

SUBPLANES OF ORDER 3 AND MAXIMAL ARCS IN PROJECTIVE PLANES OF ORDER 12

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ABSTRACT. The existence of a projective plane of order 12 is open. A subplane of order 3 of such a plane attains Bruck’s bound $n \geq m^2 + m$ with equality. The counts in this equality case show that the 27 points on no line of the subplane form a maximal arc of degree 3, and that together with the subplane they form a maximal arc of degree 4. Its secants carry a Steiner system $S(2, 4, 40)$, and Thas’s construction yields partial geometries $\text{pg}(8, 9, 6)$ and $\text{pg}(9, 8, 6)$, whose existence is listed as open. If the 27 points form the affine space $\text{AG}(3, 3)$ with the tangent classes as directions, the plane extends the points and lines of $\text{PG}(3, 3)$, and Hall and Roth, and Prince, showed that no such plane exists. We show that in this case already three lines of the subplane lead to a contradiction. More generally, $\text{AG}(3, 3)$ is not the design of a maximal arc of degree 3 in any plane of order 12. The incidence structure formed by all points and the lines that meet the subplane has no automorphism of prime order other than 3 that preserves the subplane, and one type of order 3 is excluded as well. For maximal arcs of degree 3 or 4, with no subplane assumed, the analogous structure has no automorphism of order 13. With the classification of Kirkman triple systems of order 27 with an automorphism of order 13 by Janko and van Trung, whose count we correct from 661 to 657, we also exclude the case where the 27 points carry such an automorphism compatible with the labelling of their parallel classes. Finally, suppose that every line ℓ of the subplane splits the 27 points into three sets of 9, two points lying in the same set exactly when their joining line meets ℓ in a point of the subplane. Then the 27 points form a Hall triple system, hence $\text{AG}(3, 3)$, and the plane does not exist. The computer-assisted steps, including a proof of Prince’s theorem that $\text{PG}(3, 3)$ has no covering set of spreads, are checked in Lean 4: the SAT refutations through LRAT proofs, and the remaining exhaustive searches by search procedures whose completeness is proved in Lean.

1. INTRODUCTION

Projective planes of order n exist for all prime powers n , and no plane of any other order is known. The Bruck–Ryser theorem [BR49] excludes infinitely many orders, and a large computer search excluded order 10 [Lam91]. The smallest open order is 12. Most of the work on order 12 concerns collineations. A series of papers by Janko and van Trung, by Horvatić-Baldasari, Kramer and Matulić-Bedenić, by Suetake, and by Akiyama, Suetake and Tanaka shows that a collineation group of a plane of order 12 has order 1, 2 or 3 [AST23]. The introduction of [AST19] surveys the earlier steps. Hall and Wilkinson [HW84] studied the ternary code of such a plane.

This paper concerns subplanes. Bruck [Bru55] proved that a proper subplane of order m of a projective plane of order n satisfies $n = m^2$ or $n \geq m^2 + m$. For $n = 12$ this leaves $m \in \{2, 3\}$, and $m = 3$ is a case of equality: $12 = 3^2 + 3$. Janko and van Trung [JvT80] showed that a plane of order 12 has no collineation of order 13 that leaves a subplane of order 3 invariant. Their proof uses their classification of the Kirkman triple systems of order 27 with an automorphism of order 13 [JvT81], and they later excluded collineations of order 13 altogether [JvT82]. We do not assume any collineations.

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Throughout, π is a projective plane of order 12 and π_0 is a subplane of order 3. A point of π is *free* if it lies on no line of π_0 .

Maximal arcs. A *maximal arc of degree d* in a projective plane of order n is a set of points that meets each line in 0 or d points. Such an arc has $(d-1)(n+1)+1$ points, and d divides n unless $d = n+1$. Denniston [Den69] constructed maximal arcs of all degrees d dividing q in $\text{PG}(2, q)$ for q even, and Ball, Blokhuis and Mazzocca [BBM97] proved that $\text{PG}(2, q)$ has none with $1 < d < q$ for q odd. Thas [Tha74] and Wallis [Wal73] showed that maximal arcs give partial geometries. The counts in Bruck's equality case [Bru55] place the subplane case in this setting.

Theorem A (Section 3). *Let π be a projective plane of order 12 with a subplane π_0 of order 3, and let F be the set of free points.*

- (a) $|F| = 27$, and F is a maximal arc of degree 3.
- (b) $K = \pi_0 \cup F$ is a maximal arc of degree 4. If $\pi_1 \subseteq K$ is any subplane of order 3, then K is the union of π_1 and its free points.
- (c) The secants of K , restricted to K , form a Steiner system $S(2, 4, 40)$ that contains π_0 as a subdesign. Each of the 117 points off K gives a parallel class of this system, and any two disjoint blocks lie in exactly one of these classes.
- (d) There exist a partial geometry $\text{pg}(8, 9, 6)$ and its dual $\text{pg}(9, 8, 6)$.

Theorem A follows directly from these counts. Brouwer's tables of strongly regular graphs list both partial geometries in (d) with a question mark [Bro], so their existence is open. A plane of order 12 with a subplane of order 3 would therefore settle these two entries. In the same way, a plane of order 6 with a subplane of order 2 would give a $\text{pg}(4, 3, 2)$ on 35 points. No $\text{pg}(4, 3, 2)$ exists, by a result of De Clerck recorded in [Bro], and by the Bruck–Ryser theorem there is no plane of order 6 at all.

The affine case. The free points carry a Kirkman triple system whose 13 parallel classes T_P are labelled by the points P of π_0 (Corollary 3.2). The most symmetric candidate is the affine space $\text{AG}(3, 3)$ with T_P the lines of one direction. Then π_0 plays the role of the plane at infinity, and the Steiner system of Theorem A(c) is the design of points and lines of $\text{PG}(3, 3)$. We call this the *affine case*. In this case the plane extends the points and lines of $\text{PG}(3, 3)$, and the 117 points off K give a *covering set of spreads* of $\text{PG}(3, 3)$, that is, a family of spreads such that every pair of skew lines lies in exactly one of them. Bruck [Bru69, Bru73] asked whether the points and lines of $\text{PG}(3, q)$ extend to a projective plane of order $q(q+1)$, and proposed to build such planes from covering sets of spreads. For $q = 3$, Hall and Roth [HR84] showed by a search, with some cases omitted, that no extension exists, and Prince [Pri99] gave a complete proof by showing that $\text{PG}(3, 3)$ has no covering set of spreads. For $q \geq 4$ the existence of covering sets of spreads is open, with a partial result for $q = 4$ [Pri01]. We show that in the affine case three lines of π_0 already give a contradiction.

Theorem B (Section 4). (a) † (Prince [Pri99]) *The lines of $\text{PG}(3, 3)$ admit no covering set of spreads.*

- (b) † (Hall and Roth [HR84], Prince [Pri99], with Corollary 4.2) *The affine case does not occur: no projective plane of order 12 contains a subplane of order 3 whose free points, with the triples cut out by the tangents, form $\text{AG}(3, 3)$ with each tangent class a direction class.*
- (c) † *In the affine case, already the incidences of the free points and of the non-free points on three lines of π_0 with the tangents are impossible, whether the three lines are concurrent or form a triangle.*

We give a computer proof of (a): a symmetry reduction leaves three exact-cover problems, which have no solution, and their SAT encodings are unsatisfiable. Part (c) does not depend on (a). Under the Klein correspondence, part (a) says that the non-collinearity graph of the hyperbolic quadric $Q^+(5, 3)$ has no edge decomposition into ovoids. The following result removes the subplane and the labelling.

Theorem C (Section 5). *† In no projective plane of order 12 is there a maximal arc of degree 3 whose points and secants form $\text{AG}(3, 3)$.*

Since the free points in the affine case form such an arc, Theorem C gives another proof of Theorem B(b), independent of (a). It rests on a much larger computation.

Symmetric cores. A collineation of π that maps π_0 onto itself has order 1 or 3 (Proposition 3.4). We therefore consider a weaker kind of symmetry. The *core* of π relative to π_0 is the incidence structure formed by all points of π and the 130 lines that meet π_0 . It omits the 27 lines that miss π_0 . Similarly, the core of a maximal arc consists of all points and the secants of the arc. An automorphism of a core is a permutation of the points that maps each line of the core onto a line of the core. For the core relative to π_0 we also require that it maps π_0 onto itself. Such a permutation need not extend to a collineation of π .

Theorem D (Section 6). *Let π be a projective plane of order 12.*

- (a) *† If K is a maximal arc of degree 3 or 4 in π , the core of K has no automorphism of order 13.*
- (b) *† If F is a maximal arc of degree 3 in π , the triple system formed by F and its secants has no automorphism of order 13.*
- (c) *† If π_0 is a subplane of order 3, every automorphism of prime order of the core relative to π_0 has order 3.*
- (d) *If π_0 is a subplane of order 3, no automorphism of order 3 of the core relative to π_0 fixes a free point and induces an elation on π_0 .*
- (e) *† If π_0 is a subplane of order 3, no permutation of order 13 of the free points maps each class T_P onto a class T_{gP} with g a collineation of π_0 . Here T_P is the set of free triples of the tangents at P , as in Section 3.*

The core relative to π_0 is the core of the arc $\pi_0 \cup F$, so (a) excludes automorphisms of order 13 in (c). We also give a separate proof of that case. By (c) and (d), a nontrivial core automorphism of prime order has order 3, and it either fixes no free point or induces on π_0 a collineation with exactly one fixed point (Corollary 6.9). These cases remain open.

A reduction. For a line L of π_0 , let Γ_L be the graph on F in which two free points are adjacent if the line joining them meets L in a point of π_0 .

Theorem E (Section 7). *If Γ_L is the disjoint union of three complete graphs K_9 for every line L of π_0 , then the free points form $\text{AG}(3, 3)$ in the sense of the affine case. Hence, by Theorem B(b), no projective plane of order 12 has a subplane of order 3 with this property †.*

The proof shows that the free points form a Hall triple system, and uses the theorem of Hall and Bruck that Hall triple systems of order 27 are affine [Hal60, Bru58]. An orthogonal resolution for a single line, in the sense of Section 3, does not force the hypothesis for that line (Proposition 7.4). Whether compatible orthogonal resolutions for two lines force it is the *two-line problem* of Section 8. That section also states Conjecture 8.4: no Steiner triple system of order 27 has a family of parallel classes that covers every pair of disjoint triples exactly once. This conjecture would exclude maximal arcs of degree 3 and 4 in planes of order 12, and hence subplanes of order 3.

Verification. Statements marked $\dagger$ rest on computer calculations, directly or through another marked statement. These are exhaustive searches, by dancing links [Knu00] or by SAT solvers [BFF⁺24b, BFF⁺24a]. Every SAT refutation used in a proof has an LRAT certificate [CFHH⁺17] that `lrat-check` [WHH14] verifies and that Lean 4 [dMU21] checks again; the step from a plane to a formula enters the Lean theorems as a hypothesis. The searches without a formula are run in Lean as well, by search procedures whose completeness is proved there. Several hand proofs, among them the reduction behind Theorem E, are checked by the kernel of Lean with Mathlib [The20]. Independent programs reproduce most counts and conclusions. Section 9 gives the details.

