THE PPT^2 CONJECTURE THROUGH A QUTRIT: DIAGONAL-UNITARY TWIRLS WHEN THE COHERENCE MATRIX HAS RANK AT MOST TWO

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ABSTRACT. The PPT^2 conjecture asserts that composites of PPT (completely positive and completely copositive) maps are entanglement breaking. We test it on composites through a qutrit, $\Psi \circ \Phi$ with $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$, by twirling their Choi matrix W over the local diagonal unitaries $D \otimes \bar{D}$: the twirl is separable exactly when no $D \otimes \bar{D}$-invariant entanglement witness detects W. Our main result is that for every d the twirl is separable when the coherence matrix of W, its compression to the span of the vectors $|ii\rangle$, has rank at most 2. In some cases the proof uses the theorem that composites of PPT maps through a qubit are entanglement breaking (the qubit theorem); given it, invariant witnesses cannot exhibit a counterexample at coherence rank at most 2. A peeling operation, which splits an entanglement-breaking part off one factor, extends the result to higher coherence rank whenever repeated peeling and removal of zero rows reach rank at most 2 or outer dimension at most 3. At $d = 4$ and coherence rank 3 the four matrices of the Gram form satisfy one linear relation; when it has a zero coefficient, the twirl is separable, and the proof uses a rigidity theorem for rank-four PPT entangled states of two qutrits.

1. INTRODUCTION

The PPT^2 conjecture asserts that the composite of two PPT maps is entanglement breaking (EB). It holds for maps between matrix algebras of size at most 3 [1, 2] and for composites through a qubit, $\mathrm{M}_m \to \mathrm{M}_2 \to \mathrm{M}_n$, with arbitrary outer dimensions [3]; the next case is a composite through a qutrit with an outer dimension larger than 3. If such a composite were not EB, some entanglement witness would detect its Choi matrix, and we ask whether a witness invariant under the local diagonal unitaries $D \otimes \bar{D}$ can do so. Such a witness sees only the twirl of the Choi matrix over these unitaries, an operator of a simple form, so the question becomes whether this twirl is separable.

Here a linear map $\Lambda\colon \mathrm{M}_m \to \mathrm{M}_n$ is *PPT* if both Λ and $\vartheta \circ \Lambda$ are completely positive, where ϑ is the transpose, and EB if its Choi matrix

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$J(\Lambda) = \sum_{a,b} E_{ab} \otimes \Lambda(E_{ab})$ is separable. We call the result of [3] the qubit theorem; results whose proofs use it, directly or through another result, are labelled 'given [3]'.

The conjecture also holds for all Choi-type maps [7]; for background and further special classes see [1, 7]. For maps covariant under diagonal orthogonal matrices (DOC maps), it holds when one factor is (conjugate) covariant under diagonal unitaries, as Choi-type maps are [8], and for two random DOC channels it holds almost surely as the dimension grows [8]. In general it is open for DOC maps from dimension 4 on, and a sufficient condition in every dimension is given in [9]. In dimension 3 the composite remains EB when PPT is weakened to 1-undistillability of the Choi matrices [4]. In every dimension, every PPT map $\mathrm{M}_n \to \mathrm{M}_n$ has finite entanglement-breaking index, that is, some power of it is EB [5].

This paper concerns composites through a qutrit,

$$\Phi\colon \mathrm{M}_d \to \mathrm{M}_3, \qquad \Psi\colon \mathrm{M}_3 \to \mathrm{M}_d, \qquad W := J(\Psi \circ \Phi) \in \mathrm{M}_d \otimes \mathrm{M}_d,$$

with Φ and Ψ arbitrary PPT maps and d arbitrary. Instead of W itself we study its twirl over local diagonal unitaries,

$$\mathrm{tw}(W) := \int_{[0,2\pi)^d} (D_\theta \otimes \bar{D}_\theta)\, W\, (D_\theta \otimes \bar{D}_\theta)^* \, \frac{d\theta}{(2\pi)^d},$$

where $D_\theta = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d})$. Since tw is the trace-preserving conditional expectation onto the $D \otimes \bar{D}$-invariant operators, $\mathrm{Tr}(Z\ \mathrm{tw}(W)) = \mathrm{Tr}(Z\,W)$ for every invariant Z. Hence $\mathrm{tw}(W)$ is separable if and only if no $D \otimes \bar{D}$-invariant entanglement witness detects W (a witness for $\mathrm{tw}(W)$ may be replaced by its twirl). Separability of W implies separability of $\mathrm{tw}(W)$.

The twirl has the form $\mathrm{tw}(W) = X(A, B)$, where for a matrix A with nonnegative entries and $B \succeq 0$ with $A_{ii} = B_{ii}$ we write

$$X(A,B) := \sum_{i \neq j} A_{ij}\, |ij\rangle\langle ij| + \sum_{i,k} B_{ik}\, |ii\rangle\langle kk|, \tag{1}$$

with $A_{ij} = \langle ij|W|ij\rangle$ and $B_{ik} = \langle ii|W|kk\rangle$. We call B the *coherence matrix* and $\operatorname{rank} B$ the *coherence rank* of the pair (Φ, Ψ). The positive semidefinite $D \otimes \bar{D}$-invariant operators are exactly the operators $X(A, B)$; [23] calls them conjugate local diagonal unitary invariant.

Theorem 1.1 (given [3])**.** *Let $d \geq 1$ and let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ be PPT. If the coherence rank is at most 2, then $\mathrm{tw}(J(\Psi \circ \Phi))$ is separable.*

Given the qubit theorem, Theorem 1.1 thus says that $D \otimes \bar{D}$-invariant entanglement witnesses cannot exhibit a counterexample to PPT^2 through a qutrit at coherence rank at most 2.

The proof works with Gram data. Writing $\Phi(E_{ab}) = G_a G_b^*$ and $\Psi^*(E_{ab}) = H_a H_b^*$ with $G_a \in \mathbb{C}^{3\times P}$ and $H_a \in \mathbb{C}^{3\times Q}$, the entries of W are Hilbert–Schmidt inner products of the matrices $G_a^* H_b$ (Lemma 2.1). In particular $B_{ik} = \langle T_i, T_k\rangle$ with $T_a := G_a^* H_a$, so the coherence rank is the dimension of

the *coherent space* $\mathcal{T} = \operatorname{span}\{T_a\}$. Coherence rank 1 is treated by a separate argument (Theorem 4.1). At coherence rank 2 the space $\mathcal{T}$ is a pencil, a two-dimensional space of $P \times Q$ matrices, and its rank-one members determine five types (Lemma 2.4): type (IV) has exactly one rank-one direction, and type (V) has none. The proof of Theorem 1.1 is a case analysis over these types, outlined at the start of Section 4. Its two general tools are the range lemma (Lemma 2.2) and the Schur-complement lemma behind peeling (Lemma 3.1); the other inputs are results from the literature: PPT^2 in dimension 3 [1, 2], the qubit theorem [3], the low-rank criteria of [10, 11], the rank-four two-qutrit result of [12], the Peres–Horodecki criterion in dimensions $2 \otimes 2$ and $2 \otimes 3$ [17, 18, 19], and the Fuglede–Putnam theorem [20].

For higher coherence rank we use a peeling operation (Definition 3.2). A peel at an index c splits one factor into an EB map and a PPT remainder whose coherence vanishes at c, and a zero-row compression (Lemma 2.3) removes the indices a with $B_{aa} = 0$.

Theorem 1.2 (given [3])**.** *Let $d \geq 1$ and let (Φ, Ψ) be a PPT pair as above. Suppose that a finite sequence of peels and zero-row compressions, each applied to the current pair, ends at a pair of coherence rank at most 2 or of outer dimension at most 3. Then $\operatorname{tw}(J(\Psi \circ \Phi))$ is separable.*

Corollary 1.3. (a) *(given [3]) For every d: if the coherence rank is 3 and some index c with $B_{cc} > 0$ peels, then $\operatorname{tw}(J(\Psi \circ \Phi))$ is separable.*
(b) *For $d = 4$: if some index peels, then $\operatorname{tw}(J(\Psi \circ \Phi))$ is separable, whatever the coherence rank.*
(c) *An index c with $\operatorname{rank} G_c = 1$ or $\operatorname{rank} H_c = 1$ (Section 2) always peels.*

Theorem 1.2 uses the qubit theorem only through Theorem 1.1, when the sequence ends at coherence rank at most 2; a sequence that ends at outer dimension at most 3 needs only PPT^2 in dimension 3 [1, 2]. So Corollary 1.3(b) and (c) do not use the qubit theorem.

