

BOUNDS FOR THE SIGN UNCERTAINTY CONSTANT IN DIMENSION ONE

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ABSTRACT. Bourgain, Clozel and Kahane proved that a nonzero even real function f on the real line with f and $\widehat{f}$ integrable, $f(0) \leq 0$ and $\widehat{f}(0) \leq 0$ cannot have f and $\widehat{f}$ both nonnegative outside an arbitrarily small neighborhood of the origin. Let $A(f)$ be the radius beyond which f is nonnegative, and let $A_+(1)^2$ be the largest constant such that $A(f)A(\widehat{f}) \geq A_+(1)^2$ for all such f . This is the constant of Problem 6.11 in the list of Georgiev, Gómez-Serrano, Tao and Wagner, where $f(0) < 0$ and $\widehat{f}(0) < 0$ are required. The previous bounds are $0.2025 \leq A_+(1)^2 \leq 0.3089$, due to Gonçalves, Oliveira e Silva and Steinerberger and to a computer search reported by Chung, Du and Wesley. We prove $\frac{11}{20} < A_+(1) \leq \frac{551}{1000}$, so that $A_+(1)^2$ lies in $(0.3025, 0.303601]$. For the lower bound, since extremizers exist, it suffices to show that every even real integrable g with $\widehat{g} = g$, $g(0) \leq 0$ and $g \geq 0$ on $|x| > r$ vanishes. For such g we write $g(0) = \int_{|y|>r} gW + \sum_k d_k g(\beta_k) + \int_{-r}^r g\rho$ with $d_k \geq 0$ and $\beta_k \geq r$. The weight W comes from explicit differences between Gaussians and their Fourier transforms and from cosines $\cos(2\pi\beta_k x)$, corrected on $[-r, r]$ by an approximate solution of a Fredholm integral equation, and ρ is the residual of this approximation. If $W \geq m > 0$ on $|y| > r$ and ρ is small compared with m , a bound for the Fredholm operator forces $g = 0$. Two separately written programs verify the required inequalities for $r = \frac{11}{20}$ in ball arithmetic. The upper bound comes from an explicit self-dual combination of Gaussians whose positivity beyond $\frac{551}{1000}$ is verified in the same way.

1. INTRODUCTION

We normalize the Fourier transform as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi} dx$. For a function f on $\mathbb{R}$ let

$$A(f) = \inf\{r > 0 : f(x) \geq 0 \text{ for all } |x| \geq r\},$$

with $\inf \emptyset = \infty$, where $f(x) \geq 0$ includes that $f(x)$ is real. Let $\mathcal{A}$ be the set of nonzero even functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that f and $\widehat{f}$ are integrable, $f(0) \leq 0$ and $\widehat{f}(0) \leq 0$. Such f and $\widehat{f}$ agree almost everywhere with continuous functions, and we always use the continuous versions. Bourgain, Clozel and Kahane proved that $A(f)A(\widehat{f}) \geq 0.1687$ for all $f \in \mathcal{A}$ [2, Théorème 1.1]. Following [4] we write

$$A_+(1) = \inf_{f \in \mathcal{A}} \sqrt{A(f)A(\widehat{f})}.$$

Georgiev, Gómez-Serrano, Tao and Wagner [6, Problem 6.11] ask for the largest constant $C_{6.11}$ such that $A(f)A(\widehat{f}) \geq C_{6.11}$ for every even $f \in L^1(\mathbb{R})$ with $f(0) < 0$ and $\widehat{f}(0) < 0$. We read this for f continuous at 0, so that $f(0)$ is defined. Lemma 2.4 below, together with a result of Gonçalves, Oliveira e Silva and Ramos [8], shows that $C_{6.11} = A_+(1)^2$.

Gonçalves, Oliveira e Silva and Steinerberger [9, Theorem 2] improved the constant 0.1687 to 0.2025, that is, $A_+(1) \geq 0.45$. Their proof uses rearrangement inequalities, and they remark that it can be refined to give $A_+(1) \geq 0.453$ [9, Section 4]. Explicit functions give the upper bounds $A_+(1)^2 < 0.353$ [9, Theorem 2] and $A_+(1) \leq 0.572990$ [4, Table 4.1]. Computer searches

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[1, 6] lowered the latter to $A_+(1)^2 \leq 0.321591$ [6, Table 3]. A further computer search, reported by Chung, Du and Wesley [3, Section 6.5], produced an even function $f = \hat{f}$ with $f(0) < 0$ and $A(f)^2 \leq 0.3089$, so that $C_{6.11} \leq 0.3089$. Gonçalves, Oliveira e Silva and Ramos conjecture that $A_+(1) < 0.555$ and that every minimizer vanishes identically on a sequence of nonempty intervals after its last sign change [7, Conjecture 1.7]. A least-squares fit to their numerical data suggests $A_+(1) \approx 0.550$, under a further conjecture of theirs [7, Section 6.1].

Theorem 1.1. $\frac{11}{20} < A_+(1) \leq \frac{551}{1000}$. Consequently $\frac{121}{400} < C_{6.11} \leq \frac{303601}{1000000}$, that is, $0.3025 < C_{6.11} \leq 0.303601$.