Organization. Section 2 collects definitions. Section 3 derives Theorem A from the counts in the equality case. Sections 4 and 5 treat the affine case and Theorem C. Section 6 proves Theorem D, and Section 7 proves Theorem E. Section 8 discusses open problems, and Section 9 describes the verification.

2. PRELIMINARIES

2.1. Planes. A projective plane of order n has $n^2 + n + 1$ points and as many lines. Each line has $n + 1$ points, each point lies on $n + 1$ lines, two points lie on exactly one line, and two lines meet in exactly one point. A subplane of order m is a set of points and lines that forms a projective plane of order m under the induced incidence. For $n = 12$ there are 157 points and 157 lines, each line has 13 points, and each point lies on 13 lines. Since the projective plane of order 3 is unique, every subplane of order 3 is isomorphic to $\text{PG}(2, 3)$ [HP73].

The group $\text{PGL}(3, 3)$ has order $5616 = 2^4 \cdot 3^3 \cdot 13$. We use three facts about it.

- Every involution of $\text{PGL}(3, 3)$ is a homology: it fixes a line pointwise and a point off that line. Indeed, a matrix A with $A^2 = -I$ does not exist over $\mathbb{F}_3$ in dimension 3, since $\det(A)^2 = -1$ is not a square in $\mathbb{F}_3$. Hence $A^2 = I$, and A is diagonalizable with eigenvalues ± 1 .
- An element of order 3 is either an elation, which fixes a line (its axis) pointwise and all lines through a point of the axis (its center), or it fixes exactly one point and one line, which are incident.
- The subgroups of order 13 are Sylow 13-subgroups and therefore conjugate. They are generated by Singer cycles, which act regularly on the points and on the lines.

2.2. Maximal arcs. Let K be a maximal arc of degree d in a plane of order n , with $1 < d \leq n$. Lines meeting K are *secants*, the others are *external lines*. A point of K lies on $(|K| - 1)/(d - 1) = n + 1$ secants. A point off K lies on $|K|/d$ secants. The *design* of K is the set K with the secants restricted to K as blocks. It is a Steiner system $S(2, d, |K|)$.

2.3. Designs. A Steiner system $S(2, k, v)$ is a set of v points with blocks of size k such that two points lie in exactly one block. A Steiner triple system $\text{STS}(v)$ is an $S(2, 3, v)$. A *parallel class* is a set of blocks that partitions the points, and a *resolution* is a partition of the blocks into parallel classes. A Kirkman triple system $\text{KTS}(v)$ is an $\text{STS}(v)$ with a resolution. An $\text{STS}(27)$ has 117 triples, a parallel class consists of 9 triples, and a resolution has 13 classes.

The affine space $\text{AG}(3, 3)$ is the $\text{STS}(27)$ on $\mathbb{F}_3^3$ whose triples are the lines $\{p, p + d, p + 2d\}$. Three distinct points form a line if and only if their sum is 0. The lines with a given direction $\langle d \rangle$ form a parallel class, and the 13 directions form the plane at infinity $\text{PG}(2, 3)$. The projective space $\text{PG}(3, 3)$ has 40 points and 130 lines. A *spread* of $\text{PG}(3, 3)$ is a set of 10 lines that partitions the points.

A Steiner triple system is a *Hall triple system* if any three points not in a common triple lie in a subsystem isomorphic to $\text{AG}(2, 3)$, the $\text{STS}(9)$. Hall [Hal60] showed that Hall triple systems

correspond to commutative Moufang loops of exponent 3. Such a loop of order at most 27 is an elementary abelian group, since the smallest non-associative one has order 81 [Bru58]; see also [BL85]. Hence a Hall triple system on 27 points is isomorphic to $\text{AG}(3, 3)$.

A *partial geometry* $\text{pg}(s, t, \alpha)$ is a set of points and lines such that each line has $s + 1$ points, each point lies on $t + 1$ lines, two points lie on at most one line, and for each point x and line $\ell \not\ni x$ exactly α points of ℓ are collinear with x . Its collinearity graph is strongly regular with parameters $v = (s + 1)(st + \alpha)/\alpha$, $k = s(t + 1)$, $\lambda = s - 1 + t(\alpha - 1)$ and $\mu = \alpha(t + 1)$. The dual structure is a $\text{pg}(t, s, \alpha)$.

3. STRUCTURE

The structure described in this section follows from the counts in Bruck's equality case [Bru55, Lemma 3.1]; Dembowski [Dem58, pp. 76–77] gives the incidence numbers of the three classes of points and lines. We give the short proofs to fix notation. We fix π and π_0 and call the 13 lines of π_0 the π_0 -lines. A point outside π_0 that lies on a π_0 -line is *non-free*. A line that meets π_0 in exactly one point P is a *tangent at P* , and a line that misses π_0 is *external*.

Theorem 3.1. *The points of π are the 13 points of π_0 , 117 non-free points (9 on each π_0 -line L , forming the set N_L), and 27 free points. The lines of π are the 13 π_0 -lines, 117 tangents (9 at each point of π_0), and 27 external lines. Moreover:*

- (i) *a tangent at P carries P , one non-free point of each π_0 -line not through P , and 3 free points;*
- (ii) *an external line carries exactly one non-free point of each π_0 -line;*
- (iii) *a non-free point $X \in N_L$ lies on L , on one tangent at each point of π_0 off L , and on 3 external lines;*
- (iv) *a free point lies on one tangent at each point of π_0 , and on no other line.*

Proof. A point Q on no π_0 -line is joined to the 13 points of π_0 by 13 distinct lines, because a line through Q with two points of π_0 would be a π_0 -line. These are all the lines through Q , which proves (iv). A tangent at P meets each of the 9 π_0 -lines not through P in a point outside π_0 , which is non-free, and these 9 points are distinct, since two π_0 -lines meet in a point of π_0 . The remaining $13 - 1 - 9 = 3$ points lie on no π_0 -line, which proves (i). A non-free point $X \in N_L$ lies on L and on the 9 lines XP with $P \notin L$. These lines are distinct tangents, since a line through X with two points of π_0 is a π_0 -line and hence equal to L . The remaining 3 lines through X miss π_0 , which proves (iii). Each π_0 -line has $13 - 4 = 9$ non-free points, and a non-free point lies on only one π_0 -line. Hence there are 117 non-free points and $157 - 13 - 117 = 27$ free points. By (iv) no external line contains a free point. An external line meets each π_0 -line in a point outside π_0 , and these 13 points are distinct, which proves (ii). The line counts follow by duality, since the dual of π_0 is a subplane of order 3 of the dual plane. $\square$

Two free points lie on a unique line, which is a tangent at a unique point P of π_0 . We call them *partners at P* if they lie on a common tangent at P . For $P \in \pi_0$, let T_P be the set of 9 triples of free points cut out by the tangents at P .

Corollary 3.2. (i) *The free points with the 117 triples of $\bigcup_P T_P$ form a Kirkman triple system $\mathcal{K}$ of order 27 with resolution $\{T_P : P \in \pi_0\}$.*

- (ii) *For a π_0 -line L and $X \in N_L$, let C_X be the set of free triples of the 9 tangents through X . Then C_X is a parallel class of $\mathcal{K}$ that contains one triple of each T_P with $P \notin L$, and the 9 classes C_X with $X \in N_L$ partition $\bigcup_{P \notin L} T_P$.*
- (iii) *If $X \in N_L$ and $X' \in N_{L'}$ with $L \neq L'$, then $|C_X \cap C_{X'}| \leq 1$.*
- (iv) *Each external line e gives a resolution $\{C_{e \cap L} : L \text{ a } \pi_0\text{-line}\}$ of $\mathcal{K}$.*
- (v) *The graph Γ_L is 8-regular and is the union of the 4 triangle factors T_P with $P \in L$.*

Proof. (i) Two free points lie in exactly one triple, namely the free triple of their joining tangent. By Theorem 3.1(iv) each free point lies on exactly one tangent at P , so T_P is a parallel class.

(ii) The 9 tangents through X are tangents at the 9 points of π_0 off L , and they meet only in X , which is not free. Hence their free triples are pairwise disjoint and cover all 27 free points. A tangent at $P \notin L$ meets L in exactly one point of N_L , so each triple of T_P lies in exactly one class C_X with $X \in N_L$.

(iii) The points X and X' are distinct and lie on at most one common line. A common triple of C_X and $C_{X'}$ lies on a common tangent.

(iv) A triple $t \in T_P$ lies on a tangent at P , which meets e in a non-free point X . If L is the π_0 -line through X , then $P \notin L$ and $t \in C_X$. If t lay in two of the classes, its tangent would contain two points of e and would equal e .

(v) Each free point has 2 partners at each of the 4 points of L . $\square$

Definition 3.3. Let L be a π_0 -line. A parallel class of $\mathcal{K}$ that contains one triple of T_P for each $P \notin L$ is a *transversal class* for L . A set of 9 transversal classes for L that partitions $\bigcup_{P \notin L} T_P$ is an *orthogonal resolution* for L . Orthogonal resolutions for distinct lines L and L' are *compatible* if a class of one and a class of the other share at most one triple.

An orthogonal resolution for L is a partial form of the usual notion of orthogonal resolutions of a design: its 9 classes are orthogonal to the 9 classes T_P with $P \notin L$. Our use of “compatible” differs from that of Tonchev [Ton17], where it refers to resolutions of the whole design. By Corollary 3.2, the classes C_X ($X \in N_L$) form an orthogonal resolution for L , and these resolutions are pairwise compatible. Throughout, the labels P of the classes T_P carry the incidence structure of π_0 .

Proposition 3.4. *The stabilizer of π_0 in the collineation group of π has order 1 or 3.*

Proof. By [AST23] the collineation group of π has order 1, 2 or 3. An involution of a projective plane of order n is an elation, a homology or a Baer involution [HP73]. A Baer involution requires n to be a square, and a homology of order 2 requires n to be odd. So the involutions of π are elations. Suppose an involution σ maps π_0 onto itself. It does not fix π_0 pointwise, since the fixed points of an elation lie on its axis and π_0 contains a quadrangle. Hence σ induces an involution of $\text{PGL}(3, 3)$, which is a homology. It fixes a line and a point off that line, hence five points of π_0 , not all collinear. But the fixed points of the elation σ lie on its axis, which meets π_0 in at most four points. $\square$

Let F be the set of free points and $K = \pi_0 \cup F$.

Proposition 3.5. (i) *Every line meets F in 0 or 3 points, so F is a maximal arc of degree 3. Its secants are the 117 tangents.*

(ii) *Every line meets K in 0 or 4 points, so K is a maximal arc of degree 4 with $(4-1) \cdot 13 + 1 = 40$ points. Its secants are the 130 lines meeting π_0 , and its external lines are the 27 external lines of π . These form a maximal arc of degree 3 in the dual plane.*

(iii) *The secants of K , restricted to K , form a Steiner system $\mathcal{D} = S(2, 4, 40)$ containing π_0 as a subdesign. For $X \notin K$, the 10 secants through X form a parallel class S_X of $\mathcal{D}$, and S_X contains exactly one block inside π_0 . Two disjoint blocks of $\mathcal{D}$ lie in exactly one class S_X .*

(iv) *If $\pi_1 \subseteq K$ is a subplane of order 3, then K is the union of π_1 and its free points.*

(v) *The 117 points off K and the 130 secants form a partial geometry $\text{pg}(8, 9, 6)$. Its dual is a $\text{pg}(9, 8, 6)$ whose points are the blocks of $\mathcal{D}$ and whose lines are the classes S_X .*

Proof. (i) and (ii) follow from Theorem 3.1: a π_0 -line meets K in $4 + 0$ points, a tangent in $1 + 3$, and an external line in none. A point off K is non-free and lies on 3 external lines, while a point of K lies on none. Thus the external lines form a dual maximal arc of degree 3.

