

NEW LOWER BOUNDS FOR ROMANOFF'S CONSTANT

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ABSTRACT. Let d be the lower density of the set of integers of the form $p + 2^a$ with p prime and $a \geq 1$. We prove that $d \geq 0.109406$, and, assuming a bound that Johnston and Trudgian deduce from Lichtman's Theorems 1.2 and 1.7, that $d \geq 0.117953$. The best lower bound in the literature was 0.10788, from the method of Elsholtz and Schlage-Puchta. That method splits the integers into residue classes modulo a fixed integer l and bounds, in each class, the number of pairs of representations of the same integer by a sieve bound with the constant 8. We use the modulus $l = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 41 \cdot 241$ and bound the singular series with the prime factors of $2^t - 1$ for $t \leq 1200$, which gives the first bound. The bound of Johnston and Trudgian gives the constant 6.7814 for the total over all classes, but their argument does not give it for a single class. For the second bound we use both constants. A linear program finds the distribution of the pairs over the classes that minimizes the resulting bound for d , and an exact Lagrangian certificate bounds its minimum from below.

1. INTRODUCTION

For a positive integer n let $r(n)$ be the number of pairs (p, a) with p prime, $a \geq 1$ and $n = p + 2^a$, and let

$$d = \liminf_{N \rightarrow \infty} \frac{1}{N} \#\{n \leq N : r(n) > 0\}.$$

Allowing $a = 0$ adds the integers $p + 1$, which have density 0, so it does not change d . Romanoff [16] proved that $d > 0$. Erdős [6] and van der Corput [17] showed that a positive proportion of the odd integers are not of this form. Habsieger and Roblot [8] showed that the upper density is less than 0.490941. Chen, Dai and Li [2] lowered this to 0.49035, and a certificate of Yoo [19] lowers it to 0.49019. Heuristic and numerical studies suggest a density of about 0.434 to 0.437 [15, 4].

Explicit lower bounds for d start from the second moment of $r(n)$. Chen and Sun [3] proved $d \geq 0.0868$. Habsieger and Roblot [8] improved this to 0.0933, Pintz [13] to 0.093626, and Habsieger and Sivak-Fischler [10] to 0.0936275. Elsholtz and Schlage-Puchta [5] split the integers into residue classes modulo $2^{24} - 1$, where $r(n)$ is distributed very unevenly, and proved $d \geq 0.107648$. Johnston and Trudgian [11, footnote 2] obtained the slightly better value 0.10788 by repeating these computations, and note that it is also stated at the end of page 721 of [5]. With the sieve weights of Wu [18] in place of the constant 8 in (2.1) below, the method would give 0.11011, but this extension of Wu's work has not been checked, and Elsholtz and Schlage-Puchta did not state it as a theorem [11, Section 4.2 and footnote 6]. Theorem 1.1 is below this value, and our results do not use Wu's weights. Johnston and Trudgian also obtained 0.10695 by Pintz's method with Lichtman's constant 6.7814 [12], a bound that rests on the same input as our second result.

Theorem 1.1. *We have $d \geq 0.109406$.*

2020 *Mathematics Subject Classification.* 11P32, 11N36, 11Y16; 90C05.

Key words and phrases. Romanoff's theorem, primes and powers of two, Selberg sieve, Bombieri–Vinogradov theorem, linear programming, computer-assisted proof.

Theorem 1.2. *Assume the bound (2.2) below, which Johnston and Trudgian [11, Proposition 2] deduce from Lichtman's Theorems 1.2 and 1.7 of [12]. Then $d \geq 0.117953$.*

Lichtman's paper [12] and the paper of Johnston and Trudgian [11] are preprints. Theorem 1.2 uses them only through (2.2), and Theorem 1.1 uses neither them nor (2.2).

