and $\ell \notin G(P)$ a label of these kinds. The *children* of P via ℓ are the members other than P of the lists $L(P, C)$, C a component of ℓ. If P itself is such a member, we add ℓ to $G(P)$. Each child P' is a flat. The set $G(P')$ consists of the labels whose x-polynomials lie in P', together with $G(P)$ and ℓ. We keep $G(P') \supseteq G(P) \cup \{\ell\}$ when $G(P)$ grows. The zero set of a child lies in those of P and of C. So these labels hold at its zeros in Σ_F.

Lemma 8.4. *Let P be a flat of F, let $m \in \Sigma_F$ be a zero of P, and let $\ell \in \Lambda(m) \setminus (Q(F) \cup G(P))$.*

(a) *The point m is a zero of $P + C$ for a component C of ℓ, and $P + C$ passes the filters.*
(b) *If the children of P via ℓ are computed, m is a zero of one of them.*

Proof. (a) Since ℓ holds at m, the point m is a zero of a branch ideal B of ℓ, hence of a member C of $L((0), B)$. So m is a zero of $P + C$ in Σ_F, and $P + C$ passes the filters. (b) The point m is a zero of a member of $L(P, C)$. This member is not P, since otherwise ℓ would lie in $G(P)$. So it is a child of P via ℓ. □

Expansion and closure. We call a label $\ell \notin G(P)$ of the kinds E, R, H *possible* on P if $Q(F) \cup G(P) \cup \{\ell\}$ contains no curve set and $P + C$ passes the filters for some component C of ℓ. To *expand* P, we first find its possible labels. If the set $U_A(P)$ of the labels of $Q(F)$ and $G(P)$ and of the possible labels is certified, we call P *closed* and do not compute its children. Otherwise we compute the children of P via all possible labels. If then the set $U(P)$ of the labels of $Q(F)$ and $G(P)$ and of the labels via which P has a child is certified, we call P closed as well. We call P *expanded* if its children via all possible labels are computed.

Lemma 8.5. *Let P be a closed flat of F, and let $m \in \Sigma_F$ be a zero of P. Then $\Lambda(m)$ is certified or contains a curve set.*

Proof. The labels of $Q(F)$ and $G(P)$ lie in $\Lambda(m)$. Suppose that $\Lambda(m)$ contains no curve set, and let $\ell \in \Lambda(m) \setminus (Q(F) \cup G(P))$. Then $Q(F) \cup G(P) \cup \{\ell\}$ contains no curve set, and ℓ is possible on P by Lemma 8.4(a). If the children of P are computed, P has a child via ℓ, by Lemma 8.4(b). Hence $\Lambda(m)$ lies in $U_A(P)$, and in $U(P)$ if the children of P are computed. Since P is closed, $U_A(P)$ is certified, or the children of P are computed and $U(P)$ is certified. A subset of a certified set is certified (§4.2). □

8.6. **The computation.** On each face F we expand the root and then every flat that we obtain, except the flats P for which $Q(F) \cup G(P)$ contains a curve set and the *covered* flats,

the children of closed or covered flats. We decide whether a label set is certified in the sense of §8.2 by the search of §4.3. We check every certificate exactly as there. A label set for which the search finds no certificate counts as uncertified. This only adds work. Because the choice of the decompositions that are redone with saturations depends on the time limit and on the load of the machine, the proof concerns the computed flats, which are stored in the supplementary material.

Proposition 8.6. *For each face F the computation gives finitely many flats. Table 6 gives their numbers summed over the faces. Every flat P of F satisfies one of the following:*

(a) *$Q(F) \cup G(P)$ contains a curve set;*
(b) *P is closed or covered;*
(c) *$Q(F) \cup G(P)$ is certified, and P is expanded;*
(d) *P is zero-dimensional and expanded, and each of its zeros in Σ_F is $\mathbf{q}_*$ or $\mathbf{p}_*$.*

dimension	flats	(a)	(b)	(c)	(d)
4	1			1	
3	24			24	
2	177		14	163	
1	538	6	126	406	
0	1466	12	1377		77
all	2206	18	1517	594	77

TABLE 6. The flats of the sixteen faces, by dimension and by the first of the cases (a)–(d) of Proposition 8.6 that they satisfy. Of the 1517 flats of (b), 1510 are closed. The empty face has 1979 flats.

Proof. The flats, their sets G, their lists of children and the closures are the outcome of the computation. The proof of Theorem 8.2 uses the lists of children of 35545 pairs (P, ℓ), many of them empty. On the empty face 22 of the 55 labels have a component: the ten labels E, the ten labels $\mathsf{R}(ab; s)$ with $s < a < b$, and $\mathsf{H}(25|34)$ and $\mathsf{H}(15|34)$. On this face 110 decompositions are redone with saturations, and 60 flats are saturated ideals. They all have dimension 1 and satisfy (b) or (c). The computation decides 2599 label sets: 1840 are certified, 703 have an exact witness in the sense of §9.8, and 56 remain undecided.

The 77 flats outside (a)–(c) are zero-dimensional and expanded. Of them 73 belong to the empty face, three to 12|3|4|5 and one to 1|2|3|45. We decompose each of them into primes. We add the squarefree parts of its univariate eliminants, which gives its radical (Seidenberg's lemma). We take a linear form whose minimal polynomial on the radical has as many roots as the radical has zeros, and we factor this polynomial over $\mathbb{Q}$. This gives 334 primes. Of them 310 fail (F2) and 18 fail (F3). For each of the other six we compute a shape basis for a separating linear form and isolate the real roots of its univariate polynomial exactly (class (C2)). Two of these roots give points of Σ_F: the point $\mathbf{q}_*$, a zero of a prime of degree 48 of the empty face, and the point $\mathbf{p}_*$ of 1|2|3|45. The other four primes have no real zero in Σ_F. □

8.7. **The two exceptional points.**

Proposition 8.7. (a) *Exactly one mass vector with $0 < m_1 < m_2 < m_3 < m_4 < m_5 = 1$, the point $\mathbf{q}_*$, has $\mathsf{E}(125)$ and $\mathsf{H}(25|34)$ in its label set. Moreover $\Lambda(\mathbf{q}_*) = \{\mathsf{E}(125), \mathsf{H}(25|34)\}$ and*

$$\Lambda(\mathbf{p}_*) = \{\mathsf{E}(124), \mathsf{E}(125), \mathsf{E}(234), \mathsf{E}(235), \mathsf{E}(345), \mathsf{R}(45; 3), \mathsf{EQ}(4, 5)\}.$$

(b) *The hull V_N is valid at $\mathbf{q}_*$ and at $\mathbf{p}_*$.*
(c) *Let S be $\Lambda(\mathbf{q}_*)$ or $\Lambda(\mathbf{p}_*)$. Then S has a certificate relative to V_N for the list $\mathcal{A}_{V_N}(S)$ without the attained faces of the pieces* $\mathsf{d2}[T]$, $\mathsf{d9}(T; ax, by)$, $\mathsf{d14b}$ *and* $\mathsf{d15}$.

Proof. (a) The mass vectors in question form the stratum $\Sigma_{1|2|3|4|5}$. There $\mathsf{E}(125)$ has one component, the hyperplane $y_1 = y_2 + 1$, and $\mathsf{H}(25|34)$ has one component, a curve. Every point of $\Sigma_{1|2|3|4|5}$ at which both labels hold is a zero of each component, hence of their sum. This sum is a prime ideal with 48 zeros, and exactly one of them lies in $\Sigma_{1|2|3|4|5}$ (proof of Proposition 8.6). The x-polynomials of $\mathsf{E}(125)$ and $\mathsf{H}(25|34)$ lie in this ideal. For every other label a ball enclosure at $\mathbf{q}_*$ of one of its x-polynomials excludes 0. At $\mathbf{p}_*$ we have $x = (4, 3, 2, 1, 1)$, and the label set follows by exact evaluation.

(b) Both points pass the check of §12.2. At $\mathbf{p}_*$ the triangle $\{3, 4, 5\}$ has the masses $(\frac{1}{4}, 1, 1)$ and is in the centred case.

(c) The search of §4.3 finds these certificates, and we check them exactly as there. □

Exact witnesses as in §9.8 show that neither $\Lambda(\mathbf{q}_*)$ nor $\Lambda(\mathbf{p}_*)$ is certified; but the certificates of (c) keep all pieces $\mathsf{d14a}$, so the two points need no point certificate of §6.

8.8. **Proofs of Theorem 8.2 and Corollary 8.3.**

Proof of Theorem 8.2. The labels are homogeneous in m. So we may scale m to $m_5 = 1$. Then m lies in Σ_F for one face F. We construct flats $P_0, P_1, \dots$ of F with the zero m such that $G(P_0) \subsetneq G(P_1) \subsetneq \cdots$. Let P_0 be the root, and let $P = P_r$ be constructed. The labels of $Q(F)$ and $G(P)$ lie in $\Lambda(m)$. We apply Proposition 8.6 to P.

- In case (a), $\Lambda(m)$ contains a curve set.
- In case (b), P is closed or covered. The zero set of a child lies in that of its parent, and a child strictly contains its parent. So, following parents, we reach a closed flat with the zero m. Lemma 8.5 gives (i) or (ii).
- In cases (c) and (d), suppose first that $\Lambda(m) = Q(F) \cup G(P)$. Then $\Lambda(m)$ is certified in case (c), and m is $\mathbf{q}_*$ or $\mathbf{p}_*$ in case (d). Otherwise pick $\ell \in \Lambda(m) \setminus (Q(F) \cup G(P))$. If $Q(F) \cup G(P) \cup \{\ell\}$ contains a curve set, so does $\Lambda(m)$. If not, ℓ is possible on P, by Lemma 8.4(a). Since P is expanded, m is a zero of a child P' of P via ℓ, by Lemma 8.4(b). Then $G(P')$ contains $G(P) \cup \{\ell\}$, and we put $P_{r+1} = P'$.

The sets $G(P_r)$ increase strictly, and they contain at most 55 labels. Hence the construction ends in one of the cases above. □

Proof of Corollary 8.3. Positive scaling and relabelling of the masses preserve finiteness and settledness, by the isomorphisms in the proof of Lemma 2.2. So we may assume $m = \hat{m}$. We apply Theorem 8.2 to m.

(i) If $\Lambda(m)$ is certified, Proposition 4.4 settles m (§8.2).
(ii) Let $\Lambda(m)$ contain $\sigma(D_T)$ for a name T of Table 5 and a relabelling σ. If m is a point of T_9 up to scaling and relabelling, Proposition 9.14 settles m. Otherwise Theorem 9.15 applies, since m lies on no special ray. So m has finitely many central configurations up to similarity, and it is settled unless it is a counted point.
(iii) If m is $\mathbf{q}_*$ or $\mathbf{p}_*$, then V_N is valid at m, and $\Lambda(m)$ has a certificate relative to V_N as in Proposition 8.7(c). We may leave out the faces of the pieces left out there, by §4.2. So Proposition 4.4 settles m.

A settled mass vector has finitely many central configurations up to similarity (Corollary 2.5). □

9. Curve families and the RL line

Case (ii) of Theorem 8.2 concerns the mass vectors whose label set contains a curve set, that is, one of the nine sets D_T of Table 5 after a relabelling. Each D_T defines a curve of mass vectors up to scaling (Lemma 9.1), along which a generic label set S_C holds identically. The pieces enabled by S_C include the pieces `d14a`, `d14b` and `d15` of the clusters. We settle the points of these curves by excluding such pieces point by point along each curve. In this section we do so along the whole curves. We prove that every point of T_9 is settled (Proposition 9.14), and that every point of these curves other than the points of T_9 and of the special rays has finitely many central configurations (Theorem 9.15).

9.1. **The families.** Table 7 lists the families of mass vectors defined by the sets D_T, with their parametrizations and generic label sets. The labels follow the supplementary files. The line RL plays the role of a sixth family. We define the family T_9, with the constant ρ_0, in §9.10 and treat it there.

	masses $(m_1,\dots,m_5)$; parameter	generic label set S_C
T_1	$(t^3,\,(1+t^2)^2,\,t^2(1+t^2)^2,\,t^2(1+t^2)^2,\,(1+t^2)^2)$; $t>0$	$\mathsf{R}(23;1)$ $\mathsf{R}(24;1)$ $\mathsf{R}(35;1)$ $\mathsf{R}(45;1)$ $\mathsf{EQ}(2,5)$ $\mathsf{EQ}(3,4)$
T_2	$(t^3,\,t^2(1+t^2)^2,\,(1+t^2)^2,\,(1+t^2)^2,\,(1+t^2)^2)$; $t>0$	$\mathsf{R}(23;1)$ $\mathsf{R}(24;1)$ $\mathsf{R}(25;1)$ $\mathsf{EQ}(3,4)$ $\mathsf{EQ}(3,5)$ $\mathsf{EQ}(4,5)$
T_3	$(a,a,a,a,1)$; $a>0$	$\mathsf{EQ}(i,j)$ $(1\le i<j\le 4)$
T_4	$(yr^5B,\,r^4g^2,\,rg^2B,\,rg^2B,\,r^5g^2B)$; $r>0$, $y=f^{1/2}$	$\mathsf{R}(23;1)$ $\mathsf{R}(24;1)$ $\mathsf{R}(25;1)$ $\mathsf{EQ}(3,4)$
T_5	$(q^6A^2,\,(1+q^4)^2A^2,\,q^6(1+q^4)^2,\,q^4(1+q^4)^2A^2,\,q^4(1+q^4)^2A^2)$; $q\neq 0$	$\mathsf{E}(123)$ $\mathsf{R}(24;1)$ $\mathsf{R}(25;1)$ $\mathsf{EQ}(4,5)$
T_7	$((1-s)^2,\,s^2(1-s)^2,\,s^2,\,s^2,\,s^2(1-s)^2)$; $s\neq 0,1$	$\mathsf{E}(123)$ $\mathsf{E}(124)$ $\mathsf{E}(135)$ $\mathsf{E}(145)$ $\mathsf{EQ}(2,5)$ $\mathsf{EQ}(3,4)$
T_8	$(q^2r^4fC^2,\,r^4g^2C^2,\,q^2r^4fg^2,\,fg^2C^2,\,r^4fg^2C^2)$; $r>0$, $q^4=f$	$\mathsf{E}(123)$ $\mathsf{R}(24;1)$ $\mathsf{R}(25;1)$
RL	$(4,1,4,4,\mu)$; $\mu>0$	$\mathsf{E}(123)$ $\mathsf{R}(13;2)$ $\mathsf{E}(124)$ $\mathsf{R}(14;2)$ $\mathsf{E}(234)$ $\mathsf{R}(34;2)$ $\mathsf{EQ}(1,3)$ $\mathsf{EQ}(1,4)$ $\mathsf{EQ}(3,4)$

Table 7. The families. Here $A=1+q^3+q^4$, $B=1+r+r^2$, $f=r^3+r^2+r$, $g=(1+r)(1+r^2)=1+f$ and $C=g+q^3$, with q real. On T_5 and T_8 both signs of q occur. Each S_C also contains the labels $\mathsf{D6}(ab|cd)$ with $\mathsf{EQ}(a,b)$ and $\mathsf{EQ}(c,d)$ in S_C (on T_1, T_3 and T_7), as $S(m)$ does (§3.6).

All masses in Table 7 are positive: $A\ge\frac{229}{256}$ for real q, and on T_8, $C=1+f+q^3\ge 1+f-f^{3/4}>0$. The *transversal cluster* of a family is $Q_1=\{2,3,4,5\}$ for T_1, T_2, T_4, T_5, T_7 and T_8, with masses proportional to

$$(1,c,c,1),\ c=t^2 \text{ on } T_1;\qquad (t^2,1,1,1) \text{ on } T_2;\qquad (r^3,B,B,r^4B) \text{ on } T_4;$$
$$(A^2,q^6,q^4A^2,q^4A^2) \text{ on } T_5;\qquad (1,c,c,1),\ c=(1-s)^{-2} \text{ on } T_7;$$
$$(r^4C^2,\,q^6r^4,\,q^4C^2,\,r^4q^4C^2) \text{ on } T_8.$$

On T_3 the clusters are Q_k, $k\le 4$, with masses proportional to $(1,1,1,1/a)$, and Q_5 with masses $(1,1,1,1)$. On RL it is $Q_2=\{1,3,4,5\}$ with masses $(4,4,4,\mu)$. Two cluster families recur, up to relabelling and scaling: $F_A(\nu)=(1,1,1,\nu)$ and $F_B(c)=(1,c,c,1)$.

Lemma 9.1. *Let T be one of the nine names of Table 5, and let $m\in(\mathbb{R}_{>0})^5$ satisfy every label of D_T. Then a positive multiple of m is a point of the family T, or of T_2 if $T=T_4$, or of T_5 if*

$T = T_8$. If $T = \mathrm{RL}$, a positive multiple of m is the image of a point of RL *under the exchange of the bodies* 1 *and* 2*. If $T = T_9$, it is a point of T_9 or such an image.*

Proof. With x_i as in §3.4, for positive masses $\mathsf{R}(ab;s)$ says $x_s = (x_a^2 + x_b^2)/(x_a x_b)^{1/2}$, and $\mathsf{E}(abc)$ says that one of x_a, x_b, x_c is the sum of the other two. The case T_3 is immediate.

T_1 and T_2: we scale so that $m_2 = m_5 = 1$ and $m_3 = m_4 = t^2$ with $t > 0$ (T_1), resp. $m_3 = m_4 = m_5 = 1$ and $m_2 = t^2$ (T_2). Then $\mathsf{R}(23;1)$ gives $m_1 = t^3/(1+t^2)^2$. So $(1+t^2)^2 m$ is the point t of the family.

T_7: scale so that $x_2 = x_5 = 1$. Also $x_3 = x_4$. If $x_1 + x_3 = 1$ or $x_1 = 1 + x_3$, put $s = x_1$. If $x_3 = 1 + x_1$, put $s = -x_1$. Then $s \neq 0, 1$, $x_1 = |s|$ and $x_3 = |1-s|$. So $s^2(1-s)^2 m$ is the point s of T_7.

T_4: we scale so that $x_5 = 1$, and write $x_3 = x_4 = r^2$, $x_2 = y$ with $r, y > 0$. Then $\mathsf{R}(23;1)$ and $\mathsf{R}(25;1)$ give $x_1 = (y^2 + r^4)/(y^{1/2} r) = (y^2+1)/y^{1/2}$, that is, $y^2(1-r) = r(1-r)(1+r+r^2)$. If $r = 1$, then $m_3 = m_4 = m_5$ and m satisfies D_{T_2}. Otherwise $y^2 = f(r)$ and m is a multiple of the point r of T_4.

T_5 and T_8: scale so that $x_5 = 1$ and write $x_2 = q^2$, $x_4 = r^2$ with $q, r > 0$. On T_5, $\mathsf{EQ}(4,5)$ gives $r = 1$. On T_8, $\mathsf{R}(24;1)$ and $\mathsf{R}(25;1)$ give $(q^4 + r^4)/(qr) = (q^4+1)/q$, that is, $q^4(1-r) = r(1-r)(1+r+r^2)$. Hence $r = 1$, and m satisfies D_{T_5}, or $q^4 = f(r)$. In both cases $x_1 = (1+q^4)/q$ and $x_1 - x_2 = (1 - q^3 + q^4)/q > 0$, since $1 - u^3 + u^4 \geq \frac{229}{256}$ for $u > 0$. Hence $\mathsf{E}(123)$ means $x_3 = (1 + q^4 \pm q^3)/q$. Replacing q by $-q$ in the case of the minus sign, $x_2 = q^2$, $x_1 = (1+q^4)/|q|$ and $x_3 = (1+q^3+q^4)/|q|$ with $q \in \mathbb{R} \setminus \{0\}$. Hence m is a multiple of the point q of T_5, resp. of the point (r, q) of T_8.

RL: since $x_2 = x_3 = x_4$, $\mathsf{E}(123)$ forces $x_1 = 2x_2$. Hence m is a multiple of $(1,4,4,4,\mu)$; the exchange of 1 and 2 maps the point μ of RL to this vector.

T_9: since $x_2 = x_3$, $\mathsf{E}(123)$ forces $x_1 = 2x_2$. Scale so that $x_4 = 1$ and put $x_2 = u^2$, $u > 0$. Then $\mathsf{R}(24;1)$ reads $2u^2 = (u^4+1)/u$, that is, $(u-1)(u^3 - u^2 - u - 1) = 0$. For $u = 1$, m is a multiple of $(1,4,4,4,\mu)$ as for RL. Otherwise $u = \theta_0$, the real root of $u^3 = u^2 + u + 1$, and $m = (\theta_0^{-4}/4, \theta_0^{-4}, \theta_0^{-4}, 1, m_5)$ with $\theta_0^{-4} = \rho_0$. □

9.2. **Settling a point of a family.** A piece is *excluded* at $m \in (\mathbb{R}_{>0})^5$ if no singular place P of a nontrivial curve $C \subseteq Y_m$ has its type $x(P)$ in the attained set of the piece. Let S be a set of labels, E a set of pieces and V a D5 hull. The pair (S, E) is *certified* relative to V if there are $l_1, \ldots, l_r \in \mathbb{Z}^{10}$ that satisfy (a) and (b) of Definition 4.2 for the list $\mathcal{A}_V(S)$ without the attained faces of the pieces of E and of all pieces d2 and d9. A certificate for (S, E) is one for every (S', E') with $S' \subseteq S$ and $E' \supseteq E$ (§4.2). A *minimal exclusion set* of S is a set E of pieces d14a, d14b, d15, minimal under inclusion, such that (S, E) is certified. In this section all certificates are relative to the hull V_2^{w} or to V_N (Remark 4.5).

Lemma 9.2. *Let $m \in (\mathbb{R}_{>0})^5$, let V be a D5 hull valid at m, let $S \supseteq S(m)$, and let E be a set of pieces each of which is excluded at m. If (S, E) is certified relative to V, then m is settled.*

Proof. This is the proof of Proposition 4.4, applied to $\mathcal{A}_V(S)$ without the faces of the pieces of E and of the pieces d2 and d9, which are excluded at every positive mass (§4.2): by Theorem 3.12, the validity of V and the definition of an excluded piece, at each singular place of a curve as there, the type lies in the relative interior of a face in this list. □

We exclude pieces d14a, d14b and d15 by the following results. We carry out the computations behind them for solutions of the rows over the field of Puiseux series with prescribed valuations. The expansions of the coordinates at a place are such solutions.

(X1) By Corollary 3.14, if K14(m_{Q_j}) has no solution, $\mathsf{d14a}(j)$ and all $\mathsf{d14b}(j,k)$ are excluded. At a mass point given exactly over a number field we decide this system by Theorem 6.2, one run of which covers all conjugates of the point.
(X2) By Theorem 5.21, each $\mathsf{d14b}(j,k)$ is excluded at every $m \in (\mathbb{R}_{>0})^5$.
(X3) At a place of type $\mathsf{b}_k(j)$, the leading coefficients of the cluster form a K15(m_{Q_j}, k) solution with a bad shape (Lemma 5.2 and Theorem 5.4). Thus $\mathsf{d15}(j,k)$ is excluded at m if no K15(m_{Q_j}, k) solution has a bad shape. If the three masses of $Q_j \setminus \{k\}$ are equal, $\mathsf{d15}(j,k)$ is excluded (Corollary 5.8).
(X4) By Corollary 5.19, each $\mathsf{d15}(j,k)$ is excluded at every $m \in (\mathbb{R}_{>0})^5$.

For a family with transversal cluster Q_1 we write $\pi(1,k) = \{\mathsf{d14b}(1,k), \mathsf{d15}(1,k)\}$ and $\Pi(1)$ for the nine pieces $\mathsf{d14a}(1)$, $\mathsf{d14b}(1,k)$, $\mathsf{d15}(1,k)$, $k \in Q_1$. Similarly for other clusters. By (X2) and (X4), each pair $\pi(j,k)$ is excluded at every $m \in (\mathbb{R}_{>0})^5$. A *special point* of a family is a point at which K14 has a solution for its transversal cluster (on T_3, for one of $Q_1, \dots, Q_4$).

Theorem 9.3. *Let T be one of T_1, T_2, T_3, T_4, T_5, T_7, T_8, and read* H *as $\mathcal{D}_3$ (Theorem 3.9). On each branch of T, $S(m) = S_C$ except at the* jump points*, where a label* E, R, H *or* EQ *outside S_C holds. At each jump point a label set containing $S(m)$ is determined. The numbers of the jump points on each family are those of Table 8.*

	T_1	T_2	T_3	T_4	T_5	T_7	T_8
jump points (Theorem 9.3)	7	3	2	8	11	9	18
points of §6	2	0	1	0	2	2	0
points of Table 9	1	2	1	0	2	7	0
other jump points	4	1	0	8	7	0	18
special points	16	6	6			32	
sets in the list $\mathcal{B}$ (Theorem 9.10)				1992	108	9	108

TABLE 8. Counts along the families: the jump points, split by their treatment in Proposition 9.4; the special points; and the numbers of sets with a witness in the proof of Theorem 9.10.

Proof. Each label is a disjunction of polynomial conditions (Appendix D). We compose every polynomial exactly with the masses of the family. On T_4 and T_8 we reduce it modulo the relation $y^2 = f$, resp. $q^4 = f$, and replace it by its norm over the conjugates of y, resp. q, a nonzero polynomial in r because the relation is irreducible over $\mathbb{Q}(r)$. We factor the result over $\mathbb{Q}$. We isolate the real roots of the factors in ball arithmetic at 400 bits or more and show that each root ball lies on one side of every boundary point of a branch, unless the boundary point is itself a root. On T_4 and T_8 a root r_0 is a jump point only if the label polynomial vanishes at the conjugate of the branch. We decide this by balls at the conjugates or, where they do not separate, by an exact gcd over $\mathbb{Q}(r_0)$. We decide an H label at a root of its polynomial by its two conditions Tri in ball arithmetic, and the H labels whose polynomial vanishes identically on a family through the exact gcd of their two Tri conditions. No label outside S_C holds identically on a branch: we show each of them false in ball arithmetic at two sample points of every branch. The identifications of jump points with the points of §6 and with the special rays are exact: the mass ratios have the same minimal polynomials, with a common isolating ball. □

9.3. **The exceptional points.** Table 9 lists the jump points that we treat individually. We will count three of them directly (§10).

point	masses, up to scaling and relabelling	on the families	finiteness
$\mathbf{b}_1$	$(1,4,4,4,4)$	T_1 $t=1$; T_2 $t=1$; T_3 $a=4$; T_5 $q=-1$; T_7 $s=2$; RL $\mu=4$	Theorem 10.1
$\mathbf{b}_2$	$(1,1,4,4,4)$	T_7 $s=\frac{1}{2},-1$; RL $\mu=1$	Theorem 10.2
P_1	$(9,36,36,36,4)$	T_5 $q=1$; RL $\mu=\frac{4}{9}$	Theorem 10.7
P_2	$(9,36,36,4,9)$	T_7 $s=\frac{3}{2},3$	settled
$\mathbf{h}_1$	$((1+t_*)^{-2},t_*^{-2},t_*^{-2},1,1)$	T_7 $s\approx 1.5548,\ 2.8025$	settled
$\mathbf{h}_2$	$(\frac{25}{8},\frac{25}{2},\frac{25}{2},\frac{25}{2},1)$	T_2 $t=\frac{1}{2}$; RL $\mu=\frac{8}{25}$	settled

TABLE 9. The jump points treated individually. Here $t_*\approx 1.8024986256$ is the root greater than 1 of $2t^8+t^6-8t^5-6t^4-8t^3+t^2+2$.

At P_2 the label set contains $\mathsf{H}(12|35)$ and $\mathsf{H}(13|25)$, whose conditions Tri fail. So Theorem 3.9 removes the two pieces $\mathsf{d3}$. It also enables $\mathsf{d6}(145|234)$, which is removed by Proposition G.1. K14 of the cluster Q_4, with masses $(9,36,36,9)$, has no solution, the pieces $\mathsf{d15}(4,2)$ and $\mathsf{d15}(4,3)$ are excluded by (X4), and the remaining pair is certified relative to V_N. At $\mathbf{h}_1$ the label set contains two labels H whose conditions $\mathcal{D}_3$ hold. So Theorem 3.9 does not remove the two pieces $\mathsf{d3}$. Proposition G.2 does. K14 of the cluster of the four heavier bodies has no solution, the four pieces $\mathsf{d15}$ of this cluster are excluded by (X4), and the remaining pair is certified. $\mathbf{h}_2$ is the point $\mu=8/25$ of RL after the exchange of the bodies 1 and 2. It is settled by Theorem 9.13.