(iii) Two points of K lie on one line, which is a secant. A point $X \notin K$ lies on $13 - 3 = 10$ secants, which cover the 40 points of K , four on each, and therefore cover each point once. A block of $\mathcal{D}$ meets π_0 in 1 or 4 points. If S_X contains k blocks inside π_0 , then $4k + (10 - k) = 13$, so $k = 1$. Two disjoint blocks meet in π in a point off K , which is the unique X with both blocks in S_X .

(iv) A π_1 -line is a secant of K , and its four points in K are the points of π_1 . Hence no point of $K \setminus \pi_1$ lies on a π_1 -line, and $K \setminus \pi_1$ is contained in the set of free points of π_1 . Both sets have 27 points.

(v) A secant has $13 - 4 = 9$ points off K , and a point off K lies on 10 secants, so $s = 8$ and $t = 9$. Let $X \notin K$ and let B be a secant not through X . Four of the secants through X meet B in a point of K , and the other 6 meet B in a point off K . Hence $\alpha = 6$. The parameters of the collinearity graphs are $(117, 80, 52, 60)$ and $(130, 81, 48, 54)$. $\square$

The *core* of π relative to π_0 is the incidence structure formed by all 157 points of π and the 130 lines that meet π_0 . By Proposition 3.5(ii), it is the core of the arc K . The core determines $\mathcal{K}$, the labels of its resolution, and the orthogonal resolutions of all π_0 -lines.

Remark 3.6. Conversely, let $\mathcal{D}$ be an $S(2, 4, 40)$ with a family $\mathcal{S}$ of parallel classes such that any two disjoint blocks lie in exactly one class of $\mathcal{S}$. A block is disjoint from $130 - 1 - 4 \cdot 12 = 81$ blocks, so there are 5265 disjoint pairs. Since a class covers $\binom{10}{2} = 45$ pairs, $|\mathcal{S}| = 117$. Each block lies in $81/9 = 9$ classes, and two classes share at most one block. Adding one point for each class, incident with the blocks of the class, gives 157 points and 130 lines of size 13, in which two points lie on at most one line. This is the core of a putative plane. Of the $\binom{117}{2} = 6786$ pairs of classes, $130 \binom{9}{2} = 4680$ share a block and $2106 = 27 \binom{13}{2}$ do not. A projective plane of order 12 arises exactly when one can add 27 resolutions of $\mathcal{D}$ by classes of $\mathcal{S}$ such that two classes with no common block lie in exactly one of them. In the subplane case they correspond to the resolutions of Corollary 3.2(iv).

Proof of Theorem A. Parts (a)–(c) are Theorem 3.1 and Proposition 3.5(i)–(iv), and part (d) is Proposition 3.5(v). $\square$

4. THE AFFINE CASE

In this section we suppose that there is a bijection from F to $\mathbb{F}_3^3$ that maps each class T_P onto the set of lines of $\text{AG}(3, 3)$ with one direction d_P . We call this the *affine case*. The 13 classes T_P partition the 117 lines of $\text{AG}(3, 3)$, and $P \mapsto d_P$ is therefore a bijection from π_0 onto the set of directions. The first lemma shows that it is a collineation.

Lemma 4.1. *Let Δ be a set of four directions of $\text{AG}(3, 3)$ that are not collinear in the plane at infinity. Then $\text{AG}(3, 3)$ has no parallel class consisting of one line of each of the 9 directions outside Δ .*

Proof. Let φ be a linear form on $\mathbb{F}_3^3$. Over the 27 points, $\sum_x \varphi(x)^2 \equiv 0 \pmod{3}$. Over a line $\{p + td : t \in \mathbb{F}_3\}$,

$$\sum_t (\varphi(p) + t\varphi(d))^2 = 3\varphi(p)^2 + 6\varphi(p)\varphi(d) + 5\varphi(d)^2 \equiv 2\varphi(d)^2 \pmod{3},$$

and $\varphi(d)^2$ depends only on the direction $\langle d \rangle$. If the parallel class exists, summing over its lines gives $\sum_{d \notin \Delta} \varphi(d)^2 \equiv 0$ for every φ . If $\varphi \neq 0$, then φ vanishes on 4 directions, so the sum over all

13 directions is $9 \equiv 0$. Hence $\#\{d \in \Delta : \varphi(d) \neq 0\} \equiv 0 \pmod{3}$ for every $\varphi \neq 0$. If Δ contains no three collinear directions, the line through two of them contains no other. If exactly three are collinear, the line through the fourth and one of the three contains no other. In both cases some $\varphi \neq 0$ vanishes on exactly two directions of Δ , and the count is 2. $\square$

Corollary 4.2. *In the affine case, $P \mapsto \langle d_P \rangle$ is a collineation from π_0 onto the plane at infinity of $\text{AG}(3, 3)$.*

Proof. Let L be a π_0 -line and $X \in N_L$. By Corollary 3.2(ii), C_X is a parallel class with one line of each direction d_P , $P \notin L$. By Lemma 4.1 the four directions d_P , $P \in L$, are collinear. $\square$

We identify π_0 with the plane at infinity accordingly. Then $K = F \cup \pi_0$ is the point set of $\text{PG}(3, 3)$.

Lemma 4.3. *In the affine case, the design $\mathcal{D}$ of K is the design of points and lines of $\text{PG}(3, 3)$, and the classes S_X ($X \notin K$) form a covering set of spreads: a family of 117 spreads such that every pair of skew lines of $\text{PG}(3, 3)$ lies in exactly one of them.*

Proof. The blocks of $\mathcal{D}$ inside π_0 are the lines at infinity, by Corollary 4.2. The other blocks have the form $\{P\} \cup t$ with $t \in T_P$, that is, an affine line together with its point at infinity. These are the 130 lines of $\text{PG}(3, 3)$. A parallel class of $\mathcal{D}$ is a spread, and Proposition 3.5(iii) gives the covering property. $\square$

Conversely, a covering set of spreads of $\text{PG}(3, 3)$ gives the core of a putative plane of order 12 in the affine case, as in Remark 3.6. This is the construction that Bruck [Bru69, Bru73] proposed for planes of order $q(q+1)$, with $q=3$. The next theorem is due to Prince [Pri99], and it completes the search of Hall and Roth [HR84].

Theorem 4.4 (Prince [Pri99]). $\dagger$ *The lines of $\text{PG}(3, 3)$ admit no covering set of spreads.*

Proof. Suppose $\mathcal{S}$ is a covering set of spreads. The following data come from exhaustive computation. $\text{PG}(3, 3)$ has 8424 spreads and 5265 pairs of skew lines, and each pair of skew lines lies in exactly 72 spreads. Let $\ell_1 = \langle e_1, e_2 \rangle$ and $\ell_2 = \langle e_3, e_4 \rangle$. Exactly one member S of $\mathcal{S}$ contains both lines. The stabilizer of $\{\ell_1, \ell_2\}$ in $\text{PGL}(4, 3)$ consists of the block-diagonal and block-antidiagonal matrices. It has order 2304 and three orbits on the 72 spreads through ℓ_1 and ℓ_2 , of sizes 18, 6 and 48. It maps $\mathcal{S}$ to a family with the same property, so we may assume that S is one of three orbit representatives R_0, R_1, R_2 . The other members of $\mathcal{S}$ share at most one line with S , and they cover the remaining 5220 skew pairs exactly once. For each i , a dancing-links search over the spreads that share at most one line with R_i finds no such exact cover. The same three problems, encoded as propositional formulas, are unsatisfiable, with LRAT certificates that are also checked in Lean (Section 9). $\square$

Under the Klein correspondence, lines of $\text{PG}(3, 3)$ become points of the hyperbolic quadric $Q^+(5, 3)$, skew pairs become non-collinear pairs, and spreads become ovoids. Theorem 4.4 therefore says that the non-collinearity graph of $Q^+(5, 3)$ has no edge decomposition into ovoids.

Theorem 4.5 (Hall and Roth [HR84], Prince [Pri99], with Corollary 4.2). $\dagger$ *The affine case does not occur.*

Proof. By Lemma 4.3 the affine case gives a covering set of spreads of $\text{PG}(3, 3)$, and by Theorem 4.4 there is none. $\square$

The next proposition strengthens Theorem 4.5. It concerns only the free points and the classes C_X on three lines, and its proof does not use Theorem 4.4. Let $\mathcal{K} = \text{AG}(3, 3)$ with T_P the direction class of P , where π_0 is the plane at infinity.

Proposition 4.6. † *For each line L at infinity there are exactly 648 transversal classes and 63,666 orthogonal resolutions. For two distinct lines there are exactly 2304 compatible pairs of orthogonal resolutions. No three lines admit pairwise compatible orthogonal resolutions, whether the three lines are concurrent or form a triangle.*

Proof. The counts come from an exhaustive enumeration. For three lines, the conditions form a propositional formula that encodes the classes T_P , an orthogonal resolution for each of the three lines and the compatibility condition. A SAT solver shows that the formula is unsatisfiable, both for a concurrent triple and for a triangle. Since $\text{PGL}(3, 3)$ acts transitively on each of these two kinds of triples, this covers all triples. $\square$

Proof of Theorem B. Part (a) is Theorem 4.4, part (b) is Theorem 4.5, and part (c) is Proposition 4.6. $\square$

In the language of Theorem 4.4: the members of a covering set of spreads that contain a given line at infinity give an orthogonal resolution for that line, so the spreads through three lines of one plane of $\text{PG}(3, 3)$ cannot be chosen consistently. In Klein terms, three points of one plane of $Q^+(5, 3)$ cannot each receive 9 ovoids through them that cover their 81 non-collinear points once, with ovoids at different points sharing at most one point.

5. MAXIMAL ARCS WHOSE DESIGN IS $\text{AG}(3, 3)$

We now drop the subplane. Let F be a maximal arc of degree 3 in a projective plane of order 12. It has 27 points, and its design $\mathcal{T}$ is an $\text{STS}(27)$.

Definition 5.1. A *class cover* of an $\text{STS}(27)$ is a family of parallel classes such that any two disjoint triples lie in exactly one class of the family.

In an $\text{STS}(27)$ each triple is disjoint from $117 - 1 - 3 \cdot 12 = 80$ triples, so there are 4680 disjoint pairs. Since a parallel class covers $\binom{9}{2} = 36$ of them, a class cover has exactly 130 classes.