The method. In the second-moment method one bounds $\sum_{n \leq N} r(n)^2$ from above, and the main term counts the pairs $p_1 + 2^{a_1} = p_2 + 2^{a_2}$ with $a_1 \neq a_2$. For a fixed shift $h = 2^{a_2} - 2^{a_1}$ this is a count of prime pairs $p_1 - p_2 = h$, which the Selberg sieve with the Bombieri–Vinogradov theorem bounds by four times the Hardy–Littlewood prediction (the constant 8 in (2.1) below). Elsholtz and Schlage-Puchta observed that the same bound holds with a congruence condition on p_2 modulo a fixed l , so that the second moment can be bounded in each residue class modulo l separately. Since the classes in which $r(n)$ is often large carry most of the pairs, the resulting bound for d is much better. Theorem 1.1 is this method with two changes: a modulus l with more prime factors, for which the period of 2^a modulo l is 120 instead of 24, and sharper bounds for the singular series.

Lichtman's preprint [12] proves the level of distribution $153/256 > 1/2$ for primes in progressions with triply well-factorable weights, uniformly in residues of size up to $x^{1+\varepsilon}$. Johnston and Trudgian [11] deduce that the number of prime pairs $p_1 - p_2 = h$ is at most 3.3907 times the prediction (the constant 6.7814 in (2.2) below). They also observed that this argument does not give the constant 6.7814 for prime pairs with a congruence condition modulo l : sifting $p_2 + h$ over $p_2 \equiv k \pmod{l}$ needs residues that depend on the sifting modulus, and Lichtman's estimate allows only one residue for all moduli. For Theorem 1.2 we therefore use the constant 8 in every class and 6.7814 only for the total over all classes, which carries no congruence condition. Both bounds constrain the same counts at the same N .

The pair counts of the classes enter the bound for d through a convex function, and the two inputs only constrain them from above. We therefore minimize the bound over all distributions of the pair counts that satisfy both constraints. This is a linear program (Section 4). Its minimum is attained when the pairs fill the classes with few representations up to the cap 8 and the remaining classes share the rest. Since no class can hold more than $8/6.7814 \approx 1.18$ times its share of the total, the minimum is well above the bound obtained with the constant 8 alone. In both theorems the modulus is $l = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 41 \cdot 241$. The singular series enters through the mean values $T(c)$ of Section 3, which we bound from above with the prime factors of $2^t - 1$ for $t \leq 1200$ (Section 5).

Verification. A program checks the two certificates: it recomputes all data from the factorizations and evaluates the bound in exact rational arithmetic. Section 6 lists its external inputs and two independent checks.

2. INPUTS

Throughout, p denotes a prime, l a fixed odd squarefree integer, φ Euler's function, and

$$C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\dots, \quad S(h) = \prod_{p|h, p>2} \frac{p-1}{p-2}.$$

For odd squarefree m put $f(m) = \prod_{p|m} (p-2)^{-1}$, so that $S(h) = \sum f(m)$ over the odd squarefree divisors m of h , and let o_m be the multiplicative order of 2 modulo m . For $N \geq 1$, an integer h with $0 < |h| < N$ and an integer k put

$$R(N, h) = \#\{(p_1, p_2) : p_1, p_2 \leq N, p_1 - p_2 = h\},$$

$$R(N, h, k, l) = \#\{(p_1, p_2) : p_1, p_2 \leq N, p_2 \equiv k \pmod{l}, p_1 - p_2 = h\}.$$

The per-class bound. Johnston and Trudgian [11, Section 4.2] quote from [5, (28)] the following consequence of the Selberg sieve and the Bombieri–Vinogradov theorem [1, 9]. We use it in the form they state. For every $\varepsilon > 0$ there is $N_0(\varepsilon, l)$ such that for all $N \geq N_0$, all $0 < h < N$ and all k with $(k, l) = 1$,

$$(2.1) \quad R(N, h, k, l) \leq (8 + \varepsilon) \frac{C_0 N}{\varphi(l) \log^2 N} \prod_{p|lh, p>2} \frac{p-1}{p-2}.$$

For $l = 1$ this is the classical upper bound for prime pairs. The congruence condition changes the moduli in the Bombieri–Vinogradov theorem from d to dl , with a residue that depends on d , which the maximum over the residues in that theorem allows. Since $R(N, h, k, l) = R(N, -h, k+h, l)$ and the right-hand side of (2.1) depends only on $|h|$, the bound (2.1) also holds for $-N < h < 0$ if $(k+h, l) = 1$.