Proposition 9.4. *On T_1, T_2, T_3, T_4, T_5, T_7 and T_8, every jump point of Theorem 9.3 has finitely many central configurations, and every such point is settled except the points $\mathbf{b}_1$, $\mathbf{b}_2$ and P_1 of Table 9.*

Proof. Table 8 splits the jump points into three kinds. The points of §6 are K-certified and settled there. On T_3 this is the point of equal masses. We have treated the points of Table 9 above. Every other jump point carries a point certificate: in an exact description over a number field, K14 of the transversal cluster has no solution (Theorem 6.2), the pieces $\mathsf{d15}$ needed of this cluster are excluded by (X4), and the pair formed by the label set of the point and the pieces so excluded is certified. On T_8 the cluster is Q_1 and the pair is $(S,\Pi(1))$, with S the label set of the point. Thus Lemma 9.2 applies. Each certificate in this proof holds relative to V_2^{w} or to V_N (§12.2). □

9.4. **Four-body configurations at the masses F_A and F_B.**

Lemma 9.5. K14$(1,1,1,\nu)$, $\nu>0$, *has a solution for exactly six values of ν, and* K14$(1,c,c,1)$, $c>0$, *for exactly sixteen values of c. They are listed in Table 10.*

Proof. We enumerate the configurations exactly: the components of the K14 system in Dziobek form are eliminated by resultants and norms over their coefficient fields, all zeros are enclosed by Krawczyk boxes, each rejected candidate is shown to be degenerate or to give a value of ν, resp. c, that is not positive real, and each value found is identified as a root of an exact polynomial. □

The special points of T_1, T_2, T_3 and T_7 are therefore the points $t=\sqrt{c}$, $t=\sqrt{\nu}$, $a=1/\nu$ and $s=1\pm c^{-1/2}$, respectively, with ν and c from Table 10. Table 8 gives their numbers.

Proposition 9.6. *At each of the 32 special points of T_7 the piece $\mathsf{d14a}(1)$ is excluded, and at each of the 6 special points of T_3 every $\mathsf{d14a}(k)$, $k\le 4$, is excluded.*

	values (to 4 decimals)	polynomial
F_A, ν	0.2462, 12.1840 (collinear)	$x^8 - 116x^6 - 4810x^4 - 1044x^2 + 81$
	0.2540, 19.6837 (collinear)	$x^8 - 32x^7 + 300x^6 - 864x^5 - 5098x^4 - 4320x^3 + 7500x^2 - 4000x + 625$
	$1/\sqrt{3}$	$3x^2 - 1$
	6.0193	$3x^6 - 106x^4 - 97x^2 - 25$
F_B, c	0.1125, 3.2140 (collinear)	$x^8 + 16x^6 - 40x^5 - 2x^4 - 512x^3 + 192x^2 - 24x + 1$
	0.1387, 7.2098	$x^4 - 52x^2 + 1$
	0.1801, 5.5533 (collinear)	degree 16
	0.2490; 4.0154; 0.6012; 1.6633 (collinear)	$x^4 + 4x - 1$; $x^4 - 4x^3 - 1$; $x^4 + 4x^3 - 1$; $x^4 - 4x - 1$
	0.2510, 3.9843 (collinear)	$x^{16} - 16x^{14} - 4x^{12} + 528x^{10} - 6906x^8 + 528x^6 - 4x^4 - 16x^2 + 1$
	0.3111, 8.8913 (collinear)	$x^8 - 24x^7 + 192x^6 - 512x^5 - 2x^4 - 40x^3 + 16x^2 + 1$
	0.4429; 2.2579	$x^4 + 10x^2 - 2$; $2x^4 - 10x^2 - 1$

TABLE 10. The positive masses ν and c at which $\mathrm{K14}(1,1,1,\nu)$, resp. $\mathrm{K14}(1,c,c,1)$, has a solution, with their minimal polynomials. The entry "collinear" marks the values at which the configurations are collinear.

Proof. Since the masses of T_3 are symmetric in the bodies $1,\dots,4$, it suffices to treat $\mathtt{d14a}(1)$. At a place of type $\mathtt{u}_0(1)$, after a rotation and possibly an exchange of the colours, $\operatorname{val} z_{il} = \operatorname{val} w_{il} = 1$ inside $Q = Q_1$ and $\operatorname{val} z_{1l} = \alpha$, $\operatorname{val} w_{1l} = -\alpha$ for one $\alpha \in [0,1]$. The leading coefficients of the cluster form a K14 configuration in a gauge, together with explicit data for body 1 with signs $s_l = \pm 1$ of the leading p_{1l}. The configurations at the special values are known completely (Lemma 9.5). In a weighted form of the rows, the Jacobian J of the cluster rows has rank 10 at every configuration (a nonzero 10×10 minor in ball arithmetic). The first-order forcing must lie in the image of J. This image is described by two linear conditions, the pairings with the two cluster scalings, which lie in the kernel of the Hessian of the cluster potential (Remark 5.36). For $\alpha < 1$ this forces, according to the sign pattern and to whether $\alpha = 0$, the vanishing of a signed dipole $\sum_i s_i m_i \tilde{\zeta}_i$, the quadrupole $\sum_i m_i \tilde{\zeta}_i^2$ or $A^2 Q_{zz} Q_{ww} - B^2 I^2$, where $\tilde{\zeta}$, $\tilde{\omega}$ are the centred leading positions, $Q_{zz} = \sum m_i \tilde{\zeta}_i^2$, $Q_{ww} = \sum m_i \tilde{\omega}_i^2$, $I = \sum m_i \tilde{\zeta}_i \tilde{\omega}_i$, $A = 3m_1$ and $B = 3m_1 + 2M_Q$. In 400-bit ball arithmetic each of these quantities is nonzero at all configurations of all special values, except the dipole of the sign class $(+,-,-,+)$ at sixteen centrally symmetric configurations of T_7, where it vanishes by the symmetry (proved by a Krawczyk uniqueness argument), and the quadrupole and the dipole of $(+,+,+,-)$ at $a = \sqrt{3}$, where the configuration is an equilateral triangle with its centre. In these cases, and for $\alpha = 1$ at all points, we compute the obstruction series of a Lyapunov–Schmidt reduction to higher order in ball arithmetic at all leading points, with the vanishing of the lower orders proved exactly. A nonzero coefficient excludes each case. □

9.5. **The family T_3.**

Theorem 9.7. *Every point of T_3 has finitely many central configurations, and every point other than $a = 4$ is settled.*

Proof. For each $k \in \{1,2,3,4\}$, the pair $(S_C, \{\mathtt{d14a}(k)\} \cup \Pi(5))$ is certified. The cluster Q_5 has the masses $(1,1,1,1)$, at which K14 has no solution (§6). The pieces $\mathtt{d15}(5,k)$ are excluded by (X4). Hence $\Pi(5)$ is excluded at every point. Off the six special points $\mathrm{K14}(m_{Q_k})$ has no

solution. At the special points $\mathtt{d14a}(k)$ is excluded by Proposition 9.6. By Theorem 9.3 and Lemma 9.2 each point with label set S_C is settled. Proposition 9.4 covers the other points. □

9.6. **The families T_1 and T_7.** On T_1 and T_7 the transversal cluster is $F_B(c)$, a relabelling of $(1,1,c,c)$. By (X4) all pieces $\mathtt{d15}(1,k)$ are removed on these two families.

Theorem 9.8. *Every point of T_1 and of T_7 has finitely many central configurations. Every point of T_1 other than $t=1$, and every point of T_7 other than $s \in \{2, \frac{1}{2}, -1\}$, is settled.*

Proof. On T_7, off the sixteen special values of c, $\mathrm{K14}(m_{Q_1})$ has no solution. Thus (X1) excludes $\mathtt{d14a}(1)$. At the special values $\mathtt{d14a}(1)$ is excluded by Proposition 9.6. Thus, by §9.2, $\Pi(1)$ is excluded at all points of T_7, and $\pi(1,2) \cup \pi(1,5)$ at all points of T_1. The pairs $(S_C, \Pi(1))$ on T_7 and $(S_C, \pi(1,2) \cup \pi(1,5))$ on T_1 are certified, and $\Pi(1)$ is the only minimal exclusion set of S_C on T_7 (Theorem 9.10). We conclude as in the proof of Theorem 9.7. □

9.7. **The family T_2.**

Theorem 9.9. *Every point of T_2 has finitely many central configurations, and every point other than $t=1$ is settled. Every piece $\mathtt{d15}(1,k)$, $k \in Q_1$, is excluded at every point of T_2.*

Proof. The pair $(S_C, \pi(1,3) \cup \pi(1,4))$ is certified. Each pair $\pi(1,k)$, $k \in Q_1$, is excluded at every point of T_2 (§9.2). We conclude as in the proof of Theorem 9.7. □

9.8. **The families T_4, T_5 and T_8.** The following two results describe, for the families T_4, T_5, T_7 and T_8, all ways of settling a point with label set S_C by Lemma 9.2 with $V = V_2^{\mathrm{w}}$ and pieces $\mathtt{d14a}$, $\mathtt{d14b}$, $\mathtt{d15}$. A *witness* for (S,E) is a nonempty set W of faces in $\mathcal{A}_V(S)$ outside the pieces of E and outside the pieces $\mathtt{d2}[T]$ and $\mathtt{d9}(T; ax, by)$, together with rationals $c_v > 0$ on the vertices v of the faces in W such that $\sum_v c_v v = 0$. Any functional that is nonnegative on these vertices vanishes on all of them. So, by induction on i, every functional l_i of a certificate for (S,E) vanishes on them, against (b) of Definition 4.2 for the faces of W. Hence (S,E) is not certified relative to V, and neither is (S,E') for any $E' \subseteq E$.

Theorem 9.10. *For each of the families T_4, T_5, T_7 and T_8, the minimal exclusion sets of S_C relative to V_2^{w}, among the 45 pieces $\mathtt{d14a}(j)$, $\mathtt{d14b}(j,k)$, $\mathtt{d15}(j,k)$, are the following.*

(i) *T_4 (19 sets): $\pi(1,A) \cup \pi(1,B)$ for $A < B$ in $\{2,3,4,5\}$; $\pi(1,A) \cup \pi(A,1)$ for $A \in \{3,4,5\}$; $\pi(1,A) \cup \pi(2,B) \cup \pi(2,C)$ for $\{A,B,C\} = \{3,4,5\}$; $\Pi(2)$; and $\pi(1,A) \cup \pi(2,B) \cup \Pi(C)$ for (A,B,C) a permutation of $(3,4,5)$.*
(ii) *T_5 and T_8 (4 sets): $\pi(1,4)$; $\pi(1,5)$; $\pi(1,3) \cup \Pi(2)$; $\Pi(2) \cup \Pi(3)$.*
(iii) *T_7 (1 set): $\Pi(1)$.*

Proof. Each listed set E has a certificate for (S_C, E). For each family we give a list $\mathcal{B}$ of sets A of pieces (Table 8), each with a witness for $(S_C, \text{all pieces} \setminus A)$. The listed sets are exactly the minimal transversals of $\mathcal{B}$. A set E with (S_C, E) certified meets each $A \in \mathcal{B}$ and therefore contains a minimal transversal. Each listed set is certified, and each of its proper subsets misses some A. The face systems of T_5 and T_8 coincide. The certificates and witnesses also hold with the labels of T_8 (§12.2). □

Proposition 9.11. *For each of the families T_4, T_5, T_7 and T_8, the pair formed by S_C and the 25 pieces $\mathtt{d14a}$, $\mathtt{d14b}$ of all clusters has a witness relative to V_2^{w}. Consequently every minimal exclusion set contains a piece $\mathtt{d15}$.*

Lemma 9.2 with $V = V_2^{\mathrm{w}}$ therefore settles a point with label set S_C on these families only if some piece $\mathtt{d15}$ is excluded there.

Proposition 9.12. (a) *Every jump point of T_4, T_5 and T_8 (Theorem 9.3) has finitely many central configurations, and every such point other than $q = 1$ (P_1) and $q = -1$ ($\mathbf{b}_1$) on T_5 is settled.*

(b) *The minimal exclusion sets $\pi(1,2) \cup \pi(1,3)$ of T_4 and $\pi(1,4)$ of T_5 are excluded at every point, so every point of T_4 and of T_5 is settled, except $q = \pm 1$ on T_5, where finiteness holds by counting.*

(c) *The minimal exclusion set $\pi(1,5)$ of T_8 is excluded at every point, so every point of T_8 is settled.*

Proof. (a) is Proposition 9.4. (b) and (c) follow from §9.2, Theorem 9.3, Theorem 9.10 and Lemma 9.2, and from Proposition 9.4 for the other points. □

9.9. **The RL line.** On RL, $m = (4, 1, 4, 4, \mu)$, the cluster is $Q_2 = \{1, 3, 4, 5\}$ with masses $(4, 4, 4, \mu)$.

Theorem 9.13. *Let $\mu > 0$ and $m = (4, 1, 4, 4, \mu)$. The planar five-body problem with masses m has finitely many central configurations up to similarity, and m is settled unless $\mu \in \{1, 4, \frac{4}{9}\}$.*

The proof uses three results valid for every $\mu > 0$: Proposition 7.1 excludes the piece d2[134], and (X2) and (X4) exclude every d14b$(2, k)$ and every d15$(2, k)$, $k \in Q_2$, respectively.

Proof of Theorem 9.13. Each label polynomial restricted to the line is a polynomial in μ. The label polynomials that vanish identically give S_C. The other nonzero restrictions have 6 irreducible factors over $\mathbb{Q}$ with a positive root (Table 11). We isolate these roots in ball arithmetic. Off the positive roots of these factors $S(m) \subseteq S_C$. At a root of a factor g, $S(m) \subseteq S(g)$, the union of S_C and the labels with a polynomial divisible by g. Let $U = \pi(2,1) \cup \pi(2,3) \cup \pi(2,4) \cup \pi(2,5) \cup \{\text{d2}[134]\}$, excluded at all $\mu > 0$ by the results above. We treat the factors as follows.

(i) The pair (S_C, U) is certified, and so is $(S(g), U)$ for the factors g of the first row of Table 11.

(ii) At the root $\mu = \frac{8}{25}$ of the factor in the second row, K14 has no solution for any of the five clusters (Theorem 6.2), the pieces d15 are removed by (X4), and the pair formed by $S(g)$ and all pieces d14a, d14b, d15 together with d2[134] is certified.

(iii) The roots of the factors of the third row are 1, 4 and $\frac{4}{9}$. At $\mu = 1$ and $\mu = 4$, m is a permutation of $(1, 1, 4, 4, 4)$, resp. of $(1, 4, 4, 4, 4)$. At $\mu = \frac{4}{9}$, $9m$ is a permutation of P_1. Theorems 10.2, 10.1 and 10.7 apply.

By Lemma 9.2, m is settled at every $\mu > 0$ other than 1, 4 and $\frac{4}{9}$.

treatment	factors	degrees
(i) $(S(g), U)$ certified	2	1 (root $\mu = 16$), 3
(ii) all K14 systems fail	1	1 (root $\mu = \frac{8}{25}$)
(iii) counted	3	1, 1, 1 ($\mu - 1$, $\mu - 4$, $9\mu - 4$)

Table 11. The factors with a positive root of the label polynomials restricted to RL, by their treatment in the proof of Theorem 9.13.

□

9.10. **The family T_9.** Theorem 9.15 treats the mass vectors whose label set contains a curve set, except the points of the special rays and of the family T_9 below. In this subsection we settle the points of T_9 (Proposition 9.14). For T_9 we need Theorem 6.5. Table 12 collects the numbers of the computation.

The curve T_9. Let $\rho_0 = 0.0873780\ldots$ be the real root of $p^3 + 5p^2 + 11p - 1$, and let T_9 be the curve $m(s) = (\rho_0/4, \rho_0, \rho_0, 1, s)$, $s > 0$. The cluster Q_1 has masses $(\rho_0, \rho_0, 1, s)$, with body b in position $b - 1$. The cluster Q_5 has the constant masses $(\rho_0/4, \rho_0, \rho_0, 1)$.

Proposition 9.14. *Every point of T_9 is settled.*

Proof. Labels. We reduce each label polynomial composed with $m(s)$ modulo the minimal polynomial of ρ_0, isolate the positive roots of its norm in ball arithmetic, and decide at each of them, by a ball or by an exact gcd over $\mathbb{Q}(\rho_0)$, whether the polynomial vanishes there. As in the proof of Theorem 9.3, a label H holds only where its two conditions Tri hold as well, which we decide in the same way. At each $s > 0$ the label set is $S_C = \{\mathsf{E}(123), \mathsf{R}(23;1), \mathsf{R}(24;1), \mathsf{R}(34;1), \mathsf{EQ}(2,3)\}$, except at 13 jump points, all in $[0.0030920, 2091.64]$. They fall into 10 classes by their label sets.

Pieces removed at every s. Theorem 5.21 removes the pieces $\mathsf{d14b}(1,k)$. K14 fails at the masses of Q_5 (Theorem 6.2) and Corollary 5.19 removes the pieces $\mathsf{d15}(5,k)$. Hence no piece of the cluster Q_5 is attained.

The generic set. Two certificates for S_C relative to V_2^{w} remove $\mathsf{d14b}(1,k)$, $k = 2,\ldots,5$, and in addition $\mathsf{d15}(1,5)$, respectively $\mathsf{d15}(1,2)$, $\mathsf{d15}(1,3)$, $\mathsf{d15}(1,4)$. By (X2) and (X4) these pieces are removed at all s, so that both certificates apply. Hence $m(s)$ is settled by Proposition 4.4 with these pieces removed, as in the proof of Proposition 6.7.

The jump points. Each class has a certificate whose removed pieces are removed at its points: $\mathsf{d14b}(1,k)$ and $\mathsf{d15}(1,k)$ by (X2) and (X4), the pieces of Q_5 as above, and, in the classes whose certificate removes it, $\mathsf{d14a}(1)$ because $\mathrm{K14}(m_{Q_1})$ has no solution at the roots of their jump factors, by Theorem 6.5(b). We check the hypothesis of that theorem by a unit standard basis with the factor adjoined, or through a polynomial in s alone in the ideal of the full system that is prime to the factor. At $s = \rho_0/4$, where $m_1 = m_5$, the certificate also removes the piece $\mathsf{d6}(145|234)$. After relabelling to the masses $(1,1,4,4,4/\rho_0)$, we carry out the test for diagram 6 used at P_2 in §9.3 with $\mathbb{Q}(4/\rho_0)$ in place of $\mathbb{Q}$, with the standard basis and the closed forms exact over $\mathbb{Q}(4/\rho_0)$: the leading system is zero-dimensional, and each of its solutions has a nonzero obstruction coefficient; the leading solutions and their obstructions are certified in ball arithmetic. Hence no place of this diagram exists. The certificates at $s = \rho_0/4$ and at the two jump points $s \approx 0.0520549$ and $s \approx 0.176101$, where $\mathsf{E}(245)$ and $\mathsf{E}(345)$ hold, are relative to V_N, the others relative to V_2^{w}. □

	quantity	number
T_9	jump points; classes	13; 10
	certificates for T_9	12
	solutions of the leading system of diagram 6 at $s = \rho_0/4$	5184

TABLE 12. The computation along T_9 in numbers.

9.11. **The curve sets.**

Theorem 9.15. *Let $m \in (\mathbb{R}_{>0})^5$ satisfy every label of $\sigma(D_T)$ for one of the nine names T of Table 5 and a relabelling σ. Suppose that m is not a positive multiple of a permutation of $(1,4,4,4,4)$ or of $(1,1,4,4,4)$, and, if $T = T_9$, that m is not a point of the family T_9 up to scaling and relabelling. Then the planar five-body problem with masses m has finitely many central configurations up to similarity, and m is settled unless it is a positive multiple of a permutation of P_1 (§10.4).*

Proof. By Lemma 9.1, applied to $\sigma^{-1}m$, a positive multiple of m is obtained by a relabelling from a point on one of the following families:

(i) T itself, if T is T_1, T_2, T_3, T_5 or T_7;
(ii) T_4 or T_2, if $T = T_4$, and T_8 or T_5, if $T = T_8$;
(iii) RL, if T is RL or T_9. For T_9 the other case of Lemma 9.1, a point of T_9, is excluded by hypothesis.

As in the proof of Corollary 8.3, positive scaling and relabelling of the masses preserve finiteness and settledness. Theorems 9.7, 9.8, 9.9 and 9.13 and Proposition 9.12, which hold on the whole families, give the claim. The points of these families that are only counted are the points of the special rays, excluded here, and the positive multiples of permutations of P_1, which are counted points. □

10. The special rays

Recall the points $\mathbf{b}_1$ and $\mathbf{b}_2$ of §8.1. At these two points the separation of §§8–9 fails, and at $\mathbf{b}_1$ it has to fail (§1). On the rays through these two mass vectors and their relabellings we therefore count the real central configurations directly, following Moczurad and Zgliczyński [10, 12]: a branch and bound in a reduced system with interval arithmetic, exclusion tests derived from exact identities, and Krawczyk's test [6, 8, 13]. With the same method we also count the central configurations at one further mass point needed in §9 (§10.4).

10.1. **Statements.** In this section a configuration is $q = (q_1, \dots, q_5) \in (\mathbb{R}^2)^5$ outside the collision set Δ, with $q_i = (\xi_i, \eta_i)$, $r_{ij} = |q_i - q_j|$, $M = \sum_i m_i$ and

$$F_i(q) = q_i - \sum_{j \neq i} m_j \frac{q_i - q_j}{r_{ij}^3} \qquad (i = 1, \dots, 5).$$

A *normalized central configuration* is a zero of $F = (F_1, \dots, F_5)$, that is, a solution of (1.1) with $\lambda = 1$ and $c = 0$. As in the proof of Lemma 2.1, every central configuration is similar, by an orientation-preserving similarity, to a normalized one, and two central configurations are orientation-preserving similar if and only if their normalizations lie in the same $SO(2)$-orbit. The group $SO(2)$ acts freely on normalized central configurations. Thus the central configurations up to orientation-preserving similarity correspond to the $SO(2)$-orbits of normalized central configurations. The reflection $(\xi, \eta) \mapsto (\xi, -\eta)$ acts on these orbits as an involution whose fixed orbits are the collinear ones. Hence, with N orbits of which N_c are collinear, there are $(N+N_c)/2$ central configurations up to similarity.

Rotations give $DF(q)\,Jq = 0$ at a normalized central configuration q, where J rotates each q_i by $\pi/2$. Hence $\operatorname{rank} DF(q) \leq 9$. We call q *nondegenerate* if $\operatorname{rank} DF(q) = 9$. Since $DF(q)$ is, up to the mass matrix, the Hessian of $V = \frac{1}{2}\sum_i m_i|q_i|^2 + \sum_{i<j} m_i m_j / r_{ij}$, this is the usual notion [7]: q is a nondegenerate critical point of the potential on the normalized configurations, modulo rotations.

For a non-collinear configuration let P be the convex hull of $q_1, \dots, q_5$. It is *strictly convex* if all bodies are vertices of P, and of type $(4, 1)$ or $(3, 2)$ if exactly four, resp. three, bodies are vertices of P and the others are interior points of P.

Theorem 10.1. *Let m be a positive multiple of a permutation of* $(1, 4, 4, 4, 4)$.

(i) *There are exactly* 306 *central configurations up to orientation-preserving similarity, and exactly* 183 *up to similarity.*
(ii) *Every central configuration is nondegenerate.*

(iii) *Of the* 306*, exactly* 60 *are collinear,* 24 *strictly convex,* 102 *of type* $(4,1)$ *and* 120 *of type* $(3,2)$*. In every non-collinear central configuration each body is a vertex or an interior point of the convex hull, and each of the* 24 *oriented cyclic orders of the five bodies is the counterclockwise order of exactly one strictly convex one.*

(iv) *There are exactly* 612 *positive normalized central configurations in the sense of* [2, Def. 2].

Theorem 10.2. *Let m be a positive multiple of a permutation of* $(1,1,4,4,4)$*. Then statements* (i)–(iv) *of Theorem 10.1 hold with the numbers* 294 *and* 177 *in* (i)*;* 60*,* 24*,* 90 *and* 120 *in* (iii)*; and* 588 *in* (iv)*.*

The 60 collinear configurations agree with Moulton's count [9]. Statement (iv) follows from (i) by Lemma 2.1. By (iv), the positive solutions of the system of [2] at $(4,4,4,4,1)$ are finite in number, although the system has a continuum of real solutions there (§1). At this mass, finiteness therefore rests on the signs of the distances, as anticipated in §1.

10.2. **The reduced system and the a priori bounds.** Every normalized central configuration satisfies the following identities. Summing m_iF_i gives $\sum_i m_iq_i = 0$. Taking the scalar product of $F_i = 0$ with m_iq_i and summing gives $I = U$, where $I = \sum_i m_i|q_i|^2$ and $U = \sum_{i<j} m_im_j/r_{ij}$. With Lagrange's identity $MI = \sum_{i<j} m_im_jr_{ij}^2$, this is the virial identity $\sum_{i<j} m_im_j(r_{ij}^2/M - 1/r_{ij}) = 0$. For a nonempty proper subset C of the bodies, summing m_iF_i over C gives the cluster identity $\sum_{i\in C} m_iq_i = \sum_{i\in C,\, j\notin C} m_im_j(q_i - q_j)/r_{ij}^3$. Let c_C be the centre of mass of C. The scalar products of $m_iF_i = 0$ with $q_i - c_C$, summed over C, give the virial identity of the cluster C.

Lemma 10.3. *Let q be a normalized central configuration and $R = \max_i |q_i|$.*

(a) *Some r_{ij} is at least $M^{1/3}$. Hence $R \geq M^{1/3}/2$.*

(b) *For every $\varepsilon > 0$, $R < \max\{4\varepsilon,\ \max_{1\leq s\leq 4}[(s-1)\varepsilon + W_s/\varepsilon^2]\}$, where W_s is M minus the sum of the s smallest masses.*

(c) *$r_{ij} \geq m_im_j/(MR^2)$ for all $i \neq j$.*

Proof. (a) If all $r_{ij} < M^{1/3}$, all terms of the virial identity are negative. (b) Let G be the graph with an edge ij when $r_{ij} \leq \varepsilon$, and C the component of a body i_0 with $|q_{i_0}| = R$, with s bodies. If $s = 5$, then $MR = |\sum_j m_j(q_{i_0} - q_j)| < 4M\varepsilon$. Otherwise the scalar product of the cluster identity for C with q_{i_0}/R is at least $M_C(R-(s-1)\varepsilon)$ on the left and less than $M_C(M-M_C)/\varepsilon^2$ on the right. (c) $m_im_j/r_{ij} \leq U = I \leq MR^2$. □

Part (b) sharpens [10, Thm. 11]: at $(1,4,4,4,4)$ that theorem gives about 10.11, whereas ours gives 6.352. Since the boxes B_k below have sides set by this bound, the sharper bound shrinks the region searched. Table 13 lists the values of ε that we use and the resulting bounds.

Fix real constants $c_1, \ldots, c_5$. A normalized central configuration lies in *region* k if $|q_k|^2 + c_k \geq |q_i|^2 + c_i$ for all i. Every one lies in some region. The constants only break ties: a symmetric configuration often has two bodies at the same distance from the centre of mass. With $c = 0$ it would lie on the boundary of two regions. By Lemma 10.3(a), in region k we have $|q_k|^2 \geq \mu_k := M^{2/3}/4 + \min_i c_i - c_k$, which is positive for the constants we use. Hence $q_k \neq 0$ and exactly one rotation puts q_k on the positive ξ-axis. For a second body $e \neq k$ and the other three bodies f_1, f_2, f_3, the chart $\Phi_{k,e}\colon \mathbb{R}^7 \to (\mathbb{R}^2)^5$, $v \mapsto q$ with

$$q_k = (v_0, 0), \qquad q_{f_a} = (v_{2a-1}, v_{2a})\ (a = 1,2,3), \qquad q_e = -\frac{1}{m_e}\sum_{i\neq e} m_iq_i,$$

is an isomorphism onto $\{\sum_i m_i q_i = 0,\ \eta_k = 0\}$, and the *reduced system* $RS(v)$ consists of the seven components $F_{f_1}, F_{f_2}, F_{f_3}$ and the first component of F_k at $\Phi_{k,e}(v)$ [12, Thm. 2], [11, Thm. 3].