Lemma 5.2. *The 130 points off F give a class cover of $\mathcal{T}$: the class of X consists of the 9 secants through X .*

Proof. A point $X \notin F$ lies on $27/3 = 9$ secants, and they partition F . Two disjoint secants meet in exactly one point, which lies off F . Distinct points $X \neq X'$ give distinct classes, since at most one secant passes through both. $\square$

Tonchev [Ton17, Theorem 2.1] characterizes the designs of maximal arcs by families of resolutions, and the class cover of Lemma 5.2 is the condition that appears in his proof. A class cover need not extend to a plane: the 40 external lines of F are still missing. The following theorem is therefore stronger than Theorem C.

Theorem 5.3. † *$\text{AG}(3, 3)$ has no class cover.*

Proof. By exhaustive search, $\text{AG}(3, 3)$ has 17,641 parallel classes, in agreement with [CMM92, p. 444]. Its 4680 disjoint pairs of lines are 468 parallel and 4212 skew pairs, and each pair lies in between 79 and 142 classes. Suppose $\mathcal{C}$ is a class cover. It contains exactly one class through the parallel lines $\ell_0 = \{(0, 0, t)\}$ and $\ell_1 = \{(1, 0, t)\}$, which is one of the 79 classes containing both lines. The stabilizer of $\{\ell_0, \ell_1\}$ in $\text{AGL}(3, 3)$ maps class covers to class covers. It has order 648 and 6 orbits on these 79 classes, of sizes 9, 1, 6, 3, 6 and 54. So we may assume that the class of $\mathcal{C}$ through ℓ_0 and ℓ_1 is one of six orbit representatives. This gives six cases, numbered 0 to 5 in this order, in each of which one class is fixed. In case 1 it is the class of all lines parallel to ℓ_0 , which the whole stabilizer fixes.

We split further by the same argument. Suppose classes $C_1, \dots, C_m$ are fixed. Choose a disjoint pair p covered by none of them, and let H be a group of affine maps that fix each C_i as a set of lines. In a class cover containing $C_1, \dots, C_m$, the member covering p contains both lines of p and shares no disjoint pair with any C_i . We call such classes *admissible*. The stabilizer H_p of p in H maps the cover to another cover containing $C_1, \dots, C_m$, so we may assume that the member covering p is a representative of an H_p -orbit of admissible classes. Case 1 is split in four levels, as in Table 1. This gives 23 leaves: cases 0, 2, 3, 4, 5 and the 18 leaves of Table 1.

For each leaf, a SAT solver shows that no class cover contains its fixed classes. The formula has a variable for each of the 17,641 classes, a unit clause for each fixed class, and for each disjoint pair a clause requiring at least one of its classes and a cardinality constraint allowing at most one. $\square$

TABLE 1. The split of case 1 in the proof of Theorem 5.3. At each node, H is the group used for the split, H_p the stabilizer of the chosen pair, and the children are the representatives of the H_p -orbits of admissible classes. At the first split H also fixes $\{\ell_0, \ell_1\}$.

node	admissible classes	$ H $	$ H_p $	children
case 1	36	648	72	b0–b5
b2	13	1944	108	c0–c3
c1	10	972	108	d0, d1
d1	33	972	12	e0–e8
leaves: b0, b1, b3, b4, b5, c0, c2, c3, d0, e0–e8 (18 in all)				

Proof of Theorem C. If F were such an arc, Lemma 5.2 would give a class cover of $\text{AG}(3, 3)$, contradicting Theorem 5.3. $\square$

6. SYMMETRIC CORES

6.1. Definitions. An *automorphism of the core relative to π_0* is a permutation σ of the points of π such that $\sigma(\pi_0) = \pi_0$ and σ maps each line meeting π_0 onto a line meeting π_0 . It maps π_0 -lines to π_0 -lines and tangents to tangents, so it permutes the free points and the non-free points. It induces a collineation g of π_0 , maps T_P to T_{gP} , and maps C_X to $C_{\sigma X}$. An *automorphism of the core of a maximal arc K* is a permutation of the points of π that maps each secant of K onto a secant. It maps K onto itself, because the points of K lie on 13 secants and the other points on fewer. It therefore induces an automorphism of the design of K . By Proposition 3.5(ii), an automorphism of the core relative to π_0 is an automorphism of the core of $K = \pi_0 \cup F$.

For collineations, Roth [Rot64, Theorem 7] showed that a collineation of prime order whose fixed structure is a subplane of order m with $m^2 \neq n$ forces $n \geq m^2 + m + 2$. His argument uses a line that misses the subplane, so it does not apply to automorphisms of the core. In Lemma 6.1, σ is an automorphism of the core relative to π_0 . Throughout this section, g denotes the collineation of π_0 induced by such an automorphism σ .

Lemma 6.1. (i) *The fixed free points of σ form a subsystem of $\mathcal{K}$. Each triple inside it lies in a class T_P with $gP = P$.*

(ii) *If q is a fixed free point and $gP = P$, the tangent at P through q is fixed, and σ fixes or swaps the two partners of q at P .*

(iii) *If g is the identity and σ fixes a free point, then σ is the identity.*

Proof. (i) Two fixed free points lie on a unique tangent, which is fixed. Its point $P \in \pi_0$ is fixed, and its third free point is fixed. (ii) is clear. (iii) Let q be the fixed free point. The 13 tangents through q are fixed. For a π_0 -line L , the 9 tangents through q at the points off L meet L in the 9 points of N_L , so all non-free points are fixed. Then all tangents are fixed, since each of them joins a point of π_0 to a non-free point. If a free point x were moved, the 13 tangents through x would be fixed, so all of them would contain σx as well, which is impossible. $\square$

6.2. Automorphisms of order 13. Janko and van Trung [JvT81, Theorems A and B] classified the Kirkman triple systems of order 27 whose 13 parallel classes are permuted by an automorphism ρ of order 13. They showed that none of them has 9 pairwise disjoint triples, in distinct ρ -orbits and in distinct parallel classes, such that the 13 images of the set of the 4 omitted parallel classes form the lines of a projective plane of order 3. This excludes a collineation of order 13 that leaves π_0 invariant [JvT80]. Such a collineation fixes one of the 27 external lines, since $27 \equiv 1 \pmod{13}$. Let X be a non-free point on that line. The images of C_X are the classes of the 13 non-free points on the line, and they partition the triples, so C_X meets every ρ -orbit of triples once. Its omitted classes are those at the points of a π_0 -line. An automorphism of the core need not act on the external lines, and a plane of order 12 has no collineation of order 13 at all [JvT82]. So the next theorem concerns exactly the case that these papers leave open.

Theorem 6.2. $\dagger$ *If π_0 is a subplane of order 3 of a projective plane of order 12, the core relative to π_0 has no automorphism of order 13.*

Proof. Let σ be such an automorphism. If g were the identity, σ would permute the 9 tangents at each point of π_0 in orbits of size 1 or 13, so it would fix all core lines, and all points would be fixed. Hence g has order 13. All subgroups of order 13 of $\text{PGL}(3, 3)$ are conjugate. So after replacing σ by a suitable power and relabelling π_0 , we may assume that g is a fixed Singer cycle. Since $27 \equiv 1 \pmod{13}$, σ fixes a free point q^* . A second fixed free point would lie on a tangent through q^* at a fixed point of π_0 , and g fixes no point. The other 26 free points therefore form two orbits of size 13. If one orbit contained both partners of q^* at a point P , say x and $\sigma^k x$, then σ^k would map the tangent q^*x onto itself and would fix P , so $13 \mid k$ and $\sigma^k x = x$, a contradiction. Hence we may label the partners of q^* at P as a_P and b_P with $\sigma(a_P) = a_{gP}$ and $\sigma(b_P) = b_{gP}$. A non-free point $X \in N_L$ lies on exactly one tangent through q^* . We label X by L and by the point $P \notin L$ of this tangent. Then σ acts on these labels by $(L, P) \mapsto (gL, gP)$.

With these labels, the core gives a satisfying assignment of a propositional formula. Its variables describe the classes T_P , that is, which free points are partners at P , and, for each π_0 -line L , which non-free point of L lies on the tangent at P through a free point q . Its clauses say that these data form a Kirkman triple system with pairwise compatible orthogonal resolutions for all 13 lines, that they respect the labels above, and that they are invariant under σ . A SAT solver shows that this formula is unsatisfiable. $\square$

Remark 6.3. We recounted the classification of [JvT81] with an independent program: an exact cover over the possible first parallel classes, and orbits under the normalizer of $\langle \rho \rangle$, which has order 4056. There are 657 isomorphism classes, 629 of affine type and 28 of antiaffine type in the sense of [JvT81]. Exactly three are diagonal and none is projective, as in [JvT81, Theorem B]. Each of the 661 entries of the list in [JvT81] belongs to one of the 657 classes, and each class occurs. The entries (74) and (77) are the images of (72), and the entries (75) and (76) the images of (73), under the multiplier μ of order 3 of [JvT81, Section 1]. So the list contains four repetitions, and the count in [JvT81, Theorem A] should read 657, of which 629 are of affine type. The count 661 is also quoted in later tables of Kirkman triple systems of order 27 [MR90, CMM92]. Four entries, (230), (339), (395) and (412), repeat a point as printed; each

has exactly one single-symbol correction that gives a valid first parallel class, and we used it. None of this affects the conclusions of [JvT80, JvT82].

The classification also handles the case in which only the triple system of the free points, with its labelled resolution, is symmetric.

Proposition 6.4. *† Let π_0 be a subplane of order 3 of a projective plane of order 12. Suppose that a permutation σ of order 13 of the free points maps each class T_P onto a class T_{gP} , where g is a collineation of π_0 . Then no π_0 -line has an orthogonal resolution. Such a plane therefore does not exist.*

Proof. Since $27 \equiv 1 \pmod{13}$, σ fixes a free point x . If g were the identity, σ would fix the triple of each class through x , hence each of its two other points, since σ has odd order; then σ would fix all 27 points. So g has order 13, it is a Singer cycle, and $\mathcal{K}$ with the resolution $\{T_P\}$ is one of the systems of Remark 6.3. Fix P_0 . Then $T_{g^j P_0} = \sigma^j(T_{P_0})$ for all j . All subgroups of order 13 of $\text{PGL}(3, 3)$ are conjugate, and the collineations of π_0 that normalize $\langle g \rangle$ induce the multipliers 1, 3, 9 on it. So, up to isomorphism and collineations of π_0 , each of the 657 systems carries four labellings, one for each coset of $\{1, 3, 9\}$ in $\mathbb{Z}_{13}^*$. By the action of g , all π_0 -lines behave alike. An exhaustive search over the $657 \cdot 4$ labelled systems finds that 2594 of them have no transversal class for a line, 33 have between 1 and 24 and no orthogonal resolution, and the remaining one, $\text{AG}(3, 3)$ with the natural labelling, has 648. A plane would give an orthogonal resolution for every line by Corollary 3.2(ii), so $\mathcal{K}$ would be $\text{AG}(3, 3)$ in the affine case, which does not occur by Theorem 4.5. $\square$

An automorphism of order 13 of the core induces such a permutation of the free points, so Proposition 6.4 contains Theorem 6.2; so does the next theorem, which assumes no subplane, as the special case $K = \pi_0 \cup F$. The proof of Proposition 6.4 rests on the classification of Remark 6.3 and on an exhaustive search rather than a certificate (Section 9 describes how the search is checked); that of the next theorem uses dancing links instead of SAT.