The total bound. Johnston and Trudgian [11, Proposition 2] deduce from Theorems 1.2 and 1.7 of [12] that for every $\varepsilon > 0$ there is $N_1(\varepsilon)$ such that for all $N \geq N_1$ and all $0 < |h| < N$,

$$(2.2) \quad R(N, h) \leq (6.7814 + \varepsilon) \frac{C_0 N}{\log^2 N} S(h).$$

They state it for $h > 0$, and $R(N, -h) = R(N, h)$.

For fixed even $h \neq 0$ the Hardy–Littlewood conjecture predicts $R(N, h) \sim 2C_0 N S(h) / \log^2 N$. The corresponding prediction for $R(N, h, k, l)$, when k and $k+h$ are coprime to l , is the right-hand side of (2.1) with 2 in place of $8 + \varepsilon$. So (2.1) is four times the prediction, and (2.2) is 3.3907 times the prediction.

Further inputs. We use the prime number theorem in progressions to the fixed modulus l , the bound

$$(2.3) \quad R_0 := \sum_{m \text{ odd squarefree}} \frac{f(m)}{o_m} < 1.93656$$

which follows from Pintz's bound $C_0 R_0 < 1.2784421041$ [13, Lemma 2''] and $C_0 > 0.6601618158$, as stated in [14, p. 55] (see also [11, (17)]), and the following inequality of Pintz [13, Lemma 4'].

Lemma 2.1. *Let $b_1, \dots, b_K \geq 0$ be integers with $\sum b_i = M > 0$ and $\sum b_i^2 \leq DM$. Then $\#\{i : b_i > 0\} \geq M\psi(D)$, where*

$$\psi(D) = \max_{j \geq 1} \frac{2j+1-D}{j(j+1)}.$$

Proof. For integers $b \geq 1$ and $j \geq 1$ we have $(b-j)(b-j-1) \geq 0$, that is, $b^2 \geq (2j+1)b - j(j+1)$. So $b^2 \geq (2j+1)b - j(j+1) \mathbf{1}_{b>0}$ for all integers $b \geq 0$, with equality for $b = 0$. Summing over i gives $DM \geq (2j+1)M - j(j+1)\#\{i : b_i > 0\}$. $\square$

On $[1, \infty)$ the function ψ is the piecewise linear interpolation of $1/D$ at the positive integers. It is convex and decreasing there.

3. MOMENTS IN RESIDUE CLASSES

Let P be the order of 2 modulo l , so that $2^a \bmod l$ depends only on $a \bmod P$. For a residue $k \bmod l$ let

$$U_k = \{r \in \mathbb{Z}/P\mathbb{Z} : (k - 2^r, l) = 1\}, \quad s_k = |U_k|.$$

Let $L = \lfloor \log_2 N \rfloor$. Throughout this section k is fixed and $N \rightarrow \infty$.

Lemma 3.1. $\sum_{n \leq N, n \equiv k} r(n) = (s_k / (P\varphi(l) \log 2) + o(1))N$, where $n \equiv k$ means $n \equiv k \pmod{l}$.

Proof. The sum counts the pairs (p, a) with $a \leq L$ and $p \leq N - 2^a$, $p \equiv k - 2^a \pmod{l}$. If $a \bmod P \notin U_k$, there are $O(1)$ such primes. Otherwise the prime number theorem in progressions gives $(1+o(1))(N - 2^a)/(\varphi(l) \log N)$ uniformly for $a \leq L - \log_2 L$. The remaining $O(\log L)$ values of a contribute $O(N \log L / \log N) = o(N)$. Summing over the $(s_k/P + o(1))L$ admissible a gives the claim. $\square$