Lemma 10.4. *Let $q = \Phi_{k,e}(v) \notin \Delta$.*

(a) *If $RS(v) = 0$ and $\xi_k \neq \xi_e$, then q is a normalized central configuration. Conversely, every normalized central configuration with $\eta_k = 0$ is $\Phi_{k,e}(v)$ for a zero v of RS.*
(b) *If q is a normalized central configuration with $\xi_k \neq 0$ and $DRS(v)$ is invertible, then q is nondegenerate.*

Proof. (a) For any configuration $\sum_i m_i F_i = \sum_i m_i q_i$ and $\sum_i m_i\, q_i \times F_i = 0$, with $a \times b = a_1 b_2 - a_2 b_1$. If $RS(v) = 0$, then $F_i = 0$ for $i \neq k, e$. The first identity then gives $m_k F_k + m_e F_e = 0$, and the second $m_k(q_k - q_e) \times F_k = 0$. As $q_k - q_e = (\xi_k - \xi_e, -\eta_e)$ and the first component of F_k vanishes, this is $m_k(\xi_k - \xi_e)F_{k,2} = 0$. Hence $F_k = F_e = 0$. The converse holds because the centre of mass of a normalized central configuration is 0. (b) Differentiating the first identity gives $\sum_i m_i (DF(q)u)_i = \sum_i m_i u_i$ for all u. Since $\xi_k \neq 0$, the vector Jq does not lie in the image L of $\Phi_{k,e}$. Hence $\mathbb{R}^{10}$ is the direct sum of L, the line spanned by Jq and the translations. Let $u = \ell + \alpha Jq + \tau \in \ker DF(q)$ accordingly. Then $0 = DF(q)\ell + \tau$, because $DF(q)Jq = 0$ and $DF(q)\tau = \tau$. Taking the mass-weighted sum, we obtain $M\tau = 0$. Thus $DF(q)\ell = 0$, hence $DRS(v)\,\Phi_{k,e}^{-1}(\ell) = 0$ and $\ell = 0$. Therefore $\ker DF(q)$ is spanned by Jq. □

Consequently, in region k the chart point of the rotated configuration lies in the box $B_k = [R_{\min,k}, R_{\max}] \times [-R_{\max}, R_{\max}]^6$, with $R_{\min,k} \leq \sqrt{\mu_k}$ positive and $R_{\max}$ the bound of Lemma 10.3(b), and it is a zero of RS.

10.3. **The search and the count.** We search the boxes B_k by branch and bound in outward-rounded interval arithmetic: every operation of IEEE 754 binary64 arithmetic is computed in round-to-nearest, after which the lower endpoint is moved down and the upper endpoint up by one unit in the last place, and no multiplication is fused with an addition. Every computed interval therefore contains the exact value of the corresponding real expression on the box. We discard a box if one of the following tests shows that it contains no zero of RS that is a normalized central configuration in region k:

(T0) an enclosure of $|q_i|^2 + c_i - |q_k|^2 - c_k$ is positive for some i;
(T1) the enclosure of the virial sum excludes 0;
(T2) for some proper subset C all of whose cross pairs are separated on the box, an enclosure of the difference between the two sides of the cluster identity excludes 0;
(T3) for such a C with $|C| \geq 2$, the enclosures of the two sides of the virial identity for C are disjoint;
(MV) for some component F_r of $F \circ \Phi_{k,e}$, the mean-value enclosure $F_r(c) + \sum_t \partial_t (F_r \circ \Phi_{k,e})(X)\,(X_t - c_t)$ at the midpoint c excludes 0.

Since the Newtonian kernel $(\xi, \eta) \mapsto \xi/(\xi^2 + \eta^2)^{3/2}$ has no critical point, on a rectangle not containing the origin its extrema are attained at the corners, at the points with $\eta = 0$ of its vertical sides, or at the critical points $(\pm|\beta|/\sqrt{2}, \beta)$ of its restrictions to the horizontal sides $\eta = \beta \neq 0$, where it takes the values $\pm\frac{2}{3\sqrt{3}}\beta^{-2}$. The enclosure on a rectangle is the hull of its enclosed values at those of these points that lie in the rectangle. On a box on which all pairs are separated we apply Krawczyk's test, with the interval Jacobian of RS and a floating-point approximate inverse C of the Jacobian at the midpoint (§4.4). If one component of the operator is disjoint from the box, the box contains no zero. If all components lie in the interior of the box, RS has exactly one zero v^* in it and $DRS(v^*)$ is invertible [6, 8, 13]. Boxes that are neither discarded nor certified are contracted by the operator or bisected, at an off-centre cut

so that symmetric configurations do not lie on cut faces. We tighten a certified box by further Krawczyk steps. On the tightened box the enclosure of $\xi_k - \xi_e$ excludes 0, and the enclosures of $|q_i|^2 + c_i - |q_k|^2 - c_k$ are negative for all $i \neq k$.

Proposition 10.5. *Let m be $(1,4,4,4,4)$ or $(1,1,4,4,4)$. Each certified box contains exactly one zero v^* of RS, and $\Phi_{k,e}(v^*)$ is a nondegenerate normalized central configuration lying in region k and in no other region. The map from certified boxes to $SO(2)$-orbits of normalized central configurations is a bijection. There are 306 certified boxes for $(1,4,4,4,4)$ and 294 for $(1,1,4,4,4)$, distributed over the regions as in Table 13.*

Proof. For each mass vector we search the regions in subproblems whose initial boxes tile B_k. We verify the tiling, the inequalities $R^2_{\min,k} \leq \mu_k$ and the bound $R_{\max}$ in exact rational arithmetic. In all subproblems each leaf box is discarded or certified, and no subproblem ends abnormally (§12.2). A box is discarded only if it contains no zero of RS that is a normalized central configuration in region k, contractions keep every zero of RS, and bisections cover the box. Hence every chart point of a normalized central configuration in region k lies in a certified box. By Krawczyk's test the certified box contains exactly one zero v^*, in its interior. It is a normalized central configuration by Lemma 10.4(a), since $\xi_k \neq \xi_e$ on the tightened box, and nondegenerate by Lemma 10.4(b), since $\xi_k \geq R_{\min,k} > 0$. It lies strictly in region k, hence in no other region. Each orbit is thus the image of a certified box (surjectivity). Two certified boxes with the same orbit come from the same region and, since q_k lies on the positive ξ-axis in both, give the same chart point, which lies in the interior of both boxes. The boxes of one subproblem are leaves in a tree of bisections and contractions. Boxes from different subproblems lie in different pieces of the tiling. The two boxes therefore coincide (injectivity). The counts are those of the search (Table 13). □

			region							boxes
masses	ε	bound	1	2	3	4	5	total	collinear	examined
$(1,4,4,4,4)$	$\frac{397}{250}$	$\frac{794}{125}$	168	28	38	18	54	306	60	211,263,506
$(1,1,4,4,4)$	$\frac{1587}{1000}$	$\frac{5332323001}{839523000}$	90	150	18	24	12	294	60	123,212,100

TABLE 13. The searches of Proposition 10.5: the value of ε and the bound of Lemma 10.3(b) at it, which the binary64 number $R_{\max}$ is at least; the certified boxes per region; the collinear boxes; and the number of boxes examined.

We certify the shapes in ball arithmetic (Arb, at 200 bits, through python-flint 0.9.0) on the tightened boxes, after checking that all balls in these computations are finite, that all divisors exclude 0 and that all square-root arguments are positive. We enclose the orientations of all triples of bodies on the box. If some orientation has a sign, the configuration is not collinear, and each body passes one of two tests: a body is a vertex of the convex hull if the orientations certify that the two hull edges proposed at it leave all other bodies strictly on their left. It is an interior point if they certify that it lies in the open triangle of three other bodies, or in the interior of a quadrilateral formed by the four others, as when the four bodies of mass 4 form a square with the body of mass 1 at its centre. If no orientation has a sign, we enlarge the box to one that is symmetric under the reflection $(v_2, v_4, v_6) \mapsto (-v_2, -v_4, -v_6)$ of the chart, which maps the zero set of RS to itself, and Krawczyk's test succeeds on the enlarged box. Its unique zero then coincides with its mirror image. So all five bodies lie on the ξ-axis. In this way we show that every certified box is collinear, or non-collinear with a certified shape. Table 13 gives the numbers of collinear boxes.

Proof of Theorems 10.1 and 10.2. By Proposition 10.5 there are 306, resp. 294, orbits, all nondegenerate. With the 60 collinear ones, we obtain 183 and 177 up to similarity. The shape counts are those of the classification. The reflection pairs the non-collinear orbits with equal shapes and reversed cyclic orders. As in the proof of Lemma 2.2, positive scaling of the masses multiplies the normalized central configurations by $t^{1/3}$, and permutations of the masses relabel them. Since both preserve nondegeneracy, collinearity and shapes, the statements hold on the whole rays and for all relabellings. □

Remark 10.6. By nondegeneracy and the implicit function theorem, the numbers in Theorems 10.1 and 10.2 persist on open neighbourhoods of the two mass vectors that are invariant under positive scaling.

10.4. **A further point.** We treat the mass point $P_1 = (9, 36, 36, 36, 4)$ of §9 by the same count instead of the separation.

Theorem 10.7. *Let m be a positive multiple of a permutation of P_1. Then statements* (i)–(iv) *of Theorem 10.1 hold with the same numbers.*

The proof of Theorem 10.7 is that of Theorem 10.1. Here some coefficients of the chart, $\pm 1/9$ and $\pm 10/9$, are not binary64 numbers. So we enclose them by intervals. One certified box turns out to have its zero outside its region. This does not affect the count: we do not count that box, since the surjectivity in the proof of Proposition 10.5 counts every orbit by a box of a region that contains it.

Remark 10.8. The method of [16] also gives finiteness at P_1 without the count of Theorem 10.7. At every positive multiple m of a permutation of P_1, the function U vanishes identically on every irreducible curve in the slice Y_m of §2.2 [16, Thm. 1.3]. So m is settled, and it has finitely many central configurations up to similarity.

10.5. **The complex solution sets on the rays.**

Remark 10.9. At $m = (1, 4, 4, 4, 4)$ the slice Y_m of §2.2 is the union of 36 pairwise disjoint irreducible curves and a finite set. Through the correspondence of Lemma 2.2(a) between the points of Y_m and the solutions of (2.1), the curves contain the modified Roberts continuum of §1, carried to these masses by the exchange of the bodies 1 and 5 (Lemma 2.2(b)), and no point of the curves corresponds to a positive normalized central configuration. At $m = (1, 1, 4, 4, 4)$ the slice Y_m is finite. Both statements are proved in [16, Thms. 1.1 and 1.2]. They give a second proof, by a different method, that every point of the special rays has finitely many central configurations up to similarity, but they do not give the counts.

11. Proof of Theorem 1.1

Let $m \in (\mathbb{R}_{>0})^5$. If m lies on a special ray, Theorems 10.1 and 10.2 give exactly 306 or 294 central configurations up to orientation-preserving similarity. Otherwise Corollary 8.3 gives finitely many up to similarity. In its proof we apply Theorem 8.2 to the sorted scaled image $\hat{m}$ of m:

(i) if $\Lambda(\hat{m})$ is certified in the sense of §8.2, Proposition 4.4 with $V = V_2^{\mathrm{w}}$ settles $\hat{m}$, the pieces `d2`, `d9`, `d14b` and `d15` being removed by Proposition 7.1, Theorem 5.37, Theorem 5.21 and Corollary 5.19;
(ii) if $\Lambda(\hat{m})$ contains a curve set, Proposition 9.14 or Theorem 9.15 applies, and at the counted points the latter rests on Theorem 10.7;
(iii) if $\hat{m}$ is $\mathbf{q}_*$ or $\mathbf{p}_*$, Proposition 4.4 with $V = V_N$ settles $\hat{m}$, by Proposition 8.7. □

Section 12 lists the computations on which the proof rests and how they are checked.

12. Verification

The proof of Theorem 1.1 consists of the arguments in the text and the computations they invoke, classified in §12.2. It does not use the cross-checks of §12.3 or Remarks 10.8 and 10.9, nor any re-check these arguments do not invoke. The certificates, the programs that check them, their logs and the supplementary notes are in the supplementary material, with a checksum for each file. It is available at `https://doi.org/10.5281/zenodo.23202861`.

Some objects have other names in the supplementary material. The points $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ (Tables 4 and 9) are `X52`, `X29` and 48_1 there, and the points of Table 4 appear under the point records in its last column. The relations (R1), (R2) and (R3) of §3.3 are called gen, det and tilde there and are proved as Theorems G, D and T of its module `d5fam`; the model of the centred case (§3.3) is that of its Theorem C0. Theorem 7.3 is Theorem G of its module `nod2route`. Theorem 8.2 is Theorem S of its module `strata`, in which $\mathbf{q}_*$ and $\mathbf{p}_*$ are zeros of the flats `F89` of the face 1|2|3|4|5 and `F49` of the face 1|2|3|45. The points $\mathbf{q}_*$ and $\mathbf{p}_*$ also appear under the point records 52_0 and 47_0. The jump points of T_9 (Proposition 9.14) fall there into numbered classes: the point $s = \rho_0/4$ is class 11, and the two points $s \approx 0.0520549$ and $s \approx 0.176101$ form class 14. The table of re-checks is the file `CHECKS.md`.

12.1. Literature.

We list the inputs from [2] in §2.6.

12.2. Classes of computations.

(1) *Exact rational arithmetic.* The lexicographic certificates (§4) are lists of integer functionals. We recompute the attained faces of the pieces in rational arithmetic and check every inequality in integers. The checks take the hulls of §3.3 from files. We verify in exact arithmetic that each polytope in these files is the polytope of Table 16 for its pair of modes, or its image under Lemma B.2. We make the exact algebraic decisions of class (C2) (§4.4) in exact arithmetic over $\mathbb{Q}$ and over number fields $\mathbb{Q}[\theta]/(f)$.

(2) *Ball arithmetic.* We show that polynomials, such as those of labels, do not vanish at exact mass points, and we isolate real roots. We certify isolated zeros of polynomial systems by Krawczyk's test [6, 8, 13]. We use Arb in FLINT 3.6.0, through python-flint 0.9.0.

(3) *Standard bases.* We prove that a polynomial system has no solution by a standard basis over $\mathbb{Q}$ or $\mathbb{Q}(\mathrm{j})$ (Singular 4.4.1), or modulo large primes together with a lifting argument (§6). In Theorem 5.28 we bound the number of solutions of a system by a standard basis over $\mathbb{Q}$. In §8, by standard bases, minimal primes and saturations over $\mathbb{Q}$, we describe the points of the strata at which given labels hold. We use msolve to search for solutions, version 0.9.4 for every run whose output enters a proof. Where its output enters a proof we verify it: we check the rational parametrizations against the input polynomials and prove their completeness separately.

(4) *The special rays.* We obtain the counts of §10 from a branch and bound in double precision with outward rounding.

We verify each lexicographic certificate of the paper with a program different from the one that found it. The *independent checker* of lexicographic certificates is a program written from the definitions and the tables of this paper; it shares no code with the search programs. It re-verifies in particular the certificates of the label sets in the proof of Theorem 8.2 and the two certificates of Proposition 8.7(c). The re-checks, with their counts, are tabulated in the supplementary material. They cover the computations of the proof except those in Table 14.

The following results rest also on computations stated in full in their proofs: Proposition 5.7, Theorems 5.26, 5.28 and 5.29, Lemma 5.32, Theorems 6.8 and 7.3, Propositions 8.6, 8.7 and 9.14,

computation	not re-checked
the flats of the computation (Proposition 8.6)	a second enumeration of the flats from scratch
the conditions on the primes, where (P1) is re-checked for the runs of Theorem 6.2	the other conditions, which are checked within the runs
the searches of Proposition 10.5 and Theorem 10.7	the upper bounds, which rest on the exclusion tests and on the outward rounding of the search

TABLE 14. Parts of the computations of the proof that the re-checks leave out.

Theorem 9.3, Lemma 9.5, Proposition 9.6, Theorem 9.13, and Theorems 10.1, 10.2 and 10.7. For the following results we list the computations here.

- The relations (R1)–(R3) of §3.3: an elimination whose generator has only positive coefficients for (R1), 36 eliminations by standard bases over $\mathbb{Q}$ for (R2), and a polynomial identity for (R3).
- Proposition G.1: the checks (i)–(iv) of its proof, made as stated there in exact arithmetic, by a standard basis over $\mathbb{Q}$ and in ball arithmetic.
- Proposition G.2: the checks (i)–(iii) of its proof, by exact rational bisection and in ball arithmetic.

We apply the certificates that use the hull V_N of §3.3 only at points where V_N is valid. Eleven of the certificates in the proof of Proposition 9.4, used at ten jump points of T_1, T_4 and T_5 and at ten jump points of T_8, are relative to V_N. All others there are relative to V_2^{w}. At these jump points, at the two point records of Table 4 whose final checks use V_N, and at $\mathbf{q}_*$ and $\mathbf{p}_*$, we check in ball arithmetic that the point lies off the exceptional hypersurface of §3.3 and that $m_o \neq m_a + m_b$ on every triangle $T = \{o, a, b\}$ with $\mathsf{E}(T)$ in the label set. We also show in ball arithmetic that no body of such a triangle is at its centre of mass, except at the triangles in the centred case, whose masses are exactly proportional to $(\frac{1}{4}, 1, 1)$. Each factor of the hypersurface needed at these triangles is enclosed in a ball that excludes 0.

This point check covers 88 applications of V_N, including all those used in the proof. Each of them lies off the factors of the hypersurface needed at its triangles (in the centred case of §3.3 only those of (R2) for the two other bodies of T, by the model of that case in the supplementary material) and satisfies the second condition. Thirteen of them have a triangle of masses $(\mu/4, \mu, \mu)$, in the centred case.

The enumerations of the cells of the prevarieties of §6.1 make 13609 linear-programming decisions for K14 and 32010 for each system K15. The final checks of the point records of Table 4 concern 4 distinct pairs (S, T).

12.3. **Cross-checks.**

- *Pieces* d15*, independently of Corollary 5.19.*
  (a) At the points of Table 4, and at the other mass points of §9 at which K14 is used in the proof, the system K15(m_{Q_j}, k) has no solution, by runs of Theorems 6.2 and 6.5(b), for each piece $\mathsf{d15}(j, k)$ removed by the certificate of the point. At the points of Table 4 these runs and those of K14 exclude all pieces d14a, d14b and d15 of the clusters of the transversals from the cluster systems alone, independently of §5.
  (b) The system K15$((1, 1, 1, \nu), k)$ in the normalization of §6.1, with ν as an unknown, has exactly 768 solutions when k is the body of mass ν and exactly 1448 when k

has mass 1. They lie over the roots of irreducible polynomials of degrees 128 and 724, each with exactly two positive roots. The proof counts the solutions modulo a prime and excludes the open cells of the prevariety by a local analysis, at two primes.

(c) At the four resulting values of μ on the line RL, and at the two values of t on T_2 at which t^2 is a positive root of the second polynomial, a nonzero coefficient of order at most 3 in a weighted form of the rows excludes a place of type $\mathsf{b}_k(j)$ at each leading point, 288 leading points when k is the body of mass ν and 96 otherwise, at three primes. With the previous item, this excludes the pieces $\mathsf{d15}(2,k)$ on RL and $\mathsf{d15}(1,k)$, $k=3,4,5$, on T_2.

(d) No solution of $\mathrm{K15}((1,1,t,t),k)$, $t\in\mathbb{C}^*$, has a bad shape. For $k=1$, all solutions of this system with t as an unknown, in a normalization, have unit coordinates at the prime 2147483629, by its prevariety and a local analysis at two of its cells. Two standard bases modulo that prime, of the system with the condition of a bad shape added, are the unit ideal. The other values of k reduce to $k=1$. This excludes the pieces $\mathsf{d15}$ of the transversal cluster on T_1 and on T_7.

(e) Along T_9, let Z_k be the set of s at which $\mathrm{K15}(m_{Q_1},k)$ has a torus zero. Then $Z_2=Z_1$, by the exchange of the two bodies of mass ρ_0, and $Z_4\cap Z_1=Z_4\cap Z_3=\emptyset$, by the cells of the three systems modulo 2147483629 and by polynomials u_4, u_1, u_3 in s in their ideals with $\gcd(u_4,u_1)=\gcd(u_4,u_3)=1$. Hence at every point of T_9 one of the two certificates for the generic label set applies without Corollary 5.19.

- *Proposition 5.7.* A standard basis over $\mathbb{Q}(\mathrm{j})$ of the system of Proposition 5.7, with ν as an unknown and saturated by $\prod p_{il}\cdot\nu(3+\nu)$, is $\{1\}$. This confirms the proposition whenever $\nu(3+\nu)\neq 0$. An independent encoding of the system gives the same result. With $c=d=0$ it finds the expected solutions of K14.
- *Theorem 5.21 as used in Proposition 9.12.* On the families T_4, T_5 and T_8 we confirm the exclusion of the pieces $\mathsf{d14b}(1,k)$ of the transversal cluster Q_1, for $k=2,\dots,5$ on T_4 and $k=4,5$ on T_5 and T_8, with Lemma 5.20, the description of the configurations that leads to (5.8), and Lemma 5.24, but without the rest of the proof of Theorem 5.21. These are the pieces $\mathsf{d14b}$ of the minimal exclusion sets of §9.8 that use only Q_1. By Lemma 5.20, at a place with type in the attained set of $\mathsf{d14b}(1,k)$ the leading coefficients of the cluster form a $\mathrm{K14}(m_{Q_1})$ configuration that satisfies $\mathrm{X3}_k$ (§5.5), after the exchange of the colours if necessary: body k lies at the z- or the w-centre of mass of Q_1. We show that at no point of these families does a $\mathrm{K14}(m_{Q_1})$ configuration satisfy $\mathrm{X3}_k$ in z or in w. With $d_{il}=z_{il}w_{il}$ and M_Q the mass of the cluster, the parallel-axis identity shows that $\mathrm{X3}_k$, in z or in w, is equivalent to $M_Q\sum_l m_l d_{kl}=\sum_{i<l} m_i m_l d_{il}$. By Lemma 5.24, a non-collinear configuration at positive masses has a pattern of type I or II. By complex conjugation four of these seven classes suffice. At positive masses $\kappa\neq 0$ in (5.8) and the masses are then rational functions of t. A collinear configuration solves one of 32 tournament systems $\sum_{l\neq i} m_l s_{il}/z_{il}^2=0$, $s_{il}=\pm 1$, linear in the masses. In suitable labels T_4 and T_5 lie in the plane $m_3=m_4$ of the cluster masses. An explicit resultant P_T, of degree 6 for T_4 and 12 for T_5, vanishes on each family. For each of the four classes, every needed k and each tournament we compute with Singular 4.4.1, in characteristic 0, a standard basis over $\mathbb{Q}(\mathrm{j})$ of the ideal J of the configurations with $\mathrm{X3}_k$ in this plane, and show that P_T is a unit modulo J. We establish each unit statement by an exact standard basis of $J+P_T$ in characteristic 0. Those for the four classes we establish also modulo the prime $p'=536870879$, by the following modular test: if a Gröbner basis of J has p'-integral coefficients (H2) and leading coefficients that are nonzero modulo p' (H3), then its reduction is a Gröbner basis of the reduced ideal with the same staircase,

the multiplication matrices reduce accordingly, and a unit modulo p' is a unit. On T_8, where no two masses are identically equal, every point is a zero of an explicit polynomial P_{T_8} in the masses of the bodies 3 and 5 and the parameter r. We write the systems in the same way, with the mass relations of T_8 in place of the plane. The ideal J is then not saturated, so that $V(J)$ may contain points that are not configurations. For the collinear configurations and for three of the four classes, P_{T_8} is a unit modulo J, by a standard basis in characteristic 0 followed by the modular test at p' and, except for two of the classes with $\mathrm{X3}_4$, at the prime $p = 536870909$. For the fourth class P_{T_8} is not a unit modulo either prime, but it vanishes on $V(J)$ only where a coordinate t_i vanishes. At each configuration $t_i \neq 0$. For this class we use the following extension of the modular test, with $f = t_i$. Let $A = \mathbb{Q}(\mathrm{j})[x]/J$, and let $\bar{a}$ be the dimension of the generalized eigenspace of multiplication by $\bar{f}$ for the eigenvalue 0 on the reduction of A. If (F1) $\dim A/f^N A = \bar{a}$ for some N and (F2) $\bar{f}$ is nilpotent modulo $\bar{J} + (\bar{P}_{T_8})$, then P_{T_8} has no zero on $V(J) \setminus V(f)$. Indeed, the characteristic polynomial of multiplication by f is $\lambda^a \chi_1$ with integral coefficients and $\chi_1(0) \neq 0$. Condition (F1) gives $a = \bar{a}$, so that $\chi_1(0)$ is a unit at the prime. So the idempotent of A on which f is invertible is integral and reduces to the corresponding idempotent of the reduction, on which $\bar{P}_{T_8}$ is invertible by (F2). Hence P_{T_8} is invertible on the part of A where f is invertible. Here (F1) holds with $\bar{a} = 6$ and $N = 3$, for $\mathrm{X3}_4$ and for $\mathrm{X3}_5$. Conditions (F2), (H2) and (H3) are checked at both primes. Each unit statement that we establish modulo a prime has a modular test in which we check (H2) and (H3) element by element. We derive the systems twice, independently. The two derivations agree.

- *Lemma 5.24.* We check the identity for h_{abc} in the proof of Lemma 5.24 by machine for each of the four sets $\{a, b, c\}$ of three bodies. A second proof of the lemma, by an explicit parametrization of the solutions of (5.8) of type III, is in the supplementary material. We also ran the computations of the previous item for the classes of type III on T_4 and T_5, and on T_8 for $\mathrm{X3}_5$. They agree with the lemma. On T_4 and T_5, P_T is a unit modulo J. On T_8, for $\mathrm{X3}_5$, the case $D = 0$ for a polynomial D of the class gives a unit. In the case $D \neq 0$, where one unknown is eliminated through $D \neq 0$, the extension of the modular test applies with $f = D$, $\bar{a} = 192$ and $N = 4$.
- *Theorem 5.21 as used in Theorems 9.8, 9.9, 9.13 and Proposition 9.14.* With Lemma 5.20 and the condition $\mathrm{X3}_k$, in z or in w, but without the rest of the proof of Theorem 5.21, we exclude the pieces `d14b` of the transversal cluster used in Theorems 9.8, 9.9 and 9.13 and Proposition 9.14. On T_1 and T_7, off the sixteen special values of c, $\mathrm{K14}(m_{Q_1})$ has no solution. At every configuration of Lemma 9.5 for F_B we evaluate $\mathrm{X3}_k$ in z and in w in ball arithmetic for the four positions k; no ball contains 0. If a $\mathrm{K14}(1, 1, 1, \nu)$ configuration with $\nu \neq 0$ satisfies $\mathrm{X3}_k$, then k is the body of mass ν and $3\nu^2 = \pm 1$. Indeed, in a normalization of the gauge either $z = w$ or the matrix A of the derivation of (5.8) has rank one. In each case $\mathrm{X3}_k$ adds one equation. We compute the eliminations with Singular 4.4.1, over $\mathbb{Q}(\mathrm{j})$ for the $2 \times 9 \times 2$ systems of the second case and over $\mathbb{Q}$ for the 2×32 of the first, and recompute all of them with Macaulay2 1.25.11. Only $k = 4$ with $3\nu^2 = 1$ (one system) and $3\nu^2 = -1$ (four systems) survive. This excludes $\mathtt{d14b}(1, k)$, $k = 3, 4, 5$, on T_2, and all $\mathtt{d14b}(2, k)$ on RL except $\mathtt{d14b}(2, 5)$ at $\mu = \mu^* = 4/\sqrt{3}$. There K14 at $(4, 4, 4, \mu^*)$ has exactly two configurations up to the gauge, the equilateral triangle of the bodies 1, 3, 4 with body 5 at its centre and its mirror image. The places on the segment of $\mathtt{d14b}(2, 5)$ have exactly 96 leading points, 48 over each configuration. The exchange of the bodies 3 and 4 maps one half to the other. A weighted form of the rows that does not depend on the parameter $\sigma \in (1, 2)$ of the segment, followed by a Lyapunov–Schmidt reduction, gives at each of

the 48 leading points over one configuration a series $g(\delta)$ that must vanish. Modulo two primes, with all inverses checked to be units, g has a nonzero coefficient of order at most 5. Along T_9, K14(m_{Q_1}) together with the condition of Lemma 5.20 for body k has no torus zero for any $s \in \mathbb{C}^*$, by Theorem 6.5(a) (four systems, two primes).