Here $C_{6.11} = A_+(1)^2$ by Lemma 2.4. The lower bound improves $A_+(1) \geq 0.45$ [9] to $A_+(1) > 0.55$. The upper bound improves $C_{6.11} \leq 0.3089$ of [3], and it proves the first assertion of [7, Conjecture 1.7], namely $A_+(1) < 0.555$. The second assertion, about the zeros of minimizers, remains open.

The method. The upper bound needs only an explicit function: an even self-dual (that is, $\hat{g} = g$) combination g of Gaussians with $g(0) < 0$ and $g \geq 0$ for $|x| \geq r$ gives $A_+(1) \leq r$ (Lemma 5.1). For the lower bound we use a theorem of Gonçalves, Oliveira e Silva and Steinerberger [9, Theorem 3]: some nonzero even real $g \in L^1(\mathbb{R})$ with $\hat{g} = g$ and $g(0) = 0$ satisfies $A(g) = A_+(1)$. It is therefore enough to show that for $r = \frac{11}{20}$ every even real self-dual $g \in L^1(\mathbb{R})$ with $g(0) \leq 0$ that is nonnegative on $O = \{|x| > r\}$ vanishes (Lemma 2.2). A self-dual g is orthogonal to every integrable u with $\hat{u} = -u$, so $g(0) = \int g = \int g(1 - u)$. If $1 - u$ vanished on $I = [-r, r]$ and were positive on O , then $g(0) \leq 0$ would force $g = 0$ on O , and hence $g = 0$.

We build u in two parts. The first part v is an explicit combination of differences $b - \hat{b}$ of Gaussian pairs and of cosines $\cos(2\pi\beta x)$ with $\beta \geq r$ and nonnegative weights. A cosine is not integrable, but it pairs with g to give $g(\beta) \geq 0$. The second part is $\varphi - \hat{\varphi}$ for an even function φ supported in I . The requirement $1 - u = 0$ on I becomes the Fredholm equation $(\text{Id} - K)\varphi = 1 - v$ on I , where K is the integral operator on I with kernel $\cos(2\pi xy)$. For $r = \frac{11}{20}$ the Hilbert–Schmidt norm of K is less than 1, so this equation has a unique solution. We use an explicit approximate solution, and an a priori bound for g on I absorbs the residual (Lemma 3.3). What remains to be checked is that an explicit function W is bounded below by some $m > 0$ on O and that the residual is small compared with m (Theorem 3.4).

The method is an absolutely continuous analogue of [4, Proposition 2.4]. Cohn and Gonçalves consider constants $A_\sigma(d)$, where $\sigma = \pm 1$ and $A_{+1}(1) = A_+(1)$. If $0 < t_0 < t_1 < \dots$ with $t_{j+1}/t_j \rightarrow 1$, $\lambda_j > 0$, and the summation formula $f(0) = \sigma \sum_j \lambda_j f(t_j)$ holds for all radial Schwartz functions f on $\mathbb{R}^d$ with $\hat{f} = \sigma f$, then $A_\sigma(d) \geq t_0$. They conjecture that for each (σ, d) except perhaps $(\sigma, d) = (1, 1)$ some formula of this kind proves a sharp lower bound [4, Conjecture 2.5]. In our identity (3.2) a weight W on O and finitely many point masses take the place of the discrete measure, and the identity holds up to a residual term on I .

Verification. The certificate for $r = \frac{11}{20}$ consists of 455 Gaussian pairs, 7 cosines and an even polynomial of degree 40, all with rational coefficients (Section 4). Two programs verify the hypotheses of Theorem 3.4 for these data rigorously, in ball arithmetic [11] and in exact rational arithmetic. The first evaluates $\hat{\varphi}$ by Gauss–Legendre quadrature with a rigorous error bound, and the second uses closed formulas with the complex error function. The two programs were written separately and share no code beyond one-line helpers. Both rely on Arb, but they compute $\hat{\varphi}$, the bounds on intervals, ρ and $\kappa(r)$ differently, and both treat large y by integration by parts. Table 1 lists their results. For the upper bound we give a self-dual function g with $g(0) < 0$ and $g \geq 0$ beyond $\frac{551}{1000}$ (Section 5), and two further programs check these inequalities.

2. REDUCTION TO SELF-DUAL FUNCTIONS

We first reduce Theorem 1.1 to a statement about self-dual functions. Throughout, $r > 0$, $I = [-r, r]$ and $O = \mathbb{R} \setminus I$. For $g \in L^1(\mathbb{R})$ with $\widehat{g} \in L^1(\mathbb{R})$ we use the continuous version of g , and then $g \geq 0$ almost everywhere on O if and only if $g \geq 0$ everywhere on O .

Definition 2.1. We say that (S_r) holds if every even real function $g \in L^1(\mathbb{R})$ with $\widehat{g} = g$, $g(0) \leq 0$ and $g \geq 0$ on O vanishes.