The second moment is $\sum_{n \leq N, n \equiv k} r(n)^2 = \sum_{n \leq N, n \equiv k} r(n) + \sum_{a_1 \neq a_2} X_k(a_1, a_2)$, where for $1 \leq a_1, a_2 \leq L$ with $a_1 \neq a_2$ and $h = 2^{a_2} - 2^{a_1}$,

$$X_k(a_1, a_2) = \#\{p_2 \leq N - 2^{a_2} : p_2 \equiv k - 2^{a_2} \pmod{l}, p_2 + h \text{ prime}\}$$

counts the pairs of representations $p_1 + 2^{a_1} = p_2 + 2^{a_2} \leq N$ in the class of k . It is $O(1)$ unless $a_1 \bmod P$ and $a_2 \bmod P$ lie in U_k . Both primes of such a pair are at most N , and $0 < |h| < N$, so

$$(3.1) \quad X_k(a_1, a_2) \leq R(N, h, k - 2^{a_2}, l), \quad \sum_{k \bmod l} X_k(a_1, a_2) \leq R(N, h).$$

If $a_1 \bmod P$ and $a_2 \bmod P$ lie in U_k , then $k - 2^{a_2}$ and $k - 2^{a_2} + h = k - 2^{a_1}$ are coprime to l , so (2.1) bounds the first right-hand side, for either sign of h . The bound (2.2) bounds the second.

We group the pairs (a_1, a_2) by $G = (a_1 \bmod P, a_2 \bmod P)$ and write $c = c(G) = a_2 - a_1 \bmod P$. For an odd prime p we have $p \mid h$ exactly when $\text{ord}_p(2)$ divides $a_2 - a_1$. Hence $S(h) = Q(c)S'(h)$, where

$$Q(c) = \prod_{p \mid l, \text{ord}_p(2) \mid c} \frac{p-1}{p-2}, \quad S'(h) = \prod_{p \mid h, p \nmid 2l} \frac{p-1}{p-2},$$

and the product in (2.1) equals $P_l S'(h)$ with $P_l = \prod_{p \mid l} (p-1)/(p-2)$.

Lemma 3.2. For $r_1, r_2 \in \mathbb{Z}/P\mathbb{Z}$ and $c = r_2 - r_1$,

$$\sum_{\substack{1 \leq a_1 \neq a_2 \leq L \\ a_1 \equiv r_1, a_2 \equiv r_2}} S'(2^{a_2} - 2^{a_1}) \leq \left(\frac{L}{P} + 1\right) \frac{L}{P} T(c) + O(L \log L),$$

with

$$T(c) = \sum_{\substack{m \text{ odd squarefree} \\ (m, l) = 1}} f(m) [g_m \mid c] \frac{g_m}{o_m},$$

where the congruences are modulo P , $g_m = \gcd(o_m, P)$, and $[\cdot]$ is 1 if the condition holds and 0 otherwise.

Proof. We have $S'(2^{a_2} - 2^{a_1}) = \sum f(m)$ over the squarefree m with $(m, 2l) = 1$ and $m \mid 2^{|a_2 - a_1|} - 1$, that is, $o_m \mid a_2 - a_1$. For fixed m , the conditions $a_2 - a_1 \equiv c \pmod{P}$ and $o_m \mid a_2 - a_1$ are compatible only if $g_m \mid c$, and then they fix a_2 modulo $\text{lcm}(P, o_m) = P o_m / g_m$ for given a_1 . Since $0 < |a_2 - a_1| < L$, there is no such pair if $o_m \geq L$. So for each of the at most $L/P + 1$ values of a_1 there are at most $L g_m / (P o_m) + [o_m < L]$ values of a_2 . Summing over m gives the main term and an error at most $(L/P + 1) \sum_{o_m < L} f(m)$. The primes p with $\text{ord}_p(2) < L$ divide $M_L = \prod_{t < L} (2^t - 1)$, so

$$\sum_{o_m < L} f(m) \leq \prod_{p \mid M_L} \frac{p-1}{p-2} \leq \frac{1}{C_0} \frac{M_L}{\varphi(M_L)} \ll \log \log M_L \ll \log L. \quad \square$$