- *The inclusions in the proof of Theorem 3.12, all labellings.* For all 120 labellings, each type set of Table 1 lies in the attained set of its piece in Table 3, with diagram 7 refined by Table 2. Three programs check this in exact rational arithmetic: for diagrams 1, 4, 7, 8 and 10–13, for the refined diagram 7, and for diagrams 2, 3, 6, 9 and 14–16.
- *Theorem B.1.* Exact computations confirm Theorem B.1. For each assertion of Propositions B.4–B.8, and for each main step of their proofs, a list of relations of the model that contains those used in it, together with its negated conclusion, forms a system of linear relations and disjunctions. Two programs show that each of these systems is infeasible, by exact certificates. One program takes the relations from the programs that define the models; the other builds the models anew and uses its own exact linear programming. Independently, for every pair of modes of V_2^{w}, V_2 and V_N, a complete branching over the disjunctions of the model, with an exact certificate at every node, shows that the projection of the model lies in the polytope of Table 16, or that the model is empty. The same holds for the model of the centred case on its four pairs of modes. A separate checker extracts each model again and verifies all these certificates. Moreover, every vertex v of each polytope of Table 16 is a limit of projections of model points: exact points x_s of the model and x_c of its closure, with x_c projecting to v, have the half-open segment $(x_c, x_s]$ in the model. So each polytope is the convex hull of the closure of the projection of its model.

12.4. **Formalization.** Theorem 1.1 is proved in Lean 4 with Mathlib (project `lean_full`, commit `52858e1`; toolchain v4.35.0-rc3, Mathlib commit `c55e6e7`). The formal statement asserts that for every $m \in (\mathbb{R}_{>0})^5$ there is a finite set of configurations in $\mathbb{C}^5$ such that every solution of (1.1) by distinct points, with $\lambda \in \mathbb{R}$ and $c \in \mathbb{C}$, is the image of one of them under a map $z \mapsto az+b$ or $z \mapsto a\bar{z}+b$ with $a \neq 0$. Admitting reflections does not change finiteness. The proof uses only the axioms propext, Classical.choice and Quot.sound, and no evaluation by compiled code. It is also accepted by nanoda and lean4lean, two independent implementations of the Lean kernel.

The formal proof does not assume the statements of §2.6. It proves at singular places what it needs of them, among them the rules of [2] and the classification of [2, §7]: a search in the kernel finds 1376 labelled diagrams that satisfy the rules, which are, up to relabelling and the exchange of the colours, the sixteen diagrams of [2, Fig. 11] and eight others; it excludes the eight at places. Data produced by our programs enter only as certificates checked in the kernel, among them the lexicographic certificates, the flats of Theorem 8.2, the runs and enumerations of K14, the resultants of Theorem 7.3 and the branch and bound of §10, which 83454 generated modules replay, with local repairs where a test fails in the arithmetic of the kernel. The formal proof of Theorem 5.21 does not use Proposition 5.11, and it replaces Propositions 5.34 and 5.35 by arguments of the same kind for $\sigma < 5/3$ and for $\sigma > 11/7$: these ranges cover $(1, 2)$, so the resonant values of σ play no role. In the formal proof, relation (R2) of §3.3 follows from one polynomial identity and 36 factorizations instead of eliminations. Building the project from a fresh copy of its repository takes about 860 CPU-hours; the supplementary material gives the commands.

12.5. **Hand proofs and computations.** The arguments we give by hand do not depend on any machine check; programs re-verify steps of many of these arguments (see the supplementary material).

Arguments that need no computer, among others:

- the proofs of Lemma 3.8, Propositions 5.15 and 5.18 and Theorems 5.17, 5.37 and B.1;
- the exclusion of diagram 15 (Corollary 5.19) apart from the identities marked (†) in the proof of Proposition 5.7;
- steps in the proof of Lemma 5.27;
- the levels of the expansion in Lemmas 5.22 and 5.23;
- the identities in the proofs of Propositions 5.34 and 5.35.

Computations without a hand proof: the computations used in the proof, among them

- the computations of Theorem 8.2 (Proposition 8.6) and of Proposition 8.7;
- the computational steps of the proof of Theorem 7.3 (Appendix E);
- the point certificates of §6;
- the certificates of §9;
- the computational steps of the proof of Theorem 5.21.

The formal proof of §12.4 depends neither on the statements of §2.6 nor on the correctness of the mathematics software named above. It depends on the Lean kernel and on the formal statement, which the reader can compare with Theorem 1.1 in the file `CC5Full/System.lean` of the project (with `Masses` from `CC5Full/Basic.lean`); the proof is the theorem `CC5Full.Core.Compose.MainTheorem_holds`.

Appendix A. The sixteen diagrams and their types

Table 15 describes the diagrams of [2, Fig. 11] in the numbering of the figure. In diagram 3 the colours of the four sides alternate around the square, as in the figure and in the description of [2, p. 582] (bodies $1, 3$ and $2, 4$ on the diagonals, body 5 at the centre) and of §8.3 there. See also §2.6.

diagram	edges and circles
1	z-edge 12, z-circles $1, 2$; w-triangle 345, no circles
2	as 1, with w-circles $3, 4, 5$
3	z-edges $14, 23$, w-edges $12, 34$; 1–4 circled in both colours; 5 isolated, uncircled
4	z-edge 12, z-circles $1, 2$; zw-triangle 345
5	$1, 2$ isolated, uncircled; zw-triangle 345
6	z-triangle 125, z-circles $1, 2$; w-triangle 345, w-circles $3, 4$
7	z-triangle 125, w-triangle 345, no circles
8	z-triangle 125, no circles; w-triangle 345, w-circles $3, 4$
9	zw-triangle 123, all w-circled; w-edges $14, 25$; $4, 5$ w-circled
10	zw-triangle 125, w-triangle 345, no circles
11	zw-triangle 125, no circles; w-triangle 345, w-circles $3, 4$
12	zw-triangle 125, w-circles $1, 2, 5$; w-triangle 345, w-circle 3
13	as 12, with w-circles $3, 4$
14	zw-edges on the six pairs of $\{1, 2, 3, 4\}$; 5 isolated
15	as 14, all w-circled, with the w-edge 15
16	zw-edges on all ten pairs

Table 15. The sixteen diagrams of [2, Fig. 11].

We derive the entries of Table 1 for diagrams 1–4, 6–13 and 16, using Lemma 3.2 and the rules quoted after it. We use Lemma 3.3 only where stated, and Lemma 2.4 and [2, §8.9] only for diagram 9. We treat diagrams 5, 14 and 15 in §§3.2–3.3.

Diagram 1. The ends of the z-stroke 12 are z-circled. By Rule 1d they are not z-close. So $x_{12} = 0$ by (O4) and $\omega_{12} = 2$ by (O3). By Rule 2e the three exponents ω of the triangle $T = \{3, 4, 5\}$ are equal. The triangle T is an isolated w-component without circles. By Rule 1e it has no maximal w-stroke. Hence $x = a > 0$ on T. By (O4), $a < 1$. On a cross pair $1t$, $\zeta_{1t} = -2$ by (O6). A z-stroke $1t$ would need $\omega_{1t} = 2$, which shows $\mu_t := \omega_{1t} < 2$. In the triangle $12t$, (O5) gives $\omega_{2t} = \omega_{1t}$. Hence $x_{1t} = x_{2t} = \frac{\mu_t - 2}{2}$. By (O5) in the triangles $1tt'$, where $\omega_{tt'} = 3a - 2$, either all μ_t equal some $\mu \le 3a - 2$, or two of them equal $3a - 2$ and the third is larger. Lemma 3.6 leaves $\mu_t = 3a - 2$.

Diagram 2. As for diagram 1, $x_{12} = 0$. The w-circled bodies of T are not all w-close (Rule 1d). By Rule 2e the triangle is maximal: $x = 0$ on T. On the cross pairs (O6) gives $\zeta = \omega = -2$.

Diagram 3. Since the four sides are not close by Rule 1d, they are maximal and $x = 0$. On the diagonal 13, $\zeta_{12} = 2$ (12 is a maximal w-stroke) and $\zeta_{23} = -2$ (23 is a maximal z-stroke). Then (O5) gives $\zeta_{13} = -2$. Likewise $\omega_{13} = -2$. On the pairs $k5$, (O6) applies in both colours.

Diagram 4. As for diagram 1, with $x = 1$ on T and $3a - 2$ replaced by 1. Lemma 3.6 gives $\mu_t = 1$.

Diagram 6. The pairs 12 and 34 of circled bodies are not close by Rule 1d. By Rule 2e each triangle is maximal, $x = 0$. On the four cross pairs (O6) applies in both colours.

Diagram 7. By Rule 2e the exponents are equal on each of the triangles 125 and 345. By Rule 1e, $x = a > 0$ on 125 and $x = b > 0$ on 345. By (O4), $a, b < 1$. On a cross pair, (O5) gives $\zeta_{13} = \min(\zeta_{15}, \zeta_{53}) = \min(3a - 2, 2 - b) = 3a - 2$ and likewise $\omega_{13} = 3b - 2$, which gives $x_{13} = \frac{3}{2}(a + b) - 2$. Lemma 3.3 gives $a + b \le \frac{4}{3}$.

Diagram 8. On 125, $x = a \in (0, 1)$ as for diagram 7. The edge 34 is maximal by Rule 1d. The edges 35, 45 have $\omega = -2$ by (O6). Hence $x = 0$ on 345. On the cross pairs $\zeta = 3a - 2$ by (O5) and $\omega = -2$ by (O6).

Diagram 9. Each body is w-circled. If body 4 clustered with w_1, w_2, w_3 (case §8.9.1 of [2]), the potential would have exactly one unbounded term, $m_1 m_4 r_{14}^{-1}$ [2, §8.9.1], which is impossible because U is constant on C (Lemma 2.4). The same holds with $1 \leftrightarrow 2$, $4 \leftrightarrow 5$. We are therefore in case §8.9.2, where 14 and 25 are maximal w-strokes: $\zeta = 2$, $x = 0$. In the equation $z_4 = \sum_l m_l Z_{4l}$, $e(Z_{41}) = e(Z_{52}) = 2$ and $Z_{42}, Z_{43}, Z_{51}, Z_{53}$ have exponent $\frac{5}{2}$, and $z_{45} = (m_4 + m_5)Z_{45} + O(\varepsilon^2)$. If $e(p_{45}) > 0$ all terms of z_4 and z_5 would have exponent larger than 1, and if $e(p_{45}) < 0$ then $e(z_{45}) = e(Z_{45}) < 1$, both contradicting $e(z_{45}) = 1$, which follows from $z_{45} = z_{41} + z_{12} + z_{25}$. Thus $x_{45} = 0$. Then (O5) gives the rest: the type is $\mathsf{n}(\{1, 2, 3\}; 14, 25)$, with $x = 1$ on $12, 13, 23$, 0 on $14, 25, 45$ and $-\frac{1}{2}$ on $15, 24, 34, 35$.

Diagrams 10–13. On the zw-triangle 125, $x = 1$. On 345, $x = c$ by Rule 2e. On a cross pair (O5) gives $\zeta_{13} = \min(1, 2 - c) = 1$ and $\omega_{13} = \min(1, 3c - 2) = 3c - 2$, hence $x = \frac{3c-1}{2}$. For diagram 10, $0 < c < 1$ by Rule 1e, and Lemma 3.3 gives $c \le \frac{1}{3}$. For diagrams 11–13, $c = 0$ by Rule 1d, (O6) and Rule 2e.

Diagram 16. Since every pair is a zw-edge, $x = 1$, which is excluded by Lemma 3.3.

Appendix B. The hulls of diagram 5

We prove Theorem B.1, which gives the hulls of §3.3 explicitly. Throughout, P is a place of diagram 5 with zw-triangle T and outer bodies A, B. The points z_c, w_c are the centres of mass of T. We use the coordinates $y = (x_{At_0}, x_{At}, x_{Bt_0}, x_{Bt}, x_{AB})$ of §3.3. We put $e_A = (1, 1, 0, 0, 0)$ and $e_B = (0, 0, 1, 1, 0)$.

Theorem B.1. *Consider the model of V_2^{w}, V_2 or V_N (§3.3) for a pair of modes. The projection to y of each point of the model lies in the polytope that Table 16 gives for this hull and this pair of modes, up to the exchanges of Lemma B.2; where the table gives $\emptyset$, the model is empty. In the centred case, the model of Theorem C0 of the supplementary material is empty for the pairs*

of modes $(\mathrm{Z1}, \mathrm{Zs}_{t_0})$ *and* $(\mathrm{W1}, \mathrm{Ws}_{t_0})$ *and their exchanges, where* t_0 *is the body of* T *at the centre of mass of* T.

pair of modes	V_2^{w}	V_2	V_N
(F, F)	$\mathrm{conv}\{0, e_A, e_B\}$	$\{0\}$	$\{0\}$
(F, Z1)	$\mathrm{conv}\{0, e_A, e_B\}$	$[0, e_B]$	$\emptyset$
(Z1, Z1)	P_{11}	P_{11}	$[-\frac{1}{2}(e_A + e_B), 0]$
(Z1, W1)	$[e_A, e_B]$	$[e_A, e_B]$	$\emptyset$
(Z1, Zs)	P_{1s}	P_{1s}	$\emptyset$
(Zs, Zs)	P_{ss}	P_{ss}	P_{ss}
(F, Zs), (Z1, Ws), (Zs, Ws)	$\emptyset$	$\emptyset$	$\emptyset$

TABLE 16. The hulls of §3.3 for nine pairs of modes, in the coordinates y. Here $P_{11} = \{ae_A + be_B : \ -\frac{1}{2} \le a, b \le 1, \ a + b \le 1\}$, with the vertices $-\frac{1}{2}(e_A + e_B)$, $-\frac{1}{2}e_A + e_B$, e_B, $e_A - \frac{1}{2}e_B$, e_A. The polytope P_{1s}, of dimension 4, has the vertices $-\frac{1}{2}(e_A + e_B)$, $(-\frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, 0)$, 0, $e_A - \frac{1}{2}e_B$, $(1, 1, 0, -\frac{1}{2}, -\frac{1}{2})$, $(1, 1, 0, -\frac{1}{2}, 0)$, e_A, $(1, 1, \frac{1}{3}, 0, \frac{1}{3})$. The polytope P_{ss}, of dimension 5, has the vertices $-\frac{1}{2}(e_A + e_B)$, $(-\frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, 0)$, $(-\frac{1}{3}, -\frac{1}{3}, 0, -\frac{1}{2}, 0)$, $(0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0)$, $(0, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, 0)$, $(0, -\frac{1}{2}, 0, -\frac{1}{2}, 0)$, 0, $(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})$.

B.1. **Orders and relations.** For $X \in \{A, B\}$ let Y be the other outer body. We write $c_X = e(z_X - z_c)$, $d_X = e(w_X - w_c)$, $p_X^z = e(z_X)$, $p_X^w = e(w_X)$, $p_c^z = e(z_c)$, $p_c^w = e(w_c)$ and

$$t_X^z = e\Big(\sum_{t \in T} m_t Z_{tX}\Big), \qquad t_X^w = e\Big(\sum_{t \in T} m_t W_{tX}\Big),$$

and we put $\zeta = \zeta_{AB}$, $\omega = \omega_{AB}$ and $r = x_{AB}$, so that $e(Z_{AB}) = \zeta - 3r$ and $e(W_{AB}) = \omega - 3r$. A point of a model is a real vector, with one entry for each modelled order, that satisfies the relations of the model. We denote its entries by the names of the orders.

Inside T the differences $z_t - z_c$ and $w_t - w_c$ have exponent at least 1. So, by Lemma 3.4, a body X in mode F has $c_X, d_X < 1$ and $x_{Xt} = \bar{x}_X := \frac{1}{2}(c_X + d_X)$ for all $t \in T$. A body X in mode Z1 or Zs_{t_0} has $c_X \ge 1$, the far order $f_X := d_X < 1$ and the order $s_X = \zeta_{Xt_0}$, which is 1 in mode Z1. In mode Zs_{t_0}, $s_X > 1$, $s_X + 3f_X < 4$ and $3s_X + f_X < 4$. Then $x_{Xt} = \bar{x}_X := \frac{1}{2}(1 + f_X)$ for $t \ne t_0$, and $x_{Xt_0} = x_X^0 := \frac{1}{2}(s_X + f_X)$. In the modes F and Z1 we put $x_X^0 = \bar{x}_X$, so that $y = (x_A^0, \bar{x}_A, x_B^0, \bar{x}_B, r)$. The modes W1 and Ws_{t_0} are the same with the colours exchanged. The orders c_X, d_X, ζ, ω and those of the positions are at least -2, since all components of $\mathcal{Z}$ and $\mathcal{W}$ have exponent at least -2 (§3.1). So $\bar{x}_X \ge -\frac{1}{2}$ in the modes other than F. As A and B have no strokes, $\zeta + 3\omega < 4$ and $3\zeta + \omega < 4$.

The terms of $\sum_t m_t Z_{tX}$ and $\sum_t m_t W_{tX}$ have the orders $\zeta_{Xt} - 3x_{Xt}$ and $\omega_{Xt} - 3x_{Xt}$. In mode Zs_{t_0} the term of t_0 has the least order. These two facts give the relations (S):

(S) In mode F, $t_X^z \ge c_X - 3\bar{x}_X$ and $t_X^w \ge d_X - 3\bar{x}_X$, with equality in the models of V_2 and V_N. In mode Z1, $t_X^z \ge 1 - 3\bar{x}_X$ and $t_X^w \ge f_X - 3\bar{x}_X$. In mode Zs_{t_0}, $t_X^z = s_X - 3x_X^0$ and $t_X^w = f_X - 3x_X^0$.

Each of the following relations says that, among the orders of the terms of an exact identity with nonzero constant coefficients, the minimum is attained at least twice. We list the orders and then the identity.

(P) Any three of the six orders c_A, c_B, ζ, p_A^z, p_B^z, p_c^z, and likewise in w. By the centre of mass, $Mz_c + m_A(z_A - z_c) + m_B(z_B - z_c) = 0$. So the six quantities $z_A - z_c$, $z_B - z_c$, z_{AB}, z_A, z_B, z_c are linear combinations of the first two and no two of them are proportional.

(E) p_X^z, $\zeta - 3r$ and t_X^z, and likewise in w: the equation $z_X = m_Y Z_{XY} - \sum_t m_t Z_{tX}$ of (2.1).

(F) For X not in mode F: c_X, $\zeta + \beta$, t_X^z, and also d_X, $\omega + \beta$, t_X^w, where β is the order of $1 - Mp_{AB}^3$. The identity is $M_T(z_X - z_c) = -M \sum_t m_t Z_{tX} - m_Y(1 - Mp_{AB}^3)\, z_{XY}$, with its counterpart in w.

(I) $2r$, $c_A + d_A$, $c_B + d_B$ and 0: the identity $MI = MI_T + M_T \sum_X m_X(z_X - z_c)(w_X - w_c) + m_A m_B z_{AB} w_{AB}$. Here $I_T = \sum_t m_t(z_t - z_c)(w_t - w_c)$ has order at least 2. The quantity I is a nonzero constant on C (Lemma 2.4).

(V) $d_A + t_A^z$, $d_B + t_B^z$, $-r$ and 0: the identity $U = I_T - \sum_X m_X(w_X - w_c) \sum_t m_t Z_{tX} + m_A m_B p_{AB}$. It follows from $p_{kl} = Z_{kl} w_{kl}$ and from the equations of the bodies $t \in T$ multiplied by $m_t(w_t - w_c)$. Here $U = I$ is a nonzero constant.

(V′) $c_A + t_A^w$, $c_B + t_B^w$, $-r$ and 0: the same identity with the colours exchanged.

In (I), (V) and (V′) we leave out the term I_T, since the minimum is at most $0 < 2$. We also use the following relations, in which X is in mode Z1.

(Q) The torque of an outer body X is $\Theta_X = \sum_{t \in T} m_t\big((z_t - z_c)W_{tX} - (w_t - w_c)Z_{tX}\big)$. Its order is τ_X. We sum over $t \in T$ the equation of w_t in (2.1) multiplied by $m_t(z_t - z_c)$, minus that of z_t multiplied by $m_t(w_t - w_c)$. The left sides give $w_c \sum_t m_t(z_t - z_c) - z_c \sum_t m_t(w_t - w_c) = 0$ and the terms of the pairs inside T cancel. Hence $m_A\Theta_A + m_B\Theta_B = 0$ and $\tau_A = \tau_B$. For X in mode Z1 the terms $m_t(z_t - z_c)W_{tX}$ have order at least $1 + f_X - 3\bar{x}_X = -\bar{x}_X$ and the terms $m_t(w_t - w_c)Z_{tX}$ order at least $2 - 3\bar{x}_X > -\bar{x}_X$. Thus $\tau_X \geq -\bar{x}_X$.

(R) In the model of V_N: at most one of the following *events* occurs: $c_X > 1$; $t_X^z > 1 - 3\bar{x}_X$; $t_X^w > f_X - 3\bar{x}_X$; $\tau_X > -\bar{x}_X$; and, if Y is in mode Zs, $\max(p_A^z, p_B^z, p_c^z) > 1$ and $\tilde{t}_X > -\bar{x}_X$. This is the content of (R1) and (R2) of §3.3, which are proved in the supplementary material.

(T) In the model of V_N, if Y is in mode Zs_{t_0}: the orders $\tilde{t}_X$, $\tilde{t}_Y$ of the two quantities of (R3) (proved in the supplementary material) satisfy $\tilde{t}_X \geq -\bar{x}_X$, and $\tilde{t}_Y = \min(e_1, e_2)$ if $e_1 \neq e_2$, where $e_1 = 1 - \frac{1}{2}(s_Y + 3f_Y)$ and $e_2 = -\bar{x}_Y$. The minimum of $\tilde{t}_X$, $\tilde{t}_Y$ and $p_c^w + 1$ is attained at least twice.

In the centred case, for X in mode Z1 and Y in mode Zs_{t_0} with t_0 at the centre of mass of T, the model of the centred case in the supplementary material has the relations of the model of V_N other than (T) and the event $\tilde{t}_X > -\bar{x}_X$, and moreover $c_Y > 1$. In the modes W1 and Ws the relations are the same with the colours exchanged. The model of V_2 is that of V_2^{w} together with the equalities of (S) in mode F. The models of V_N and of the centred case contain the relations of the model of V_2. So a statement proved for the model of V_2^{w} holds for the other models, and one proved for the model of V_2 holds for those of V_N and of the centred case. The models have further relations, which we do not use.

B.2. **Symmetries and rules.**

Lemma B.2. *The exchange of the colours maps the model of each hull, and that of the centred case, for a pair of modes onto the model of the same kind for the pair with Z1, Zs and W1, Ws exchanged, and it fixes y. The exchange of A and B maps the model for a pair of modes (a, b) onto the model for (b, a), and y to $(y_3, y_4, y_1, y_2, y_5)$. Every pair of modes is the image of one of the nine pairs of Table 16 under these exchanges.*

Proof. The system (2.1) and the type are invariant under the exchange of z and w. This exchange maps each relation of a model to a relation of the model for the pair of modes with the colours exchanged: (P), (E) and (F) in z go to their counterparts in w, (V) goes to (V′), and (S), (Q),

(R), (T) go to their mirror images. The exchange of A and B renames the outer bodies. The two exchanges generate a group of order 4, whose orbits on the 25 pairs of modes are those of the nine pairs of the table. □

By Lemma B.2 we treat the nine pairs of Table 16. For their images under the exchange of the colours, the proofs below hold with (V′) in place of (V).

Lemma B.3. *At every point of the model of V_2^{w}, for $X \in \{A, B\}$:*

(a) *In each colour, five of the six orders c_A, c_B, ζ, p_A^z, p_B^z, p_c^z are equal to their minimum μ_z, and the sixth, the* exception*, is at least μ_z; likewise in w, with the minimum μ_w. So if two of the six orders differ, the larger is the exception and the other five are equal to the smaller. If $c_A, c_B \geq 1$, then $\zeta \geq 1$ and $p_A^z, p_B^z, p_c^z \geq 1$.*

(b) *$t_X^z \geq \min(p_X^z, \zeta - 3r)$, with equality if $p_X^z \neq \zeta - 3r$. If $r < 0$ and $p_X^z = \mu_z$, then $t_X^z = \mu_z$. The same holds in w.*

(c) *In the modes F and Z1, $d_X + t_X^z \geq -\bar{x}_X$, with equality in mode F in the models of V_2 and V_N. In mode Zs, $d_X + t_X^z = -x_X^0$.*

(d) *In mode Z1, $3\bar{x}_X < 2 + f_X$; in mode Zs, $3x_X^0 < 2 + f_X$. Moreover $3r < 2 + \zeta$ and $3r < 2 + \omega$.*

(e) *In the modes Z1 and Zs, $c_X + d_X \geq 1 + f_X = 2\bar{x}_X$. If $r > 0$, the minimum of $c_A + d_A$, $c_B + d_B$ and 0 is attained at least twice. If $r < 0$, the minimum of $2r$, $c_A + d_A$ and $c_B + d_B$ is attained at least twice, and it is at most $2r$.*

Proof. (a) If two of the six orders exceeded the minimum, they would form, with an order equal to the minimum, a triple whose minimum is attained once, against (P). If $c_A, c_B \geq 1$ and $\mu_z < 1$, both c_A and c_B would be exceptions. (b) is (E): the two smallest of p_X^z, $\zeta - 3r$ and t_X^z are equal. If $r < 0$, then $\zeta - 3r > \zeta \geq \mu_z$. (c) follows from (S); for example, in mode F, $d_X + t_X^z \geq d_X + c_X - 3\bar{x}_X = -\bar{x}_X$. (d) restates $f_X < 1$, $3s_X + f_X < 4$, $\zeta + 3\omega < 4$ and $3\zeta + \omega < 4$. (e) The inequality holds as $c_X \geq 1$. The rest is (I), whose minimum is at most $0 < 2r$ if $r > 0$, and at most $2r < 0$ if $r < 0$. □

B.3. **The pairs of modes.**

Proposition B.4. *The models of the pairs of modes* (F, Zs), (Z1, Ws) *and* (Zs, Ws) *are empty.*

Proof. Let B be in mode Ws_{t_0}, with the far order $g := c_B < 1$, so that $d_B \geq 1$, $s_B > 1$, $x_B^0 = \frac{1}{2}(s_B + g)$, $t_B^z = g - 3x_B^0$ and $t_B^w = s_B - 3x_B^0$. Let A be in mode Z1 or Zs. By Lemma B.3(a), $c_B < 1 \leq c_A$ gives $\zeta = p_A^z = p_B^z = p_c^z = g$, and $d_A < 1 \leq d_B$ gives $\omega = p_A^w = p_B^w = p_c^w = f_A$. So $2r = f_A + g$. In z, relation (E) for B says that the maximum of 0, r, x_B^0 is attained twice. In w, since $t_B^w - f_A = -x_B^0 - 2r$, it says that the maximum of 0, $3r$, $x_B^0 + 2r$ is attained twice. If $r > 0$, the first statement gives $x_B^0 = r$, hence $s_B = f_A < 1$. If $r < 0$, it gives $x_B^0 = 0$, and then 0 is the only maximum in the second. If $r = 0$, it gives $x_B^0 \leq 0$, hence $s_B \leq -g = f_A < 1$. Each case contradicts $s_B > 1$.