Lemma 2.2. *If (S_r) holds, then $A_+(1) > r$.*

Proof. By [9, Theorem 3] (see also [4, Theorem 1.4]) there is a nonzero even real $g \in L^1(\mathbb{R})$ with $\widehat{g} = g$, $g(0) = 0$ and $A(g) = A_+(1)$. Suppose that $A_+(1) \leq r$. Then $g(x) \geq 0$ for all $|x| > A_+(1)$, in particular on O , and (S_r) implies $g = 0$, a contradiction. $\square$

Remark 2.3. The dilation argument of [2, Section 1] (see also [9, Section 3.3]) gives $A_+(1) \geq r$ without the existence of extremizers. Let $f \in \mathcal{A}$ with $A(f)$ and $A(\widehat{f})$ finite. Both numbers are positive [9, Section 1.1]. After a dilation, $A(f) = A(\widehat{f}) = \tau$ with $\tau^2 = A(f)A(\widehat{f})$. Then $g = f + \widehat{f}$ is even, self-dual and nonnegative for $|x| > \tau$, and $g(0) \leq 0$. It is also nonzero: if $\widehat{f} = -f$, then f and $\widehat{f}$ both vanish for $|x| > \tau$, and a compactly supported integrable function with compactly supported Fourier transform is zero. If $\tau \leq r$, then (S_r) would force $g = 0$, a contradiction. Hence $\sqrt{A(f)A(\widehat{f})} > r$ for every $f \in \mathcal{A}$ (trivially when $A(f)$ or $A(\widehat{f})$ is infinite), and $A_+(1) \geq r$.

The class of [6, Problem 6.11] does not require $\widehat{f} \in L^1(\mathbb{R})$. The next lemma shows that this makes no difference.

Lemma 2.4. *Let $f \in L^1(\mathbb{R})$ be continuous at 0, and suppose that $\widehat{f}$ is real and that $\widehat{f}(\xi) \geq 0$ for all $|\xi| \geq R$, for some $R > 0$. Then $\widehat{f} \in L^1(\mathbb{R})$. Consequently $C_{6.11} \geq A_+(1)^2$, and in fact $C_{6.11} = A_+(1)^2$.*

Proof. For $L > 0$ let $\Lambda_L(\xi) = (1 - |\xi|/L)_+$ and $F_L(x) = L(\sin(\pi Lx)/(\pi Lx))^2$, so that $F_L \geq 0$, $\int F_L = 1$, $F_L(x) \leq 1/(\pi^2 Lx^2)$ and $\widehat{\Lambda}_L = F_L$. By Fubini's theorem $\int \widehat{f}\Lambda_L = \int fF_L$, and the right side tends to $f(0)$ as $L \rightarrow \infty$, because f is integrable and continuous at 0. On $|\xi| < R$ dominated convergence gives $\int_{|\xi| < R} \widehat{f}\Lambda_L \rightarrow \int_{|\xi| < R} \widehat{f}$. On $|\xi| \geq R$ the integrand is nonnegative and increases with L , so monotone convergence gives $\int_{|\xi| \geq R} \widehat{f} = f(0) - \int_{|\xi| < R} \widehat{f} < \infty$.

Now let $f \in L^1(\mathbb{R})$ be even and continuous at 0, with $f(0) < 0$ and $\widehat{f}(0) < 0$. Then $A(f) > 0$, and $A(\widehat{f}) > 0$ because $\widehat{f}$ is continuous. If $A(f)$ or $A(\widehat{f})$ is infinite, there is nothing to prove. Otherwise f and $\widehat{f}$ are real outside bounded sets. The imaginary part w of f is even and integrable, and $\widehat{w} = \text{Im } \widehat{f}$ because f is even. So w and $\widehat{w}$ vanish outside bounded sets, and $w = 0$. Thus f and $\widehat{f}$ are real, and $\widehat{f} \in L^1(\mathbb{R})$ by the first part. Then f agrees almost everywhere with a continuous function $\tilde{f}$, and $\tilde{f}(0) = f(0)$ because f is continuous at 0. Hence $\tilde{f} \in \mathcal{A}$. If $f \geq 0$ for $|x| \geq R'$, then $\tilde{f} \geq 0$ almost everywhere there, and hence everywhere by continuity, so $A(\tilde{f}) \leq A(f)$. Therefore $A(f)A(\widehat{f}) \geq A(\tilde{f})A(\widehat{f}) \geq A_+(1)^2$, and $C_{6.11} \geq A_+(1)^2$.