Theorem 6.5. *† Let K be a maximal arc of degree 4 or 3 in a projective plane of order 12. The core of K has no automorphism of order 13.*

Proof. We treat degree 4 first. Let σ be an automorphism of order 13 of the core, and let $\mathcal{D}$ be the design of K , an $S(2, 4, 40)$. If σ fixed all points of K , it would fix all secants and then all points. The fixed points of σ in K form a closed set: the block through two fixed points is fixed, and its other two points are fixed because σ has order 13. A closed set with $w \geq 2$ points is a block or an $S(2, 4, w)$, so $w \equiv 1$ or $4 \pmod{12}$. The number of fixed points is $\equiv 40 \equiv 1 \pmod{13}$, so it is 1, 14, 27 or 40. The values 14 and 27 are not $\equiv 1$ or $4 \pmod{12}$, and 40 is excluded by the first observation. Hence it is 1. Call the fixed point ∞ and write the other points as (o, i) with $o \in \{0, 1, 2\}$, $i \in \mathbb{Z}_{13}$ and $\sigma(o, i) = (o, i + 1)$.

No block is fixed, since a fixed block would consist of fixed points. Hence the 13 blocks through ∞ form one orbit. Its base block meets each point orbit once, since otherwise two blocks through ∞ would share a second point. Translating each point orbit separately commutes with σ , so we may take the base block $B_\infty = \{\infty, (0, 0), (1, 0), (2, 0)\}$. The pairs of points fall into 60 orbits under σ : 6 inside each point orbit, 13 between any two point orbits, and 3 through ∞ . The orbit of B_∞ covers 6 of them. The other 117 blocks form 9 orbits, whose base blocks cover each of the remaining 54 pair orbits exactly once. An exact-cover search over the 3762 candidate base blocks gives 3,291,704 labelled designs.

Now consider the parallel classes S_X for $X \notin K$. A fixed X would give a fixed class, whose block through ∞ would be fixed. Hence the 117 points off K form 9 orbits of size 13. Within an orbit, the 13 classes contain the 13 blocks through ∞ , one each, so exactly one class of each

orbit contains B_∞ . Call these classes $S_0, \dots, S_8$. They partition the 81 blocks disjoint from B_∞ , and since two classes share at most one block,

$$(6.1) \quad |S_k \cap \sigma^d S_l| \leq 1 \quad \text{for all } k, l \text{ and } d \neq 0.$$

Conversely, if parallel classes $S_0, \dots, S_8$ through B_∞ partition the blocks disjoint from B_∞ and satisfy (6.1), then the 117 classes $\sigma^d S_k$ cover every pair of disjoint blocks exactly once. Indeed, each point of a block B not through ∞ lies in B_∞ after exactly one translation, and these 4 translations are distinct, since a translate with two points of B_∞ would equal B_∞ , which contains ∞ . So B has exactly 4 translates that meet B_∞ , and it lies in 9 of the classes, which pairwise share only this block and so cover its 81 disjoint blocks once. A dancing-links search over all 3,291,704 designs, with condition (6.1) checked during the search, finds no design that admits such classes.

For degree 3 the argument is the same, with an STS(27). The number of fixed points is 1, 14 or 27, a closed set with $w \geq 2$ points has $w \equiv 1$ or $3 \pmod{6}$, and 27 is excluded as before, so it is 1. There are two point orbits, the base triple $T_\infty = \{\infty, (0, 0), (1, 0)\}$ and $130 = 10 \cdot 13$ points off K . There are 164 candidate base triples and 38,288 labelled STS(27), and none admits 10 classes through T_∞ with the analogous properties. $\square$

For degree 3 a symmetry of the design alone suffices.

Theorem 6.6. $\dagger$ *In no projective plane of order 12 is there a maximal arc of degree 3 whose design has an automorphism of order 13.*

Proof. Let F be such an arc and $\mathcal{T}$ its design. The degree-3 version of the first two paragraphs of the proof of Theorem 6.5 uses only the design and its automorphism of order 13. Hence $\mathcal{T}$ is isomorphic to one of the 38,288 labelled systems. By Lemma 5.2, $\mathcal{T}$ has a class cover. Four of the systems are AG(3, 3), which has none by Theorem 5.3. For each of the other 38,284 systems we list all parallel classes and search for a class cover by dancing links, with no symmetry assumed. No system has one, and each search ends after at most 5 nodes. In 35,540 of these systems some disjoint pair lies in no parallel class at all. $\square$

6.3. Automorphisms of order 2 and 3.

Lemma 6.7. *No automorphism of order 3 of the core relative to π_0 fixes a free point and induces an elation on π_0 .*

Proof. Let σ be such an automorphism, q a fixed free point, and g the induced elation with axis a and center c . The fixed points of g are the four points of a .

The fixed free points. For $P \in a$, σ fixes the two partners of q at P by Lemma 6.1(ii), since an element of order 3 cannot swap two points. A fixed free point $x \neq q$ lies on a fixed tangent with q , at a fixed point of π_0 , that is, at a point of a . Thus the fixed free points are q and its 8 partners at the points of a . By Lemma 6.1(i) they form a subsystem $C_0 \cong \text{AG}(2, 3)$. For $P \in a$, the tangent at P through a fixed free point is fixed, so its other two free points are fixed, again because σ has order 3. Hence the classes T_P , $P \in a$, restrict to the four parallel classes of C_0 .

A fixed class C_X . Let $\ell \neq a$ be a π_0 -line through c , so $g\ell = \ell$, and let $P \in a \setminus \{c\}$, so $P \notin \ell$. The tangent at P through q is fixed and meets ℓ in a fixed point $X \in N_\ell$. Hence C_X is invariant under σ .

Contradiction. Let $t \in C_X$ lie in a class $T_{P'}$ with $gP' \neq P'$. Then $\sigma t \in C_X$ lies in a different class, so $\sigma t \neq t$ and $t \cap \sigma t = \emptyset$, and t contains no fixed point. The fixed points of g off ℓ are the three points of $a \setminus \{c\}$. So C_0 is covered by the three triples of C_X in the classes T_P with $P \in a \setminus \{c\}$, and each of these triples lies in C_0 . They are pairwise disjoint lines of the affine plane C_0 from three different parallel classes. But lines from different parallel classes meet. $\square$

Proposition 6.8. † *The core relative to π_0 has no automorphism of order 2.*

Proof. Let σ be an involution of the core. Since 27 is odd, σ fixes a free point, which we call 0. The induced collineation g is the identity or a homology. Let s be the set of fixed points of g at which σ swaps the two partners of 0. By Lemma 6.1, the fixed free points other than 0 are the partners of 0 at the fixed points of g outside s , and they form, with 0, a subsystem of size $1 + 2(|\text{Fix } g| - |s|)$. A proper subsystem of an STS(27) has 1, 3, 7, 9 or 13 points.

If g is the identity, σ is the identity by Lemma 6.1(iii). Hence g is a homology with axis a and center $c \notin a$, and $|\text{Fix } g| = 5$. The subsystem size forces $|s| \in \{1, 2, 4, 5\}$. Up to the centralizer of g there are seven cases: $s = \{c\}$ or $s = \{P_1\}$ with $P_1 \in a$; two cases with $|s| = 2$; two cases with $|s| = 4$; and $s = a \cup \{c\}$.

$|s| = 2$. The 7 fixed free points form a Fano plane. By Lemma 6.1(i) its 7 lines lie in the 5 classes T_P with $gP = P$. Two triples of one class are disjoint, while two Fano lines meet, so each class contains at most one of them, which is impossible for 7 lines and 5 classes.

$|s| = 5$. The only fixed free point is 0. For a free point $x \neq 0$, the tangent through x and σx is fixed, so its point in π_0 is fixed, and its third free point is fixed, so it is 0. Then x is a partner of 0 at a fixed point of g . There are only 10 such partners, but 26 free points $x \neq 0$.

$|s| = 1$. The 9 fixed free points form a subsystem $C_0 \cong \text{AG}(2, 3)$. As in the proof of Lemma 6.7, a triple of a class C_X with $\sigma X = X$ that lies in a class $T_{P'}$ with $gP' \neq P'$ contains no fixed point.

- Let $s = \{c\}$. For $P \in a$, the tangent at P through 0 is a line of C_0 in T_P . These four lines pass through 0, so they have four different directions in C_0 , and the lines of C_0 in T_P are pairwise disjoint, hence all of the direction δ_P of the first one. Let ℓ be a π_0 -line through c , let $P_0 = \ell \cap a$, and let X be the point where the tangent at some $P \in a \setminus \{P_0\}$ through 0 meets ℓ . Then X is fixed. The classes T_c and T_{P_0} do not occur in C_X , since $c, P_0 \in \ell$. So C_0 is covered by the three triples of C_X in the classes T_P with $P \in a \setminus \{P_0\}$, and each of them lies in C_0 . These are three pairwise disjoint lines of C_0 with three different directions, which is impossible.
- Let $s = \{P_1\}$ with $P_1 \in a$. Let ℓ be the π_0 -line through c and a point $P_0 \in a \setminus \{P_1\}$, and let X be the point where the tangent at P_1 through 0 meets ℓ . Then X is fixed, and the fixed points of g off ℓ are the three points of $a \setminus \{P_0\}$. The triple of C_X in T_{P_1} consists of 0 and its two partners at P_1 , which are swapped, so it contains one fixed point. The two other triples of C_X in fixed classes contain at most 3 fixed points each. Hence C_X covers at most $1 + 3 + 3 = 7$ of the 9 fixed free points, although it is a parallel class.

$|s| = 4$. All homologies of $\text{PGL}(3, 3)$ are conjugate, so we may fix g . Along each 2-cycle of g we label the partners of 0 so that σ maps the first partner at P to the first partner at gP . At the fixed points of g , σ fixes the two partners outside s and swaps them on s . With these labels, and with the non-free points labelled through 0 as in the proof of Theorem 6.2, the action of σ is determined by s . In each of the two cases, a SAT solver shows that the core encoding of Theorem 6.2, with all 13 lines and with the clauses for invariance under σ instead of those for the Singer cycle, is unsatisfiable. $\square$

Corollary 6.9. † *Let σ be an automorphism of prime order p of the core relative to π_0 . Then $p = 3$. Moreover σ either fixes no free point and induces an elation or a collineation with exactly one fixed point on π_0 , or it fixes a free point and induces a collineation with exactly one fixed point.*

Proof. If $p \geq 5$ and $p \neq 13$, then p does not divide $|\text{PGL}(3, 3)| = 2^4 \cdot 3^3 \cdot 13$, so g is the identity. Since $p \nmid 27$, σ fixes a free point, and σ is the identity by Lemma 6.1(iii). The cases $p = 13$ and $p = 2$ are Theorem 6.2 and Proposition 6.8. Let $p = 3$. If g is the identity, then σ fixes no free

point by Lemma 6.1(iii). Then for a free point q , the tangent through q and σq at some point P is mapped to the tangent through σq and $\sigma^2 q$ at P . These two tangents share σq , so they coincide, and each orbit of σ on F is a triple of $\mathcal{K}$ whose tangent is fixed. Such a fixed tangent at P meets each π_0 -line $L \not\ni P$ in a fixed point X . The 9 tangents through a fixed $X \in N_L$ are fixed, so C_X consists of 9 orbits of σ , one in each class $T_{P'}$ with $P' \notin L$. As there are only 9 orbits, the points P of the fixed tangents are exactly the 9 points off L . This holds for each π_0 -line L that carries a fixed non-free point, and there are at least two such lines. But only 6 points of π_0 lie off two distinct lines. Therefore g is not the identity. It is an elation or has exactly one fixed point. Lemma 6.7 excludes an elation together with a fixed free point. $\square$

The cases left open by Corollary 6.9 have a rigid fixed structure.