Now let A be in mode F and B in mode Zs_{t_0}. By Lemma B.3(a), $c_A < 1 \leq c_B$ gives $\zeta = c_A = p_A^z = p_B^z = p_c^z =: \kappa < 1$. Thus $\omega = 2r - \kappa$.

If $r > 0$, then $\zeta - 3r < \kappa$, and Lemma B.3(b) gives $t_A^z = t_B^z = \kappa - 3r$. By (S), $t_A^z \geq \kappa - 3\bar{x}_A$. Hence $\bar{x}_A \geq r$, that is, $d_A \geq \omega$. And $t_B^z = -\frac{1}{2}(s_B + 3f_B) = \kappa - 3r$ with $s_B > 1$ gives $f_B < 2r - \frac{1}{3}(2\kappa + 1) < \omega$. So ω and d_A both exceed f_B and both are exceptions in w.

If $r < 0$, then $\zeta - 3r > \kappa$, and $t_A^z = t_B^z = \kappa$. For A this gives $\bar{x}_A \geq 0$. Thus $d_A \geq -\kappa > \omega$. So d_A is the exception in w and $f_B = \omega = p_B^w$. For B it gives $f_B = -\frac{1}{3}(2\kappa + s_B)$. As $\omega - 3r > p_B^w$, Lemma B.3(b) gives $t_B^w = p_B^w = f_B$. Then $x_B^0 = 0$, $s_B = -f_B$ and $f_B = -\kappa$, whereas $f_B = \omega < -\kappa$.

If $r = 0$, then $t_B^z \geq \kappa$, so $s_B + 3f_B \leq -2\kappa$ and $f_B < -\frac{1}{3}(2\kappa+1) < -\kappa = \omega$. So ω is the exception in w and $d_A = f_B = p_A^w < \omega$. Hence $t_A^w = f_B$, while (S) gives $t_A^w \geq d_A - 3\bar{x}_A = -\frac{1}{2}(3\kappa + f_B)$, that is, $f_B \geq -\kappa$. □

Proposition B.5. *At every point of the model of V_2^{w} for each of the following pairs of modes, $r = 0$, and:*

(a) $(\mathrm{Z1}, \mathrm{W1})$*: $\bar{x}_A + \bar{x}_B = 1$;*
(b) (F, F)*: $\bar{x}_A, \bar{x}_B \geq 0$ and $\bar{x}_A + \bar{x}_B < 1$; in the models of V_2 and V_N, $\bar{x}_A = \bar{x}_B = 0$;*
(c) $(\mathrm{F}, \mathrm{Z1})$*: $\bar{x}_A, \bar{x}_B \geq 0$ and $\bar{x}_A + \bar{x}_B < 1$; in the model of V_2, $\bar{x}_A = 0$;*
(d) $(\mathrm{Z1}, \mathrm{Z1})$*: one of f_A, f_B is equal to μ_w, and $\mu_w \leq \omega \leq -1$; so $\bar{x}_A + \bar{x}_B < 1$.*

Proof. (a) Let $g = c_B < 1$ be the far order of B. By Lemma B.3(a), $\zeta = g$ and $\omega = f_A$. Thus $2r = f_A + g$. Since $c_A + d_A \geq 1 + f_A > 2r$ and $c_B + d_B \geq 1 + g > 2r$, the minimum in (I) is attained by $2r$ and 0. So $r = 0$ and $\bar{x}_A + \bar{x}_B = 1 + \frac{1}{2}(f_A + g) = 1$.

(b) Suppose that $r > 0$. Since the minimum in (V) is at most $-r < 0$, it is attained by $d_X + t_X^z$ for some X, and Lemma B.3(c) gives $\bar{x}_X \geq r > 0$. By Lemma B.3(e), the other body Y has $c_Y + d_Y = 0$. The minimum of c_A, c_B, ζ is attained twice. So $c_Y \geq \min(c_X, \zeta)$, and likewise $d_Y \geq \min(d_X, \omega)$; hence $0 = c_Y + d_Y \geq \min(c_X, \zeta) + \min(d_X, \omega)$. As $c_X + d_X > 0$ and $\zeta + \omega > 0$, either $c_X \leq \zeta$ and $\omega \leq d_X$, or $\zeta \leq c_X$ and $d_X \leq \omega$. In the first case also $\omega \leq d_Y$. Thus $\omega = \mu_w$. Then $p_Y^w \geq \omega > \omega - 3r$ and $t_Y^w \geq d_Y - 3\bar{x}_Y = d_Y \geq \omega$ by (S). Hence the minimum $\omega - 3r$ in (E) is attained once. The second case is the same in z.

Suppose that $r < 0$. By Lemma B.3(e), $c_X + d_X \leq 2r$ for some X. Then $\bar{x}_X < 0$ and $d_X + t_X^z \geq -\bar{x}_X > 0$. As $-r > 0$, the minimum in (V) is 0, attained also by $d_Y + t_Y^z \geq -\bar{x}_Y$. Hence $\bar{x}_Y \geq 0$. Then c_Y and d_Y are not both minimal in their colours, since otherwise $2\bar{x}_Y \leq c_X + d_X < 0$. Say that c_Y is the exception in z. Then $p_X^z = c_X = \mu_z$. Lemma B.3(b) gives $t_X^z = c_X$, against $t_X^z \geq c_X - 3\bar{x}_X > c_X$. The case of d_Y is the same in w.

So $r = 0$. If $\bar{x}_A < 0$, then $c_A + d_A < \zeta + \omega$; say $c_A < \zeta$. Then ζ is the exception in z and $p_A^z = c_A < \zeta - 3r$. So $t_A^z = c_A$, against $t_A^z \geq c_A - 3\bar{x}_A > c_A$. Hence $\bar{x}_A, \bar{x}_B \geq 0$. If ζ were the exception in z, then $c_A = c_B = \mu_z$ and $\omega = -\zeta < -\mu_z \leq d_A, d_B$, which would give two exceptions in w. So $\zeta = \mu_z$, likewise $\omega = \mu_w$, and $\mu_z + \mu_w = 0$. One of c_A, c_B is equal to μ_z and one of d_A, d_B to μ_w. Hence $\bar{x}_A + \bar{x}_B$ is half the sum of the two other orders among c_A, d_A, c_B, d_B, which are less than 1. In the models of V_2 and V_N, (S) and Lemma B.3(b) give $c_A - 3\bar{x}_A = t_A^z \geq \mu_z$ and $d_A - 3\bar{x}_A \geq \mu_w$. Adding, we obtain $-4\bar{x}_A \geq 0$. So $\bar{x}_A = 0$ and likewise $\bar{x}_B = 0$.

(c) By Lemma B.3(a), $c_A < 1 \leq c_B$ gives $\zeta = c_A = p_A^z = p_B^z = p_c^z =: \kappa < 1$. If $r > 0$, Lemma B.3(b) gives $t_A^z = t_B^z = \kappa - 3r$. For A, (S) then gives $\bar{x}_A \geq r > 0$. Thus Lemma B.3(e) gives $c_B + f_B = 0$ and $f_B \leq -1$. For B, (S) gives $\kappa - 3r \geq 1 - 3\bar{x}_B$, that is, $f_B \geq 2r - \frac{1}{3}(2\kappa+1) > -1$. If $r < 0$, then $t_A^z = t_B^z = \kappa$, so $\bar{x}_A \geq 0$ and $f_B \geq -\frac{1}{3}(2\kappa + 1) > -1$. Then $c_A + d_A > 2r$ and $c_B + f_B > 0 > 2r$, against Lemma B.3(e). So $r = 0$ and $\omega = -\kappa$. As in (b), $\bar{x}_A \geq 0$. So ω is not the exception in w, since otherwise $d_A = \mu_w < \omega$ and $\bar{x}_A < 0$. Hence $\omega = \mu_w = -\kappa > -1$. Then $f_B \geq -\kappa$ gives $\bar{x}_B > 0$. One of d_A, f_B is equal to $-\kappa$. If $d_A = -\kappa$, then $\bar{x}_A = 0$ and $\bar{x}_B < 1$. If $f_B = -\kappa$, then $\bar{x}_A + \bar{x}_B = \frac{1}{2}(\kappa + d_A) + \frac{1}{2}(1 - \kappa) = \frac{1}{2}(1 + d_A) < 1$. In the model of V_2, (S) and Lemma B.3(b) give $\kappa - 3\bar{x}_A = t_A^z \geq \kappa$, so $\bar{x}_A = 0$.

(d) By Lemma B.3(a), $\zeta \geq 1$. Suppose that $r > 0$. By Lemma B.3(e), $c_X + f_X \leq 0$ for some X. Then $f_X \leq -1$ and $\bar{x}_X \leq 0$. As in (b), (V) gives $d_Y + t_Y^z \leq -r$ for some Y. Hence $\bar{x}_Y \geq r > 0$. Thus $Y \neq X$ and $f_X < f_Y$. By Lemma B.3(a), $f_X = \omega = p_X^w = \mu_w$. Lemma B.3(b) gives $t_X^w = \omega - 3r < f_X$, against $t_X^w \geq f_X - 3\bar{x}_X \geq f_X$. Suppose that $r < 0$. By Lemma B.3(e), $c_X + f_X \leq 2r$ for some X. Then $f_X < -1$ and $\bar{x}_X < 0$. As in (b), the minimum in (V) is $0 = d_Y + t_Y^z$. Hence $\bar{x}_Y \geq 0$ and $f_Y \geq -1 > f_X$. Then $p_X^w = f_X = \mu_w$. Lemma B.3(b) gives

$t^w_X = f_X$, against $t^w_X \geq f_X - 3\bar{x}_X > f_X$. So $r = 0$ and $\omega = -\zeta \leq -1$. By Lemma B.3(a), one of f_A, f_B is equal to $\mu_w \leq \omega$, and the other is less than 1. □

Proposition B.6. *Let both outer bodies be in mode Zs. At every point of the model of V_2^{w}, $r \geq 0$ and $x^0_A, x^0_B \leq r$. If $r > 0$, then y lies in the triangle $\mathcal{T} = \mathrm{conv}\{(0, -\frac{1}{2}, 0, -\frac{1}{2}, 0),\ 0,\ (\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})\}$. If $r = 0$, then y lies in the polytope $\mathcal{R}$ of the points with $y_5 = 0$, $y_2 \leq y_1 \leq 0$, $y_4 \leq y_3 \leq 0$, $y_2, y_4 \geq -\frac{1}{2}$ and*

$$3y_4 \leq y_1 + 2y_2 \leq 2y_4, \qquad 3y_2 \leq y_3 + 2y_4 \leq 2y_2.$$

Proof. For a body X in mode Zs, $t^z_X = s_X - 3x^0_X = 1 - x^0_X - 2\bar{x}_X$ and $t^w_X = f_X - 3x^0_X$. By Lemma B.3(c), relation (V) says that the maximum of x^0_A, x^0_B, r and 0 is attained at least twice. By Lemma B.3(a), $\zeta \geq 1$ and $p^z_A, p^z_B, p^z_c \geq 1$.

$r \geq 0$. Suppose that $r < 0$. If $p^w_A = p^w_B = \mu_w$, Lemma B.3(b) gives $t^w_X = \mu_w \leq f_X$, so $3x^0_X = f_X - \mu_w \geq 0$ for both X. If moreover $f_A = f_B = \mu_w$, then $x^0_X = 0$ and $t^z_X = s_X = -f_X = -\mu_w$. One of p^z_A, p^z_B is equal to μ_z. Hence $t^z_X = \mu_z$ for that X, and $2r = \zeta + \omega \geq \mu_z + \mu_w = 0$, against $r < 0$. If instead $f_A > \mu_w$, then $f_B = \mu_w$ and $x^0_B = 0 < x^0_A$, so the maximum of x^0_A, x^0_B, r, 0 is attained once; likewise if $f_B > \mu_w$. If $p^w_A > \mu_w$, then $f_A = f_B = \omega = p^w_B = \mu_w$ and $t^w_B = f_B$, so $x^0_B = 0$ and $t^z_B = s_B = -\omega > \zeta \geq \mu_z$. By Lemma B.3(b), $p^z_B \neq \mu_z$, that is, the position of B is the exception in z, and $c_A = \zeta = p^z_A = \mu_z$. Then $t^z_A = \zeta$. Thus $x^0_A = -\zeta - f_A = -2r > 0$, and the maximum is attained once. The case $p^w_B > \mu_w$ is the same.

$x^0_X \leq r$. Suppose that $x^0_A > r$. As $r \geq 0$, the maximum of x^0_A, x^0_B, r, 0 exceeds r and 0. Hence $x^0_A = x^0_B =: q > r$. If $t^w_X < \omega - 3r$ for some X, Lemma B.3(b) gives $p^w_X = t^w_X = f_X - 3q < f_X$, so f_X is the exception in w and $p^w_X = \omega$. Then $\omega < \omega - 3r$, against $r \geq 0$. So $t^w_X \geq \omega - 3r$ for both X, that is, $f_X \geq \omega + 3(q - r) > \omega$, which gives two exceptions in w.

$r > 0$. Then $x^0_Y = r$ for some Y, by the maximum rule. In (E) for Y in w, $t^w_Y = f_Y - 3r$. We use Lemma B.3(b). If $f_Y > \omega$, then $t^w_Y > \omega - 3r$ and $p^w_Y = \omega - 3r < \omega$, while f_Y is the exception and $p^w_Y = \omega$. If $f_Y < \omega$, then $p^w_Y = t^w_Y < f_Y$, while ω is the exception and $p^w_Y = f_Y$. So $f_Y = \omega = \mu_w$. For the other body X, $x^0_X < r$ would give $t^w_X > f_X - 3r \geq \omega - 3r$, hence $p^w_X = \omega - 3r < \mu_w$. So $x^0_X = r$ and $f_X = \omega$ as for Y. Put $f = \omega$. If $1 + f > 0$, then $c_A + f > 0$ and $c_B + f > 0$, and the minimum 0 in Lemma B.3(e) is attained once. So $f \leq -1$, that is, $\bar{x}_A = \bar{x}_B = \frac{1}{2}(1 + f) \leq 0$. By Lemma B.3(d), $3r < 2 + \omega = 1 + 2\bar{x}_A$. So $y = (r, \bar{x}_A, r, \bar{x}_A, r)$ with $0 < 3r < 1 + 2\bar{x}_A$ and $\bar{x}_A \leq 0$, which lies in $\mathcal{T}$.

$r = 0$. Put $L = \zeta = -\omega$, so that $L \geq 1$, $\zeta - 3r = L$ and $\omega - 3r = -L$. As $x^0_X \leq 0$, $t^w_X \geq f_X$ and $t^z_X \geq s_X > 1$. By Lemma B.3(a) and (b), in z either (z1) ζ is the exception, $c_A = c_B = p^z_A = p^z_B = p^z_c = \mu_z < L$ and $t^z_A = t^z_B = \mu_z$, or (z2) $\zeta = \mu_z = L$ and $t^z_X \geq L$ for both X. In w there are four cases:

(w1) ω is the exception, $f_A = f_B = p^w_A = p^w_B = \mu_w < -L$; then Lemma B.3(b) gives $t^w_X = f_X$, so $x^0_X = 0$ and $t^z_X = s_X = -f_X$;
(w2) f_A is the exception, and $f_B = \omega = -L < f_A$;
(w3) the same with A and B exchanged;
(w4) $f_A = f_B = \omega = -L$.

In the cases (w2)–(w4), (z1) is impossible: for X with $f_X = -L$, $t^z_X = \frac{1}{2}(3L - s_X) = \mu_z$ would give $x^0_X = \frac{1}{2}(s_X - L) = L - \mu_z > 0$, against $x^0_X \leq 0$. So $t^z_X \geq L$ for both X in these cases. We show $y \in \mathcal{R}$. The inequalities $y_2 \leq y_1 \leq 0$, $y_4 \leq y_3 \leq 0$ and $y_2, y_4 \geq -\frac{1}{2}$ hold since $s_X > 1$, $x^0_X \leq 0$ and $\bar{x}_X \geq -\frac{1}{2}$. As $t^z_X = 1 - x^0_X - 2\bar{x}_X$ and $f_Y = 2\bar{x}_Y - 1$, the other four inequalities say that

$$-f_Y \leq t^z_X \leq -\tfrac{1}{2}(1 + 3f_Y) \qquad \text{for } \{X, Y\} = \{A, B\}.$$

The lower bound holds with equality in (w1). In (w2), $t^z_A \geq L = -f_B$ and $t^z_B \geq L > -f_A$; (w3) is the same; and in (w4), $t^z_X \geq L = -f_Y$. For the upper bound, $s_X > 1$ gives $t^z_X < -\frac{1}{2}(1 + 3f_X)$,

which suffices if $f_X \geq f_Y$. It remains to treat (w2) with $X = B$, and (w3) with $X = A$, which is the same. In (F) for A in w, the terms are f_A, $-L+\beta$ and $t_A^w \geq f_A$. Hence $-L+\beta \geq f_A > -L$, that is, $\beta > 0$. In (F) for B in w, the terms are $f_B = -L$, $-L+\beta > -L$ and $t_B^w \geq -L$. So $t_B^w = -L = f_B$. Hence $x_B^0 = 0$, $s_B = L$ and $t_B^z = L$. Finally, $t_A^z \geq L$ gives $3f_A \leq -2L - s_A < -2L-1$, that is, $t_B^z = L < -\frac{1}{2}(1+3f_A)$. □

Proposition B.7. *Let A be in mode Z1 and B in mode Zs. At every point of the model of V_2^{w}, y lies in $\mathcal{S}^0$ if $r=0$, in $\mathcal{S}^+$ if $r>0$, and in $\mathcal{S}^-$ if $r<0$, where*

$$\mathcal{S}^0 = \{y:\ y_1 = y_2,\ y_5 = 0,\ y_4 \leq y_3 \leq 0,\ -\tfrac{1}{2} \leq y_4 \leq y_1 \leq 1\},$$
$$\mathcal{S}^+ = \{y:\ y_1 = y_2,\ y_3 = y_5 \geq 0,\ -\tfrac{1}{2} \leq y_4 \leq 0,\ 3y_5 \leq y_1 + 2y_4,\ y_1 \leq 1\},$$
$$\mathcal{S}^- = \{y:\ y_1 = y_2,\ y_3 = 0,\ y_4 \leq y_5 \leq 0,\ y_1 + 2y_4 \geq 0,\ y_1 \leq 1\}.$$

Moreover $x_B^0 \leq \max(r,0)$. If $r>0$, then $f_B = \omega \leq -1$ and $t_A^w = \omega - 3r$. If $r=0$, then $f_B \leq f_A$.

Proof. Here $y = (\bar{x}_A, \bar{x}_A, x_B^0, \bar{x}_B, r)$, with $\bar{x}_A = \frac{1}{2}(1+f_A) < 1$, $\bar{x}_B \geq -\frac{1}{2}$ and $x_B^0 > \bar{x}_B$. By Lemma B.3(a), $\zeta \geq 1$ and $p_A^z, p_B^z, p_c^z \geq 1$. By Lemma B.3(c), (V) says that the minimum of $f_A + t_A^z$, $-x_B^0$, $-r$ and 0 is attained at least twice, and $f_A + t_A^z \geq -\bar{x}_A$.

Step 1: $x_B^0 \leq \max(r,0)$. Suppose not. Then the minimum in (V) is attained by $-x_B^0$ and $f_A + t_A^z$. Hence $\bar{x}_A \geq x_B^0 > 0$ and $f_A > -1$. If $t_B^w < \omega - 3r$, Lemma B.3(b) gives $p_B^w = t_B^w < f_B$, so f_B is the exception in w, $p_B^w = \omega = f_A$ and $\omega < \omega - 3r$. Then $r<0$ and $\zeta = 2r - \omega < 1$, against $\zeta \geq 1$. Otherwise $f_B \geq \omega + 3(x_B^0 - r) > \omega$. Thus again f_B is the exception and $f_A = \omega$. If $r \leq 0$, then $\zeta = 2r - \omega < 1$ as before. If $r>0$, then $\zeta - 3r = -r - \omega > -x_B^0 - \omega = t_A^z$, and Lemma B.3(b) gives $p_A^z = t_A^z = -x_B^0 - \omega$. As $p_A^z \geq 1$, we obtain $\omega < -1$, against $\omega = f_A > -1$.

Step 2: if $r>0$, then $x_B^0 = r$. If $x_B^0 < r$, the minimum in (V) is attained by $-r$ and $f_A + t_A^z$, so $\bar{x}_A \geq r > 0$. By Lemma B.3(e), $c_B + f_B = 0$. Thus $f_B \leq -1 < f_A$. So f_A is the exception in w and $f_B = \omega = p_B^w$. In (E) for B in w, the term $\omega - 3r$ is less than $p_B^w = \omega$ and than $t_B^w = \omega - 3x_B^0$. Hence the minimum is attained once.

Step 3: if $r>0$, then $f_B = \omega \leq -1$, $t_A^w = \omega - 3r$ and $3r \leq \bar{x}_A + 2\bar{x}_B$. As $x_B^0 = r$, (E) for B in w gives $f_B = \omega = \mu_w$, as in the proof of Proposition B.6. If $c_B + f_B > 0$, Lemma B.3(e) gives $c_A + f_A = 0$, so $f_B = \mu_w \leq f_A \leq -1$; otherwise $f_B \leq -c_B \leq -1$. In (F) for B in w, the terms are ω, $\omega+\beta$ and $t_B^w = \omega - 3r$. Hence $\beta = -3r$. In (F) for A in w, the terms are $f_A \geq \omega > \omega - 3r = \omega + \beta$ and t_A^w. Thus $t_A^w = \omega - 3r$. By (S), $t_A^w \geq f_A - 3\bar{x}_A = -\frac{1}{2}(3+f_A)$. Hence $6r \leq 3 + f_A + 2f_B$, which is $3r \leq \bar{x}_A + 2\bar{x}_B$.

Step 4: if $r<0$, then $x_B^0 = 0$, $\bar{x}_B \leq r$ and $\bar{x}_A + 2\bar{x}_B \geq 0$. Since $x_B^0 \leq 0$ by step 1, $t_B^w \geq f_B$. If $p_B^w = \mu_w$, Lemma B.3(b) gives $t_B^w = \mu_w \leq f_B \leq t_B^w$, so $x_B^0 = 0$ and $f_B = \mu_w$. If $p_B^w > \mu_w$, then $f_A = f_B = \omega = p_A^w = \mu_w$, and Lemma B.3(b) gives $t_A^w = f_A$. By (S), $f_A \geq -\frac{1}{2}(3+f_A)$. Thus $\omega = f_A \geq -1$ and $\zeta = 2r - \omega < 1$, which is impossible. So $x_B^0 = 0$ and $f_B = \mu_w$.

Next, $c_B + f_B \leq 2r$. Otherwise, by Lemma B.3(e), $c_A + f_A = 2r$. Hence $f_A < -1$, $\bar{x}_A < 0$ and $t_A^w \geq f_A - 3\bar{x}_A > f_A$. Then $p_A^w \neq \mu_w$ by Lemma B.3(b). So $f_A = f_B = \omega$. Now $c_B > c_A$ makes c_B the exception in z. Hence $p_B^z = \mu_z = \zeta$, and Lemma B.3(b) gives $t_B^z = \zeta$. As $x_B^0 = 0$, this gives $s_B = t_B^z = \zeta$ and $x_B^0 = \frac{1}{2}(\zeta + \omega) = r < 0$, a contradiction. So $f_B \leq 2r - c_B \leq 2r - 1$, that is, $\bar{x}_B \leq r$.

Finally, if $p_A^w = \mu_w$, then Lemma B.3(b) gives $t_A^w = \mu_w = f_B$, and (S) gives $f_B \geq -\frac{1}{2}(3+f_A)$, that is, $\bar{x}_A + 2\bar{x}_B \geq 0$. If $p_A^w > \mu_w$, then $f_A = f_B = \omega$, and $s_B = -f_B$ gives $t_B^z = s_B = -\omega = \zeta - 2r$. As $\zeta - 2r < \zeta - 3r$, (E) for B in z and Lemma B.3(b) give $p_B^z = t_B^z > \zeta$. So the position of B is the exception in z and $p_A^z = \zeta = \mu_z$. Then $t_A^z = \zeta$. By (S), $\zeta \geq -\frac{1}{2}(1+3\omega)$, that is, $\omega \geq -1 - 4r$. But $\omega = f_B \leq 2r - 1 < -1 - 4r$.

Step 5: if $r=0$, then $f_B \leq f_A$. Otherwise $f_A = \omega = \mu_w$. In (F) for B in w, the terms are f_B, $\omega + \beta$ and $t_B^w \geq f_B$. Hence $\beta \geq f_B - \omega > 0$. In (F) for A in w, the terms are ω, $\omega + \beta > \omega$

and t^w_A. Thus $t^w_A = \omega = f_A$, and (S) gives $f_A \geq -1$. Then $\bar{x}_B > \bar{x}_A \geq 0$. Hence $x^0_B > \bar{x}_B > 0$, against step 1.

Steps 1–5, with $\bar{x}_A < 1$, $\bar{x}_B \geq -\frac{1}{2}$ and $x^0_B > \bar{x}_B$, give the three cases. □

Proposition B.8. (a) *The models of V_N for* (Z1, W1), (F, Z1) *and* (Z1, Zs) *are empty, and so is the model of Theorem C0 of the supplementary material for* $(\mathrm{Z1}, \mathrm{Zs}_{t_0})$ *in the centred case.*

(b) *At every point of the model of V_N for* (Z1, Z1), $\bar{x}_A = \bar{x}_B \leq 0$.

Proof. Since these models contain the relations of the model of V_2, Propositions B.5 and B.7 apply to them. By (R), a body X in mode Z1 has at most one event. For each of the other quantities of (R) the opposite inequality holds. With (S), (Q) and (T), this means that $c_X = 1$, that t^z_X and t^w_X take their lower bounds $1 - 3\bar{x}_X$ and $f_X - 3\bar{x}_X$, that $\tau_X = -\bar{x}_X$ and, if Y is in mode Zs, that $p^z_A, p^z_B, p^z_c \leq 1$ and (in the model of V_N) that $\tilde{t}_X = -\bar{x}_X$, except for the event.

(a) *(Z1, W1).* By Proposition B.5(a) and its proof, $r = 0$, $\zeta = g := c_B$, $\omega = f_A$ and $f_A + g = 0$. Lemma B.3(b) gives $t^z_B \geq g$ and $t^w_B \geq f_A$. If t^z_B took its lower bound $-\frac{1}{2}(3+g)$, then $g \leq -1$. If t^w_B took its lower bound $-\frac{1}{2}(1+3g)$, then $-\frac{1}{2}(1+3g) \geq -g$, and again $g \leq -1$. As $g = -f_A > -1$, both are events of B, against (R).

(F, Z1). By Proposition B.5(c) and its proof, $r = 0$, $\bar{x}_A = 0$, $\zeta = c_A =: \kappa < 1$, $\omega = \mu_w = -\kappa$ and $f_B \geq -\kappa$. Lemma B.3(b) gives $t^z_B \geq \kappa$ and $t^w_B \geq -\kappa$. If $t^z_B = -\frac{1}{2}(1 + 3f_B)$, then $f_B \leq -\frac{1}{3}(2\kappa + 1) < -\kappa$. So t^z_B is the event of B. Then $t^w_B = -\frac{1}{2}(3 + f_B) \geq -\kappa$ gives $f_B \leq 2\kappa - 3 < -1$, against $f_B \geq -\kappa > -1$.

(Z1, Zs). Let A be in mode Z1 and B in mode Zs_{t_0}, in the model of V_N or in that of the centred case. By Proposition B.7, y lies in $\mathcal{S}^0$, $\mathcal{S}^+$ or $\mathcal{S}^-$. We say that *det holds* if $p^z_A, p^z_B, p^z_c \leq 1$; it fails only if this is the event of A. As these positions are at least 1, det means that they are equal to $1 = \mu_z$. If det fails, then $c_A = 1 = \mu_z$.