Further results and organization. Section 2 sets up the Gram form, the twirl, zero-row compression, the range lemma and the types. Section 3 proves the peeling lemma, Theorem 1.2 and Corollary 1.3, and Section 4 proves Theorem 1.1; the case analyses of Section 4 and the proofs of several lemmas of Section 5 are given in the supplementary material, referred to as [Supp]. Section 5 shows that $\operatorname{tw}(W) + \frac{d-3}{2} D_B$ is separable for every $d \geq 3$, where $D_B = \sum_i B_{ii} |ii\rangle\langle ii|$ (Theorem 5.1), and then treats $d = 4$ at coherence rank 3, where the T_a satisfy a single linear relation $\sum_a \lambda_a T_a = 0$, up to scale. If some λ_a vanishes, the twirl is separable, and $\Psi \circ \Phi$ is EB when exactly one λ_a vanishes (Theorem 5.13); the proof works with the compressions of $J(\Phi)$ and $J(\Psi^*)$ to sets of outer indices, which we call corners, and uses the classification of PPT states of rank four on $\mathbb{C}^3 \otimes \mathbb{C}^3$ [13] together with a rigidity theorem for the entangled ones (Theorem 5.30). Relations without zero coefficients remain open (Section 5.11). Section 6 gives a cycle

criterion for explicit product-state certificates (Lemma 6.3). Section 7 reduces the question to extreme pairs, that is, pairs (Φ, Ψ) that span extreme rays of the cones of PPT maps (Lemma 7.1), shows that certificates built from Kraus operators fail at some extreme pairs for every choice of Kraus operators (Theorem 7.10), and proves Proposition 7.21 below. Section 8 gives separability criteria at $d \le 4$ for ferromagnetic coherence matrices and for self-composites. Section 9 lists the computations, Section 10 summarizes the companion paper [34], and Section 11 asks open questions. Results stated without a citation are new, with two exceptions: results whose source is given in the text that follows them, and elementary or standard facts that we prove for completeness. The latter include Lemmas 2.1–2.3, 7.9 and 7.13, Proposition 7.12(a), the identity of Lemma 5.39 and the Schur-complement steps in several proofs.

The constant $\frac{1}{5}$. Write $M := \Phi^*(I)^\top \otimes \Psi(I)$, the Choi matrix of $\Psi \circ \tau \circ \Phi$ with $\tau(X) = \mathrm{Tr}(X)\, I_3$. PPT^2 through a qutrit is the statement that $W + \lambda M$ is separable for $\lambda = 0$ and all pairs. By averaging over compressions of the middle qutrit to a qubit, Proposition 7.21 shows, given the qubit theorem, that $\lambda = \frac{1}{5}$ suffices for every d. How far this constant can be lowered is the subject of the companion paper [34]; Section 10 states its main results.

Computations. Random semidefinite searches find configurations that no peel reaches, called residual points in Section 9, at $d = 4$ and $d = 5$; at every tested point the twirl has a separable fit. At 19 of the 42 points at $d = 4$ both factors are entangled by the realignment criterion; at the others, and at $d = 5$, a factor may be EB. These observations are numerical (Section 9). The proofs of Theorems 1.1 and 1.2 use no computation. Where later results use exact computations, Section 9 lists them; except for the parts listed there, they are also proved in Lean 4 with Mathlib on the deposited data, partly by evaluating compiled code.

2. Preliminaries

2.1. **Gram form.** Let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ be completely positive with Kraus operators $K_1, \dots, K_P \in \mathbb{C}^{3\times d}$, and put

$$G_a := [K_1 e_a \ \cdots \ K_P e_a] \in \mathbb{C}^{3\times P} \qquad (a = 1, \dots, d),$$

so that $\Phi(E_{ab}) = G_a G_b^*$. Similarly, if $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ has Kraus operators $L_1, \dots, L_Q \in \mathbb{C}^{d\times 3}$, put $H_a := [L_1^* e_a \ \cdots \ L_Q^* e_a] \in \mathbb{C}^{3\times Q}$, so that the adjoint satisfies $\Psi^*(E_{ab}) = H_a H_b^*$. We call (G_a) and (H_a) *Gram data* of Φ and Ψ. Any family (G_a) with $\Phi(E_{ab}) = G_a G_b^*$ arises in this way. We write

$$X_a := \mathrm{col}\, G_a, \quad Y_a := \mathrm{col}\, H_a, \quad E_a := \mathrm{col}\, G_a^*, \quad F_a := \mathrm{col}\, H_a^*, \quad T_a := G_a^* H_a,$$

so that $E_a \subseteq \mathbb{C}^P$, $F_a \subseteq \mathbb{C}^Q$ and $T_a \in \mathbb{C}^{P\times Q}$. Inner products of matrices are $\langle X, Y\rangle = \mathrm{Tr}(X^*Y)$.

Lemma 2.1. *$\langle ab|W|cd\rangle = \langle G_a^* H_b,\, G_c^* H_d\rangle$ for all a,b,c,d. In particular $A_{ij} = \|G_i^* H_j\|^2$ and $B_{ik} = \langle T_i, T_k\rangle$, so B is the Gram matrix of $T_1,\dots,T_d$ and $\operatorname{rank} B = \dim \mathcal{T}$ with $\mathcal{T} := \operatorname{span}\{T_a\}$.*

Proof. We have

$$\langle ab|W|cd\rangle = \langle b|\Psi(\Phi(E_{ac}))|d\rangle = \operatorname{Tr}(\Psi^*(E_{db})\Phi(E_{ac})) = \operatorname{Tr}(H_d H_b^* G_a G_c^*),$$

and $\operatorname{Tr}(H_d H_b^* G_a G_c^*) = \operatorname{Tr}((G_a^* H_b)^* G_c^* H_d)$. □

The formula for $\operatorname{tw}(W)$ in the introduction follows: the phase of $\langle ab|W|cd\rangle$ under $D_\theta \otimes \bar{D}_\theta$ is $e^{i(\theta_a-\theta_b-\theta_c+\theta_d)}$, whose average vanishes unless $a=b, c=d$ or $a=c, b=d$.

2.2. **PPT in Gram form.** The Choi matrix of Φ is $J(\Phi) = [G_a G_b^*]_{a,b}$, and its partial transpose on the input factor is $J(\Phi)^\Gamma = [G_b G_a^*]_{a,b}$. Thus Φ is PPT if and only if $[G_b G_a^*]_{a,b} \succeq 0$. Since Ψ is PPT if and only if Ψ^* is, Ψ is PPT if and only if $K' := [H_y H_x^*]_{x,y} \succeq 0$.

Lemma 2.2 (range lemma)**.** *If Ψ is PPT, then $H_x(F_c) \subseteq Y_c$ for all indices x, c. If Φ is PPT, then $G_x(E_c) \subseteq X_c$ for all x, c.*

Proof. In the positive semidefinite block matrix K', the range of the block $K'_{cx} = H_x H_c^*$ lies in the range of the diagonal block $K'_{cc} = H_c H_c^*$, which is Y_c. Since H_c^* maps $\mathbb{C}^3$ onto F_c, the range of $H_x H_c^*$ is $H_x(F_c)$. The second statement is the same argument for $[G_b G_a^*]_{a,b}$. □

2.3. **Zero rows and compression.** For $I \subseteq \{1,\dots,d\}$ let $\Pi_I \colon \mathbb{C}^d \to \mathbb{C}^I$ be the coordinate projection. The *compression* of (Φ,Ψ) to I is the pair

$$\Phi_I := \Phi \circ \operatorname{Ad}_{\Pi_I^*} \colon \mathrm{M}_I \to \mathrm{M}_3, \qquad \Psi_I := \operatorname{Ad}_{\Pi_I} \circ \Psi \colon \mathrm{M}_3 \to \mathrm{M}_I.$$

Both maps are PPT, because $\operatorname{Ad}_{\Pi_I}$ and $\operatorname{Ad}_{\Pi_I^*}$ are completely positive and commute with the transpose (Π_I is real). Their Gram data are $(G_a)_{a\in I}$ and $(H_a)_{a\in I}$, so the matrices T_a, $a \in I$, are unchanged, and the coherence matrix of the compressed pair is the principal submatrix B_I.

Lemma 2.3 (zero-row compression)**.** *Let $I = \{a : B_{aa} > 0\}$ and $W_I := J(\Psi_I \circ \Phi_I)$. Then $\operatorname{rank} B_I = \operatorname{rank} B$ and*

$$\operatorname{tw}(W) = \iota_I\big(\operatorname{tw}_I(W_I)\big) + \sum_{i\neq j,\ \{i,j\}\not\subseteq I} A_{ij}\, |ij\rangle\langle ij|,$$

where tw_I is the twirl on $\mathbb{C}^I \otimes \mathbb{C}^I$ and $\iota_I = \operatorname{Ad}(\Pi_I^* \otimes \Pi_I^*)$. In particular, $\operatorname{tw}(W)$ is separable if $\operatorname{tw}_I(W_I)$ is.

Proof. B is positive semidefinite, so $B_{aa} = 0$ forces the a-th row and column of B to vanish; this gives $\operatorname{rank} B_I = \operatorname{rank} B$ and leaves only the diagonal terms $A_{ij}|ij\rangle\langle ij|$ with $\{i,j\} \not\subseteq I$. The entries of $\operatorname{tw}(W)$ with all indices in I are those of $\operatorname{tw}_I(W_I)$ by Lemma 2.1, because the Gram data of the compressed pair are the restricted Gram data. The map ι_I is a local isometric embedding, and the remaining terms are nonnegative multiples of product projectors. □

Throughout, a reducible $B = B_I \oplus B_J$ is treated in the same way: then $\mathrm{tw}(W) = \iota_I \,\mathrm{tw}_I(W_I) + \iota_J \,\mathrm{tw}_J(W_J)$ plus nonnegative diagonal terms, and each block is the twirl of a compressed PPT pair.