4. THE LINEAR PROGRAM

We measure counts in the unit $N/(P\varphi(l)\log 2)$. For a class k with $s_k > 0$ and a group $G = (r_1, r_2) \in U_k \times U_k$, let $Y_k(G)$ be the sum of $X_k(a_1, a_2)$ over the pairs in G , in this unit, and let $Y_k = \sum_G Y_k(G)$. Put $\kappa = C_0 P_l / (P \log 2)$, and let A_G be the number of classes $k \bmod l$ with $r_1, r_2 \in U_k$. A class k is counted in A_G exactly when, for every $p \mid l$, the residue $k \bmod p$ avoids 2^{r_1} and 2^{r_2} , which coincide modulo p when $\text{ord}_p(2) \mid c$. Hence

$$A_G = \varphi(l) \prod_{p \mid l, \text{ord}_p(2) \nmid c} \frac{p-2}{p-1}, \quad P_l \frac{A_G}{\varphi(l)} = Q(c).$$

Lemma 4.1. *Let $T^+(c) \geq T(c)$ for all c , and let $\delta > 0$. For all large N , all k with $s_k > 0$ and all $G \in U_k \times U_k$,*

$$(4.1) \quad Y_k(G) \leq (8 + \delta) \kappa T^+(c)$$

and

$$(4.2) \quad \frac{1}{N} \#\{n \leq N : n \equiv k, r(n) > 0\} \geq \frac{s_k - \delta}{P\varphi(l)\log 2} \psi\left(1 + \frac{Y_k}{s_k} + \delta\right).$$

If (2.2) holds, then for all large N and all G also

$$(4.3) \quad \sum_{k: G \in U_k \times U_k} Y_k(G) \leq (6.7814 + \delta) \kappa T^+(c) A_G.$$

Proof. By (3.1), (2.1) and Lemma 3.2, the sum of $X_k(a_1, a_2)$ over the pairs in G is at most $(8 + \varepsilon)C_0 N P_l (\varphi(l) \log^2 N)^{-1} ((L/P)^2 T(c) + O(L \log L))$. Since $L^2 / \log^2 N \rightarrow 1 / \log^2 2$ and $T(c) \geq 1$ (the term $m = 1$), this is at most $(8 + \delta) \kappa T(c)$ in the unit $N/(P\varphi(l)\log 2)$ for large N . For (4.2) we apply Lemma 2.1 to the numbers $r(n)$, $n \leq N$, $n \equiv k$. By Lemma 3.1 their sum is $s_k + o(1)$ units. Their sum of squares exceeds their sum by the sum of $X_k(a_1, a_2)$ over all $a_1 \neq a_2$, which is at most $Y_k + o(1)$ units, since the pairs with $a_1 \bmod P$ or $a_2 \bmod P$ outside U_k contribute $O(L^2) = o(N)$. Since ψ is decreasing, (4.2) follows. Finally, in the same way as for (4.1), the bounds (3.1), (2.2) and Lemma 3.2 bound the left-hand side of (4.3) by $(6.7814 + \delta)C_0 Q(c)T(c)\varphi(l)/(P \log 2)$, which is the right-hand side of (4.3) since $Q(c)\varphi(l) = P_l A_G$. $\square$

Put $\nu = (l/\varphi(l))/(P \log 2)$ and

$$(4.4) \quad F(Y) = \frac{\nu}{l} \sum_{k: s_k > 0} s_k \psi\left(1 + \frac{Y_k}{s_k}\right).$$