If $r = 0$, then $x^0_B \leq 0$ and $t^z_B \geq s_B > 1$. If det holds, (E) for B in z has the terms $p^z_B = 1$, $\zeta \geq 1$ and $t^z_B > 1$, so $\zeta = 1$. If det fails, $\zeta > 1$ would make ζ the exception in z, and Lemma B.3(b) would give $t^z_B = p^z_B = 1$; so again $\zeta = 1$ and $\omega = -1$. If ω were the exception in w, then $p^w_A = f_A < -1$, and Lemma B.3(b) would give $t^w_A = f_A$, against $t^w_A \geq -\frac{1}{2}(3 + f_A) > f_A$. So $\omega = \mu_w = -1$ and one of f_A, f_B is equal to -1. As $f_B \leq f_A$, this gives $f_B = -1$. Then $\bar{x}_B = 0 < x^0_B$, against $x^0_B \leq 0$.

If $r < 0$, then $x^0_B = 0$ and $t^z_B = s_B > 1$. If det holds, then $p^z_B = 1 = \mu_z$, and Lemma B.3(b) gives $t^z_B = 1$, against $t^z_B > 1$. If det fails, $p^z_B = \mu_z$ would give the same, so the position of B is the exception in z and $c_A = \zeta = p^z_A = 1$. Then $t^z_A = 1$ by Lemma B.3(b). As t^z_A takes its lower bound $-\frac{1}{2}(1 + 3f_A)$, $f_A = -1$, that is, $\bar{x}_A = 0$. Then $\bar{x}_A + 2\bar{x}_B \geq 0$ gives $\bar{x}_B \geq 0 > r$, against $\bar{x}_B \leq r$.

If $r > 0$, then $x^0_B = r$ and $f_B = \omega \leq -1$, so $s_B = 2r - \omega = \zeta > 1$ and $t^z_B = \zeta - 3r$. In the centred case $c_B > 1$. If det holds, $\zeta > 1 = \mu_z$ makes ζ the exception in z, so $c_B = 1$. If det fails, $c_A = 1 = \mu_z$ makes c_B the exception, so $\zeta = 1$. Both are impossible. In the model of V_N, suppose first that det holds and that t^z_A and t^w_A take their lower bounds. By (E) for B in z, with the terms $p^z_B = 1$ and $t^z_B = \zeta - 3r$, we have $\zeta - 3r \leq 1$. The equality $t^w_A = \omega - 3r = -\frac{1}{2}(3 + f_A)$ gives $f_A = 6r - 3 - 2\omega$. If $\zeta - 3r < 1 = p^z_A$, Lemma B.3(b) gives $t^z_A = \zeta - 3r = -r - \omega$, and $t^z_A = -\frac{1}{2}(1 + 3f_A)$ then gives $\omega = 2r - 1$ and hence $r = \frac{1}{2}(1 + \omega) \leq 0$. If $\zeta - 3r = 1$, then $\omega = -1 - r$, $f_A = 8r - 1$, and $t^z_A \geq 1$ gives $f_A \leq -1$ and hence $r \leq 0$. Hence the event of A is the failure of det, or t^z_A, or t^w_A. In each case $\tilde{t}_A = -\bar{x}_A$. In (T), $e_1 - e_2 = \frac{1}{2}(3 - 2r - \omega) > 0$, as $\omega \leq -1$ and $r < 1$. Hence $\tilde{t}_B = e_2 = -\frac{1}{2}(1 + \omega)$. If t^w_A takes its lower bound, then $\bar{x}_A = 3r - 1 - \omega$ and $\tilde{t}_A = 1 + \omega - 3r < \tilde{t}_B$, so (T) gives $p^w_c + 1 = \tilde{t}_A$, that is, $p^w_c = \omega - 3r < \mu_w$, which is impossible. If t^z_A takes its lower bound and the event is t^w_A, then det holds, and as above $t^z_A = \zeta - 3r$ if

$\zeta-3r<1$ and $t_A^z\ge 1$ if $\zeta-3r=1$. In the first case $\bar x_A=\frac13(1+r+\omega)$. Then $3r\le \bar x_A+2\bar x_B$ gives $8r\le 4+4\omega\le 0$. In the second case $\bar x_A\le 0$ and $\bar x_B=\frac12(1+\omega)=-\frac12 r$. Then $3r\le \bar x_A+2\bar x_B$ gives $4r\le \bar x_A\le 0$. Both contradict $r>0$.

(b) By Proposition B.5(d), $r=0$, and one of f_A, f_B is equal to $\mu_w\le -1$. By (Q) and (R), $\tau_A=\tau_B$ and $\tau_X\ge -\bar x_X$, with equality unless τ_X is the event of X. Suppose that $f_B<f_A$. Then $\tau_A=\tau_B\ge -\bar x_B>-\bar x_A$. So τ_A is the event of A. Hence $c_A=1$ and t_A^z, t_A^w take their lower bounds. By Lemma B.3(a), $f_B=\omega=-\zeta$ and $\mu_z=1$. Lemma B.3(b) gives $t_A^z\ge 1$, that is, $f_A\le -1$. If $\zeta>1$, then ζ is the exception in z, and Lemma B.3(b) gives $t_A^z=t_B^z=1$. So $f_A=-1$. Then $t_B^z=1\ge -\frac12(1+3f_B)$ contradicts $f_B<-1$. If $\zeta=1$, then $f_B=-1<f_A\le -1$. By symmetry $f_A=f_B$ and both are equal to $\mu_w\le -1$, that is, $\bar x_A=\bar x_B\le 0$. □

Proof of Theorem B.1. By Lemma B.2 it suffices to treat the nine pairs of Table 16. Proposition B.4 gives the empty pairs. For (Z1, W1), (F, F), (F, Z1) and (Z1, Z1), Proposition B.5 gives $y=\bar x_A e_A+\bar x_B e_B$ with the stated conditions, and $\bar x_A,\bar x_B\in[-\frac12,1]$; these points lie in the polytopes of the table. For (Zs, Zs), Proposition B.6 gives $y\in\mathcal R\cup\mathcal T$. For (Z1, Zs), Proposition B.7 gives $y\in\mathcal S^0\cup\mathcal S^+\cup\mathcal S^-$. Proposition B.8 gives the column of V_N and the centred case; for (F, F) and (Zs, Zs) the model of V_N lies in that of V_2. We find the vertices of $\mathcal R$, $\mathcal S^0$, $\mathcal S^+$ and $\mathcal S^-$ by the following remark. Let Δ be a polytope and $q_1,\dots,q_k$ affine maps on Δ. If $\delta=\sum_v\lambda_v v$ is a convex combination of the vertices v of Δ and $u=\sum_j\mu_j q_j(\delta)$ is a convex combination, then $(\delta,u)=\sum_{v,j}\lambda_v\mu_j\,(v,q_j(v))$. So every point (δ,u) with $\delta\in\Delta$ and $u\in\operatorname{conv}\{q_1(\delta),\dots,q_k(\delta)\}$ lies in the convex hull of the points $(v,q_j(v))$.

On $\{y_5=0\}$ we put $u_1=y_1+2y_2$ and $u_2=y_3+2y_4$. Since $y_2\le y_1\le 0$ means $3y_2\le u_1\le 2y_2$, and likewise for y_3 and y_4, the polytope $\mathcal R$ is the set of the points with $y_2,y_4\ge -\frac12$ and $3\max(y_2,y_4)\le u_1,u_2\le 2\min(y_2,y_4)$. Over the triangle $-\frac12\le y_2\le y_4\le\frac23 y_2$, with the vertices $(y_2,y_4)=(-\frac12,-\frac12)$, $(-\frac12,-\frac13)$ and $(0,0)$, the bounds are $3y_4$ and $2y_2$, and (u_1,u_2) lies in the square spanned by the four pairs of bounds; over the triangle with y_2 and y_4 exchanged the same holds. By the remark, $\mathcal R$ lies in the convex hull of the points with $(y_2,y_4)=(-\frac12,-\frac12)$ and $u_1,u_2\in\{-\frac32,-1\}$, the points with $(y_2,y_4)=(-\frac12,-\frac13)$ or $(-\frac13,-\frac12)$ and $u_1=u_2=-1$, and 0. These seven points lie in $\mathcal R$. They are the seven vertices of P_{ss} with $y_5=0$. The triangle $\mathcal T$ is the convex hull of two of them and $(\frac13,0,\frac13,0,\frac13)$. So $P_{ss}=\operatorname{conv}(\mathcal R\cup\mathcal T)$, a pyramid over $\mathcal R$ with apex $(\frac13,0,\frac13,0,\frac13)$.

The points of $\mathcal S^0$, $\mathcal S^+$ and $\mathcal S^-$ have $y_1=y_2$ and $-\frac12\le y_4\le 0$. They are determined by y_4 and the pair (y_1,y_3), (y_1,y_5), resp. (y_1,y_5). For fixed y_4 this pair ranges over the rectangle $[y_4,1]\times[y_4,0]$ for $\mathcal S^0$ (where $y_5=0$), over the triangle with the vertices $(-2y_4,0)$, $(1,0)$ and $(1,\frac13(1+2y_4))$ for $\mathcal S^+$ (where $y_3=y_5$), and over the rectangle $[-2y_4,1]\times[y_4,0]$ for $\mathcal S^-$ (where $y_3=0$). By the remark, applied over the segment $-\frac12\le y_4\le 0$, $\mathcal S^0$ lies in the convex hull of $-\frac12(e_A+e_B)$, $(-\frac12,-\frac12,0,-\frac12,0)$, 0, $e_A-\frac12 e_B$, $(1,1,0,-\frac12,0)$ and e_A, $\mathcal S^+$ in that of 0, $(1,1,0,-\frac12,0)$, e_A and $(1,1,\frac13,0,\frac13)$, and $\mathcal S^-$ in that of 0, $(1,1,0,-\frac12,-\frac12)$, $(1,1,0,-\frac12,0)$ and e_A. These points lie in the respective sets. Together they are the eight vertices of P_{1s}. So $P_{1s}=\operatorname{conv}(\mathcal S^0\cup\mathcal S^+\cup\mathcal S^-)$. □

Appendix C. Proof of Theorem 3.9

By relabelling we take $s=5$, $D_1=\{1,3\}$, $D_2=\{2,4\}$. The sides are $12,14,23,34$. Let P have type $\mathsf{h}(5;13|24)$.

Orders and leading solutions. By Table 1 and (O3)–(O6), after the rotation the coordinates relative to body 5 have $e(u_k)=e(v_k)=-2$ for $k\le 4$, $e(p)=0$ on the sides and $e(p)=2$ on the diagonals and on the pairs $k5$. We write the coordinates as τ-powers times y and divide each row by its lowest power. This gives a weighted system $G(\tau,y)=G_0(y)+\tau^2G_2(y)+\tau^3G_3(y)$, to be solved with $y(0)$ a *leading solution*, $G_0(y(0))=0$ with all coordinates nonzero. We normalize

$y_{u_1} = y_{v_1} = 1$. The leading solutions then exist only when $m_1 m_3 = m_2 m_4$. They are explicit: in case A, $y_u = (1, U, U, 1)$, $y_v = (1, 1, V, V)$ with $U = -m_1/m_2$, $V = -m_2/m_3$, $k_e^3 = 1/(m_k + m_l)$ on each side $e = kl$, and $q_{13}^2 = m_3/M_Q$, $q_{24}^2 = -m_3/M_Q$, $q_{15}^2 = 1$, $q_{25}^2 = -m_2/m_1$, $q_{35}^2 = m_3/m_1$, $q_{45}^2 = -m_4/m_1$, where k_e, q_e are the leading coefficients of p_e on the sides and on the other pairs. Case B exchanges u and v.

Rank and obstruction. At each leading solution the Jacobian J_0 of G_0 has rank 16: one checks by a direct elimination that on the side rows and the rows Z_k, W_k the reduced Jacobian in the variables $y_{u_2}, y_{u_3}, y_{u_4}, y_{v_2}, y_{v_3}, y_{v_4}$ and the four side values is injective. The six other rows have the invertible diagonal block $\operatorname{diag}(2/q_e)$. The left kernel of J_0 is spanned by $\lambda_Z = -Z_1 - \frac{m_2}{m_1} Z_2 + \frac{m_2}{m_1} Z_3 + Z_4$ and $\lambda_W = \frac{m_2}{m_3} W_1 - \frac{m_2}{m_3} W_2 - W_3 + W_4$ (in case A). Since λ vanishes on the side rows, $\lambda \cdot G_2 = 0$. The first obstruction is

$$\lambda_Z \cdot G_3(y_0) = \frac{O}{m_1 m_2},$$

$$O = m_2\big(m_1 m_3 q_{13} + m_2 m_4 q_{24} + m_1 m_5 q_{15} + m_2 m_5 q_{25} + m_3 m_5 q_{35} + m_4 m_5 q_{45}\big),$$

and $\lambda_W \cdot G_3(y_0) = -O/(m_1 m_3)$. If $O \neq 0$, we compare coefficients in $G(\tau, y(\tau)) \equiv 0$ through the order τ^3, using that J_0 restricted to the rows other than Z_4, W_2 is invertible, and obtain a contradiction. Hence $O = 0$.

The condition $O = 0$. The values q_{13}, q_{15}, q_{35} are real and q_{24}, q_{25}, q_{45} are imaginary, with all signs occurring among the leading solutions. Therefore $O = 0$ for some leading solution if and only if both $m_1 m_3 q_{13} \pm m_1 m_5 q_{15} \pm m_3 m_5 q_{35}$ and $m_2 m_4 q_{24} \pm m_2 m_5 q_{25} \pm m_4 m_5 q_{45}$ vanish for some signs. We divide the first triple of absolute values $m_1 m_3^{3/2} M_Q^{-1/2}$, $m_1 m_5$, $m_3^{3/2} m_5 m_1^{-1/2}$ by $m_1 m_3^{3/2} m_5$ and obtain $x_5^2/\sqrt{M_Q}$, x_3^3, x_1^3. A signed sum of three positive numbers vanishes for some signs if and only if one is the sum of the other two. For the first triple this is $\mathrm{Tri}(13; 5)$. The second triple gives $\mathrm{Tri}(24; 5)$ in the same way. □

Appendix D. The labels

The polynomials are those of [2, §8] (see §2.6). The files of the supplementary material contain them with checksums. With x_i as in §3.4, they read as follows.

- $\mathsf{E}(\{a, b, c\})$: $m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2 - 2 m_a m_b m_c (m_a + m_b + m_c) = 0$, the product over the signs of $x_a \pm x_b \pm x_c$ times $(m_a m_b m_c)^2$.
- $\mathsf{R}(ab; s)$: $m_s^2 (m_a + m_b)^4 - m_a^3 m_b^3 = 0$.
- $\mathsf{H}(ij|kl)$: $m_i m_j - m_k m_l = 0$; and $\mathrm{Tri}(ij; s)$: for positive masses equivalent to

$$\Big(\frac{1}{m_s^2 M_{Q_s}} - \frac{1}{m_i^3} - \frac{1}{m_j^3}\Big)^2 - \frac{4}{m_i^3 m_j^3} = 0.$$

Appendix E. Diagram 2 beyond the torque

In this appendix we prove Lemma 7.2 and Theorem 7.3. Let P be a singular place of diagram 2 of a curve C as in Theorem 3.12. Exchanging the colours and relabelling the bodies (with their masses) we may assume that the segment is $\{1, 2\}$, z-coloured, and the triangle $\{3, 4, 5\}$, w-coloured. Then the ends of the segment are z-circled and the vertices of the triangle are w-circled. Put $M_t = m_3 + m_4 + m_5$, $\rho = m_2/m_1$, and fix j_ρ with $\mathrm{j}_\rho^2 = -\rho$.

E.1. **The weighted system.** By Table 1 the type is $\mathsf{a}_1(\{3, 4, 5\})$: $x_{kl} = 0$ inside the segment and the triangle and $x_{kl} = -2$ across. By (O3) and (O6) of Lemma 3.2, $\zeta_{12} = -2$, $\omega_{12} = 2$, $\omega_{kl} = -2$ and $\zeta_{kl} = 2$ on the sides of the triangle, and $\zeta = \omega = -2$ across. Hence, in the coordinates $u_k = z_k - z_5$, $v_k = w_k - w_5$ of §2.2, the exponents of Lemma 3.1 are $e(u) = (-2, -2, 2, 2)$, $e(v) = (-2, -2, -2, -2)$, $e(p_{kl}) = 0$ for $kl \in \{12, 34, 35, 45\}$ and $e(p_{kl}) = 2$ for the six cross pairs.

Let $w = e/2$ be the *weights*. For each of the eighteen rows (2.2), we substitute $q = T^{w_q} y_q$ for each coordinate q and divide by the lowest power of T. This gives $G_r(T, y) = \sum_{i\geq 0} T^i G_{r,i}(y)$. Since its coefficients are ± 1, $\pm m_l$ and $-(m_k + m_5)$, the same monomials occur at all positive masses. A *leading solution* is a zero y_0 of $G_0 = (G_{r,0})_r$ with $y_{u_1} = y_{v_1} = 1$ and all coordinates nonzero. The matrix J_0 is the 18×16 Jacobian matrix of G_0 at y_0 in the other sixteen unknowns.

Lemma E.1. *After a rotation depending on the local parameter and a change of the parameter, the coordinates near P are $q = T^{w_q} y_q(\xi)$ with $T = \xi^L$ for an integer $L \geq 1$, where the y_q are convergent power series in ξ with $y_{u_1} \equiv y_{v_1} \equiv 1$, $G(\xi^L, y(\xi)) \equiv 0$, and $y(0)$ is a leading solution.*

Proof. By Lemma 3.1, after the rotation there, each coordinate is $\vartheta^{d\,e_q}\tilde{y}_q(\vartheta)$ with $\tilde{y}_q(0) \neq 0$. We put $L = 2d$ (a positive integer) and $T = \vartheta^L$, so that $q = T^{w_q}\tilde{y}_q$. A further rotation by the unit $\tilde{y}_{u_1}^{-1}$ makes $y_{u_1} \equiv 1$ and multiplies all other $\tilde{y}_q$ by units. The zero set of G_0 is invariant under $y_q \mapsto \lambda^{e'_q} y_q$, with $e'_q = w_q - w_{u_1}$, $w_q + w_{u_1}$ or w_q for q a u_k, a v_k or a p_{kl}, which comes from $T \mapsto \lambda T$ combined with a rotation. In particular, $e'_{u_1} = 0$ and $e'_{v_1} = -2$. Replacing T by $T'\phi^L$ and rotating accordingly turns y_q into $\phi^{Le'_q} y_q$. The equation $\phi^{-2L} y_{v_1}(\xi\phi) = 1$ is solved by a convergent $\phi(\xi)$, $\phi(0)^{2L} = y_{v_1}(0)$, by the implicit function theorem. Since all rows vanish on X_m, $G(\xi^L, y(\xi)) \equiv 0$. At $\xi = 0$ this gives $G_0(y(0)) = 0$. □

E.2. **The leading solutions and the torque.** For $h \in \mathbb{C}^{\{3,4,5\}}$ let $F'_k(h) = h_k - \sum_{l\in\{3,4,5\}\setminus k} m_l p_{kl}^3 (h_k - h_l)$, the rows of the three bodies of the triangle alone. The rows of G_0 are the least-weight terms of the rows (2.2).

Lemma E.2. *At a leading solution: $u_2 = -m_1/m_2$, $p_{12}^3 = 1/(m_1 + m_2)$, $v_2 = v_c := (m_3 v_3 + m_4 v_4)/M_t = 1$; with $\omega_k = v_k - v_c$ for $k \in \{3, 4, 5\}$ (so $\omega_5 = -1$ and $\sum_k m_k\omega_k = 0$), $p_{1k}^2 = -1/\omega_k$ and $p_{2k} = \sigma_k \mathrm{j}_\rho p_{1k}$ with $\sigma_k = \pm 1$; and the triangle coordinates $u_3, u_4, v_3, v_4, p_{34}, p_{35}, p_{45}$ satisfy $F'_3 = F'_4 = F'_5$ for $h = u$ and for $h = v$ and $p_{kl}^2 u_{kl} v_{kl} = 1$, so they form a complex central configuration of three bodies: of Lagrange type ($p_{kl}^3 = 1/M_t$) or of Euler type (u parallel to v).*

Proof. The least-weight parts of Z_1, Z_2 are $u_1 - m_2 p_{12}^3 (u_1 - u_2)$ and its mirror, which give u_2 and p_{12}. That of D_{12} gives $v_2 = v_1 = 1$. Those of W_1 and of $m_3 W_3 + m_4 W_4$ give $v_2 = \bar{v}$ and $\bar{v} = v_c$, where $\bar{v} = m_3 v_3 p_{35}^3 + m_4 v_4 p_{45}^3$. Those of D_{1k}, D_{2k} are $p_{1k}^2(1 - v_k) - 1$ and $p_{2k}^2 u_2 (1 - v_k) - 1$. The remaining seven rows are the three-body rows of the triangle. For the dichotomy let $\Psi(h)_k = \sum_l m_l p_{kl}^3 (h_k - h_l)$. By antisymmetry Ψ maps the plane $\sum_k m_k h_k = 0$ to itself. The vectors $u - u_{\mathrm{cm}}\Vdash$ and ω are fixed vectors of Ψ in it. If they are independent, Ψ is the identity on the plane, and comparing the quadratic forms $\sum_k m_k h_k \Psi(h)_k$ and $\sum_k m_k h_k^2$, we obtain $p_{kl}^3 = 1/M_t$. Otherwise ω is a multiple of $u - u_{\mathrm{cm}}\Vdash$. □

A *triple point* is a leading solution whose triangle is of Lagrange type with $u = \Lambda v$. Since the shape of a Lagrange triangle with u parallel to v is equilateral, triple points have an equilateral ω.

Let $\ell(y) \in \mathbb{C}^{18}$ have the entries $\ell_{Z_k} = m_k(M_t v_k - s_v)$ and $\ell_{W_k} = -m_k(M_t u_k - s_u)$ for $k \in \{3, 4\}$, with $s_u = m_3 u_3 + m_4 u_4$, $s_v = m_3 v_3 + m_4 v_4$, and 0 elsewhere. Put $\lambda = \ell(y_0)$, $\tau(y) = \ell(y) \cdot G_1(y)$ (the *torque*) and $\sigma_3(y) = \ell(y) \cdot G_3(y)$. At a leading solution put, for $k \in \{3, 4, 5\}$,

$$\tau_k = m_k p_{1k}(m_1 + \sigma_k \mathrm{j}_\rho m_2), \qquad \theta_k = \frac{\tau_k}{m_k \omega_k} = \frac{m_1 p_{1k}(1 + \sigma_k \mathrm{j}_\rho \rho)}{\omega_k}.$$

We say that θ is *constant* if $\theta_3 = \theta_4 = \theta_5$. Since $|m_1 + \sigma_k \mathrm{j}_\rho m_2|^2 = m_1^2(1 + \rho^3)$ and $|p_{1k}| = |\omega_k|^{-1/2}$, $|\theta_k| = m_1(1 + \rho^3)^{1/2}|\omega_k|^{-3/2}$. Thus constant θ forces $|\omega_3| = |\omega_4| = |\omega_5|$.

Lemma E.3. (a) $\ell(y) \cdot G_0(y) = 0$ *identically.*

(b) *G_1 vanishes on every row except Z_3, Z_4, where $G_{1,Z_k} = \sum_{j=1,2} m_j u_j (p_{kj}^3 - p_{5j}^3)$. Moreover, $\ell \cdot G_2 \equiv 0$ and $\ell \cdot G_n \equiv 0$ for $n \geq 4$.*

(c) *At a leading solution, $G_{1,Z_k} = -(\theta_k - \theta_5)$ and $\tau = -M_t \sum_{k\in\{3,4,5\}} \tau_k$.*

(d) *$\ell(y) \cdot G(T,y) = T\tau(y) + T^3\sigma_3(y)$, so along the series of Lemma E.1, $\tau(y(\xi)) = -T^2\sigma_3(y(\xi))$. In particular $\tau(y_0) = 0$.*

Proof. (a) The rows Z_k, W_k of G_0, $k \in \{3,4\}$, are $F_k' - F_5'$ at u and at v (Lemma E.2). By antisymmetry $\sum_k m_k F_k'(u) = s_u$, $\sum_k m_k F_k'(v) = s_v$ and $\sum_k m_k\big(v_k F_k'(u) - u_k F_k'(v)\big) = 0$. Inserting these into $\ell \cdot G_0$ gives 0. Note that this is the conservation of angular momentum of the triangle. (b) Each row is a polynomial in T. In Z_k, $k \in \{3,4\}$, the terms with a factor p_{kj} or p_{5j}, $j \in \{1,2\}$, have weight 2 (those with u_j) or 4, one or three more than the least weight 1. Hence Z_k has only the relative exponents 0, 1, 3. In W_k these terms have weight 2, three more than -1. So W_k has only 0 and 3. The other rows have the relative exponents 0, 2, 3 and 5 (Z_1, Z_2), 0 and 3 (W_1, W_2), 0 and 2 (D_{12} and D_{jk}, $j \in \{1,2\}$, $k \in \{3,4\}$) and 0 (D_{15}, D_{25} and the sides of the triangle). Since ℓ is supported on Z_3, Z_4, W_3, W_4, the coefficients of T^2 and of T^n, $n \geq 4$, vanish there. (c) With $u_1 = 1$, $u_2 = -m_1/m_2$, $p_{2k}^3 = -\rho\sigma_k \mathrm{j}_\rho p_{1k}^3$ and $p_{1k}^3 = -p_{1k}/\omega_k$ we get $m_1(p_{1k}^3 - p_{2k}^3) = -\theta_k$. Thus $G_{1,Z_k} = -(\theta_k - \theta_5)$. With $\ell_{Z_k} = M_t m_k \omega_k$ and $\sum_k m_k\omega_k = 0$, we obtain $\tau = -M_t \sum_k m_k\omega_k\theta_k$. (d) follows from (a), (b), and $G(\xi^L, y(\xi)) \equiv 0$. □

Lemma E.4. *At a leading solution with $\tau = 0$, $\theta_3 = \theta_4$ holds if and only if θ is constant, if and only if $G_1(y_0) = 0$.*

Proof. With $a_k = m_k\omega_k \neq 0$, $\sum_k a_k = 0$ and $\sum_k a_k\theta_k = 0$ by Lemma E.3(c). Hence $a_4(\theta_3 - \theta_4) = a_5(\theta_5 - \theta_3)$ and $a_4(\theta_4 - \theta_5) = a_3(\theta_5 - \theta_3)$. The last equivalence is Lemma E.3(b), (c). □

E.3. **Simple points.**

Lemma E.5. *Let y_0 be a leading solution with $\tau(y_0) = 0$ and* $\operatorname{rank} J_0 = 16$.