Conversely, [8, Proposition 3] shows that $A_+(1)$ is also the infimum of $\sqrt{A(f)A(\widehat{f})}$ over the functions $f \in \mathcal{A}$ with $\widehat{f} = f$ and $f(0) < 0$. These functions satisfy the conditions of Problem 6.11, so $C_{6.11} \leq A_+(1)^2$. $\square$

3. A SUFFICIENT CONDITION

We now state a sufficient condition for (S_r) in terms of explicit data. For $u \in L^2(I)$ let $(Ku)(x) = \int_I u(t) \cos(2\pi xt) dt$ for $x \in I$. Since $\cos^2(2\pi xt) = \frac{1}{2}(1 + \cos(4\pi xt))$ and $\int_I \cos(4\pi xt) dt = \sin(4\pi rx)/(2\pi x)$, the Hilbert–Schmidt norm of K is

$$(3.1) \quad \kappa(r) = \left(\int_I \int_I \cos^2(2\pi xt) dx dt \right)^{1/2} = \left(2r^2 + \frac{\text{Si}(4\pi r^2)}{2\pi} \right)^{1/2}, \quad \text{Si}(z) = \int_0^z \frac{\sin t}{t} dt.$$

Definition 3.1. A *certificate* for r consists of finitely many triples (a_j, s_j, c_j) with $a_j > 0$, $s_j \geq 0$ and $c_j \in \mathbb{R}$, finitely many pairs (β_k, d_k) with $\beta_k \geq r$ and $d_k \geq 0$, and an even polynomial ψ . With

$$b_j(x) = e^{-\pi a_j(x-s_j)^2} + e^{-\pi a_j(x+s_j)^2}, \quad \widehat{b}_j(y) = \frac{2}{\sqrt{a_j}} e^{-\pi y^2/a_j} \cos(2\pi s_j y),$$

we associate with it the functions

$$\begin{aligned} v(x) &= \sum_j c_j (b_j(x) - \widehat{b}_j(x)) + \sum_k d_k \cos(2\pi \beta_k x), & \varphi &= (1 - v + \psi) \mathbf{1}_I, \\ W(y) &= 1 - v(y) + \widehat{\varphi}(y) \quad (y \in \mathbb{R}), & \rho(x) &= \widehat{\varphi}(x) - \psi(x) \quad (x \in I), \end{aligned}$$

where $\widehat{\varphi}(y) = \int_I \varphi(t) \cos(2\pi yt) dt$.

The functions v , $\widehat{\varphi}$, W and ρ are even, bounded and continuous, and $\widehat{\varphi} = K\varphi$ on I .

Lemma 3.2. For every certificate and every even real $g \in L^1(\mathbb{R})$ with $\widehat{g} = g$,

$$(3.2) \quad g(0) = \int_O g(y) W(y) dy + \sum_k d_k g(\beta_k) + \int_I g(x) \rho(x) dx.$$

Proof. Since g and $\widehat{g} = g$ are integrable, g is continuous and bounded, and since g is even, $g(x) = \int g(t) \cos(2\pi xt) dt$ for all x . In particular $g(0) = \int g$. The products of g with the bounded functions v , φ and $\widehat{\varphi}$ are integrable. As b_j and φ are integrable, Fubini's theorem gives $\int g \widehat{b}_j = \int \widehat{g} b_j = \int g b_j$ and $\int g \widehat{\varphi} = \int \widehat{g} \varphi = \int g \varphi$. The inversion formula gives $\int g(x) \cos(2\pi \beta_k x) dx = g(\beta_k)$. Hence $\int g v = \sum_k d_k g(\beta_k)$ and $\int g(\varphi - \widehat{\varphi}) = 0$, and therefore

$$g(0) = \int g(1 - v - \varphi + \widehat{\varphi}) + \sum_k d_k g(\beta_k).$$

On O we have $\varphi = 0$, so $1 - v - \varphi + \widehat{\varphi} = W$. On I we have $1 - v - \varphi = -\psi$, so $1 - v - \varphi + \widehat{\varphi} = \rho$. $\square$

Lemma 3.3. Suppose that $\kappa(r) < 1$. Let $g \in L^1(\mathbb{R})$ be even and real with $\widehat{g} = g$ and $g \geq 0$ on O , and put $P = \int_O g$. Then

$$\|g\|_{L^2(I)} \leq \frac{\sqrt{2r} P}{1 - \kappa(r)}.$$

Proof. For $x \in I$ the inversion formula gives $g(x) = (Kg_I)(x) + h(x)$, where g_I is the restriction of g to I and $h(x) = \int_O g(t) \cos(2\pi xt) dt$. Since $g \geq 0$ on O , we have $|h| \leq P$ on I , so $\|h\|_{L^2(I)} \leq \sqrt{2r} P$. The operator norm of K is at most $\kappa(r) < 1$, so $\text{Id} - K$ is invertible with $\|(\text{Id} - K)^{-1}\| \leq 1/(1 - \kappa(r))$ by the Neumann series, and $g_I = (\text{Id} - K)^{-1}h$. $\square$

Theorem 3.4. Let a certificate for r be given, and let $m > 0$ and $\delta \geq 0$ satisfy

- (i) $\kappa(r) < 1$,
- (ii) $W(y) \geq m$ for all $y > r$,
- (iii) $|\rho(x)| \leq \delta$ for all $x \in [0, r]$,
- (iv) $m > 2r\delta/(1 - \kappa(r))$.

Then (S_r) holds, and therefore $A_+(1) > r$.