2.4. **Two-dimensional coherent spaces.**

Lemma 2.4 (types)**.** *Let $\mathcal{T} \subseteq \mathbb{C}^{P\times Q}$ be a two-dimensional subspace. Exactly one of the following holds.*

(I) *All members of $\mathcal{T}$ have a common one-dimensional column space.*
(II) *All members of $\mathcal{T}$ have a common one-dimensional row space.*
(III) *$\mathcal{T}$ contains exactly two rank-one directions, $\mathcal{T} = \mathrm{span}\{u_1 w_1^*, u_2 w_2^*\}$ with $u_1 \nparallel u_2$ and $w_1 \nparallel w_2$.*
(IV) *$\mathcal{T}$ contains exactly one rank-one direction.*
(V) *$\mathcal{T}$ contains no rank-one member.*

Proof. On the projective line $\mathbb{P}(\mathcal{T})$ the 2×2 minors restrict to binary quadratic forms. If all of them vanish, every member has rank at most one, and a two-dimensional space of matrices of rank at most one has a common column space or a common row space: if $xy^* + x'y'^*$ has rank at most one with $xy^*, x'y'^* \neq 0$, then $x \parallel x'$ or $y \parallel y'$. Otherwise a nonzero binary quadric has at most two zeros, which gives (III), (IV) or (V). In case (III), $u_1 \parallel u_2$ or $w_1 \parallel w_2$ would give a common column or row space, hence (I) or (II). □

In type (III) we write I_1 and I_2 for the sets of indices a with $T_a \parallel u_1 w_1^*$ and $T_a \parallel u_2 w_2^*$ (the *axes*), and I_0 for the remaining indices, whose T_a have rank two. The axes are *orthogonal* if $\langle u_1 w_1^*, u_2 w_2^*\rangle = (u_1^* u_2)(w_2^* w_1) = 0$.

3. The peeling lemma

Let Ψ be PPT and fix an index c. By Lemma 2.2, $H_y(F_c) \subseteq Y_c$ for every y, and H_c maps F_c bijectively onto Y_c. Hence

$$\tilde{N}_y := (H_c|_{F_c})^{-1} H_y|_{F_c} \in \mathrm{End}(F_c) \qquad (y = 1, \dots, d)$$

is well defined, $\tilde{N}_c = I$, and $H_y P_{F_c} = H_c \tilde{N}_y$ when $\tilde{N}_y$ is extended by 0 on $F_c^{\perp}$.

Lemma 3.1 (Schur complement at a pivot)**.** *Let Ψ be PPT and let c be an index.*

(a) *The following block matrix is positive semidefinite:*

$$\mathcal{S} := \left[H_c(\tilde{N}_y \tilde{N}_x^* - \tilde{N}_x^* \tilde{N}_y) H_c^* + H_y(I - P_{F_c}) H_x^*\right]_{x,y}.$$

(b) *If $\mathrm{col}\, H_x^* \subseteq F_c$, then $\tilde{N}_x$ is normal and $\tilde{N}_y \tilde{N}_x^* = \tilde{N}_x^* \tilde{N}_y$ for every y.*
(c) *If the operators $\tilde{N}_y$ $(y = 1, \dots, d)$ are normal and pairwise commuting, then $\Psi = \Psi_{F_c} + \Psi_{(c)}$, where $\Psi_{F_c}^*(E_{xy}) = H_x P_{F_c} H_y^*$ and $\Psi_{(c)}^*(E_{xy}) = H_x(I - P_{F_c}) H_y^*$; the map Ψ_{F_c} is EB and the map $\Psi_{(c)}$ is PPT.*

The same statements hold on the Φ side, with E_c, $\tilde{N}_y^{\Phi} := (G_c|_{E_c})^{-1} G_y|_{E_c}$, and the PSD matrix $[G_b G_a^*]_{a,b}$.

Proof. (a) In $K' = [H_y H_x^*]_{x,y} \succeq 0$ the blocks in row and column c are $K'_{xc} = H_c H_x^* = H_c \tilde{N}_x^* H_c^*$ and $K'_{cy} = H_y H_c^* = H_c \tilde{N}_y H_c^*$, because $H_c = H_c P_{F_c}$. Also $K'_{cc} = H_c H_c^*$, and $H_c^*(H_c H_c^*)^+ H_c = P_{F_c}$. The generalized Schur complement of a PSD block matrix with respect to a diagonal block is PSD. Here it equals

$$K'_{xy} - K'_{xc}(K'_{cc})^+ K'_{cy} = H_y H_x^* - H_c \tilde{N}_x^* \tilde{N}_y H_c^*,$$

and $H_y H_x^* = H_c \tilde{N}_y \tilde{N}_x^* H_c^* + H_y(I - P_{F_c}) H_x^*$ gives $\mathcal{S}$.

(b) If $H_x(I - P_{F_c}) = 0$, the diagonal block $\mathcal{S}_{xx} = H_c[\tilde{N}_x, \tilde{N}_x^*]H_c^*$ is PSD. Since H_c^* maps $\mathbb{C}^3$ onto F_c, the traceless operator $[\tilde{N}_x, \tilde{N}_x^*]$ is PSD, hence zero. A PSD block matrix with a zero diagonal block has zero row and column there, and $H_x(I - P_{F_c}) = 0$ removes the second term of $\mathcal{S}_{yx}$. So $H_c(\tilde{N}_x \tilde{N}_y^* - \tilde{N}_y^* \tilde{N}_x) H_c^* = 0$, and H_c is injective on F_c.

(c) A commuting family of normal operators is simultaneously diagonal in an orthonormal basis (f_i) of F_c, say $\tilde{N}_y = \sum_i \nu_i(y) f_i f_i^*$. Then $\tilde{N}_y \tilde{N}_x^* = \tilde{N}_x^* \tilde{N}_y$, so $\mathcal{S} = [H_y(I - P_{F_c})H_x^*]_{x,y} = J(\Psi_{(c)}^*)^\Gamma \succeq 0$; hence $\Psi_{(c)}^*$ is PPT (it is completely positive, being in Gram form), and so is $\Psi_{(c)}$. Moreover

$$J(\Psi_{F_c}^*) = [H_c \tilde{N}_x \tilde{N}_y^* H_c^*]_{x,y} = \sum_i |\nu_i\rangle\langle\nu_i| \otimes |H_c f_i\rangle\langle H_c f_i|,$$

with $|\nu_i\rangle := \sum_x \nu_i(x)|x\rangle$, is separable, so $\Psi_{F_c}^*$ and Ψ_{F_c} are EB. The decomposition $\Psi^*(E_{xy}) = H_x P_{F_c} H_y^* + H_x(I - P_{F_c}) H_y^*$ is exact. □

By the Fuglede–Putnam theorem [20], the family $(\tilde{N}_y)$ is commuting and normal if and only if $\tilde{N}_y \tilde{N}_x^* = \tilde{N}_x^* \tilde{N}_y$ for all x, y. Equivalently, F_c has an orthonormal basis (f_i) with $H_y f_i \in \mathbb{C}\, H_c f_i$ for all y and i. This condition depends only on the map Ψ, not on the choice of Gram data.

Definition 3.2. An index c *peels on the* Ψ *side* if the operators $\tilde{N}_y$ are normal and pairwise commuting, and *on the* Φ *side* if the operators $\tilde{N}_y^\Phi$ are. The *peel* at c on the Ψ side replaces (Φ, Ψ) by the *remainder* $(\Phi, \Psi_{(c)})$; on the Φ side it replaces (Φ, Ψ) by $(\Phi_{(c)}, \Psi)$.

Lemma 3.3 (peel)**.** *Let c peel on the Ψ side. Then:*

(a) *the remainder $(\Phi, \Psi_{(c)})$ is a PPT pair $\mathrm{M}_d \to \mathrm{M}_3 \to \mathrm{M}_d$ with Gram data G_a and $H_a(I - P_{F_c})$;*
(b) *its coherent matrices are $T'_a = T_a(I - P_{F_c})$, and $T'_c = 0$; in particular the remainder has a zero row at c;*
(c) *if $T_c \neq 0$, the coherence rank drops by at least one;*
(d) $\mathrm{tw}(J(\Psi \circ \Phi)) = \mathrm{tw}(J(\Psi_{F_c} \circ \Phi)) + \mathrm{tw}(J(\Psi_{(c)} \circ \Phi))$*, and the first term is separable.*

The same holds on the Φ side, with $G_a(I - P_{E_c})$ and $T'_a = (I - P_{E_c})T_a$.

Proof. (a) is Lemma 3.1(c). (b) $H_c(I - P_{F_c}) = 0$ because $F_c^\perp = \ker H_c$. (c) The linear map $\tau \mapsto \tau(I - P_{F_c})$ sends $\mathcal{T}$ onto $\mathrm{span}\{T'_a\}$ and kills T_c. (d) The composite $\Psi_{F_c} \circ \Phi$ is EB, so its Choi matrix is separable; tw is linear and

maps separable operators to separable operators, being an average of local unitary conjugations. □

Remark 3.4. If $T_c = 0$, a peel at c need not lower the coherence rank; it can, since $\tau \mapsto \tau(I - P_{F_c})$ may kill other members of $\mathcal{T}$. A single peel can also lower the coherence rank by more than one.