(a) *The left kernel of J_0 is spanned by λ and $l_0 = e_{W_1} - e_{W_2}$, and $J_0 y_1 = -G_1(y_0)$ has exactly one solution y_1.*

(b) *y_1 vanishes on u_2, p_{12}, v_2; $(y_1)_{p_{jk}} = -p_{jk}(y_1)_{v_k}/(2\omega_k)$ for $j \in \{1,2\}$, $k \in \{3,4\}$, and $(y_1)_{p_{j5}} = 0$; and $m_3(y_1)_{v_3} + m_4(y_1)_{v_4} = 0$. Write $(y_1)_{v_3} = t/m_3$, $(y_1)_{v_4} = -t/m_4$.*

(c) *A series as in Lemma E.1 with $y(0) = y_0$ exists only if*

$$\eta_2 := D\tau(y_0) \cdot y_1 = \tfrac{1}{2} M_t\, t\, (\theta_3 - \theta_4) = 0.$$

(d) *Suppose that θ is constant. Then $y_1 = 0$, and, with y_2 the solution of $J_0 y_2 = -G_2(y_0)$, a series as in* (c) *exists only if*

$$\Omega_3 := \lambda \cdot \big(DG_1(y_0)\, y_2 + G_3(y_0)\big) = D\tau(y_0) \cdot y_2 + \sigma_3(y_0) = 0.$$

Proof. (a) $\lambda J_0 = 0$ by Lemma E.3(a), and $l_0 J_0 = 0$ because the rows W_1, W_2 of G_0 differ by $(1 - v_2)(1 - (m_1 + m_2)p_{12}^3)$, whose gradient vanishes at a leading solution. The two vectors are independent ($\lambda_{Z_k} = M_t m_k \omega_k \neq 0$). Thus they span the left kernel, of dimension 2. Since $\lambda \cdot G_1 = \tau = 0$ and $l_0 \cdot G_1 = 0$, a solution y_1 exists. It is unique by the rank. (b) The rows Z_1, Z_2, D_{12} of G_0 involve only u_2, p_{12}, v_2, with an invertible Jacobian there, and zero right side. The rows D_{jk} give the across entries. The row W_1 and the identity $m_3 W_3 + m_4 W_4 = s_v - M_t \bar{v}$ of the rows of G_0 give $Ds_v \cdot y_1 = 0$. (c) Since J_0 is injective, the series is a power series in T with first coefficient y_1 (the coefficient of a non-integral power, if any, would be in the kernel of J_0). So $\tau(y(T)) = T\, D\tau(y_0) y_1 + O(T^2)$, which is $O(T^2)$ by Lemma E.3(d). For the formula: $\tau = \sum_{k=3,4} \ell_{Z_k} G_{1,Z_k}$. By (b), $DG_{1,Z_k} \cdot y_1 = \frac{3(y_1)_{v_k}}{2\omega_k}\theta_k$ and $D\omega_k \cdot y_1 = (y_1)_{v_k}$. Hence we obtain $D\tau \cdot y_1 = M_t \sum_{k=3,4} m_k\big(-(\theta_k - \theta_5) + \frac{3}{2}\theta_k\big)(y_1)_{v_k} = \frac{1}{2} M_t \sum_{k=3,4} m_k\theta_k (y_1)_{v_k}$, using $\sum_{k=3,4} m_k (y_1)_{v_k} = 0$. (d) Since $G_1(y_0) = 0$ by Lemma E.4, $y_1 = 0$. The equation at order T^2

is solvable because $\lambda \cdot G_2 \equiv 0$ and G_2 vanishes on the rows W. At order T^3 the equation is $J_0 y_3 = -(DG_1(y_0)y_2 + G_3(y_0))$. Its l_0-component vanishes (it is the cross part of $F_1'(v) - F_2'(v)$, a multiple of τ). Its λ-component is Ω_3. The second expression for Ω_3 is Lemma E.3(d). □

E.4. **Triple points.**

Lemma E.6. *Let y_0 be a triple point, $u = \Lambda v$ on the triangle, with $\tau(y_0) = 0$.*

(a) $\ker J_0 = \mathbb{C}\,\kappa$, *where κ has $\delta v = (1, -m_3/m_4)$ on (v_3, v_4), $\delta u = -\Lambda\,\delta v$ on the triangle, $\delta p_{kl} = 0$ on the triangle, 0 on u_2, p_{12}, v_2, and across entries as in Lemma E.5(b). So* $\operatorname{rank} J_0 = 15$. *The rows R other than Z_3, Z_4, W_2 and the columns other than v_3 give an invertible 15×15 minor.*

(b) *A series as in Lemma E.1 with $y(0) = y_0$ exists only if θ is constant.*

(c) *Every triple point has the equilateral shape of ω, so constant θ forces $m_3 = m_4 = m_5$. Then, if $\tau = 0$, σ is constant (Lemma E.8(c)). Let y_0 have θ and σ constant, let $y(T, s)$ solve the rows R with $y_{v_3} = v_3 + s$, and put $f_3(T, s) = Z_3(T, y(T, s))$ and $F(T, s) = \tau(y(T, s)) + T^2\sigma_3(y(T, s))$. Then $f_3 = b_3 sT + O(|(T, s)|^3)$ and $F = F_{ss}s^2 + F_{sT}sT + F_{TT}T^2 + O(|(T, s)|^3)$, with $b_3 = \frac{3\theta_3}{2\omega_3}$, $F_{ss} = -\frac{3}{8}M_t m_1(1+\sigma \mathrm{j}_\rho \rho)\, c \sum_k m_k(\delta\omega_k)^2/\omega_k$ ($c = p_{1k}/\omega_k$, $\delta\omega = (1, -m_3/m_4, 0)$), and F_{TT} the value of* (E.1) *below. A series as above exists only if $b_3 F_{ss} F_{TT} = 0$.*

Proof. (a) For $J_0 x = 0$, (b) of Lemma E.5 holds with zero right sides. At a Lagrange triangle the triangle rows of J_0 involve only δP_{kl}, $P = p^3$, and read $\sum_l m_l u_{kl}\delta P_{kl} = 0 = \sum_l m_l v_{kl}\delta P_{kl}$. For $u = \Lambda v$ they coincide and force $\delta P = \kappa'\,\delta^0$ with $\delta^0 = (m_5 v_{35} v_{45}, -m_4 v_{34} v_{45}, m_3 v_{34} v_{35})$ on $(34, 35, 45)$. The rows D_{kl}, times Λv_{kl}, read $\delta u_{kl} + \Lambda\delta v_{kl} = -\frac{2}{3}M_t\kappa'\Lambda v_{kl}\delta^0_{kl}$. When we sum over the cycle of the three sides, the left sides cancel and the right side is $\pm\frac{2}{3}M_t\kappa'\Lambda M_t v_{34} v_{35} v_{45}$. Hence $\kappa' = 0$, $\delta u = -\Lambda\delta v$, and the row W_1 gives $m_3\delta v_3 + m_4\delta v_4 = 0$. The left kernel of J_0 is spanned by $l_0 = e_{W_1} - e_{W_2}$, $e_{Z_3} - \Lambda e_{W_3}$ and $e_{Z_4} - \Lambda e_{W_4}$, since $Z_k = \Lambda W_k$ as rows of J_0. Removing one row from the support of each and the column v_3, where κ does not vanish, leaves an invertible minor. (b) The rows R have a solution $y(T, s)$ with $y_{v_3} = v_3 + s$ and no T^1 term, since G_1 lives on Z_3, Z_4. A series gives $s(T) \to 0$ with $y = y(T, s(T))$. Here $y(T, s) = y_0 + s\kappa + O(s^2 + T^2)$. The proof of Lemma E.5(c) applied to κ gives $D\tau \cdot \kappa = \frac{1}{2}M_t m_3(\theta_3 - \theta_4)$. If $\theta_3 \neq \theta_4$, Lemma E.3(d) forces $s(T) = O(T^2)$, and then the rows Z_3, Z_4 give $G_1(y_0) = 0$, that is θ constant (Lemma E.4), a contradiction. (c) The rows Z_3 and τ restricted to $y(T, s)$ satisfy $\ell_{Z_3} f_3 + \ell_{Z_4} f_4 = TF$. Hence a series gives a curve germ on which $f_3 = F = 0$. The coefficient of T in f_3 is $G_{1,Z_3}(y_0) = 0$. That of T^2 vanishes because $Z_3 = \Lambda W_3$ as rows of J_0 and G_2 vanishes on W_3. That of s^2 vanishes because along κ, $\delta P = 0$ and the rows are bilinear in (P, h). The coefficient of sT is $b_3 = DG_{1,Z_3} \cdot \kappa$. At $T = 0$ the across rows hold exactly and $F(0, s) = -M_t m_1(1+\sigma \mathrm{j}_\rho \rho)\sum_k m_k p_{1k}$, whose first-order terms are multiples of $\sum_k m_k\delta\omega_k = 0$, which gives F_{ss}. The number F_{TT} is the coefficient of T^2 in $F(T, 0)$, computed as in Lemma E.7. The tangent of the germ is a common zero of the quadratic parts. The equation $b_3 sT = 0$ leaves the two axes, which are excluded by $F_{TT} \neq 0$ and $F_{ss} \neq 0$. □

E.5. **The order-three obstruction and the equilateral realizations.**

Lemma E.7. *In Lemma E.5(d) let σ be constant, let $A = p_{1k}^3$ (independent of k) and $J_{uv} = \sum_k m_k\omega_k(u_k - u_{\mathrm{cm}})$, $u_{\mathrm{cm}} = \sum_k m_k u_k/M_t$. Then $\sigma_3(y_0) = 0$ and*

$$\Omega_3 = \tfrac{3}{2}M_t m_1 A\Big[(1 - \sigma\rho^2 \mathrm{j}_\rho)\, J_{uv} + \frac{M_t\rho^2 m_1 p_{12}(1 - \sigma \mathrm{j}_\rho)}{1 + \rho}\Big]. \tag{E.1}$$

For a triangle of Lagrange type, $J_{uv} = M_t^{-1/3}\sum_{k<l} m_k m_l \zeta_{kl}$, where $p_{kl} = M_t^{-1/3}\zeta_{kl}$ and $\zeta_{kl}^3 = 1$.

Proof. By the weights, $G_2(y_0)$ is -1 on D_{12}, $-p_{jk}^2 u_k(v_j - v_k)$ on D_{jk} ($j \in \{1,2\}$, $k \in \{3,4\}$), $-\Phi$ with $\Phi = m_3 P_{35} u_3 + m_4 P_{45} u_4$ on Z_1 and Z_2, and 0 on the other rows. We solve $J_0 y_2 = -G_2$: on the segment $\delta p_{12} = 0$, $\delta u_2 = \Phi(m_1 + m_2)/m_2$ and $\delta v_2 = -m_2 p_{12}$. The triangle part is κ'' times the rotation, with $\kappa'' = m_2^2 p_{12}/(m_1 + m_2)$ from the row W_1. The rows D_{jk} give the across entries. Since θ is constant, $p_{1k}^3 = A$ and $p_{2k}^3 = -\sigma\rho \mathrm{j}_\rho A$ for all k. Hence the terms in δu_2 cancel in $D\tau \cdot y_2 = \sum_{k=3,4} \ell_{Z_k} DG_{1,Z_k} \cdot y_2$. Collecting terms, we obtain (E.1). The rows of G_3 on Z_k and W_k are $-\alpha u_k$ and $-\alpha(\omega_k + 1)$ with a common α, which gives $\sigma_3 = 0$. For a Lagrange triangle $u_{kl} v_{kl} = M_t^{2/3} \zeta_{kl}$, and $\sum_k m_k \omega_k (u_k - u_{\mathrm{cm}}) = M_t^{-1} \sum_{k<l} m_k m_l u_{kl} v_{kl}$. □

Lemma E.8. *For every positive mass vector, every σ, every $\rho > 0$ and either labelling of the segment, no leading solution with $\tau = 0$ whose triangle has an equilateral ω satisfies* (i)–(iv) *of Lemma 7.2.*

(a) *At $m_3 = m_4 = m_5$ and constant σ, the Lagrange realization of the equilateral shape is simple with θ constant, and with $m_1 = 1$ and unit triangle masses, $\Omega_3 = \mp\frac{9}{2}\big[3^{2/3}\xi(1 - i\sigma\rho^{5/2}) + 3\zeta_{12}\rho^2(1+\rho)^{-4/3}(1 - i\sigma\rho^{1/2})\big]$ with ξ, ζ_{12} cube roots of unity, and $|\Omega_3| \geq \frac{9}{2}\, 3^{2/3}(1 - 0.909) > 0.85$.*

(b) *At $m_3 = m_4 = m_5$ and constant σ, the triple realization has $J_{uv} = 0$, $F_{TT} = \mp\frac{27}{2}\rho^2(1 + \rho)^{-4/3}\zeta_{12}(1 - i\sigma\sqrt{\rho}) \neq 0$, $F_{ss} = \mp\frac{9}{8}(1 + \sigma \mathrm{j}_\rho \rho) \neq 0$ and $b_3 = \pm\frac{3}{2}(1 + \sigma \mathrm{j}_\rho \rho) e^{-i\pi/3} \neq 0$.*

(c) *At $m_3 = m_4 = m_5$ and nonconstant σ, the equilateral realizations have $\tau \neq 0$.*

(d) *At unequal triangle masses, θ is not constant, the Lagrange realization is simple with $\eta_2 \neq 0$, and the triple realization is excluded by Lemma E.6(b).*

Proof. By Lemma E.11(a), a triangle of Lagrange type is equilateral only in the classes with equal or distinct cube roots, and in the second case y_0 is a triple point. By Lemma E.12(c), a triangle of Euler type is equilateral only at a triple point. So the triangles with an equilateral ω are those of the Lagrange realization of the class with equal cube roots and those of the triple points, and (a)–(d) treat all of them. (a) At equal masses $\omega = (e^{i\pi/3}, e^{-i\pi/3}, -1)$ and $p_{1k} = \pm\omega_k$ for the signs $\epsilon = \pm(1, 1, -1)$ of the square roots. So θ is constant and, for constant σ, $\tau = 0$. Inserting the equilateral data into (E.1), we obtain the formula. With $s = \sqrt{\rho}$, the ratio of the moduli of the two terms is $r(s) = 3^{1/3} s^4 (1+s^2)^{-5/6} (1+s^{10})^{-1/2}$. Then $r^2 \leq (9/256)^{1/3} s^{4/3} \leq (9/16)^{1/3} < 0.826$ for $s \leq 2$ (by the inequality between the arithmetic and geometric means), $r^2 \leq 3^{2/3} s^{-16/3} < 0.06$ for $s \geq 2$. Hence $r < 0.909$ and $|\Omega_3| \geq \frac{9}{2}\, 3^{2/3}(1 + s^{10})^{1/2}(1 - r)$. (b) For the class with distinct cube roots, $J_{uv} = 3^{-1/3}(1 + \varpi + \varpi^2) = 0$ with $\varpi = e^{2\pi i/3}$. The formulas of Lemma E.6(c) give the values, with $\sum_k (\delta\omega_k)^2/\omega_k = e^{-i\pi/3} + e^{i\pi/3} = 1$. Lemma E.6(c) excludes a series. (c) The numbers $q_k = \epsilon_k(-1/\omega_k)^{1/2}$ are sixth roots of unity. With $1 + \sigma_k \mathrm{j}_\rho \rho$ proportional to $e^{i\sigma_k \varphi}$, $\tan\varphi = \rho^{3/2}$, zero torque reads $e^{i\varphi} \sum_{k \neq o} q_k + e^{-i\varphi} q_o = 0$ for the body o of the other sign, which forces $|\sum_{k \neq o} q_k| = 1$. We enumerate the twelve cases and obtain $e^{2i\varphi} = \pm 1$, i.e. $\varphi \in \{0, \pi/2\}$, impossible for $\rho > 0$. (d) Equal moduli $|\omega_k|$ would make the centre of mass of the triangle its circumcentre, which for an equilateral triangle is the centroid. Then the masses would be equal. Hence θ is not constant. The Lagrange realization of the class with equal cube roots is not a triple point. At a Lagrange triangle that is not a triple point the triangle block of J_0 has rank 6. Hence the realization is simple. On the triangle, δP in the solution y_1 is a multiple of $(m_5 v_{35} v_{45}, -m_4 v_{34} v_{45}, m_3 v_{34} v_{35})$ with a factor proportional to $\theta_3 - \sum_k m_k \theta_k / M_t$. The rows D_{kl} give t as a nonzero multiple of this factor times $m_5 r_{34} + m_4 r_{35} + m_3 r_{45}$, $r_{kl} = u_{kl}/v_{kl}$. In this class r_{kl} is proportional to $1/v_{kl}^2$. Thus the last sum is proportional to $m_3 + m_4 \bar{\varpi} + m_5 \varpi$, which vanishes only at equal masses. So $t \neq 0$, $\theta_3 \neq \theta_4$ (Lemma E.4), and $\eta_2 \neq 0$. □

Proof of Lemma 7.2. Let $y_0 = y(0)$ for the series of Lemma E.1. By Lemma E.3(d), $\tau(y_0) = 0$. If $\operatorname{rank} J_0 = 16$, Lemma E.5(c) gives $\eta_2 = 0$, which is (i) if θ is not constant. If $\operatorname{rank} J_0 = 16$

and θ is constant, Lemma E.5(d) gives $\Omega_3 = 0$, which is (ii). If $\operatorname{rank} J_0 \le 15$ and y_0 is a triple point, Lemma E.6(b) and (c) give (iii). Otherwise (iv) holds. The last statement of Lemma 7.2 is Lemma E.8. □

E.6. **The obstruction on the set $\tau = 0$.** In the rest of this appendix y_0 is a leading solution with $\tau(y_0) = 0$, and $P_{kl} = p^3_{kl}$ for $k, l \in \{3, 4, 5\}$. On $\mathbb{C}^{\{3,4,5\}}$ we use the nondegenerate symmetric bilinear form $\langle a, b\rangle = \sum_k m_k a_k b_k$, the vector $\mathbb{1} = (1, 1, 1)$, the mean $\bar\theta = \langle \mathbb{1}, \theta\rangle / M_t$ and the vector

$$e_3 = \Big(\frac{v_{45}}{m_3}, \frac{v_{53}}{m_4}, \frac{v_{34}}{m_5}\Big).$$

Then $\langle \mathbb{1}, e_3\rangle = v_{45} + v_{53} + v_{34} = 0$ and $\langle \omega, e_3\rangle = v_3 v_{45} + v_4 v_{53} + v_5 v_{34} = 0$.

Lemma E.9. *There is a number γ with $\theta - \bar\theta\mathbb{1} = \gamma e_3$. Moreover $\theta_3 - \theta_4 = -\gamma M_t \omega_5/(m_3 m_4)$, and $\gamma = 0$ if and only if θ is constant.*

Proof. By Lemma E.3(c), $\langle \omega, \theta\rangle = \sum_k \tau_k = -\tau/M_t = 0$. So $\psi = \theta - \bar\theta\mathbb{1}$ satisfies $\langle \mathbb{1}, \psi\rangle = \langle \omega, \psi\rangle = 0$. Since $\langle \mathbb{1}, \omega\rangle = 0$ and $\omega_5 = -1$, ω is not a multiple of $\mathbb{1}$. Hence these two conditions define a line. It contains $e_3 \ne 0$. The formula for $\theta_3 - \theta_4$ follows from $m_4 v_{45} - m_3 v_{53} = m_3\omega_3 + m_4\omega_4 - (m_3 + m_4)\omega_5 = -M_t\omega_5$. Finally $\gamma = 0$ if and only if $\psi = 0$. □

Theorem E.10. (a) *Let the triangle be of Lagrange type, and let y_0 not be a triple point. Put $r_{kl} = u_{kl}/v_{kl}$, $\beta = m_5 r_{34} + m_4 r_{35} + m_3 r_{45}$, $\Delta = u_{34}v_{35} - u_{35}v_{34}$ and $d_r = (r_{34} - r_{35})/m_3 + (r_{34} - r_{45})/m_4$. Then $\Delta \ne 0$, $d_r \ne 0$, y_0 is simple, and*

$$\eta_2 = -\gamma^2 \frac{M_t^3\, \omega_5\, v_{34} v_{35} v_{45}\, \beta}{3\,\Delta\, d_r\, m_3^2 m_4^2 m_5}.$$

(b) *Let the triangle be of Euler type, $u = \Lambda v$, with $\langle \omega, \omega\rangle \ne 0$, and put $\mu = \sum_{k<l}(m_k + m_l)P_{kl} - 1$. If $\mu = 1$, then y_0 is a triple point. If $\mu \notin \{1, -\frac12\}$, then y_0 is simple, and*

$$\eta_2 = -\gamma^2 \frac{3M_t\, \mu\, \langle e_3, e_3\rangle}{4\Lambda(1-\mu)(1+2\mu)}.$$

Proof. Let J_T be the triangle block of J_0. It has the rows $Z_3, Z_4, W_3, W_4, D_{34}, D_{35}, D_{45}$ and the columns of $u_3, u_4, v_3, v_4, p_{34}, p_{35}, p_{45}$. These rows of G_0 contain no other coordinates (Lemma E.2). For a vector x in these columns we write δ for the derivative along x. Then $J_T x = 0$ says that $\delta F'_k(u)$ and $\delta F'_k(v)$ do not depend on $k \in \{3, 4, 5\}$ and that $\delta P_{kl}/P_{kl} = -\frac32(\delta u_{kl}/u_{kl} + \delta v_{kl}/v_{kl})$. By Lemma E.3(c), the triangle part of y_1 satisfies the same conditions with $\delta F'_k(u) - \theta_k$ in place of $\delta F'_k(u)$. The rotation R, with $\delta u_k = u_k$, $\delta v_k = -v_k$ and $\delta p = 0$, lies in $\ker J_T$, since the rows are invariant under $(u, v) \mapsto (\alpha u, \alpha^{-1} v)$. If $\ker J_T = \mathbb{C}R$, then y_0 is simple. Indeed, let $J_0 x = 0$. Lemma E.5(b) holds for x with zero right sides. So the triangle part of x is cR for a number c, and $0 = m_3\delta v_3 + m_4\delta v_4 = -c(m_3 v_3 + m_4 v_4) = -cM_t$. Hence $c = 0$, and then $x = 0$.

(a) Here $P_{kl} = 1/M_t$, so that $\delta F'_k(h) = \langle \mathbb{1}, \delta h\rangle/M_t - \sum_l m_l h_{kl}\delta P_{kl}$ for $h = u, v$. Since $\sum_k m_k \sum_l m_l h_{kl}\delta P_{kl} = 0$, the triangle part of y_1 satisfies $\sum_l m_l v_{kl}\delta P_{kl} = 0$ and $\sum_l m_l u_{kl}\delta P_{kl} = \bar\theta - \theta_k$ for $k \in \{3, 4, 5\}$. A vector of $\ker J_T$ satisfies the same equations with 0 in place of $\bar\theta - \theta_k$. The equations in v have rank 2 and give $\delta P = \kappa_L \delta^0$ with $\delta^0 = (m_5 v_{35} v_{45}, -m_4 v_{34} v_{45}, m_3 v_{34} v_{35})$ on $(34, 35, 45)$. The equation in u for $k = 3$ then reads $\kappa_L m_4 m_5 v_{45}\Delta = \bar\theta - \theta_3$. Here $\Delta \ne 0$. Otherwise (u_{34}, u_{35}) is a multiple of (v_{34}, v_{35}), so that u is parallel to v and y_0 is a triple point. The rows D_{kl} give $\delta u_{kl} = -\frac23 M_t u_{kl}\delta P_{kl} - r_{kl}\delta v_{kl}$. We sum over the cycle $34, 45, 53$ and put $\phi(\delta v) = (r_{34} - r_{35})\delta v_3 + (r_{45} - r_{34})\delta v_4$. This gives $\phi(\delta v) = -\frac23 M_t \kappa_L v_{34} v_{35} v_{45}\beta$. The linear form ϕ vanishes at (v_3, v_4). It is not zero, since otherwise $r_{34} = r_{35} = r_{45}$ and y_0 is a triple point.

For a vector of $\ker J_T$ we obtain $\kappa_L = 0$ and $\delta P = 0$, and then $\phi(\delta v) = 0$. So δv is a multiple cv of v, $\delta u_{kl} = -r_{kl}\delta v_{kl} = -cu_{kl}$, and the vector lies in $\mathbb{C}R$. Thus y_0 is simple.

Next, $d_r = \phi(1/m_3, -1/m_4)$. The vector $(1/m_3, -1/m_4)$ is not a multiple of (v_3, v_4), since $m_3v_3+m_4v_4 = M_t \neq 0$. Hence $d_r \neq 0$. For y_1, Lemma E.5(b) gives $\delta v_3 = t/m_3$ and $\delta v_4 = -t/m_4$, so that $\phi(\delta v) = t\, d_r$. Lemma E.9 gives $\bar\theta - \theta_3 = -\gamma v_{45}/m_3$, hence $\kappa_L = -\gamma/(m_3m_4m_5\Delta)$ and

$$t = \frac{2M_t\gamma\, v_{34}v_{35}v_{45}\,\beta}{3m_3m_4m_5\Delta\, d_r}.$$

Lemma E.5(c) and the formula for $\theta_3 - \theta_4$ in Lemma E.9 give the formula for η_2.

(b) We write $\delta u = \Lambda(\delta v + \varepsilon)$ on the triangle, with $\varepsilon_5 = 0$. Then $\delta P_{kl}/P_{kl} = -3\delta v_{kl}/v_{kl} - \frac{3}{2}\varepsilon_{kl}/v_{kl}$. With $\Psi(h)_k = \sum_l m_l P_{kl}(h_k - h_l)$ as in the proof of Lemma E.2 we obtain

$$\delta F'(v) = (1 + 2\Psi)\delta v + \tfrac{3}{2}\Psi\varepsilon, \qquad \delta F'(u) = \Lambda\big(\delta F'(v) + (1 - \Psi)\varepsilon\big).$$

The map Ψ is self-adjoint for $\langle\cdot,\cdot\rangle$, $\Psi\mathbb{1} = 0$, and $\Psi\omega = \omega$ by the rows of v in Lemma E.2. Since $\langle\mathbb{1},\mathbb{1}\rangle = M_t$, $\langle\mathbb{1},\omega\rangle = 0$ and $\langle\omega,\omega\rangle \neq 0$, the vectors $\mathbb{1}$, ω, e_3 form a basis, and $\Psi e_3 = \mu' e_3$ for a number μ'. The trace of Ψ is $\sum_{k<l}(m_k + m_l)P_{kl} = 1 + \mu'$. So $\mu' = \mu$.

If $\mu = 1$, then Ψ agrees on this basis with the map $h \mapsto h - \langle\mathbb{1}, h\rangle\mathbb{1}/M_t$. Comparing the entries off the diagonal gives $P_{kl} = 1/M_t$. So the triangle is also of Lagrange type, and y_0 is a triple point.

Let $\mu \notin \{1, -\frac{1}{2}\}$. For a vector of $\ker J_T$, both $\Lambda(1 - \Psi)\varepsilon = \delta F'(u) - \Lambda\,\delta F'(v)$ and $\delta F'(v)$ are multiples of $\mathbb{1}$. In the basis $\mathbb{1}, \omega, e_3$ this puts ε and δv into $\mathbb{C}\mathbb{1} + \mathbb{C}\omega$. Since $\varepsilon_5 = \delta v_5 = 0$ and $v = \mathbb{1} + \omega$, both are multiples of v. The equation for $\delta F'(v)$ then gives $\varepsilon = -2\,\delta v$. Thus $\delta u = -\Lambda\,\delta v$ and $\delta P = 0$, and the vector lies in $\mathbb{C}R$. So y_0 is simple. For y_1, Lemma E.9 gives $\theta = \bar\theta\mathbb{1} + \gamma e_3$. In the basis $\mathbb{1}, \omega, e_3$ the coefficient of e_3 is $\gamma/(\Lambda(1-\mu))$ in ε and $-\frac{3}{2}\mu\gamma/(\Lambda(1-\mu)(1+2\mu))$ in δv. By Lemma E.5(b), $\delta v = (t/m_3, -t/m_4, 0)$. So $\langle\mathbb{1}, \delta v\rangle = 0$, and $t(\theta_3 - \theta_4) = \langle\theta, \delta v\rangle = \gamma\langle e_3, \delta v\rangle$. This is $\gamma\langle e_3, e_3\rangle$ times the coefficient of e_3 in δv. Lemma E.5(c) gives the formula for η_2. □

E.7. **Triangles of Lagrange type.** Let the triangle be of Lagrange type. Then $p_{kl} = M_t^{-1/3}\zeta_{kl}$ with $\zeta_{kl}^3 = 1$ (Lemma E.2), and $p_{kl}^2 u_{kl}v_{kl} = 1$ gives $u_{kl} = M_t^{2/3}\zeta_{kl}/v_{kl}$. Since $u_{34} + u_{45} + u_{53} = 0$, the shape $\chi = v_3/v_4$ is a root of

$$\zeta_{45}\chi^2 - (\zeta_{35} + \zeta_{45} - \zeta_{34})\chi + \zeta_{35}.$$

So the shape does not depend on the masses, and $\beta = M_t^{2/3}(m_3\zeta_{45}/v_{45}^2 + m_4\zeta_{35}/v_{35}^2 + m_5\zeta_{34}/v_{34}^2)$. A common factor of the three ζ_{kl} changes neither the shape nor the vanishing of β. We put $\varpi = e^{2\pi i/3}$.