Lemma 6.10. *Let σ be an automorphism of order 3 of the core relative to π_0 that fixes no free point, and let g be the induced collineation of π_0 . For a point P of π_0 with $gP = P$, let k_P be the number of triples of T_P fixed by σ .*

- (i) *A triple fixed by σ lies in a class T_P with $gP = P$, and it is an orbit of σ . Each k_P is 0, 3, 6 or 9, and the sum of the k_P is at most 9.*
- (ii) *If $gP = P$ and $g\ell = \ell$ for a π_0 -line ℓ not through P , then k_P is the number of fixed points of σ in N_ℓ .*
- (iii) *If g is an elation with axis a and center c , then k_P has the same value $k \in \{0, 3\}$ for the three points $P \in a \setminus \{c\}$, and $k = 3$ implies $k_c = 0$.*

Proof. (i) Since σ maps T_P onto T_{gP} , a fixed triple lies in a class with $gP = P$. It contains no fixed point, so it is a union of orbits of size 3, that is, one orbit. The automorphism permutes the 9 triples of T_P in orbits of size 1 or 3, so $k_P \equiv 9 \equiv 0 \pmod{3}$. An orbit is a single triple, so it lies in one class, and there are only 9 orbits.

(ii) The classes C_X , $X \in N_\ell$, partition the triples of the classes $T_{P'}$ with $P' \notin \ell$. Hence $X \mapsto (\text{the triple of } C_X \text{ in } T_P)$ is a bijection from N_ℓ onto T_P . It commutes with σ , because $\sigma C_X = C_{\sigma X}$ and $gP = P$. So it maps the fixed points onto the fixed triples.

(iii) The elation fixes every line through c . For $P \in a \setminus \{c\}$ and a π_0 -line $\ell \neq a$ through c we have $P \notin \ell$, so by (ii) k_P is the number of fixed points in N_ℓ , which does not depend on P . By (i), $k_c + 3k \leq 9$, so $k \in \{0, 3\}$, and $k = 3$ forces $k_c = 0$. $\square$

Lemma 6.10 is among the formalized results of Section 9. By part (i), the elation case without fixed free points splits into 35 cases, one for each choice of the numbers k_P with $P \in a$. Part (iii) leaves 5 of them. A SAT solver refutes each of the other 30 cases in less than a minute, with LRAT certificates. The 5 remaining cases, the 4 cases (by k_P at the unique fixed point) of the other type without fixed free points, and the type with a fixed free point remain open.

Proof of Theorem D. Part (a) is Theorem 6.5, part (b) is Theorem 6.6, part (c) is Corollary 6.9, part (d) is Lemma 6.7, and part (e) is Proposition 6.4. $\square$

7. A REDUCTION OF THE GENERAL CASE

In this section $\mathcal{K}$ is a Kirkman triple system of order 27 whose resolution $\{T_P\}$ is labelled by the points P of $\text{PG}(2, 3)$. For a line L of $\text{PG}(2, 3)$, the graph Γ_L joins two points if their triple lies in T_P for some $P \in L$. In the situation of Corollary 3.2 this is the graph of Theorem E.

Theorem 7.1. *Suppose that Γ_L is a disjoint union of three copies of K_9 for every line L . Then there is an isomorphism from $\mathcal{K}$ to $\text{AG}(3, 3)$ that maps each class T_P onto the set of lines of one direction d_P , and $P \mapsto d_P$ is a collineation onto the plane at infinity.*

Proof. Step 1: the classes of a triangle are collinear. Let q_1, q_2, q_3 be points not in a common triple, and let P_{ij} be the class of the triple through q_i and q_j . These three classes are distinct, since each point lies in only one triple of each class. Let L be the line through P_{12} and P_{13} . In Γ_L we have $q_1 \sim q_2$ and $q_1 \sim q_3$. Since Γ_L is a union of cliques, $q_2 \sim q_3$, so $P_{23} \in L$.

Step 2: $\mathcal{K}$ is a Hall triple system. With L as in Step 1, let C be the component of Γ_L containing q_1, q_2 and q_3 . If two points of a triple of T_P with $P \in L$ lie in C , so does the third. Hence the triples of the classes T_P , $P \in L$, inside C cover each pair of points of C exactly once, and C is a closed subsystem $\text{STS}(9) \cong \text{AG}(2, 3)$. Thus any three points not in a common triple lie in a subsystem $\text{AG}(2, 3)$.

Step 3: $\mathcal{K} \cong \text{AG}(3, 3)$. This follows from Step 2 and the theorem of Hall and Bruck (Section 2).

Step 4: the labelling. A closed set of 9 points of $\text{AG}(3, 3)$ is a plane. By the argument of Step 2, every component of Γ_L is closed. So the three components of Γ_L are pairwise disjoint planes, and they are the cosets of a 2-dimensional subspace W_L . For $P \in L$, each triple of T_P lies in a component of Γ_L , so its direction lies in W_L . For distinct lines $L \neq L'$ we have $W_L \neq W_{L'}$. Otherwise the 7 classes T_P with $P \in L \cup L'$ would contribute 63 triples inside the three cosets, which contain only $3 \cdot 12 = 36$ triples. Now let P be a point and $L \neq L'$ two lines through P . Every triple of T_P has its direction in the 1-dimensional space $W_L \cap W_{L'}$. Hence T_P is the direction class of $d_P = W_L \cap W_{L'}$. Distinct classes have distinct directions, since a direction class consists of 9 lines. Finally $d_P \subseteq W_L$ for all $P \in L$, so the injective map $P \mapsto d_P$ maps lines to lines. $\square$

Proof of Theorem E. By Corollary 3.2 the free points satisfy the setting of this section. Theorem 7.1 shows that we are in the affine case, which does not occur by Theorem 4.5. $\square$

The components of Γ_L have a geometric meaning. Let $K = \pi_0 \cup F$ be the maximal arc of degree 4 of Proposition 3.5.

Lemma 7.2. *In a plane of order 12 with a subplane π_0 of order 3, let L be a π_0 -line and C a component of Γ_L with 9 points. Then C and the four points of π_0 on L are the points of a subplane π_1 of order 3, and $\pi_0 \cap \pi_1$ consists of the points of L . Conversely, every subplane of order 3 contained in K that meets π_0 in the points of a line L arises in this way.*

Proof. Since Γ_L is 8-regular, C is a 9-clique. Two points of C lie on a tangent at a point of L , whose third free point is adjacent to both, so it lies in C . Take as lines L and the 12 tangents at points of L that carry a triple inside C . Two points of C lie on one of these tangents, and two points of L lie on L . A point of C and a point $P \in L$ lie on the tangent at P through that point, which carries a triple inside C because C is a component. Each of the 13 lines contains 4 of the 13 points, so these points and lines form a projective plane of order 3. Conversely, let $\pi_1 \subseteq K$ meet π_0 in the points of L . Its other 9 points are free, and its lines other than L meet π_0 in one point of L , so they are tangents at points of L . Hence these 9 points are pairwise adjacent in Γ_L and form a component. $\square$

Lemma 7.3. *In a plane of order 12 with a subplane π_0 of order 3, two distinct subplanes of order 3 contained in K meet in the four points of a line of both.*

Proof. Let $\pi' \subseteq K$ be a subplane of order 3. A line of π' meets K only in its 4 points of π' , since K is an arc of degree 4. So no point of $K \setminus \pi'$ lies on a line of π' , and since both sets have 27 points, $K \setminus \pi'$ is the set of free points of π' . By Proposition 3.5 applied to π' , every line meets $K \setminus \pi'$ in 0 or 3 points. A line that meets K meets it in 4 points, so it meets π' in 1 or 4 points. Now let $\pi'' \subseteq K$ be another subplane of order 3. Each line of π' meets K , hence meets π'' , in a point of π' . So $\pi' \cap \pi''$ meets every line of π' . It is also closed: the line through two common

points is a line of both, and its 4 points in K lie in both. A closed set of points of $\text{PG}(2, 3)$ that meets every line is a line or the whole plane, and $\pi' \neq \pi''$. $\square$

So Γ_L is a disjoint union of three copies of K_9 if and only if L lies in four subplanes of order 3 contained in K , as a line of $\text{PG}(3, 3)$ lies in four planes. In these terms, Theorem E excludes planes in which every line of π_0 lies in four subplanes of order 3 contained in K , and by Lemma 7.3 the subplanes of order 3 in K meet pairwise in lines, as the planes of $\text{PG}(3, 3)$ do.

The hypothesis of Theorem 7.1 is not a consequence of a single orthogonal resolution.

Proposition 7.4. $\dagger$ *There is a Kirkman triple system of order 27 with a resolution labelled by the points of $\text{PG}(2, 3)$ and an orthogonal resolution for one line L , in which Γ_L is connected.*

Proof. In a plane, a free point q determines the map from the points off L to N_L that sends P to the point where the tangent at P through q meets L . We describe the example by such maps. Identify the 9 points of $\text{PG}(2, 3)$ off L , and a set N of 9 further symbols, with $\mathbb{Z}_3^2$, where (x, y) has index $3x + y$. Let f_1, f_2, f_3 be the maps on $\{0, \dots, 8\}$ whose lists of values are

$$(0, 2, 6, 8, 5, 3, 1, 4, 7), \quad (0, 6, 7, 2, 8, 4, 3, 5, 1), \quad (5, 1, 3, 0, 7, 8, 4, 6, 2),$$

and for $t \in \mathbb{Z}_3^2$ let $f_{i,t}(P) = f_i(P - t) + t$. The points of the system are the 27 maps $f_{i,t}$. These maps are permutations, each pair (P, X) satisfies $f(P) = X$ for exactly three of them, and two of them agree in at most one argument. For $P \notin L$, the class T_P consists of the triples $\{f : f(P) = X\}$ with $X \in N$. The graph Γ that joins two maps agreeing nowhere is 8-regular and connected, and it has 81 triangles, whereas $3K_9$ has 252. A search finds a decomposition of Γ into four triangle factors, and these are the classes T_P with $P \in L$. Then two maps lie in exactly one triple, $\Gamma_L = \Gamma$, and the classes $C_X = \{\{f : f(P) = X\} : P \notin L\}$, $X \in N$, form an orthogonal resolution for L . $\square$

8. OPEN PROBLEMS

Theorems 4.5 and 7.1 leave the following question about triple systems. In it, $\mathcal{K}$ is a Kirkman triple system of order 27 with a resolution labelled by the points of $\text{PG}(2, 3)$, as in Section 7.