Proof of Theorem 1.2. By Lemma 3.3(d), each peel writes the current twirl as a separable operator plus the twirl of the remainder, which is again a PPT pair with middle dimension 3 and the same outer dimension. By Lemma 2.3, each zero-row compression writes the current twirl as an embedded twirl of a compressed PPT pair plus nonnegative diagonal terms. Composing these identities expresses $\mathrm{tw}(J(\Psi \circ \Phi))$ as a sum of separable operators and of $\iota(\mathrm{tw}(W_{\mathrm{fin}}))$, where W_{fin} is the Choi matrix of the final pair and ι a local isometric embedding. If the final pair has coherence rank at most 2, $\mathrm{tw}(W_{\mathrm{fin}})$ is separable by Theorem 1.1, applied at the final outer dimension. If its outer dimension d_f is at most 3, let $\Pi\colon \mathbb{C}^{d_f} \to \mathbb{C}^3$ be the coordinate isometry; the maps $\Phi_f \circ \mathrm{Ad}_{\Pi^*}$ and $\mathrm{Ad}_{\Pi} \circ \Psi_f$ are PPT maps $\mathrm{M}_3 \to \mathrm{M}_3$, their composite is EB by [1, 2], and $\Psi_f \circ \Phi_f = \mathrm{Ad}_{\Pi^*} \circ (\mathrm{Ad}_{\Pi} \circ \Psi_f \circ \Phi_f \circ \mathrm{Ad}_{\Pi^*}) \circ \mathrm{Ad}_{\Pi}$ is EB. (For $d_f \le 2$ one may instead use that a PPT operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is separable [17, 18].) If a compression leaves no index, the twirl is diagonal. □

Proof of Corollary 1.3. (a) A peel at c with $B_{cc} > 0$ lowers the coherence rank from 3 to at most 2 (Lemma 3.3(c)); apply Theorem 1.2. (b) After a peel at c, the index c has a zero row (Lemma 3.3(b)); the compression leaves at most 3 indices, and Theorem 1.2 applies. (c) If $\operatorname{rank} H_c = 1$, then $\dim F_c = 1$ and the $\tilde{N}_y$ are scalars; similarly on the Φ side. □

Remark 3.5. In Theorem 1.2, compressions may be interleaved with peels. A compression can make new peels available, because the family $(\tilde{N}_y)_{y \in I}$ at c in the compressed pair is a subfamily of the original one while F_c is unchanged. Exact examples at $d = 5$ and $d = 6$ in which every chain without intermediate compressions stops at four or more nonzero rows, while a chain with compressions ends at outer dimension at most 3, are verified in exact arithmetic (†19). For $d = 4$ a different argument [Supp, S8] gives the stronger conclusion that $\Psi \circ \Phi$ is EB whenever some $\operatorname{rank} G_c = 1$, unless $J(\Psi^*)$ has rank at least 6 and no product vector in its kernel, and likewise with the roles of Φ and Ψ exchanged. Corollary 1.3(b), (c) give separability of the twirl also in these excluded cases.

4. Coherence rank at most two

Throughout this section $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ are PPT. By Lemma 2.3 we may assume $B_{aa} > 0$ for all a: the compressed pair has the same coherent matrices on its indices, and every statement below holds for every outer dimension. We put $U := \sum_a \operatorname{col} T_a$, $V := \sum_a \operatorname{col} T_a^*$, $u := \dim U$, $v := \dim V$, and write Φ_U for the completely positive map with Gram data $G_a P_U$.

The proof of Theorem 1.1 is a case analysis over the coherence rank and, at coherence rank 2, over the five types of Lemma 2.4. At coherence rank 1 and for types (I)–(III) the proof uses only [10, 11, 17, 18, 19]; for types (IV) and (V) it also uses [1, 2, 12], and for some pencils of type (V) it uses the qubit theorem (Table 1). Its steps are as follows; the details are in [Supp, S1–S7].

- Coherence rank 0 is trivial. At coherence rank 1 every T_a is a multiple of one matrix T. If $\operatorname{rank} T \geq 2$, the composite is EB; if $\operatorname{rank} T = 1$, the twirl satisfies the cycle criterion of Lemma CYC (Theorem 4.1; [Supp, S1]).
- Types (I) and (II): all T_a have one column line, or one row line. Then the coherent pencil has a frame of members of rank at most one, so the twirl is a sum of twirls of product vectors and a nonnegative diagonal operator (Proposition 4.2; [Supp, S2]).
- Type (III): $\mathcal{T} = \operatorname{span}\{u_1 w_1^*, u_2 w_2^*\}$, and Proposition 4.5 has four cases. If no index lies on an axis, all T_a have the same column and row spaces and $\Psi \circ \Phi$ is EB [Supp, S3]. If exactly one axis is occupied, a Schur complement at a rank-2 index, as in Lemma 3.1, and a product-term certificate show that the twirl is separable [Supp, S4]. If both axes are occupied and orthogonal, the indices on the axes give pencil members of rank at most one that form a frame [Supp, S5]. If the axes are not orthogonal, the compressions of Lemmas 4.3 and 4.4, together with peeling, settle the cases; the proof is outlined after Proposition 4.5 [Supp, S6].
- Types (IV) and (V): Proposition 4.6 runs over the Kronecker types of the pencil; Table 1 lists them with the argument for each. For four Kronecker types of type (V) and their transposes the argument uses the qubit theorem [3], applied after one factor splits into a PPT part with two-dimensional output support and parts with product Choi matrices [Supp, S7].

4.1. **Coherence rank zero and one.** If $B = 0$, then $\operatorname{tw}(W)$ is diagonal with nonnegative entries, hence separable.

Theorem 4.1. *If the coherence rank is* 1*, then* $\operatorname{tw}(W)$ *is separable, for every* d*.*

The proof, given in [Supp, S1], uses the range lemma, the low-rank criterion of [10] and the Peres–Horodecki criterion in dimension $2 \otimes 2$.

4.2. **Types (I) and (II).**

Proposition 4.2. *If the column spaces of all* T_a *lie in one line, or the row spaces of all* T_a *lie in one line, then* $\operatorname{tw}(W)$ *is separable. This holds at every coherence rank and for all outer dimensions.*

We prove it in [Supp, S2]. In particular it covers types (I) and (II) of Lemma 2.4.

4.3. **Two tools for type (III).** The following compression lemma and its normal-compression criterion are used for type (III).

Lemma 4.3 (coherent compression)**.** *Let* $\Pi_\Phi := \sum_a G_a(U) \subseteq \mathbb{C}^3$.

(a) *If* Φ_U *is EB, then* $\mathrm{tw}(W)$ *is separable.*
(b) *If* $\Pi \subseteq \mathbb{C}^3$ *is a plane with* $\Pi_\Phi \subseteq \Pi$ *and* $\Psi \circ \mathrm{Ad}_{P_\Pi}$ *is EB, then* $\mathrm{tw}(W)$ *is separable.*

Proof. Split $\Phi = \Phi_U + \Phi_{U^\perp}$ along $\mathbb{C}^P = U \oplus U^\perp$; both parts are completely positive. The pair $(\Phi_{U^\perp}, \Psi)$ has coherent matrices $P_{U^\perp} T_a = 0$, so the twirl of $J(\Psi \circ \Phi_{U^\perp})$ is diagonal and nonnegative. It remains to show that $\Psi \circ \Phi_U$ is EB. In case (a) this is immediate. In case (b), the range and co-range of $\Phi_U(E_{ab}) = G_a P_U G_b^*$ lie in Π, so $\Phi_U = \mathrm{Ad}_{P_\Pi} \circ \Phi_U$ and $\Psi \circ \Phi_U = (\Psi \circ \mathrm{Ad}_{P_\Pi}) \circ \Phi_U$ is EB. □

Lemma 4.4 (normal compression)**.** *Let* $v_{x,0}, v_{x,1} \in \mathbb{C}^Q$ $(x = 1, \dots, d)$*, and let* J *be the operator on* $\mathbb{C}^d \otimes \mathbb{C}^2$ *with entries* $\langle x, i|J|y, k\rangle = \langle v_{x,i}, v_{y,k}\rangle$*. Assume that*

(a) $V_0 := \mathrm{span}\{v_{x,0}\}$ *has dimension* 2*, and*
(b) *some* x_0 *has* $v_{x_0,0} \neq 0$ *and* $v_{x_0,1} \in \mathbb{C} v_{x_0,0}$.

If J *is PPT, then* J *is separable.*

Proof. Let $\Gamma_i \colon \mathbb{C}^d \to \mathbb{C}^Q$, $|x\rangle \mapsto v_{x,i}$. The blocks of J with respect to the qubit are $\Gamma_i^* \Gamma_k$, and

$$J^\Gamma = \begin{pmatrix} \Gamma_0^*\Gamma_0 & \Gamma_1^*\Gamma_0 \\ \Gamma_0^*\Gamma_1 & \Gamma_1^*\Gamma_1 \end{pmatrix} \succeq 0.$$

The range of $\Gamma_1^*\Gamma_0$ lies in the range of $\Gamma_0^*\Gamma_0$, which is $\Gamma_0^*(V_0)$. Since Γ_0^* is injective on V_0, there is a unique operator N on V_0 with

$$P_{V_0} v_{x,1} = N v_{x,0} \qquad (x = 1, \dots, d). \tag{2}$$

With $Z := \Gamma_1^* P_{V_0^\perp} \Gamma_1 \succeq 0$, the Schur complement of the upper left block is

$$\Gamma_0^*(N^*N - NN^*)\Gamma_0 + Z \succeq 0. \tag{3}$$

At x_0, assumption (b) gives $Z_{x_0 x_0} = 0$ and $Nf = \lambda f$ for $f := v_{x_0,0}/\|v_{x_0,0}\|$. The (x_0, x_0) entry of (3) gives $\|Nf\| \geq \|N^* f\|$, while $\|N^* f\| \geq |\langle f, N^* f\rangle| = |\lambda| = \|Nf\|$. Equality in the Cauchy–Schwarz inequality gives $N^* f = \bar{\lambda} f$, so the line $f^\perp \cap V_0$ is N-invariant ($\dim V_0 = 2$), and N is diagonal in an orthonormal basis (f_1, f_2) of V_0, with eigenvalues λ_1, λ_2. Finally

$$J = \Gamma^* P_{V_0} \Gamma + \Gamma^* P_{V_0^\perp} \Gamma = \sum_{k=1,2} |\varphi_k\rangle\langle\varphi_k| \otimes |\beta_k\rangle\langle\beta_k| + Z \otimes |1\rangle\langle 1|,$$

with $\varphi_k = \sum_x \langle v_{x,0}, f_k\rangle |x\rangle$ and $\beta_k = |0\rangle + \bar{\lambda}_k |1\rangle$, by (2) and $N^* f_k = \bar{\lambda}_k f_k$. Every term is a product. □