Proof. Let g be as in Definition 2.1, and put $P = \int_O g$. The function g is continuous and nonnegative on O . In particular $g(\beta_k) \geq 0$ for every k , as a limit from the right when $\beta_k = r$. Since W and ρ are even, (ii) holds on O and (iii) holds on I . We apply Lemma 3.2, the Cauchy–Schwarz inequality and Lemma 3.3:

$$0 \geq g(0) \geq mP - \|g\|_{L^2(I)} \|\rho\|_{L^2(I)} \geq mP - \frac{\sqrt{2r} P}{1 - \kappa(r)} \sqrt{2r} \delta = P \left(m - \frac{2r\delta}{1 - \kappa(r)} \right).$$

By (iv), $P = 0$. Hence $g = 0$ on O . Then g has compact support, so $\hat{g}$ extends to an entire function. Since $\hat{g} = g$ vanishes on O , we get $\hat{g} = 0$ and $g = 0$. By Lemma 2.2, $A_+(1) > r$. $\square$

Remark 3.5. The proofs above use only that ψ is even, bounded and continuous. If φ solves $(\text{Id} - K)\varphi = 1 - v$ on I and $\psi = K\varphi$, then $\rho = 0$, and (iv) reduces to $m > 0$. In a certificate, ψ is a polynomial close to $K\varphi$ and ρ is the error of this approximation. Since W and ρ depend affinely and continuously on the coefficients, a small change of the coefficients changes them only slightly, and the data can be rounded to rationals.

4. PROOF OF THE LOWER BOUND

4.1. The certificate. The file `cert_r055_U30.json` contains a certificate for $r = \frac{11}{20}$, and all its numbers are exact fractions. It has 455 Gaussian pairs. Of these, 449 have widths $a_j \in \{2, 8, 32\}$ and centers $s_j \in \{\frac{1}{5}, \frac{2}{5}, \dots, 30\}$ (all combinations except $a_j = 32$, $s_j = \frac{1}{5}$), and 6 are narrow pairs with $a_j \in \{128, 512, 2048\}$ and $s_j \in [\frac{1}{2}, \frac{11}{20}]$. The coefficients c_j are multiples of 10^{-8} with $|c_j| < 1000$. The 7 cosines have frequencies $\beta_k \in \{\frac{11}{20}, \frac{57}{100}, \frac{63}{100}, \frac{97}{100}, \frac{231}{100}, \frac{49}{20}, \frac{62}{25}\}$ and weights d_k that are multiples of 10^{-8} . The weight at $\beta_k = r$ is 0.89174379, and $\sum_k d_k = \frac{44999999}{50000000}$. The polynomial ψ has degree 40 and terminating decimal coefficients.

The function φ jumps at $\pm r$, with $\varphi(r) > 53.38$. Hence $W(r) = \varphi(r) + \rho(r)$ is large, and $\hat{\varphi}(y)$ behaves like $\varphi(r) \sin(2\pi r y) / (\pi y)$ for large y . Condition (ii) must therefore be checked far out: both programs subdivide $[r, 1500]$ and treat $[1500, \infty)$ by integration by parts.

4.2. The programs. The programs are written in Python and use python-flint, the Python interface to FLINT [5], which contains Arb. Both programs read the data as exact fractions and accept an inequality only when ball arithmetic or exact rational arithmetic proves it. They also check that $\beta_k \geq r$ and $d_k \geq 0$ for every k .

Program 1 (`check_cert_pm.py`, with 192-bit balls) evaluates $\hat{\varphi}$ and its first two derivatives by N -point Gauss–Legendre quadrature on $[0, r]$, with $N = 1600$ for the primary certificate ($N = 500$ for the other) and nodes and weights enclosed by Arb. Its error bound rests on [13, Theorem 8.1], applied after the change of variables $t \mapsto r(1+t)/2$ from $[-1, 1]$ to $[0, r]$: if F is analytic in the open Bernstein ellipse E_θ and $|F| \leq M$ there, then the Chebyshev coefficients of F satisfy $|a_k| \leq 2M\theta^{-k}$. The rule integrates polynomials of degree less than $2N$ exactly and has positive weights with sum 2, so its error on $[-1, 1]$ is at most $4 \sum_{k \geq 2N} 2M\theta^{-k} = 8M\theta^{-2N}/(1-\theta^{-1})$. The program bounds M term by term on the ellipse with $\theta = \frac{11}{10}$ (respectively $\theta = \frac{5}{4}$). For $0 \leq y \leq Y_T$ the factor $|\cos(2\pi y t)| \leq \cosh(2\pi Y_T |\text{Im } t|)$ dominates M , and this is why N must be large. It covers $[r, Y_T]$ by intervals $[y_0 - \eta, y_0 + \eta]$ of initial width $\frac{1}{500}$ and checks $W(y_0) - |W'(y_0)|\eta - \eta^2 \sup |W''|/2 \geq m$, where the supremum is enclosed by evaluating the Taylor coefficients of W on the whole interval. It bisects an interval when the check fails, up to depth 24. For $y \geq Y_T$ it uses $W \geq 1 - \sum_k d_k - V_T - 2B$, where V_T bounds the Gaussian part of $|v|$ on $[Y_T, \infty)$ (each Gaussian term decreases there, because all $s_j \leq Y_T$, which the program