Let V_8 be the minimum of F over all $Y \geq 0$ that satisfy (4.1) with $\delta = 0$, and let V be its minimum over all $Y \geq 0$ that satisfy (4.1) and (4.3) with $\delta = 0$. Summing (4.2) over the classes k with $s_k > 0$ shows that $d \geq V_8$, and that $d \geq V$ if (2.2) holds. Indeed, for large N the counts $Y_k(G)$ satisfy the constraints of Lemma 4.1 with δ , and dividing them by $1 + \delta$ gives a point that satisfies them with $\delta = 0$. Since ψ is $\frac{1}{2}$ -Lipschitz and the Y_k are bounded by (4.1), this division, the replacement of $s_k - \delta$ by s_k and the removal of the $+\delta$ in the argument of ψ change the sum of the right-hand sides of (4.2) by $O(\delta)$. So $d \geq V_8 - O(\delta)$ for every $\delta > 0$, and in the same way $d \geq V - O(\delta)$. Since the Y_k are bounded, only finitely many j in the definition of ψ matter, so V_8 and V are values of linear programs.

Reduction to rotation orbits. We call U_k the pattern of the class k . For a rotation orbit u of nonempty patterns $U \subset \mathbb{Z}/P\mathbb{Z}$, let K_u be the set of classes k with $U_k \in u$, and put $m_u = |K_u|$, $\rho_u = m_u/l$, $s_u = |U|$, $w_u = \nu \rho_u s_u$ and $n_u(c) = \#\{r \in U : r + c \in U\}$. These do not depend on

the choice of $U \in u$. For a feasible point Y put

$$y_u(c) = \frac{1}{m_u} \sum_{k \in K_u} \sum_{G \in U_k \times U_k, c(G)=c} Y_k(G).$$

For each $k \in K_u$ there are $n_u(c)$ groups $G \in U_k \times U_k$ with $c(G) = c$, so (4.1) gives $0 \leq y_u(c) \leq y_u^{\max}(c)$ with $y_u^{\max}(c) = 8\kappa n_u(c) T^+(c)$. Summing (4.3) over the P groups with difference c and dividing by l gives $\sum_u \rho_u y_u(c) \leq 6.7814 \kappa T^+(c) \mathcal{N}(c)$, where

$$\mathcal{N}(c) = \frac{1}{l} \sum_{c(G)=c} A_G = \sum_u \rho_u n_u(c).$$

Since ψ is convex, Jensen's inequality gives $\sum_{k \in K_u} s_u \psi(1 + Y_k/s_u) \geq m_u s_u \psi(1 + \sum_c y_u(c)/s_u)$. Hence V is at least the minimum of

$$\sum_u w_u \psi\left(1 + \frac{\sum_c y_u(c)}{s_u}\right)$$

subject to

$$0 \leq y_u(c) \leq y_u^{\max}(c), \quad \sum_u \rho_u y_u(c) \leq 6.7814 \kappa T^+(c) \mathcal{N}(c),$$

and V_8 is at least the minimum subject to the first family of constraints alone. For our modulus there are 10,815,600 nonempty patterns, in 92,206 rotation orbits, and one empty pattern.

Certificate. For multipliers $\mu_c \geq 0$, weak duality for the Lagrangian relaxation of the budget rows gives

$$(4.5) \quad V \geq \sum_u \min_{0 \leq y \leq y_u^{\max}} \left(w_u \psi\left(1 + \frac{\sum_c y(c)}{s_u}\right) + \rho_u \sum_c \mu_c y(c) \right) - \sum_c \mu_c 6.7814 \kappa T^+(c) \mathcal{N}(c).$$

For a fixed total $z = \sum_c y(c)$ the linear part is smallest when the $y(c)$ with the smallest μ_c are filled first. So each inner minimum is the minimum of a convex piecewise linear function of z , which is attained at one of its breakpoints and can be computed exactly.

Without budget rows, the same bound with all $\mu_c = 0$ holds for V_8 . Since ψ is decreasing, each inner minimum is then attained at $y = y_u^{\max}$, so every class carries its full cap, and

$$d \geq V_8 \geq \sum_u w_u \psi\left(1 + \frac{\sum_c y_u^{\max}(c)}{s_u}\right) \geq 0.10940622.$$

This proves Theorem 1.1.