4.4. **Type (III).**

Proposition 4.5. *Let the coherent space be of type (III), with axes I_1, I_2. Then* $\mathrm{tw}(W)$ *is separable, for every d, in each of the following cases, which exhaust type (III):*

(a) *no axis is occupied ($I_1 = I_2 = \varnothing$); then $\Psi \circ \Phi$ is EB when all $B_{aa} > 0$;*
(b) *exactly one axis is occupied;*
(c) *both axes are occupied and the axes are orthogonal;*
(d) *both axes are occupied and the axes are not orthogonal.*

Cases (a)–(c) are proved in [Supp, S3–S5] and case (d) in [Supp, S6]. We outline the proof of case (d). Let $\Pi_j := \mathrm{span}_a\{G_a u_j\}$ for $j = 1, 2$, so that $\Pi_\Phi = \Pi_1 + \Pi_2$, and similarly on the Ψ side.

- If $I_0 \neq \varnothing$, then at an off-axis index c either $G_c(U^\perp) = 0$ or $H_c(V^\perp) = 0$ (a per-index fact for $\operatorname{rank} T_c = 2 = u$), and an inclusion lemma for PPT pairs [Supp, S6] makes Π_Φ or its Ψ-side analogue a plane.
- If Π_Φ is a plane Π, the compressed operator $J(\mathrm{Ad}_{P_\Pi} \circ \Psi^*)$ lives on $\mathbb{C}^d \otimes \mathbb{C}^2$ and is PPT. A case analysis that uses the non-orthogonality of the axes shows that either its rank is at most its $\mathbb{C}^d$-local rank, so that [11, 10] apply, or the hypotheses of Lemma 4.4 hold. In both cases $\Psi \circ \mathrm{Ad}_{P_\Pi}$ is EB, and Lemma 4.3(b) applies. The mirror statement holds on the Ψ side.
- If $I_0 = \varnothing$ and $\dim \Pi_\Phi = 3$, either some on-axis index c has $\operatorname{rank} G_c = 1$ or $\operatorname{rank} H_c = 1$, so that it peels (Corollary 1.3(c)) and the remainder has coherence rank at most one (Lemma 3.3(c), Theorem 4.1), or each Π_j is a plane and the Kraus operators of Φ split by input class into two pieces and an EB part. The compressed piece on each class has a Choi matrix of rank at most one, because $H_a^*(\Pi_j) = \operatorname{col} T_a^* = \mathbb{C} w_j$ for $a \in I_j$. Being PPT, it is a product.
- If $\dim \Pi_\Phi \leq 1$, then Φ_U maps every input into multiples of one rank-one projector, so it is EB, and Lemma 4.3(a) applies.

Non-orthogonality enters only through the fact that, for $a \in I_j$, the vector $G_a u_{j'}$ $(j' \neq j)$ is not in $G_a(u_j^\perp)$.

4.5. **Types (IV) and (V).** For types (IV) and (V) we use the Kronecker canonical form of the matrix pencil spanned by two members of $\mathcal{T}$. Here L_k is the $k \times (k+1)$ singular block, $L_k^\top$ its transpose, and $J_k(\lambda)$ a Jordan block; S_0 is the set of indices a with T_a on the rank-one direction (type (IV)).

Proposition 4.6 (given [3])**.** *If the coherent space is of type (IV) or (V), then* $\mathrm{tw}(W)$ *is separable, for every d.*

For the proof we refer to [Supp, S7]. It runs over the Kronecker types listed in Table 1; the list is complete [Supp, S7]. The table names the argument in each case. Transposed types follow by passing to the pair (Ψ^*, Φ^*).

TABLE 1. Types (IV) and (V): Kronecker type of $\mathcal{T}$ and the argument that settles it. "EB" means that the stated map is entanglement breaking.

Kronecker type	Argument
L_2, $L_2^\top$	common column (row) space; a compression of $J(\Psi^*)$ is PPT of rank ≤ 3, so the coherent sub-composite is EB
$L_1 \oplus L_1$ and transpose	as above if $\dim \Pi_\Phi = 2$; if $\dim \Pi_\Phi = 3$, Ψ is EB (in one case by [12]), or the indices form three classes on which H_a and T_a are proportional, and repeated splitting as in Corollary 5.15 and [1, 2] give $\Psi \circ \Phi$ EB
$L_1 \oplus L_1^\top$; $J_3(\lambda)$, $J_2(\lambda) \oplus J_1(\mu)$, $J_1(\lambda) \oplus J_1(\mu) \oplus J_1(\nu)$ with distinct eigenvalues ($u = v = 3$, all $\operatorname{rank} T_a \geq 2$)	a commuting normal family, as in Lemma 3.1, if some T_a has rank 3; a split and an explicit two-term certificate if all have rank 2
J_2	$S_0 = \varnothing$: all T_a have rank 2 with common column and row spaces, and $\Psi \circ \Phi$ is EB; $S_0 \neq \varnothing$: a Schur complement at a rank-2 index, as in Lemma 3.1, and a product-term certificate
$L_1 \oplus J_1$ and transpose	$S_0 = \varnothing$: common column space; $S_0 \neq \varnothing$: peeling (Lemma 3.1) and case analysis
$J_2(\lambda) \oplus J_1(\lambda)$	$S_0 \neq \varnothing$: peeling and case analysis; $S_0 = \varnothing$: Φ has Choi rank ≤ 3, hence EB
$\operatorname{diag}(s, s, t)$	peeling and case analysis
$L_1 \oplus J_1(\lambda) \oplus J_1(\lambda)$ and transpose	$S_0 \neq \varnothing$: case analysis; $S_0 = \varnothing$: Φ EB
L_3, $L_2 \oplus L_1$, $L_1^{\oplus 3}$ and transposes	Φ (or Ψ) has Choi rank ≤ 3, hence EB
$L_2 \oplus J_1$, $L_1 \oplus J_2$, $L_1 \oplus L_1 \oplus J_1$, $L_1 \oplus J_1(\lambda) \oplus J_1(\mu)$ ($\lambda \neq \mu$) and transposes	case analysis, using [3]
$L_2 \oplus L_1^\top$ and transpose	$\Psi \circ \Phi$ EB (Kraus ranks ≤ 4, [10] or [1, 2])
$L_1 \oplus L_1 \oplus L_1^\top$ and transpose	Ψ EB
$L_1 \oplus L_1^\top \oplus J_1$	Ψ or Φ EB
generic rank ≥ 4 (T_a on finitely many lines)	splits as in Lemma 3.1 at an index of each line; explicit two-term certificates; reduction to at most three indices and [1, 2]

4.6. **Proof of Theorem 1.1.** By Lemma 2.3 we may assume $B_{aa} > 0$ for all a; the compressed pair has the same coherent space and coherence rank, and the statements of this section hold for every outer dimension. Coherence rank 0 is trivial and coherence rank 1 is Theorem 4.1. For coherence rank

2, Lemma 2.4 gives five types: (I) and (II) are Proposition 4.2, (III) is Proposition 4.5, and (IV) and (V) are Proposition 4.6. □

5. Near-separability, corners and relations

Throughout this section $\rho := J(\Phi) = [G_a G_b^*]_{a,b}$ and $\sigma := J(\Psi^*) = [H_a H_b^*]_{a,b}$, both on $\mathbb{C}^d \otimes \mathbb{C}^3$, and Γ is the partial transpose on the outer factor $\mathbb{C}^d$. For $I \subseteq \{1,\dots,d\}$ the *corners* are $\rho_I := [G_a G_b^*]_{a,b\in I}$ and $\sigma_I := [H_a H_b^*]_{a,b\in I}$; they are PPT, being compressions of PPT operators by $\Pi_I \otimes I$. We write $D_B := \sum_i B_{ii}|ii\rangle\langle ii|$ and $E\colon \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$, $E|a\rangle = |aa\rangle$, so that $B = E^* W E$.

Theorem 5.1 (near-separability) holds for every $d \geq 3$. Most of the rest of the section concerns $d = 4$ and coherence rank 3, where the T_a satisfy a single relation $\sum_a \lambda_a T_a = 0$. The first subsections give tools for corners: after a diagonal filter, a relation makes the corners on its support orthogonal (Lemma 5.3), and a corner that is a sum of product states with independent outer factors splits off (Lemma 5.6 and its variants in Section 5.4). Theorem 5.13 treats relations with a zero coefficient; its proof divides into cases, which Sections 5.6–5.10 treat according to the number of zero coefficients and the ranks of the corners. A first reading can take these cases on trust and continue with Section 5.11, which shows, under a condition (G) on the pair, why relations of full support are beyond the reach of these methods. The last subsection collects three further tools: a sandwich criterion, coordinate decompositions and a middle-block form of the twirl.