checks) and

$$B = \sum_{k < 6} \frac{|\varphi^{(k)}(r)|}{\omega^{k+1}} + \frac{1}{\omega^6} \int_0^r |\varphi^{(6)}|, \quad \omega = 2\pi Y_T,$$

bounds $|\int_0^r \varphi(t) \cos(2\pi y t) dt|$ for all $y \geq Y_T$ by six integrations by parts. The boundary terms at 0 vanish, because $\sin 0 = 0$ and the odd derivatives of φ vanish at 0. The integral of $|\varphi^{(6)}|$ is bounded by the sum, over 20,000 subintervals, of the length times an enclosure of $\sup |\varphi^{(6)}|$; each term is replaced by a larger rational number, with a relative margin of 10^{-6} . For (iii) the program writes $\rho(x) = \sum_k \rho_k x^{2k}$ with $\rho_k = (-1)^k (2\pi)^{2k} \mu_{2k} / (2k)! - q_k$, where q_k is the coefficient of x^{2k} in ψ and $\mu_{2k} = \int_I \varphi(t) t^{2k} dt$, and bounds $\sup_{[0,r]} |\rho|$ by $\sum_k |\rho_k| r^{2k}$. It computes the moments μ_{2k} for $k \leq 60$ by the same quadrature and bounds the terms with $k > 60$ by a geometric series.

Program 2 (`indep_check_pm.py`, with 256-bit balls) evaluates $\widehat{\varphi}$ in closed form: the Gaussian terms through the complex error function, the polynomial term through $\int_I \psi(t) e^{2\pi i y t} dt = 2 \operatorname{Re}[e^{2\pi i r y} \sum_j (-1)^j \psi^{(j)}(r) / (2\pi i y)^{j+1}]$ (by its power series for small y), and the cosines through $\int_I \cos(2\pi \beta t) \cos(2\pi y t) dt = r [\operatorname{sinc}(2\pi r(\beta - y)) + \operatorname{sinc}(2\pi r(\beta + y))]$ with $\operatorname{sinc} z = (\sin z)/z$. On an interval $[y_0, y_1]$ it uses $W \geq \min(W(y_0), W(y_1)) - (y_1 - y_0)^2 \sup |W''|/8$, with $\sup |W''| \leq \sup |v''| + 4\pi^2 \int_I |\varphi(t)| t^2 dt$ (and an integration-by-parts bound for $y \geq 20$). It bounds $|\rho|$ on a grid of $[0, r]$ by the same interpolation bound, with $\sup |\rho''| \leq 4\pi^2 \int_I |\varphi(t)| t^2 dt + \sup |\psi''|$, treats $[Y_T, \infty)$ by eight integrations by parts, and bounds $\kappa(r)$ in exact rational arithmetic. For certificates without cosines we run `check_cert.py` and `indep_check.py`: Program 1 without the cosine terms, and a version of Program 2 that has neither the cosine terms nor the separate bound for $y \geq 20$.

Proposition 4.1. *The certificate `cert_r055_U30.json` satisfies (i)–(iv) of Theorem 3.4 for $r = \frac{11}{20}$ with $(m, \delta) = (\frac{1}{1250}, 10^{-7})$, and also with $(m, \delta) = (\frac{1}{2500}, 10^{-5})$.*

Proof. Program 1 verifies (i)–(iv) for the first pair, and Program 2 for the second (Table 1). $\square$

Proof of the lower bound in Theorem 1.1. By Proposition 4.1 and Theorem 3.4, $A_+(1) > \frac{11}{20}$. The inequality $C_{6.11} \geq A_+(1)^2$ of Lemma 2.4, which does not use [8], then implies $C_{6.11} > \frac{121}{400}$. $\square$

Without cosines the method still gives $C_{6.11} > 0.3$. The certificate `cert_r05485_X30.json` has 450 Gaussian pairs with $a_j \in \{2, 8, 32\}$ and $s_j \in \{\frac{1}{5}, \frac{2}{5}, \dots, 30\}$ and no cosines, and both programs verify it for $r = \frac{1097}{2000}$ (Table 1). It follows that $A_+(1) > 0.5485$ and $C_{6.11} > 0.30085$.

5. THE UPPER BOUND

We now prove the upper bound in Theorem 1.1 with an explicit function.