For Theorem 1.2 we take all μ_c equal to $\mu^* = 1.3020458752176792 \cdot 10^{-4}$, found by maximizing the right-hand side of (4.5) over a common value of the μ_c . If (2.2) holds, then $d \geq V$, and (4.5) gives $V \geq 0.11795319$, which proves Theorem 1.2. Since (4.5) holds for all $\mu \geq 0$, the bound does not depend on how the multipliers were found. The value μ^* exceeds $\nu/272$ by a relative $2 \cdot 10^{-12}$, and $1/272 = 1/(16 \cdot 17)$ is the slope of $-\psi$ on $[16, 17]$. So with these multipliers the orbits with $1 + \sum_c y_u^{\max}(c)/s_u \leq 16$ are filled to their cap, and the others stop at $1 + z/s_u = 16$.

In the checker, κ is replaced by an upper bound and ν by a lower bound, computed from the bounds for C_0 and $\log 2$ in Section 6. This is allowed, since the right-hand side of (4.5) does not increase when κ or T^+ increases and does not decrease when ν increases.

5. THE SINGULAR SERIES

We need upper bounds $T^+(c)$ for the sums $T(c)$ of Lemma 3.2. Let $\mathcal{A}$ be the set of the primes p with $\text{ord}_p(2) \leq 300$. They divide the cyclotomic values $\Phi_t(2)$, $t \leq 300$, whose complete factorizations are known, and there are 721 of them. Let $\mathcal{M}$ be the set of the 3231 primes

$q < 10^{100}$ with $300 < \text{ord}_q(2) \leq 1200$ that occur in the known factorizations of $\Phi_t(2)$, $301 \leq t \leq 1200$. We split the odd squarefree m into three classes:

- (a) every prime factor of m lies in $\mathcal{A}$;
- (b) $m = m'q$, where m' is in class (a) and $q \in \mathcal{M}$;
- (c) all other m .

We do not need $\mathcal{M}$ to contain all primes of these orders: a missing prime only moves terms from class (b) to class (c).

Let $T_a(c)$, $T_b(c)$ and $T_c(c)$ be the parts of $T(c)$ from the three classes, and $R_{0,a}$, $R_{0,b}$, $R_{0,c}$ those of R_0 . The sums T run over the m coprime to l , and the sums R_0 over all odd squarefree m . For the m that contribute to $T_c(c)$ we have $g_m \mid c$ and $g_m \mid P$, so $g_m \leq \gcd(c, P)$, and therefore

$$T_c(c) \leq \gcd(c, P) R_{0,c} \leq \gcd(c, P)(1.93656 - R_{0,a}^- - R_{0,b}^-),$$

where $R_{0,a}^-$ and $R_{0,b}^-$ are lower bounds for $R_{0,a}$ and $R_{0,b}$.

We compute $T_a(c)$ and $R_{0,a}$ by a dynamic program over the primes of $\mathcal{A}$ (for T_a only those not dividing l) that keeps, for each value of the least common multiple of the orders, the sum of $f(m)$, in ball arithmetic. A branch is stopped when this multiple exceeds $3 \cdot 10^6$. For $R_{0,a}$ the stopped branches are dropped, which only lowers the bound. For $T_a(c)$ a term with $o_m > 3 \cdot 10^6$ is at most $f(m)P/(3 \cdot 10^6)$, since $g_m \leq P$. So the stopped branches and all their extensions contribute at most

$$E = \Delta \prod_{p \in \mathcal{A}, p \nmid l} \left(1 + \frac{1}{p-2}\right) \frac{P}{3 \cdot 10^6},$$