5.1. **Near-separability.**

Theorem 5.1. *Let $d \geq 3$ and let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ be PPT. Then* $\mathrm{tw}(W) + \frac{d-3}{2} D_B$ *is separable.*

Proof. For a 3-subset I, the compression (Φ_I, Ψ_I) of Section 2 is a PPT pair $\mathrm{M}_I \to \mathrm{M}_3 \to \mathrm{M}_I$. Its composite is EB by [1, 2], so $\mathrm{tw}_I(W_I)$ and $\iota_I \,\mathrm{tw}_I(W_I)$ are separable. By Lemma 2.1,

$$\iota_I \,\mathrm{tw}_I(W_I) = \sum_{i\neq j\in I} A_{ij}|ij\rangle\langle ij| + \sum_{i,k\in I} B_{ik}|ii\rangle\langle kk|.$$

A pair $\{i,j\}$ lies in $d-2$ of the 3-subsets and an index i lies in $\binom{d-1}{2}$ of them. Hence

$$\begin{aligned}\sum_{|I|=3} \iota_I \,\mathrm{tw}_I(W_I) &= (d-2)\ \mathrm{tw}(W) + \Big(\tbinom{d-1}{2} - (d-2)\Big) D_B \\ &= (d-2)\Big(\mathrm{tw}(W) + \tfrac{d-3}{2} D_B\Big).\end{aligned}$$

□

For $d = 3$ this is [1, 2]. For $d = 4$ it says that $\mathrm{tw}(W) + \frac{1}{2} D_B$ is separable for every PPT pair. The overshoot $\frac{1}{2} D_B$ cannot be removed inside this construction when B is frustrated, that is, when not all B_{ij} with $i \neq j$ are real and nonpositive:

Proposition 5.2. *Let $d = 4$, and let the coherence rank be 3 with a relation $\sum_a \lambda_a T_a = 0$ in which all $\lambda_a \neq 0$, normalized by a diagonal filter to $B\mathbf{1} = 0$. Suppose that*

$$\operatorname{tw}(W) = \tfrac{1}{2} \sum_{|S|=3} \iota_S \, X\big(A_{SS}, B_{SS} - \operatorname{diag} s_S\big) + N,$$

where N is a nonnegative diagonal operator, $s_S \geq 0$, $B_{SS} - \operatorname{diag} s_S \succeq 0$, and $X(A', B')$ is as in (1). Then $B_{ij} \leq 0$ for all $i \neq j$.

Proof. Comparing the coefficients of $|ii\rangle\langle ii|$ gives $\sum_{S \ni i} s_S(i) \geq B_{ii}$. Positivity of $B_{SS} - \operatorname{diag} s_S$ at $\mathbf{1}_S$ gives $\sum_{i \in S} s_S(i) \leq \mathbf{1}_S^* B_{SS} \mathbf{1}_S$, and for $S = \{1,2,3,4\} \setminus \{l\}$ the right side equals B_{ll} because $B\mathbf{1} = 0$. Summing over S gives $\operatorname{Tr} B \leq \sum_S \sum_{i \in S} s_S(i) \leq \operatorname{Tr} B$. So $\mathbf{1}_S$ lies in the kernel of $B_{SS} - \operatorname{diag} s_S$, that is, $s_S(i) = \sum_{k \in S} B_{ik} = -B_{il} \geq 0$ for $i \in S = \{1,2,3,4\} \setminus \{l\}$. □

The weights $\frac{1}{2}$ are no restriction when all $B_{ij} \neq 0$: comparing the coefficients of $|ii\rangle\langle jj|$, each pair $\{i, j\}$ lies in exactly two 3-subsets, and the six resulting equations force weight $\frac{1}{2}$ on every corner. If $B_{ij} \leq 0$ for all $i \neq j$, separability follows directly from the 2×2 blocks of the partial transpose (Remark 6.2).

5.2. **Relations and orthogonal corners.**

Lemma 5.3. *Let $I \subseteq \{1, \ldots, d\}$ and $\alpha_a \neq 0$ for $a \in I$. Let ρ'_I be the corner of the Choi matrix of the filtered map with Gram data $\bar{\alpha}_a G_a$. Then $\sum_{a \in I} \alpha_a T_a = 0$ if and only if $\rho'_I \sigma_I = 0$. In that case also $(\rho'_I)^\Gamma \sigma_I^\Gamma = 0$, and*

$$\operatorname{rank} \rho_I + \operatorname{rank} \sigma_I \leq 3|I|, \qquad \operatorname{rank} \rho_I^\Gamma + \operatorname{rank} \sigma_I^\Gamma \leq 3|I|.$$

Proof. The vanishing of $\sum_{a \in I} \alpha_a T_a$ does not depend on the Gram data, because a change of Gram data replaces T_a by $U^* T_a V$ with unitary U, V. Write $\rho'_I = \sum_p k_p k_p^*$ and $\sigma_I = \sum_q s_q s_q^*$, and take for $a \in I$ the Gram data with entries $k_p[a, i]$ for $\bar{\alpha}_a G_a$ and $s_q[a, i]$ for H_a. Then

$$\Big(\sum_{a \in I} \alpha_a T_a\Big)_{pq} = \sum_{a \in I,\, i} \overline{k_p[a,i]}\, s_q[a,i] = \langle k_p, s_q \rangle.$$

All of these vanish if and only if $\operatorname{Tr}(\rho'_I \sigma_I) = 0$, if and only if $\rho'_I \sigma_I = 0$, since both operators are positive semidefinite. Both corners are PPT, and $\operatorname{Tr}((\rho'_I)^\Gamma \sigma_I^\Gamma) = \operatorname{Tr}(\rho'_I \sigma_I) = 0$, so $(\rho'_I)^\Gamma \sigma_I^\Gamma = 0$. Filters do not change ranks, and the corners act on a space of dimension $3|I|$. □

For $I = \{1, \ldots, d\}$ and all $\alpha_a \neq 0$ this is the identity $\rho(D_\lambda \otimes I)\sigma = 0$ of Section 9; a relation with zero coefficients is handled by restricting to $I = \{a : \lambda_a \neq 0\}$. At $d = 4$ and coherence rank 3 the relation is unique up to scale, and its support need not be full, even when B is irreducible:

Example 5.4. Let $t_1 = (1,0,0)$, $t_2 = (0,1,0)$, $t_3 = (1,1,0)$, $t_4 = (1,0,1)$ and $G_a = H_a = \operatorname{diag}(t_a) \in \mathbb{C}^{3\times 3}$. Then $J(\Phi) = J(\Psi^*) = \sum_{i=1}^{3} |\tau_i\rangle\langle\tau_i| \otimes E_{ii}$ with $\tau_i = (t_a(i))_a$, so both maps are EB and in particular PPT. Here $T_a = \operatorname{diag}(t_a)$, the coherence rank is 3, the only relation is $T_1 + T_2 - T_3 = 0$, all $B_{aa} > 0$, and the support graph of B (edges $13, 23, 14, 34$) is connected.

So at $d = 4$ the pairs of coherence rank 3 (with all $B_{aa} > 0$) fall into two classes: those with a full-support relation, which after a diagonal filter satisfy $\rho\sigma = 0$, and those whose relation vanishes at one or two indices. The second class is treated in Theorem 5.13.

Corollary 5.5. *Let $d = 4$, let the coherence rank be 3 with a full-support relation, and assume that $\Phi(I)$ and $\Psi^*(I)$ have rank 3 and that the outer marginals $\operatorname{Tr}_2 \rho$ and $\operatorname{Tr}_2 \sigma$ have rank 4. Then either one factor is EB, or each of $\operatorname{rank}\rho$, $\operatorname{rank}\rho^\Gamma$, $\operatorname{rank}\sigma$, $\operatorname{rank}\sigma^\Gamma$ lies in $\{5,6,7\}$, with $\operatorname{rank}\rho + \operatorname{rank}\sigma \le 12$ and $\operatorname{rank}\rho^\Gamma + \operatorname{rank}\sigma^\Gamma \le 12$.*

Proof. If $\operatorname{rank}\rho \le 4$, then ρ is a PPT state of rank at most its larger local rank, hence separable [10], and Φ is EB. The operator ρ^Γ is PPT with partial transpose ρ, and its outer marginal has rank 4; if $\operatorname{rank}\rho^\Gamma \le 4$, then ρ^Γ and hence ρ are separable. The same holds for σ. The upper bounds follow from Lemma 5.3 with $I = \{1,2,3,4\}$. □

5.3. **Low-rank corners.** Consider the following hypothesis on a corner σ_I, $I \neq \varnothing$:

(H) $\sigma_I = \sum_{k=1}^{r} |x_k \otimes y_k\rangle\langle x_k \otimes y_k|$ with nonzero $x_k \in \mathbb{C}^I$, $y_k \in \mathbb{C}^3$, and either (H-i) $x_1, \dots, x_r$ are linearly independent, or (H-ii) $y_1, \dots, y_r$ are linearly independent and $x_1, \dots, x_r$ are pairwise non-parallel.

By [10], (H-i) holds whenever $\operatorname{rank}\sigma_I$ equals the rank of the outer marginal $[\operatorname{Tr} H_a H_b^*]_{a,b\in I} = \sum_k \|y_k\|^2 x_k x_k^*$.