Problem 8.1 (two-line problem). *If two distinct lines L and L' admit compatible orthogonal resolutions, is Γ_L a disjoint union of three copies of K_9 ?*

In a plane of order 12 with a subplane π_0 of order 3, all 13 lines of π_0 have pairwise compatible orthogonal resolutions (Corollary 3.2). A positive answer would therefore give the hypothesis of Theorem E for all lines, and *no projective plane of order 12 would contain a subplane of order 3*.

Problem 8.2 (three-line problem). *Is there a $\mathcal{K}$ with pairwise compatible orthogonal resolutions for three concurrent lines?*

A negative answer would give the same conclusion. By Proposition 4.6, $\text{AG}(3, 3)$ with the natural labelling is not an example.

Problem 8.1 is also open without the word “compatible”. The following conjecture concerns all lines at once; by Theorem 4.5 it would also exclude subplanes of order 3.

Conjecture 8.3. *If every line of $\text{PG}(2, 3)$ admits an orthogonal resolution, then $\mathcal{K}$ is $\text{AG}(3, 3)$ with each T_P a direction class.*

The evidence is as follows.

- Twelve single-line structures as in Proposition 7.4, all invariant under $\mathbb{Z}_3^2$ and with Γ_L not a union of three K_9 , admit no orthogonal resolution for any second line. For some choices of the four triangle factors on L , as many as 6 of the other 12 lines admit a transversal class.
- For any $\mathcal{K}$ as above, the points of $\text{PG}(2, 3)$ and of $\mathcal{K}$, with the lines of $\text{PG}(2, 3)$ and the sets $\{P\} \cup t$ for $t \in T_P$ as blocks, form an $S(2, 4, 40)$. For a plane this is the design $\mathcal{D}$ of Proposition 3.5. A core needs 117 parallel classes of $\mathcal{D}$ that cover the disjoint pairs of blocks exactly once, and $\text{PG}(3, 3)$ has 8424 spreads. The designs built from four of these single-line structures have only 18 to 33 parallel classes in total.
- Suppose that $\mathcal{K}$ has an orthogonal resolution for every line, and that a permutation of order 13 of its points maps T_P to T_{gP} for a Singer cycle g and the resolution of L to that of gL , for all P and L . This permutation has a unique fixed point, and its component in each graph Γ_L is a K_9 $\dagger$.

A simple linear condition does not explain this scarcity. Let G be the space of functions h from the points of $\mathcal{K}$ to $\mathbb{F}_3$ such that $\sum_{x \in t} h(x)$ depends only on the class of the triple t , and let $c_h(P)$ be its value on T_P . A transversal class for L gives $\sum_{P \notin L} c_h(P) = \sum_x h(x)$. So if every line has a transversal class, the sum $\sum_{P \in L} c_h(P)$ does not depend on L . For $\text{AG}(3, 3)$, G is the 10-dimensional space of quadratic functions, and all these sums vanish. For four of the single-line structures above, $\dim G \in \{2, 3\}$ and the condition holds for all lines, although all but at most three of the other lines admit no transversal class. The space G contains the functions whose sum over each triple vanishes, and the dimension of this subspace is 27 minus the 3-rank of the incidence matrix of $\mathcal{K}$ (see [DHV78] for these ranks).

For all 13 lines at once, a symmetry reduction turns the question whether pairwise compatible orthogonal resolutions force Γ_L to be a disjoint union of three copies of K_9 into four explicit SAT instances. A collineation of π_0 and a relabelling of the free points move a violation to a fixed line and a fixed free point. The 432 resulting configurations form one orbit under the stabilizer of the line together with swaps of partners, and a split by the class of one more tangent leaves four orbits under the remaining symmetries. Refuting the four instances, in this form or relaxed to two lines, would also exclude subplanes of order 3. General-purpose SAT solvers have not decided them in runs of up to 48 hours, and none of the 182 instances tried at the next level of the split was decided in 12 hours. By Lemmas 7.2 and 7.3, in a plane with such a violation either K contains a second subplane of order 3, meeting π_0 in a line, or π_0 is the only subplane of order 3 in K . The order-3 cases left open by Corollary 6.9 have also resisted SAT solvers, in runs of 20 to 24 hours.

Class covers. The next conjecture concerns maximal arcs directly. By Lemma 5.2 and duality it would rule out maximal arcs of degree 3 and 4 in planes of order 12, and with them subplanes of order 3. It is stronger than the nonexistence of these arcs, since a class cover need not come from a plane.

Conjecture 8.4. *No Steiner triple system of order 27 has a class cover.*

Here duality means the following. The external lines of a maximal arc of degree 3 in a plane of order 12 form a maximal arc of degree 4 in the dual plane, and conversely, as in the proof of Proposition 3.5(ii).

The lines of a partial geometry whose collinearity graph is the disjointness graph of the triples of an STS(27) are sets of 9 pairwise disjoint triples, that is, parallel classes. So Conjecture 8.4 says that this graph, which is strongly regular with parameters $(117, 80, 52, 60)$, is never the collinearity graph of a $\text{pg}(8, 9, 6)$. The evidence for Conjecture 8.4 is computational.

- $\text{AG}(3, 3)$ has no class cover by Theorem 5.3, which needs a deep search.

- Every other system tested fails after at most 5 search nodes: the 38,284 non-affine systems with an automorphism of order 13 (Theorem 6.6); 300 systems obtained by tripling, described below; the 144 labelled cyclic STS(27) over $\mathbb{Z}_{27}$, which have 526 to 814 parallel classes; and 200 random systems.
- 120 systems obtained from AG(3,3) by 1, 2 or 3 random cycle switches have 9973, at most 7009, and at most 6601 parallel classes, respectively. All of them fail at the first node: some disjoint pair lies in no parallel class.
- In the degree-4 form, an $S(2, 4, 40)$ with 117 parallel classes covering the disjoint pairs of blocks, the 8 labelled 1-rotational $S(2, 4, 40)$ over $\mathbb{Z}_{39}$ fail trivially: each has only 13 parallel classes.

A weaker property is (DP): any two disjoint triples lie in a common parallel class. A system with a class cover satisfies (DP), but (DP) does not characterize AG(3,3). The *tripling* construction gives an STS(27) from a Latin square Q of order 9: take three copies $\text{AG}(2, 3) \times \{i\}$, $i \in \mathbb{Z}_3$, with their 36 lines, and the 81 triples $\{(x, 0), (y, 1), (Q(x, y), 2)\}$. With random isotopes of the Cayley tables of $\mathbb{Z}_9$ and $\mathbb{Z}_3^2$, 81 of 300 such systems satisfy (DP) and have only 3379 to 4061 parallel classes, and therefore they are not AG(3,3). Further examples are 2744 of the non-affine systems with an automorphism of order 13, and 54 of the 144 labelled cyclic systems. None of them has a class cover, and each search ends within 5 nodes.

A necessary local condition is the following. A system is *residually resolvable* at a triple t if the 80 triples disjoint from t split into 10 parallel classes of the 24 points off t . The design of a maximal arc of degree 3 is residually resolvable at every triple: the 10 points off the arc on the secant t give such classes.

Conjecture 8.5. *An STS(27) that is residually resolvable at every triple is AG(3,3).*

AG(3,3) is residually resolvable at all 117 triples. The 81 tripled systems with (DP) fail at 76 to 115 triples each, 7708 in all, and 60 random tripled systems fail at 78 or more triples each. A failing triple is not refuted immediately: a search that first lists the classes through t and then solves an exact cover needs up to 326 nodes. For a triple t of a Hall triple system, the graph on the points off t that joins two points whose triple meets t is a disjoint union of four copies of $K_{3,3}$. Residual resolvability does not coincide with this shape at single triples: two of the tripled systems with (DP) have 18 and 8 resolvable triples without the shape, and 2 and 7 triples with the shape that are not resolvable. Traced on systems with (DP), the short searches for a class cover fail as follows: some disjoint pair lies in exactly one parallel class, and choosing this class leaves another disjoint pair with no compatible class. The minimum number of parallel classes through a disjoint pair is 79 in AG(3,3). Among the 81 tripled systems with (DP) it is 1, 2 or 3, for 67, 13 and 1 systems, so a lower bound of 2 or 3 on this number does not force AG(3,3).

9. VERIFICATION

All searches are exhaustive. Exact-cover problems are solved by Algorithm X with dancing links and the minimum-column rule [Knu00]. In the SAT encodings, a constraint “exactly one of $x_1, \dots, x_k$ ” is the clause $x_1 \vee \dots \vee x_k$ together with an encoding of “at most one”, by pairwise clauses if $k \leq 6$ and by a sequential counter otherwise. Some encodings also contain constraints “exactly 2” or “exactly 3”, which are encoded by sequential counters. Symmetry enters only through orbit representatives under explicit groups, as described in the proofs. The SAT solvers are kissat [BFF⁺24b] and CaDiCaL 3.0.1 [BFF⁺24a]. For every formula used in a proof, CaDiCaL wrote a certificate in the LRAT format [CFHH⁺17], and `lrat-check` [WHH14] verified it.

The unsatisfiability of each of these formulas is also a theorem of Lean 4 [dMU21] (version 4.35.0-rc3), proved without Mathlib. The formula file is included in the Lean source as a string

and parsed inside Lean, and a theorem states that Lean’s DIMACS printer maps the parsed formula back to the file byte for byte. The LRAT proof, trimmed to the lemmas needed for the empty clause, is checked by the verified LRAT checker of Lean’s standard library, the checker behind the `bv_decide` tactic. Its soundness theorem then gives unsatisfiability. The run of the checker is evaluated by `native_decide`, so the trusted base consists of the Lean kernel and, for this evaluation, the Lean compiler and runtime. The SAT solver, the trimming program and the LRAT parser are not trusted. The theorems use no `sorry` and no axioms beyond the standard ones and those recorded by `native_decide`, and a fresh replay by the kernel (`leanchecker --fresh`) accepts all modules. No certificate is checked by the kernel alone, without `native_decide`; for the large ones such a check is out of reach.

The passage from a geometric statement to a formula is not formalized. It enters each geometric Lean theorem as a hypothesis, that is, a proposition defined in the source and passed as an argument, not an axiom. For example, `EncodingStepZ13` asserts that a plane of order 12 with a subplane of order 3 whose core has an automorphism of order 13 yields a model of the formula of Theorem 6.2. The theorem `theorem4b_of_z13` derives from this hypothesis that no such plane exists. These hypotheses comprise the soundness of the encodings, the uniqueness of $\text{PG}(2, 3)$, the conjugacy arguments that fix the action on π_0 , and the symmetry splits. Lean checks the computational parts of some of them: the lists of spreads and parallel classes, the formulas derived from these lists, and the orbit decompositions of the first symmetry step. Table 2 names the hypotheses.