where Δ is the sum of f over the branches at the moment they are stopped, and we add E to the bound for every $T_a(c)$. For class (b) we have $f(m) = f(m')/(q-2)$ and $o_m = \text{lcm}(o_{m'}, \text{ord}_q(2))$. We bound $T_b(c)$ from above with the weights of the dynamic program for m' , adding $E \sum_{q \in \mathcal{M}} (q-2)^{-1}$, and $R_{0,b}$ from below. These sums of positive terms are evaluated in floating-point arithmetic, with every weight rounded outward and a relative margin of 10^{-9} . Each sum has fewer than $3 \cdot 10^6 + 10^3$ terms, so its relative rounding error is below $4 \cdot 10^{-10}$. This gives $R_{0,a} \geq 1.9352319$, $R_{0,b} \geq 0.0011322$, and a mass of at most $1.959 \cdot 10^{-4}$ for class (c). With the resulting upper bounds $T_a^+(c)$ and $T_b^+(c)$ we take $T^+(c) = T_a^+(c) + T_b^+(c) + \gcd(c, P)(1.93656 - R_{0,a}^- - R_{0,b}^-)$.

6. COMPUTATION AND CHECKS

A certificate lists the prime factors of l , the period P , the number of orbits, the external constants below and the multipliers μ_c , which are all zero for Theorem 1.1. A program in Python with python-flint [7] and numpy checks a certificate. It reads the factorizations of $\Phi_t(2)$ for $t \leq 300$, checks by multiplication that they are complete, and proves with FLINT that every factor is prime. It reads the known factorizations of $\Phi_t(2)$ for $301 \leq t \leq 1200$, checks that the product of the factors is $\Phi_t(2)$, and for every factor $q < 10^{100}$ coprime to t proves that q is prime and that $\text{ord}_q(2) = t$; it does not use the other factors. It then computes the bounds of Section 5, the patterns U_k , their orbits and the data of the relaxed program, and evaluates (4.5) in exact rational arithmetic. The external inputs are the per-class constant 8, the total constant 6.7814 (for Theorem 1.2) and the bounds

$$\begin{aligned} C_0 &< 0.66016181584686958, & R_0 &< 1.93656, \\ 0.693147180559945309 &< \log 2 &< 0.693147180559945310. \end{aligned}$$

For the certificates of Theorems 1.1 and 1.2 it prints 0.109406225 and 0.117953196, rounded to nine decimals. On one core the two runs take about 24 and 29 minutes, and the README of the archive gives the commands.

We checked the results in two further ways. First, separately written bounds for the singular series, computed in floating point and widened outward, and an independently written exact evaluator of (4.5) give 0.109406230 for Theorem 1.1 and 0.117953202 for Theorem 1.2, within $7 \cdot 10^{-9}$ of the certified values. Second, a separately written solver of the linear program of Section 4 agrees to within $2.1 \cdot 10^{-5}$ with the values that Johnston and Trudgian [11] report for the method of Elsholtz and Schlage-Puchta. It uses the modulus $3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$, the radical of $2^{24} - 1$, whose period is 24, and a coarser bound for the singular series: the terms of the m whose prime factors all have order at most 300, summed in floating point and widened outward, and the remaining mass of R_0 , which the linear program distributes in the worst way. Each value is an exact lower bound for this linear program. With the constant 8 for both the classes and the total it gives 0.107896, against 0.10788. With 6.7814 for both it gives 0.125340, against the value 0.12532 reported for this hypothetical case, and with Wu's constant 7.8209 for both it gives 0.110125, against 0.11011.

Dependence on the constants. The bound responds strongly to the constant for the total. The following values, computed by a script in the archive, are floating-point evaluations of (4.5) with equal multipliers for hypothetical inputs, not certified bounds. With the modulus of Theorem 1.2 and the per-class constant 8, the total constants 6.5, 6.0 and 4.0 would give 0.121009, 0.127623 and 0.174945. A per-class constant 7.8209 with the total constant 6.7814 would give 0.118891. Johnston and Trudgian [11, Table 2] list the total constant 7.8209 as implied by the work of Wu [18]. If this held uniformly in the shift h , the per-class constant 8 with this total would give 0.110119.

DATA AVAILABILITY

The checking program, the two certificates, the factorization data and the programs that produced them are archived at doi:10.5281/zenodo.23539874 and available at <https://github.com/tejasvi/romanoff-constant>.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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