Lemma 5.6 (corner lemma)**.** *Let $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ be PPT with Gram data H_a, let $I \neq \varnothing$, $J := \{1,\dots,d\} \setminus I$, and let σ_I satisfy (H). Then*

$$J(\Psi^*) = \sum_{k=1}^{r} |\hat{x}_k \otimes y_k\rangle\langle \hat{x}_k \otimes y_k| + R_J, \qquad \hat{x}_k = x_k \oplus \eta_k \ (\eta_k \in \mathbb{C}^J),$$

where $R_J \succeq 0$ is supported on $\mathbb{C}^J \otimes \mathbb{C}^3$ and $R_J^\Gamma \succeq 0$. The same holds for Φ, with G_a in place of H_a.

Proof. (1) The vectors $x_k \otimes y_k$ are linearly independent, and so are the $\bar{x}_k \otimes y_k$: under (H-i) compare coefficients in the basis (x_l), resp. $(\bar{x}_l)$; under (H-ii), $\sum_k c_k x_k \otimes y_k$ corresponds to $X \operatorname{diag}(c) Y^\top$ with $Y^\top$ right invertible, which vanishes only if $c_k x_k = 0$ for all k.

(2) Let $\mathbf{H}_I$ be the block column $[H_a]_{a\in I}$, so that $\mathbf{H}_I \mathbf{H}_I^* = \sigma_I = VV^*$ with $V = [x_1 \otimes y_1, \dots, x_r \otimes y_r]$ of full column rank. Put $\Omega^* := (V^*V)^{-1} V^* \mathbf{H}_I$. Then $V\Omega^* = \mathbf{H}_I$, because $\operatorname{col}\mathbf{H}_I = \operatorname{col} V$, and $\Omega^*\Omega = I_r$. So the columns ω_k

of Ω are orthonormal, $H_a = \sum_k x_k(a)\, y_k \omega_k^*$ for $a \in I$, and with $P := \Omega\Omega^*$ we have $H_a(I - P) = 0$ for $a \in I$.

(3) Let $K := J(\Psi^*)^\Gamma = [H_b H_a^*]_{a,b} \succeq 0$. For $a, b \in I$, $K_{II} = \sum_k |\bar{x}_k \otimes y_k\rangle\langle \bar{x}_k \otimes y_k|$. For $a \in I$, $c \in J$, the block $K_{ac} = H_c H_a^* = \sum_k \overline{x_k(a)}\, h_k^c y_k^*$ with $h_k^c := H_c \omega_k$, so the block column $(K_{ac})_{a\in I}$ maps $v \in \mathbb{C}^3$ to $\sum_l (\bar{x}_l \otimes h_l^c)\langle y_l, v\rangle$, which must lie in $\operatorname{range} K_{II} = \operatorname{span}\{\bar{x}_l \otimes y_l\}$. Under (H-i), take $v = y_k$; independence of the $\bar{x}_l$ gives $\|y_k\|^2 h_k^c \in \mathbb{C} y_k$. Under (H-ii), take v with $\langle y_l, v\rangle = \delta_{lk}$; then $\bar{x}_k (h_k^c)^\top = \bar{X} \operatorname{diag}(\beta) Y^\top$ for some β, and a right inverse of $Y^\top$ shows $\beta_l = 0$ for $l \neq k$ and $h_k^c = \beta_k y_k$. So $H_c \omega_k = \eta_k(c)\, y_k$ for $c \in J$, and $H_a P = \sum_k \hat{x}_k(a)\, y_k \omega_k^*$ for every a.

(4) Split $J(\Psi^*) = [H_a P H_b^*] + [H_a(I - P)H_b^*]$. By orthonormality of the ω_k the first term is $\sum_k |\hat{x}_k \otimes y_k\rangle\langle \hat{x}_k \otimes y_k|$. The second, R_J, is a Gram matrix whose blocks vanish unless $a, b \in J$, by (2).

(5) Let $N := \sum_k |\bar{\hat{x}}_k \otimes y_k\rangle\langle \bar{\hat{x}}_k \otimes y_k|$ be the partial transpose of the first term, so that $K = N + R_J^\Gamma$. By (3), $N_{II} = K_{II}$ and $N_{IJ} = K_{IJ}$. Since $\operatorname{rank} N = r = \operatorname{rank} N_{II}$ by (1), $N_{JJ} = N_{JI} N_{II}^+ N_{IJ}$. Hence the Schur complement $K_{JJ} - K_{JI} K_{II}^+ K_{IJ}$, which is positive semidefinite, equals $(R_J^\Gamma)_{JJ}$, and R_J^Γ is supported on $\mathbb{C}^J \otimes \mathbb{C}^3$. □

Corollary 5.7. *Let $d = 4$ and let some corner σ_I of $J(\Psi^*)$, or ρ_I of $J(\Phi)$, satisfy (H).*

(a) *If $|I| \geq 2$, that factor is EB, and W is separable.*
(b) *If $|I| = 1$, the slice has rank one, and* $\operatorname{tw}(W)$ *is separable by Corollary 1.3(b), (c).*

The same applies to the PPT pair $(\Phi \circ \vartheta, \vartheta \circ \Psi)$, that is, to corners of $J(\Phi)^\Gamma$ and $J(\Psi^*)^\Gamma$.

Proof. (a) If $|J| = 1$, $R_J = E_{cc} \otimes M$ is a product. If $|J| = 2$, R_J is a PPT operator on $\mathbb{C}^2 \otimes \mathbb{C}^3$, hence separable [17, 18]. So $J(\Psi^*)$ is separable. (b) A rank-one slice peels (Corollary 1.3(c)). □

5.4. **Corners under a rigidity condition.** Lemma 5.6 uses (H) only through two consequences. Let $\sigma_I = \sum_{k=1}^r |x_k \otimes y_k\rangle\langle x_k \otimes y_k|$ with nonzero $x_k \in \mathbb{C}^I$ and $y_k \in \mathbb{C}^3$, and consider:

(I$^+$) the vectors $x_k \otimes y_k$ ($k \leq r$) are linearly independent, and so are the vectors $\bar{x}_k \otimes y_k$;
(R) whenever $h_1, \ldots, h_r \in \mathbb{C}^3$ satisfy $\sum_k \langle y_k, v\rangle\, \bar{x}_k \otimes h_k \in \operatorname{span}\{\bar{x}_m \otimes y_m : m \leq r\}$ for every $v \in \mathbb{C}^3$, every h_k lies in $\mathbb{C} y_k$.

Lemma 5.8. *The conclusion of Lemma 5.6 holds for every corner σ_I of this form that satisfies (I$^+$) and (R), and so does Corollary 5.7(a). The same holds for ρ_I. Hypothesis (H) implies (I$^+$) and (R).*

Proof. In the proof of Lemma 5.6, step (1) is (I$^+$), and step (2) uses only the full column rank of V. In step (3) the block column $(K_{ac})_{a\in I}$ maps v to $\sum_l \langle y_l, v\rangle\, \bar{x}_l \otimes h_l^c$, which lies in $\operatorname{range} K_{II} = \operatorname{span}\{\bar{x}_l \otimes y_l\}$ for every v; by

(R), $h_l^c \in \mathbb{C}y_l$. Steps (4) and (5) are unchanged; step (5) uses $\operatorname{rank} N_{II} = r$, which is (I$^+$). Under (H), step (1) of that proof gives (I$^+$) and step (3) gives (R). □

Lemma 5.9. *If σ_I satisfies (I$^+$) and (R), then $5r \le 9|I|$.*

Proof. The condition in (R) is linear in v. For a basis v_1, v_2, v_3 of $\mathbb{C}^3$ it says that $(h_k)_k$ lies in the kernel of the map $L\colon (\mathbb{C}^3)^r \to (\mathbb{C}^{3|I|}/S)^3$, $(h_k)_k \mapsto \big([\sum_k \langle y_k, v_j\rangle\, \bar{x}_k \otimes h_k]\big)_{j=1,2,3}$, where $S := \operatorname{span}\{\bar{x}_k \otimes y_k\}$ has dimension r by (I$^+$). The families $h_k = c_k y_k$ lie in $\ker L$, so (R) says that $\dim\ker L = r$, while $\dim\ker L \ge 3r - 3(3|I| - r)$. Hence $6r - 9|I| \le r$. □

Lemma 5.10. *Let $|I| = 3$ and $r = 4$, and suppose that any three of $x_1, \dots, x_4$ and any three of $y_1, \dots, y_4$ are linearly independent. Then (I$^+$) and (R) hold.*

Proof. (I$^+$): let $\sum_k c_k \bar{x}_k \otimes y_k = 0$ and write $\bar{x}_4 = \sum_{j\le 3} d_j \bar{x}_j$; every $d_j \neq 0$. The $\bar{x}_j$-component gives $c_j y_j + c_4 d_j y_4 = 0$, so $c_j = c_4 = 0$. The same argument applies to the $x_k \otimes y_k$.