Lemma E.11. (a) *If the three ζ_{kl} are equal or pairwise distinct, the triangle is equilateral. In the second case y_0 is a triple point. If exactly two of the ζ_{kl} are equal, the triangle is obtuse.*

(b) *If exactly two of the ζ_{kl} are equal and $\beta = 0$, then (m_3, m_4, m_5) is a positive multiple of a permutation of $(X_L, Y_L, 1)$. Here $X_L + 1/X_L = (7 + \sqrt{21})/2$, $X_L < 1$ and $Y_L = 5X_L/(1 + X_L)$, so that $X_L = 0.1781535864\ldots$ and $Y_L = 0.7560711460\ldots$.*

(c) *If exactly two of the ζ_{kl} are equal, then θ is not constant.*

Proof. (a) For equal ζ_{kl} the quadratic is $\chi^2 - \chi + 1$, whose roots are $e^{\pm i\pi/3}$. For $(\zeta_{34}, \zeta_{35}, \zeta_{45}) = (\varpi^2, \varpi, 1)$ and $(1, \varpi, \varpi^2)$ it is $(\chi + \varpi^2)^2$, resp. $\varpi^2(\chi + \varpi)^2$. Every class with distinct ζ_{kl} is a multiple of one of these two. The double root gives an equilateral triangle with v_{kl}^2 proportional to ζ_{kl}. Then $r_{kl} = M_t^{2/3}\zeta_{kl}/v_{kl}^2$ does not depend on kl, so that u is parallel to v. Let exactly two of the ζ_{kl} be equal. Complex conjugation conjugates the class and the shape, and a relabelling of the triangle permutes the masses. These maps reduce the claims to the class $(\zeta_{34}, \zeta_{35}, \zeta_{45}) = (1, 1, \varpi)$. There $\chi^2 - \chi + \varpi^2 = 0$. So χ is not real, and $\operatorname{Re}\chi = \frac{1}{2}\big(1 \pm ((3 + \sqrt{21})/2)^{1/2}\big)$ is $1.47\ldots$ or $-0.47\ldots$. Hence the triangle $\chi, 1, 0$ is obtuse at 1 or at 0.

(b) As in (a) we take the class $(1,1,\varpi)$. There $\chi^2(\chi-1)^2=\varpi$, and $\beta=0$ reads $m_3+m_4/\eta+m_5\eta=0$ with $\eta=\varpi\chi^2$. Since $\chi=\varpi^2(\eta+1)$, we get $\eta^2+(2-\varpi)\eta+1=0$, and η is not real. Hence η and $\bar\eta$ are the roots of $m_5T^2+m_3T+m_4$, so that $m_4/m_5=|\eta|^2$ and $m_3/m_5=-2\operatorname{Re}\eta$. The two roots η and $1/\eta$ give mass vectors that differ by the exchange of m_4 and m_5. Moreover $|\eta|\neq 1$, since $\eta+1/\eta$ is not real. For the root with $t=|\eta|^2>1$ we write $\eta=a+bi$. Then $\eta+1/\eta=\varpi-2$ gives $a(1+t^{-1})=-\frac{5}{2}$ and $b(1-t^{-1})=\frac{\sqrt3}{2}$, and $a^2+b^2=t$ becomes $t^4-7t^3+9t^2-7t+1=0$. So $t+t^{-1}$ is a root of s^2-7s+7. It is $(7+\sqrt{21})/2$, since the other root is less than 2. Thus $(m_3:m_4:m_5)=(5t/(t+1):t:1)=(Y_L:1:X_L)$ with $X_L=1/t$ and $Y_L=5/(t+1)$.

(c) If θ is constant, the moduli $|\omega_k|$ are equal (§E.2). Then the centre of mass of the triangle v_3,v_4,v_5 in $\mathbb{C}=\mathbb{R}^2$ is its circumcentre. With positive masses it lies inside the triangle, which is therefore acute. This contradicts (a). □

E.8. **Triangles of Euler type.** Let the triangle be of Euler type, $u=\Lambda v$. From $p_{kl}^2u_{kl}v_{kl}=1$ we get $p_{kl}=s_{kl}\Lambda^{-1/2}/v_{kl}$ for a fixed square root $\Lambda^{-1/2}$ and signs $s_{kl}=-s_{lk}$. So $P_{kl}=s_{kl}\Lambda^{-3/2}/v_{kl}^3$, and the rows of v in Lemma E.2 become $\omega=\Lambda^{-3/2}g$ with $g_k=\sum_{l\neq k}m_ls_{kl}/v_{kl}^2$.

We multiply the triangle masses by $c>0$, the v_k by $\alpha\neq0$, u by $c^{2/3}\alpha^{-1}$ and the p_{kl} by $c^{-1/3}$. This maps the central configurations of the triangle to central configurations. It does not change $\chi=v_3/v_4$, μ, the relations $\omega_k^3=\omega_l^3$ and $m_o^2=m_k^2-m_km_l+m_l^2$, and the T-test of §E.9. It multiplies ω, $\langle\omega,\omega\rangle$ and $\langle e_3,e_3\rangle$ by nonzero numbers. In this subsection we may therefore take $(m_3,m_4,m_5)=(X,Y,1)$ with $X,Y>0$ and $(v_3,v_4,v_5)=(\chi,1,0)$. Then $M_t=X+Y+1$ and $M_t\omega=(\tilde\omega_3,\tilde\omega_4,\tilde\omega_5)$ with

$$\tilde\omega_3=(Y+1)\chi-Y,\qquad \tilde\omega_4=X+1-X\chi,\qquad \tilde\omega_5=-(X\chi+Y).$$

Lemma E.12. (a) *χ is a root of the quintic*

$$\mathcal{E}_s(\chi)=s_{45}\tilde\omega_3\chi^2(\chi-1)^2+s_{53}\tilde\omega_4(\chi-1)^2+s_{34}\tilde\omega_5\chi^2,\qquad s=(s_{45},s_{53},s_{34}),$$

whose leading coefficient is $s_{45}(Y+1)$. Since $\mathcal{E}_{-s}=-\mathcal{E}_s$, it suffices to take s among the patterns $p_1=(1,1,-1)$, $p_2=(1,-1,1)$, $p_3=(-1,1,1)$ *and* $p_4=(1,1,1)$.

(b) *$\mathcal{E}_{p_4}=(\chi^2-\chi+1)\,C$ with a cubic C. If $s=\pm p_4$ and χ is a root of $\chi^2-\chi+1$, then y_0 is a triple point.*

(c) *$\mathcal{E}_{p_1}(e^{i\pi/3})=-2X-Y+i\sqrt3Y$, $\mathcal{E}_{p_2}(e^{i\pi/3})=2X+1+i\sqrt3$ and $\mathcal{E}_{p_3}(e^{i\pi/3})=Y-1-i\sqrt3(Y+1)$. Hence a triangle of Euler type is equilateral only if y_0 is a triple point.*

Proof. (a) Since $\omega=\Lambda^{-3/2}g$, we have $\omega_4g_5-\omega_5g_4=0$. The identity $M_t\chi^2(\chi-1)^2(\omega_4g_5-\omega_5g_4)=X\mathcal{E}_s(\chi)$ holds in $\mathbb{Z}[\chi,X,Y]$. (b) At $\chi=e^{\pm i\pi/3}$ we have $v_{34}^3=v_{45}^3=v_{53}^3=1$. For $s=\pm p_4$ all P_{kl} are then equal. Since $\Psi\omega=\omega$, they equal $1/M_t$, and the triangle is also of Lagrange type. (c) The values are exact. Their imaginary parts do not vanish for $X,Y>0$, and $\mathcal{E}_s$ has real coefficients. The equilateral shapes are $\chi=e^{\pm i\pi/3}$. □

We put $\nu=\sum_{k<l}(m_k+m_l)s_{kl}/v_{kl}^3$. Then $\mu=\Lambda^{-3/2}\nu-1$. For $k\in\{3,4,5\}$ we put

$$f_{0,k}=(\tilde\omega_k\nu-M_tg_k)\chi^3(\chi-1)^3,\qquad f_{h,k}=(2\tilde\omega_k\nu-M_tg_k)\chi^3(\chi-1)^3,$$

$$f_\omega=X\tilde\omega_3^2+Y\tilde\omega_4^2+\tilde\omega_5^2,\qquad f_e=Y+X\chi^2+XY(\chi-1)^2,\qquad \Theta_{kl}=\tilde\omega_k^2+\tilde\omega_k\tilde\omega_l+\tilde\omega_l^2.$$

These are polynomials in $\mathbb{Z}[X,Y][\chi]$. At the shape χ of the triangle, $\tilde\omega=M_t\Lambda^{-3/2}g$. Hence $f_{0,k}=M_tg_k\mu\chi^3(\chi-1)^3$ and $f_{h,k}=M_tg_k(2\mu+1)\chi^3(\chi-1)^3$. So $\mu=0$, resp. $\mu=-\frac12$, forces $f_{0,k}$, resp. $f_{h,k}$, to vanish at χ for all three k. Moreover $f_\omega=M_t^2\langle\omega,\omega\rangle$ and $f_e=XY\langle e_3,e_3\rangle$, and $\Theta_{kl}=0$ if and only if $\omega_k=\varpi^{\pm1}\omega_l$.

Lemma E.13. *Let $E, F \in \mathbb{Z}[X,Y][\chi]$ have a common root χ_0 at $(X,Y) = (X_0, Y_0)$. Then $\mathrm{Res}_\chi(E,F)$ vanishes at (X_0, Y_0).*

Proof. By the adjugate of the Sylvester matrix, $\mathrm{Res}_\chi(E,F) = AE + BF$ with $A, B \in \mathbb{Z}[X,Y][\chi]$. We evaluate at χ_0. □

Proposition E.14. *Let the triangle be of Euler type and not equilateral, and let s be its pattern. Then $\langle\omega,\omega\rangle \neq 0$ and $\langle e_3, e_3\rangle \neq 0$. If $\mu = 0$, then (X,Y) is $(\frac{1}{5},\frac{1}{5})$, $(1,5)$, $(5,1)$ or $(1,1)$ for $s = p_1, p_2, p_3, p_4$, respectively. If $\mu = -\frac{1}{2}$, then (X,Y) is (X_*, X_*), $(1, 2+2\sqrt{2})$, $(2+2\sqrt{2}, 1)$ or $(1,1)$, respectively, where $X_* = (\sqrt{2}-1)/2$.*

Proof. By Lemma E.12, χ is a root of $\mathcal{E}_s$, and of C for $s = p_4$. We compute the 32 resultants $\mathrm{Res}_\chi(\mathcal{E}_s, F)$, with C in place of $\mathcal{E}_{p_4}$, for $F \in \{f_\omega, f_e, f_{0,k}, f_{h,k} : k = 3,4,5\}$, and factor them over $\mathbb{Z}$ (Singular 4.4.1, FLINT). Every factor of these 32 resultants is of one of the following kinds on the open quadrant $X, Y > 0$:

(1) a polynomial with positive coefficients;
(2) $X^2 - X + 1$, $Y^2 - Y + 1$ or $X^2 \pm XY + Y^2$, which are positive;
(3) $q_1 = 8X^2 + 9XY + 8Y^2 - 5X - 5Y + 1 = \frac{25}{4}(X+Y-\frac{2}{5})^2 + \frac{7}{4}(X-Y)^2$, zero only at $(\frac{1}{5},\frac{1}{5})$, and $q_2 = Y^2 q_1(X/Y, 1/Y)$ and $q_3 = X^2 q_1(1/X, Y/X)$, zero only at $(1,5)$ and $(5,1)$;
(4) the octic P_1 below, positive except at (X_*, X_*), and $P_2 = Y^8 P_1(X/Y, 1/Y)$ and $P_3 = X^8 P_1(1/X, Y/X)$, positive except at $(1, 2+2\sqrt{2})$ and $(2+2\sqrt{2}, 1)$;
(5) $X - 1$, $Y - 1$ or $X - Y$.

All factors of the resultants of f_ω and f_e are of kind (1). Lemma E.13 gives the first statement. If $\mu = 0$, the three resultants of the $f_{0,k}$ vanish at (X,Y), again by Lemma E.13. For $s = p_i$, $i \leq 3$, one of them has no factor of kind (5), and its only zero in the quadrant is the zero of q_i. For $s = p_4$ they are $Y - 1$, $X - 1$ and $X - Y$ times factors of kind (1). The case $\mu = -\frac{1}{2}$ is the same, with $f_{h,k}$ and P_i in place of $f_{0,k}$ and q_i. □

The octic is $P_1 = \mathrm{Res}_\chi(\mathcal{E}_{p_1}, \partial_\chi \mathcal{E}_{p_1})/(Y+1)$, a symmetric polynomial with 38 terms and degree 6 in each variable. We show that $P_1 > 0$ on the open quadrant except at (X_*, X_*). Put $H_0(s) = s(17s+24)^3(s^2+2s-1)^2/16$, which is positive for $s > 0$ except at $s = \sqrt{2}-1$. Then $K = (P_1 - H_0(X+Y))/(X-Y)^2$ is a polynomial. The Bernstein coefficients of K on the box $[\frac{43}{213}, \frac{45}{211}]^2$, computed exactly, are positive. So $P_1 \geq 0$ on this box, with equality only at (X_*, X_*). We substitute $X = x/(1-x)$ and $Y = y/(1-y)$. The square $[0,1]^2$ minus the image $[\frac{43}{256}, \frac{45}{256}]^2$ of the box is tiled by 39 dyadic boxes. On each of them the Bernstein coefficients of $(1-x)^6(1-y)^6 P_1$, of bidegree $(6,6)$ and computed exactly, are nonnegative. Moreover, on each of these boxes every corner other than $(0,0)$ and $(1,1)$ carries a positive coefficient, and so does every edge. Hence this polynomial is positive on each of these boxes, except possibly at $(0,0)$ and $(1,1)$, which are not images of points of the open quadrant.

Lemma E.15. *Let θ be constant, and let the triangle not be equilateral. Then σ is not constant. Let o be the body of the triangle whose sign differs from those of the other two, k and l. Then $\omega_k^3 = \omega_l^3$, $\omega_k \neq \omega_l$, and $m_o^2 = m_k^2 - m_k m_l + m_l^2$.*

Proof. For $\sigma_k = \sigma_l$, $\theta_k = \theta_l$ gives $p_{1k}/\omega_k = p_{1l}/\omega_l$, since $1 + \sigma_k \mathrm{j}_\rho \rho \neq 0$. Squaring and inserting $p_{1k}^2 = -1/\omega_k$ gives $\omega_k^3 = \omega_l^3$, and $\omega_k \neq \omega_l$ since $v_k \neq v_l$. If σ is constant, this holds for all three pairs, so that $\omega_3, \omega_4, \omega_5$ are ω_5 times the three cube roots of unity and the triangle is equilateral. Hence σ is not constant, and $\omega_k = \zeta\omega_l$ with $\zeta \in \{\varpi, \varpi^2\}$. From $\sum_j m_j\omega_j = 0$ we get $m_o\omega_o = -(m_k\zeta + m_l)\omega_l$. Since $|\omega_o| = |\omega_l|$ (§E.2), $m_o = |m_k\zeta + m_l|$, and $m_o^2 = m_k^2 - m_k m_l + m_l^2$. □

Proposition E.16. *For each pattern s and each $o \in \{3,4,5\}$, with $\{k,l\}$ the other two bodies, $\mathcal{E}_s$ (C for $s = p_4$) and Θ_{kl} have no common root at any point (X,Y) with $X,Y > 0$ of the conic $m_o^2 = m_k^2 - m_k m_l + m_l^2$.*

Proof. We compute $R = \mathrm{Res}_\chi(\mathcal{E}_s, \Theta_{kl})$ exactly and parametrize the conic by rational functions of t. For $o = 5$, for instance, $X = (t^2-1)/(t^2-t+1)$ and $Y = t(t-2)/(t^2-t+1)$, and $X,Y > 0$ exactly for $t < -1$ or $t > 2$. The parameter $t = \infty$ gives $(X,Y) = (1,1)$ on all three conics, and there $R = 2916$ for $s = p_1, p_2, p_3$ and $R = 729$ for $s = p_4$. For each of the 12 pairs (s,o), R composed with the parametrization has no real root in the range of t where $X,Y > 0$, by exact real root isolation, and again by Sturm sequences. Lemma E.13 concludes. □

E.9. **Proof of Theorem 7.3.**

Lemma E.17. *Let the triangle of y_0 not be equilateral, and let y_0 satisfy one of* (i)–(iv) *of Lemma 7.2. Then one of the following holds:*

(L) *the triangle is of Lagrange type with exactly two equal ζ_{kl}, and $\beta = 0$;*
(EI) *it is of Euler type with $\langle \omega, \omega \rangle = 0$;*
(EQ) *it is of Euler type with $\langle e_3, e_3 \rangle = 0$;*
(E0) *it is of Euler type with $\mu = 0$;*
(Eh) *it is of Euler type with $\mu = -\frac{1}{2}$;*
(Θ) *θ is constant.*

Proof. Triple points have an equilateral ω (§E.2). So y_0 is not a triple point, and (iii) does not hold. If the triangle is of Lagrange type, exactly two of the ζ_{kl} are equal, by Lemma E.11(a). If it is of Euler type with $\langle \omega, \omega \rangle = 0$, (EI) holds. So we may assume $\langle \omega, \omega \rangle \neq 0$ for the Euler type. Then $\mu \neq 1$ by Theorem E.10(b). In case (ii), (Θ) holds. In case (iv), y_0 is not simple. So Theorem E.10 excludes the Lagrange type, and it gives $\mu = -\frac{1}{2}$ for the Euler type. In case (i), $\gamma \neq 0$ by Lemma E.9, and $\eta_2 = 0$. For the Lagrange type, Theorem E.10(a) gives $\beta = 0$, since M_t, ω_5, Δ, d_r, the masses and the v_{kl} are nonzero. For the Euler type with $\mu = -\frac{1}{2}$, (Eh) holds. Otherwise Theorem E.10(b) gives $\mu \langle e_3, e_3 \rangle = 0$, since Λ is finite and nonzero. □

For $k \in \{3,4,5\}$ put $T_k = m_k |\omega_k|^{-1/2}$. We say that y_0 passes the *T-test* if $T_k > \sum_{l \neq k} T_l$ for some k. Since $|p_{1k}| = |\omega_k|^{-1/2}$ and $|m_1 + \sigma_k \mathrm{j}_\rho m_2|^2 = m_1^2(1+\rho^3)$, we have $|\tau_k| = m_1(1+\rho^3)^{1/2} T_k$. Hence the T-test gives $|\sum_k \tau_k| \geq |\tau_k| - \sum_{l \neq k} |\tau_l| > 0$, so that $\tau = -M_t \sum_k \tau_k \neq 0$. The T-test depends only on the masses of the triangle and on ω. So it excludes $\tau = 0$ for every σ, every $\rho > 0$ and either labelling of the segment.

condition	pattern	$(m_3 : m_4 : m_5)$	factor of $\mathcal{E}_s$	margin
(E0)	p_1	$(1:1:5)$	$\chi^2 - \frac{3}{2}\chi + 1$	0.6157
	p_2	$(1:5:1)$	$\chi^2 - \frac{1}{2}\chi + \frac{1}{2}$	0.6157
	p_3	$(5:1:1)$	$\chi^2 - \chi + 2$	0.6157
(Eh)	p_1	$(1:1:2+2\sqrt{2})$	$(\chi^2 + (\sqrt{2}-3)\chi + 1)^2$	0.5961
	p_2	$(1:2+2\sqrt{2}:1)$	$(\chi^2 + (1-\sqrt{2})\chi - 1 + \sqrt{2})^2$	0.5961
	p_3	$(2+2\sqrt{2}:1:1)$	$(\chi^2 - \chi + 1 + \sqrt{2})^2$	0.5961
(L)		permutations of $(X_L : Y_L : 1)$	two conjugate shapes at each	0.2768

TABLE 17. The triangles at which (E0), (Eh) or (L) of Lemma E.17 can hold, in the frame of §E.8. The last column gives the margin $(T_k - \sum_{l \neq k} T_l)/\sum_l T_l$ of the T-test for the heaviest body k.

Proof of Theorem 7.3. Let the triangle of y_0 not be equilateral, and let y_0 satisfy one of (i)–(iv). By Lemma E.17, one of (L), (EI), (EQ), (E0), (Eh), (Θ) holds. We show that (Θ), (EI) and (EQ) do not hold and that in the other cases y_0 passes the T-test, which contradicts $\tau(y_0) = 0$.

Case (Θ) is excluded for the Lagrange type by Lemma E.11(c). For the Euler type, Lemma E.15 puts (X, Y) on a conic $m_o^2 = m_k^2 - m_k m_l + m_l^2$ and makes Θ_{kl} vanish at χ, which Proposition E.16 excludes. Proposition E.14 excludes (EI) and (EQ).

In case (E0) or (Eh), Proposition E.14 puts the masses, up to a common factor, at a point of Table 17 or at $(1 : 1 : 1)$ for $s = p_4$. At $(1 : 1 : 1)$, $C = 2(\chi - 2)(\chi - \frac{1}{2})(\chi + 1)$. At its roots ω_4, ω_3, ω_5 vanish, which is impossible at a leading solution. At the other points we factor $\mathcal{E}_s$ exactly over $\mathbb{Q}(\sqrt{2})$. On every irreducible factor other than the one of the table, some $f_{0,k}$, resp. $f_{h,k}$, has a nonzero remainder. So χ is a root of the factor of the table.

In case (L), Lemma E.11(b) puts the masses, up to a common factor, at a permutation of $(X_L, Y_L, 1)$. At each of these six points, ball arithmetic shows $\beta \neq 0$ at ten of the twelve shapes of the classes with exactly two equal ζ_{kl}. The other two are the shapes of the table.

At every shape of the table we enclose the roots and the margins of the T-test in ball arithmetic. The margins are those of the table, and they are positive. □

Appendix F. The polynomial P_{11}

In terms of $e_1 = m_a + m_b + m_c$, $e_2 = m_a m_b + m_b m_c + m_c m_a$ and $e_3 = m_a m_b m_c$,

$$\begin{aligned} P_{11} = e_1^{11} - 13e_1^9 e_2 + 17e_1^8 e_3 + 63e_1^7 e_2^2 - 162e_1^6 e_2 e_3 - 135e_1^5 e_2^3 + 105e_1^5 e_3^2 + 513e_1^4 e_2^2 e_3 \\ + 108e_1^3 e_2^4 - 651e_1^3 e_2 e_3^2 - 540e_1^2 e_2^3 e_3 + 250e_1^2 e_3^3 + 1000e_1 e_2^2 e_3^2 - 675e_2 e_3^3. \end{aligned}$$

Expanded in the masses it has 78 terms, and $P_{11}(1, 1, 1) = 9$. The substitution $m_a = c + p + q$, $m_b = c + p$, $m_c = c$ gives a polynomial with 72 terms, all coefficients positive and the coefficient of c^{11} equal to 9. With the symmetry of P_{11}, this gives $P_{11}(m) \geq 9 \min(m)^{11} > 0$ on the open positive orthant.

Appendix G. Diagrams 6 and 3 at two exceptional mass points

At the mass point P_2 and at the mass point $\mathbf{h}_1$ of Table 9 a piece of diagram 6, respectively of diagram 3, is enabled by the mass conditions. We exclude these diagrams there by computing the leading solutions of a place and an obstruction at a higher order.

At a place with a fixed diagram and type, the exponents of the coordinates (relative to body 5) are determined, up to the colour exchange. We write the coordinates as $\tau^{2e(q)} y_q$ as in §5.1, divide each row by its lowest power of τ, and normalize two coordinates as there. This gives a weighted system $G(\tau, y) = \sum_{n \geq 0} \tau^n G_n(y)$. The point $y(0)$ is a *leading solution*: $G_0(y(0)) = 0$, with all coordinates nonzero. Let R be a set of rows such that the Jacobian $J_0[R]$ of G_0 restricted to these rows is invertible at y_0. Then the rows of R determine a unique formal solution $Y(\tau)$ of $G_R(\tau, Y) = 0$ with $Y(0) = y_0$, by the implicit function theorem for formal series. Then every solution $y(\tau)$ of $G = 0$ with leading solution y_0 equals Y after the normalization. Let $\eta_{R,n}(y_0)$ be the coefficient of τ^n in the remaining rows evaluated at $Y(\tau)$. If $\eta_{R,n}(y_0) \neq 0$ for some n, no place of the given type has leading solution y_0.

Proposition G.1. *Let $m = (9, 36, 36, 4, 9)$. No singular place of an irreducible curve $C \subseteq Y_m$ has diagram 6 with centre 4 and wings $\{1, 4, 5\}$, $\{2, 3, 4\}$.*

Proof. We renumber the bodies so that $m = (9, 9, 36, 36, 4)$ and the diagram has centre 5 and wings $\{1, 2, 5\}$, $\{3, 4, 5\}$. At such a place the type is $\mathsf{v}(5; 125|345)$. The exponents of the coordinates are fixed by it (after exchanging the colours if necessary). The leading solutions of

the weighted system are given by closed formulas in 10 roots (cube and square roots of explicit functions of the masses), which satisfy a square system of 10 polynomial relations. We check:

(i) in exact arithmetic, that each point of the relations, substituted into the closed formulas, is a leading solution;
(ii) by a standard basis over $\mathbb{Q}$, that the leading system has 5184 solutions counted with multiplicity;
(iii) in ball arithmetic, with Krawczyk's test, that the relations have 5184 distinct simple zeros, so that the closed-form points are all the leading solutions and each is simple, and J_0 has rank 16 at each of them;
(iv) in ball arithmetic, that at each leading solution, for a set R of 16 rows with $J_0[R]$ invertible, some $\eta_{R,n}$ with $n \leq 6$ is nonzero.

The first nonzero obstruction has order 3 at 3888 leading solutions, order 4 at 576, order 5 at 576 and order 6 at 144. By the remark above, no place has this diagram. □

Chen and Tong state the stronger result that no curve has a place of diagram 6 at any positive mass [3, Thm. 9.10].

Let $t_* \approx 1.8024986256$ be the root greater than 1 of $p(t) = 2t^8 + t^6 - 8t^5 - 6t^4 - 8t^3 + t^2 + 2$. The other positive root of p is $1/t_*$.

Proposition G.2. *Let $t \in \{t_*, 1/t_*\}$ and $m = (1, 1, t^{-2}, t^{-2}, (1+t)^{-2})$, so that $(x_1, x_3, x_5) = (1, t, 1+t)$ with $x_i = m_i^{-1/2}$. No singular place of an irreducible curve $C \subseteq Y_m$ has as its type one of the 15 points $\mathsf{h}(s; D_1|D_2)$ of diagram 3.*

Proof. Since $\mathsf{E}(\{1,3,5\})$ holds at m, Theorem 3.9 leaves only the two points $\mathsf{h}(5; 13|24)$ and $\mathsf{h}(5; 14|23)$, at which $\mathcal{D}_3$ holds. The relabelling (34) fixes m and exchanges them. At $\mathsf{h}(5; 13|24)$ the leading solutions are the 10368 points of Appendix C. The first obstruction vanishes exactly at the 648 of them with $O = 0$. We check:

(i) an enclosure of t of width 2^{-440}, by exact rational bisection on a sign change of p;
(ii) enclosures of the 10368 leading solutions in balls, from their closed forms;
(iii) in ball arithmetic at 400 bits, that at each leading solution, for a set R of 16 rows with $J_0[R]$ invertible, some $\eta_{R,n}$ with $n \leq 5$ is nonzero.

The first nonzero obstruction has order 3 at 9720 leading solutions, order 4 at 432 and order 5 at 216. □

The mass points of Proposition G.2 are, up to relabelling, the point $\mathbf{h}_1$ of Table 9.

Use of AI tools

Generative AI tools were used to find and prove the results, to write the programs and the Lean formalization, and to draft the text.

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