Lemma 5.1. *Let $r > 0$, let (a_j, s_j, c_j) be finitely many triples with $a_j > 0$, $s_j \geq 0$ and $c_j \in \mathbb{R}$, and put $g = \sum_j c_j (b_j + \widehat{b}_j)$ with b_j and $\widehat{b}_j$ as in Definition 3.1. If $g(0) < 0$ and $g(y) \geq 0$ for all $y \geq r$, then $A_+(1) \leq r$ and $C_{6.11} \leq r^2$.*

Proof. The function g is even, real and integrable. The Fourier transform of b_j is $\widehat{b}_j$, and the Fourier transform of $\widehat{b}_j$ is b_j by Fourier inversion, because b_j is even. Hence $\widehat{g} = g$. Since $g(0) = \widehat{g}(0) < 0$, the function g belongs to $\mathcal{A}$ and to the class of [6, Problem 6.11]. As g is even and nonnegative for $|x| \geq r$, we have $A(\widehat{g}) = A(g) \leq r$, and therefore $A(g)A(\widehat{g}) \leq r^2$. $\square$

The file `pcert_r0551_G2.json` contains such data for $r = \frac{551}{1000}$, and all its numbers are exact fractions. It has 1606 Gaussian pairs: all combinations of widths $a_j \in \{1, 2, 4, 8, 16\}$ and centers

TABLE 1. Verification of the certificates. A run for the lower bound verifies the certificate if (i)–(iv) of Theorem 3.4 hold with the stated m and δ , and a run for the upper bound if it proves $g(0) < 0$ and $g > 0$ on $[r, \infty)$. The value $W(y^*)$ at a numerically located minimizer y^* is computed in ball arithmetic. Here G is the Gaussian of the wide pair (Section 5); the smallest local lower bounds for g occur at the right ends of the ranges, where g decays like G . Upper bounds are rounded up and lower bounds down.

Certificate	cert_r055_U30.json		cert_r05485_X30.json	
r ; Gaussian pairs; cosines	$\frac{11}{20}$; 455; 7		$\frac{1097}{2000}$; 450; 0	
$\kappa(r) <$	0.943607		0.942138	
$W(y^*) <$	0.0009661 at $y^* = 2.4806855$		0.0022918 at $y^* \approx 4.5972503$	
Program	1	2	1	2
m ; δ	$\frac{1}{1250}$; 10^{-7}	$\frac{1}{2500}$; 10^{-5}	$\frac{1}{500}$; 10^{-7}	$\frac{1}{1000}$; $\frac{1}{40000}$
N ; θ (Program 1)	1600; $\frac{11}{10}$	—	500; $\frac{5}{4}$	—
Y_T	1500	1500	300	300
Interval evaluations on $[r, Y_T]$	790,725	159,585	187,420	209,781
$W \geq$ on $[Y_T, \infty)$	0.0880	0.0880	0.6733	0.6762
$\sup_{[0, r]} \rho \leq$	$1.004 \cdot 10^{-10}$	$9.65 \cdot 10^{-7}$	$6.98 \cdot 10^{-10}$	$2.28 \cdot 10^{-5}$
$2r\delta/(1 - \kappa(r)) <$	$1.951 \cdot 10^{-6}$	$1.951 \cdot 10^{-4}$	$1.896 \cdot 10^{-6}$	$4.740 \cdot 10^{-4}$
Time (s)	825	474	504	1074
Result	verified	verified	verified	verified
Upper bound: pcert_r0551_G2.json	Program 3		Program 4	
$g(0) <$	$-1.00005 \cdot 10^{-8}$		$-1.00005 \cdot 10^{-8}$	
$g > 0$ proved on	$[r, 92]$, 31,808 evaluations		$[r, 184]$, 103,961 evaluations	
smallest local lower bound for g	$8.62 \cdot 10^{-8}$		$3.47 \cdot 10^{-235}$	
$g/G \geq$ on $[92, \infty)$	$1.32142 \cdot 10^{-4}$		$1.32142 \cdot 10^{-4}$	
Time (s); result	88; verified		191; verified	

$s_j \in \{0, \frac{1}{4}, \frac{1}{2}, \dots, 80\}$, and one wide pair T with $a_T = \frac{1}{64}$ and $s_T = 80$, whose coefficient is $c_T > 1.32142 \cdot 10^{-4}$. All coefficients satisfy $|c_j| < 1$. Numerically, the largest zero of g is near 0.5509994, just below r .

Two further programs, written independently of each other, check $g(0) < 0$ and $g > 0$ on $[r, \infty)$, again in ball arithmetic. *Program 3* (check_primal.py, with 192-bit balls) proves $g > 0$ on $[r, Y]$ with $Y = \max_j s_j + 12 = 92$ by the Taylor bound of Program 1. For $y \geq Y$ it compares g with $G(y) = e^{-\pi a_T(y-s_T)^2}$. It bounds each cosine factor by 1 in absolute value and drops the term $c_T e^{-\pi a_T(y+s_T)^2}$, which is positive. Let $E_j(y) = e^{-\pi a_j(y-s_j)^2} + e^{-\pi a_j(y+s_j)^2} + 2a_j^{-1/2} e^{-\pi y^2/a_j}$ for $j \neq T$, and $E_T(y) = 2a_T^{-1/2} e^{-\pi y^2/a_T}$. Each E_j/G is nonincreasing on $[Y, \infty)$ if $a_T < a_j < 1/a_T$ and $Y \geq (a_j s_j - a_T s_T)/(a_j - a_T)$ for $j \neq T$, and if $a_T < 1/a_T$; the program checks these conditions. Hence $g/G \geq c_T - \sum_j |c_j| E_j(Y)/G(Y)$ on $[Y, \infty)$. *Program 4* (indep_primal.py, with 256-bit balls) evaluates $g(0)$ in Arb and in mpmath, proves $g > 0$ on $[r, 2Y]$ by the interpolation bound of Program 2 with an explicit second derivative, and bounds g/G on $[Y, \infty)$ through the vertices of the quadratic exponents. Table 1 lists the results.