Lemmas 6.7, 6.10, 7.2 and 7.3, Theorem 7.1, Corollary 3.2(i) and the point counts of Theorem 3.1 and Proposition 3.5(i)–(ii) are formalized in Lean 4 with Mathlib [The20] and checked by the kernel, with only the axioms `propext`, `Classical.choice` and `Quot.sound`. Lemma 6.7 is stated for an arbitrary projective plane of order 12 with a subplane of order 3. Theorem 7.1 is stated for an abstract labelled KTS(27) and assumes the theorem of Hall and Bruck as the hypothesis `HallBruck27`. A further theorem checks that its other hypotheses hold for $\text{AG}(3, 3)$ with the natural labelling. Corollary 3.2(i) is formalized as well: for a plane of order 12 with a subplane of order 3, the free points with the triples of the tangents form a KTS(27) whose resolution is labelled by the points of the subplane. Together with Theorem 7.1 this gives the plane form of the first sentence of Theorem E, again with `HallBruck27` as the only hypothesis.

The searches without a formula are checked in Lean as well. For Proposition 6.4 and Theorems 6.5 and 6.6, a search written in Lean enumerates the objects in question (the first parallel classes through $\{\infty, 0, 13\}$, and the Singer-invariant designs and triple systems), and a theorem proves that every such object is among its results. A second procedure, proved sound, rejects each object, and `native_decide` evaluates both. Thus `Z13Kts.z13kts` states that for every first parallel class and every labelling, an orthogonal resolution for a line forces exactly 648 transversal classes for this line; `Z13Arc.z13arc4` and `Z13Arc.z13arc3` state that no Singer-invariant design of degree 4 or 3 admits classes as in the proof of Theorem 6.5; and `Z13Design.z13design` states that every Singer-invariant STS(27) with a class cover is a translate of one of four systems, which the kernel shows to be copies of $\text{AG}(3, 3)$. These theorems need no hypothesis of the kind above. The reductions from a plane to these objects are not formalized, nor are the facts that the system with 648 transversal classes is $\text{AG}(3, 3)$ and that the four affine systems have no class cover; the latter is Theorem 5.3, which Lean proves for its own encoding.

The kernel checks Proposition 7.4. The evidence in Section 8, apart from the Singer-symmetric formula, rests on searches without certificates; independent programs reproduce it, as listed in Table 2.

The programs, formulas, certificates and Lean sources are available on Zenodo at <https://doi.org/10.5281/zenodo.23615385>. Paths in Table 2 are relative to its root. The Lean names use an internal numbering, and the README of the repository lists the correspondence;

for instance, `theorem4b_of_z13` concerns Theorem 6.2. Lean names in Table 2 are in the namespace `PP12Lrat`, except those marked G, which are in `PP12Geom`, as is `HallBruck27`.

Table 2: Verification of the computational claims and formal proofs.

Result	Computation and outcome	Certificate and formal proof	Independent checks	Location
Thm. 4.4 (Prince)	8424 spreads; 5265 skew pairs, each in 72 spreads; stabilizer of order 2304 with orbits 18, 6, 48; dancing links: no cover in the three cases (5018, 6401, 7207 nodes).	Three formulas with 382,239 variables and 1,121,446 clauses; LRAT verified. Lean: <code>PgCube0.unsat</code> , <code>PgCube1.unsat</code> , <code>PgCube2.unsat</code> , <code>corollary3c_of_pgCubes</code> , <code>theorem4_of_pgCubes</code> (hypotheses <code>EncodingStepPg</code> , <code>AgSpreadStep</code>). Lean also checks the list of spreads by its own enumeration, derives the three formulas from it, and checks the orbits.	Separate enumeration: same 8424 spreads, in agreement with the 2106 regular and 6318 Hall spreads given by their stabilizers; same orbits; no cover.	<code>sub3/pgarc/</code>
Prop. 4.6	648 transversal classes and 63,666 orthogonal resolutions per line; 2304 compatible pairs; no compatible triple (triangle, concurrent).	Two formulas with 100,359 variables and 753,570 clauses; LRAT verified. Lean: <code>Ag014.unsat</code> , <code>Ag023.unsat</code> , <code>theorem4_of_ag014</code> , <code>theorem4_of_ag023</code> (hypotheses <code>EncodingStep014</code> , <code>EncodingStep023</code>).	Enumeration in Python and in a separate C++ program (same counts); a second SAT encoding: unsatisfiable for 3 and for 13 lines; a third, smaller encoding with one variable for each line and class: unsatisfiable for both triples of lines, LRAT verified.	<code>sub3/ag/</code> , <code>sub3/agx/</code> , <code>sub3/ml/</code>

Result	Computation and outcome	Certificate and formal proof	Independent checks	Location
Thm. 5.3	17,641 parallel classes; 4680 disjoint pairs (468 parallel, 4212 skew), each in 79 to 142 classes; six cases, case 1 split as in Table 1; 23 leaves.	Formulas with 648,037 variables, 1,891,188 clauses and the unit clauses of the leaf; LRAT verified for all 23 leaves. Lean: <code>AgArcCube0.unsat, ..., AgArcCube5.unsat</code> (not case 1), the 18 leaves <code>AgArcCube1B0.unsat, ..., AgArcCube1E8.unsat, theorem4f_of_agArcCubes_split, theorem4f_arc_of_agArcCubes_split</code> (hypotheses <code>SplitStepAgArcCube1, EncodingStepAgArc, AgArcStep</code>). Lean also checks the list of classes by its own enumeration, derives the six case formulas from it, and checks the first-level orbits.	Separate enumeration: same classes and orbits; every split re-checked; LRAT proofs re-checked; dancing links: no cover in cases 0–5 (613,186 to 3,044,456 nodes); kissat: unsatisfiable in cases 0–5.	sub3/agarc/
Thm. 6.2	Singer-invariant formula with all 13 lines and compatibility: unsatisfiable.	Formula with 371,709 variables and 3,660,956 clauses; LRAT verified. Lean: <code>Z13Skel.unsat, theorem4b_of_z13</code> (hypothesis <code>EncodingStepZ13</code>).	kissat with four seeds; Theorem 6.5 by a different method.	sub3/z13/
Prop. 6.4	$657 \cdot 4$ labelled systems; transversal classes for a line: 0 in 2594, 1 in 28, 2 in 1, 12 in 1, 24 in 3, 648 in 1 ($AG(3, 3)$); no orthogonal resolution except for $AG(3, 3)$.	No certificate. Lean: <code>Z13Kts.z13kts</code> (search proved complete, evaluated by <code>native.decide</code>); Lean also checks these counts for the $657 \cdot 4$ systems.	A second program with a different search reproduces the counts and the conclusions.	sub3/jvt2/
Rem. 6.3	14,520 first parallel classes through $\{\infty, 0, 13\}$; 657 classes under the normalizer (629 affine, 28 antiaffine); 3 diagonal, 0 projective.	None (exact cover and orbit count, 20 s). Lean recounts the 14,520 classes.	Every entry of the list of [JvT81] (read from the printed pages) is matched to a class, with the four repetitions found by explicit maps.	sub3/jvt/

Result	Computation and outcome	Certificate and formal proof	Independent checks	Location
Thm. 6.5	3762 candidate base blocks, 3,291,704 labelled designs (12 copies of $\text{PG}(3, 3)$); 164 base triples, 38,288 labelled $\text{STS}(27)$ (4 copies of $\text{AG}(3, 3)$); no design admits the classes.	No certificate. Lean: Z13Arc.z13arc4 , Z13Arc.z13arc3 (searches proved complete, evaluated by native_decide); a copy of $\text{PG}(3, 3)$ meets every condition except (6.1) (kernel).	Separate program: same totals, no design admits the classes; with (6.1) dropped it finds all 16 copies. SAT encoding on samples: 301 designs under a weaker condition (1 satisfiable, $\text{PG}(3, 3)$), 21 with $d = \pm 1$ only and 21 with (6.1), all unsatisfiable; at degree 3, 301 designs with S_0, S_1, S_2 only (1 satisfiable, $\text{AG}(3, 3)$) and 21 with the full condition, all unsatisfiable.	sub3/z13arc/
Thm. 6.6	38,284 systems; each search ends after 1, 2, 3, 4, 5 nodes for 35,540, 2132, 464, 116, 32 systems.	No certificate. Lean: Z13Design.z13design (searches proved complete, evaluated by native_decide); the four exceptions are copies of $\text{AG}(3, 3)$ (kernel).	Separate exact cover agrees system by system (outcome, nodes, number of classes); a group-law test identifies the 4 copies of $\text{AG}(3, 3)$.	sub3/z13arc/
Lem. 6.10	Hand proof. The 30 elation cases excluded by part (iii): unsatisfiable.	G: lemma4h_a , lemma4h_b , lemma4h_c , lemma4h_d (kernel). The 30 cases: LRAT verified.	kissat refutes the 30 cases in 1 to 46 seconds.	sub3/ml/z3cubes.py , lean/P12Geom/
Lem. 6.7	Hand proof. SAT form (all 13 lines): unsatisfiable; with the axis line alone: satisfiable.	G: lemma4d (kernel). SAT form: LRAT verified; Lean: SymE1.unsat , lemma4d_of_symE1 (hypothesis EncodingStepSymE1).	kissat and CaDiCaL agree.	sub3/ml/z3s/ , lean/PP12Geom/
Prop. 6.8	The two cases $ s = 4$, and all 18 cases (11 with g the identity, 7 with a homology): unsatisfiable.	LRAT verified for all 18 cases. Lean: SymZ2_0.unsat , ..., SymZ2_17.unsat , prop4e_of_symZ2 (hypothesis EncodingStepSymZ2).	kissat and CaDiCaL agree; a Burnside count gives the 18 cases; the cases $ s = 1$ and $ s = 4$ are satisfiable for the triple system alone.	sub3/ml/z3s/
Prop. 7.4	The 27 maps have the stated properties; Γ is connected with 81 triangles and has 4 triangle factors.	Explicit example. Lean: Single.single (kernel).	Two separate verifiers rebuild the maps, check the properties and find the factors.	sub3/s1/
Sect. 8	Singer-invariant formula without compatibility, with a non-clique component at the fixed point: unsatisfiable.	LRAT verified.	kissat with four seeds.	sub3/z13/

Result	Computation and outcome	Certificate and formal proof	Independent checks	Location
Sect. 8	Counts for (DP), Conjectures 8.4 and 8.5, tripled, cyclic, near-affine and 1-rotational systems.	None.	Separate programs reproduce the tripled systems with (DP), their counts and failures, the residual resolvability counts, and the 144 cyclic systems with their failures.	sub3/z13arc/ sub3/cyc27/ sub3/z39/
Sect. 8	Symmetry reduction of the question for all 13 lines: the 432 configurations of a violation form one orbit, and the second split leaves four orbits.	None (orbit computation).	—	sub3/ml/tlsu b.py
Thm. 3.1, Prop. 3.5	Hand proof.	G: card_free, card_nonfree_on_line, tangent_card_free, nonfree_card_external, free_not_on_external, card_free_on_line, card_arc_on_line (kernel).	—	lean/PP12Geo m/
Thm. 7.1	Hand proof.	G: side_classes_collinear, isHall, iso_AG33 (hypothesis HallBruck27), natural_labelling, class_eq_lines, thm6_hyps_satisfiable (kernel).	—	lean/PP12Geo m/
Lem. 7.2, 7.3	Hand proofs.	G: lemma8a, lemma8a_converse, k9_component_iff, lemma8b, k9_component_i ff_second_subplane (kernel).	—	lean/PP12Geo m/

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