Proposition 5.2. *For the data in pcert_r0551_G2.json, $g(0) < 0$ and $g(y) > 0$ for all $y \geq \frac{551}{1000}$.*

Proof. Programs 3 and 4 verify both inequalities (Table 1). $\square$

Proof of the upper bound in Theorem 1.1. By Proposition 5.2 and Lemma 5.1, $A_+(1) \leq \frac{551}{1000}$ and $C_{6.11} \leq (\frac{551}{1000})^2 = \frac{303601}{1000000}$. $\square$

6. REMARKS

Finding the certificates. None of the following enters the proof. For fixed r , fixed Gaussian pairs and fixed frequencies β_k , we replace φ by the Nyström solution of $(\text{Id} - K)\varphi = 1 - v$. Then W depends affinely on the coefficients (c_j, d_k) , and we maximize t subject to $W(y) \geq t$ on a grid in $[r, 1500]$, $|c_j| \leq 1000$, $d_k \geq 0$ and $\sum_k d_k \leq 0.9$. The last constraint keeps the bound for $[Y_T, \infty)$ positive. We add the local minima of W on a finer grid as new constraints and solve again. The Gaussian terms of $\widehat{\varphi}$ are evaluated in closed form through the Faddeeva function, and the linear programs are solved with the interior point method [12] of the HiGHS solver [10]. Finally we round c_j and d_k to multiples of 10^{-8} , solve for φ again and take for ψ the degree-40 Taylor polynomial of $K\varphi$, computed from the moments of φ . Because of the jump of φ at r , the grid must contain $y = r$, be dense near r and extend far beyond the centers s_j . For the upper bound the unknowns are the coefficients of g in Lemma 5.1. We maximize ε subject to $g \geq \varepsilon w$ on a grid, $g(0) \leq 0$, $c_T \geq 0$ and a bound on the size of the coefficients, where $w = 1$ up to the centers and w decays like the wide pair beyond them. We then round the coefficients to multiples of 10^{-12} and lower the coefficient of the pair with $a_j = 1$ and $s_j = 0$ so that $g(0)$ is about -10^{-8} .

Limits of the families. With $v = 0$, that is without Gaussian pairs and cosines, φ solves $(\text{Id} - K)\varphi = 1$, and $W = 1 + \widehat{\varphi}$ is positive on O for $r < 0.4535$ in a floating-point computation. This is close to the bound 0.453 mentioned in [9, Section 4]. The Gaussian pairs and the cosines raise this threshold, but in our floating-point runs the families above do not raise it much further. The cutting-plane linear programs returned negative values of t for $r = 0.5488$ without cosines (450 pairs, centers up to 30), and for $r = 0.5502$ and $r = 0.5503$ with cosines (460 and 680 pairs, centers up to 30 and 45). Since the interior point solutions are not certified, these runs only indicate that the families leave little room beyond $r = 0.55$. The remaining range is $0.55 < A_+(1) \leq 0.551$.

The shape of W . If a certificate with $\rho = 0$ and $W \geq 0$ on O existed for $r = A_+(1)$, then (3.2) applied to an extremizer g would give $gW = 0$ on O , so that g would vanish wherever $W > 0$. In the certificate of Section 4, a floating-point scan shows $W < 0.0015$ at most points of the intervals $[1.09, 1.99]$, $[3.14, 3.75]$, $[5.04, 5.62]$ and $[6.86, 7.43]$, and on shorter intervals near 10.9 and 21.8. Between any two of the first four intervals W exceeds 3 somewhere, and it also has narrow dips; the smallest value $W(y^*)$ of Table 1 lies in one of them. This is consistent with the prediction of [7, Conjecture 1.7] that extremizers vanish identically on a sequence of intervals beyond the last sign change.

DATA AVAILABILITY

The certificates, the checking programs, the linear programming scripts, the logs of the runs in Table 1 and a script that prints the properties of the certificates stated in Sections 4 and 5 are archived at doi:10.5281/zenodo.23506074 and available at <https://github.com/tejasvi/sign-uncertainty-bound>. The SHA-256 checksums of `cert_r055_U30.json` and `pcert_r0551_G2.json` are

12bceda53300a060e4277fb1c5f03a6529761767316f4fb8c0f753e45d63f423 and
fc32c0d2c7052a1a70462f94a7b2e8fc87d9b4a96406b4c549bc910c36be392a.

We ran the programs with Python 3.12.13 and python-flint 0.9.0, and the times in Table 1 are wall-clock seconds on a shared server. The archive also contains perturbed copies of the certificates, which the programs reject, and a floating-point test of the identity (3.2).

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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