(R): fix $k \neq l$ and let m, n be the other two indices. Choose $v \neq 0$ with $\langle y_m, v\rangle = \langle y_n, v\rangle = 0$; then $\alpha := \langle y_k, v\rangle$ and $\gamma := \langle y_l, v\rangle$ are nonzero, and $\alpha\, \bar{x}_k \otimes h_k + \gamma\, \bar{x}_l \otimes h_l = \sum_i \beta_i\, \bar{x}_i \otimes y_i$ for some β. Write $\bar{x}_l = c_k \bar{x}_k + c_m \bar{x}_m + c_n \bar{x}_n$ with $c_k, c_m, c_n \neq 0$. The $\bar{x}_m$- and $\bar{x}_n$-components give $\gamma c_m h_l = \beta_m y_m + \beta_l c_m y_l$ and $\gamma c_n h_l = \beta_n y_n + \beta_l c_n y_l$, so $(\beta_m/c_m) y_m = (\beta_n/c_n) y_n$, $\beta_m = \beta_n = 0$ and $h_l \in \mathbb{C}y_l$. The $\bar{x}_k$-component then gives $\alpha h_k \in \mathbb{C}y_k$. Every index lies in some pair $\{k, l\}$. □

Remark 5.11. For $|I| = 3$, condition (R) holds for generic data with $r \le 5$ [Supp, S11], and by Lemma 5.9 it fails for every configuration with $r \ge 6$ that satisfies (I$^+$). The analogue of Lemma 5.10 for $r = 5$ is false: for

$$x = (e_1, e_2, e_3, e_1 + e_2 + e_3, e_1 + 2e_2 + 3e_3),$$
$$y = (e_1, e_2, e_3, e_1 + e_2 + e_3, e_1 + 3e_2 - 2e_3),$$

and for the Tiles configuration [14] $x = (e_1, e_1 - e_2, e_3, e_2 - e_3, e_1 + e_2 + e_3)$, $y = (e_1 - e_2, e_3, e_2 - e_3, e_1, e_1 + e_2 + e_3)$, any three x_k and any three y_k are linearly independent and (I$^+$) holds, but the solution space of (R) has dimension $6 > 5$ (exact computation, †11). Moreover (R) fails at $r = 5$ whenever the y_k are real up to phases and x_1, x_2, x_3 are independent [Supp, S11].

The next lemma replaces rank-one product terms by blocks.

Lemma 5.12 (block corner lemma)**.** *Let $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ be PPT with Gram data H_a, let j be an index and $I := \{1, \dots, d\} \setminus \{j\}$. Suppose $\sigma_I = \sum_{g=0}^{m} \sigma_g$, where each $\sigma_g \neq 0$ is positive semidefinite and supported on $\Pi_g \otimes P_g$ with $\Pi_g \subseteq \mathbb{C}^I$ and $P_g \subseteq \mathbb{C}^3$, the sum $\Pi_0 + \Pi_1 + \cdots + \Pi_m$ is direct, $\dim P_g = 1$ for $g \ge 1$, and either $\dim \Pi_0 = 1$, or $\dim \Pi_0 = 2$ and $\dim P_0 \le 2$. Then $J(\Psi^*)$ is separable, so Ψ is EB. The same holds for Φ and ρ_I.*

We follow the proof of Lemma 5.6 and give the details in [Supp, S11]. The new point is that the block σ_0 need not be normal in the sense of (R): the Schur complement in step (5) can be indefinite on P_0, and the positivity of the whole partial transpose compensates it; this compensation is exactly the PPT property of an operator supported on $(\Pi_0 + \mathbb{C}e_j) \otimes P_0$, a system of dimensions $2 \otimes 3$ or $3 \otimes 2$, which is separable [17, 18, 19]. A sum of independent PSD blocks has PSD blocks (if $\sum_g A_g \succeq 0$ with A_g Hermitian and supported on independent subspaces S_g, then every $A_g \succeq 0$), which gives $\sigma_0^\Gamma \succeq 0$. Both hypotheses are used. Without the directness of the Π_g this step fails, since an NPT block σ_0 can be compensated by σ_1 (an exact example is in [Supp, S11]); and $\dim P_0 \leq 2$ cannot be dropped when $\dim \Pi_0 = 2$, since the Tiles state on $(\Pi_0 + \mathbb{C}e_j) \otimes \mathbb{C}^3$ is PPT and entangled.

5.5. **Relations of non-full support.** At $d = 4$ and coherence rank 3 the relation λ is unique up to scale. If it vanishes at one or two indices, the corner results above and the tools of Sections 5.6–5.10 decide separability of the twirl.

Theorem 5.13. *Let $d = 4$, let the coherence rank be 3 with all $B_{aa} > 0$, and let the relation λ vanish on $Z := \{a : \lambda_a = 0\} \neq \varnothing$; put $I := \{1, 2, 3, 4\} \setminus Z$. Then $|Z| \in \{1, 2\}$ and $\mathrm{tw}(W)$ is separable. If $|Z| = 1$, then $\Psi \circ \Phi$ is EB.*

Proof. A relation with a single nonzero coefficient would force $T_a = 0$, contradicting $B_{aa} > 0$; so $|Z| \leq 2$. After the diagonal filter that makes $\lambda_a = 1$ on I (which preserves PPT, ranks and the separability of W and of $\mathrm{tw}(W)$), Lemma 5.3 gives $\rho_I \sigma_I = 0$, $\rho_I^\Gamma \sigma_I^\Gamma = 0$ and $\operatorname{rank} \rho_I + \operatorname{rank} \sigma_I \leq 3|I|$.

- $|Z| = 2$: Theorem 5.18.
- $|Z| = 1$: $\operatorname{rank} \rho_I + \operatorname{rank} \sigma_I \leq 9$. If a corner has rank at most 3, Theorem 5.23 applies. Otherwise $(\operatorname{rank} \rho_I, \operatorname{rank} \sigma_I) \in \{(4, 4), (4, 5), (5, 4)\}$, and a corner of rank 4, being PPT, is separable or PPT entangled.
  - If some corner of rank 4 is separable, Theorem 5.26 applies.
  - Otherwise every corner of rank 4 is PPT entangled, of the form described in Section 5.9. For ranks $(4, 4)$ Theorem 5.31 applies, and for ranks $(5, 4)$ and $(4, 5)$ Theorem 5.40 applies.

In every case with $|Z| = 1$ the cited result shows that $\Psi \circ \Phi$ is EB, and hence that W and $\mathrm{tw}(W)$ are separable. □

For $|Z| = 2$ the proof establishes only that the twirl is separable: some branches end with a peel (Corollary 1.3(b)). Pairs with some $B_{aa} = 0$ are outside Theorem 5.13; zero-row compression (Lemma 2.3) reduces them to outer dimension 3, where [1, 2] apply.

5.6. **Two outer indices.** We use the following splitting lemma throughout. For $n \in \mathbb{C}^d$ put $G(n) := \sum_z n_z G_z$ and $H(n) := \sum_z n_z H_z$.

Lemma 5.14 (splitting)**.** *Let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_k$ be PPT with Gram data $G_y \in \mathbb{C}^{k\times P}$, let $n \in \mathbb{C}^d$ be nonzero, and suppose $G(n)^* u = 0$ for all u in a subspace $U \subseteq \mathbb{C}^k$ of dimension at least $k-1$. Then $\Phi = \Phi' + \mathrm{Ad}_K$, where $\operatorname{rank} K \le 1$, Φ' is PPT with Gram data $G'_y = G_y(I - \hat\delta\hat\delta^*)$ for a unit vector or zero $\hat\delta \in \mathbb{C}^P$, and $G'(n) = 0$, so that $\Phi' = \Phi' \circ \mathrm{Ad}_{P_{n^\perp}}$. Moreover $G'(m)^* = (I - \hat\delta\hat\delta^*)G(m)^*$ for every $m \in \mathbb{C}^d$.*

Proof. Since $(J(\Phi)^\Gamma)_{yz} = G_z G_y^*$, we have $\langle n \otimes u, J(\Phi)^\Gamma(n \otimes u)\rangle = \|G(n)^* u\|^2 = 0$ for $u \in U$, so $J(\Phi)^\Gamma(n \otimes u) = 0$; its y-th block is $G(n)G_y^* u$. Hence $\operatorname{range} G(n) \subseteq U^\perp$, of dimension at most 1. If $G(n) = 0$, take $\hat\delta = 0$ and $K = 0$. Otherwise $U^\perp = \mathbb{C}\nu$ with $\|\nu\| = 1$, $G(n) = \nu\delta^*$ with $\delta \neq 0$, and $\delta^* G_y^* u = 0$ gives $G_y\delta \in \mathbb{C}\nu$ for every y. Put $\hat\delta := \delta/\|\delta\|$ and $K := \sum_p \hat\delta_p K_p$, where $K_p e_y = G_y e_p$ are the Kraus operators of Φ. Then $Ke_y = G_y\hat\delta \in \mathbb{C}\nu$, so $K = \nu r^\top$ with $r_y := \nu^* G_y \hat\delta$; the Kraus directions orthogonal to $\hat\delta$ give Φ', and $\Phi = \Phi' + \mathrm{Ad}_K$, $G'(n) = G(n)(I - \hat\delta\hat\delta^*) = 0$. For PPT of Φ': $J(\Phi')^\Gamma(n \otimes w) = 0$ for every w, $J(\mathrm{Ad}_K)^\Gamma = |\bar r\rangle\langle \bar r| \otimes \nu\nu^*$, and $Kn = G(n)\hat\delta = \|\delta\|\nu$ gives $r^\top n = \|\delta\| \neq 0$; for every vector x and $s \in \mathbb{C}$,

$$0 \le \langle x + s\, n \otimes \nu, J(\Phi)^\Gamma(x + s\, n \otimes \nu)\rangle = \langle x, J(\Phi')^\Gamma x\rangle + |\langle \bar r \otimes \nu, x\rangle + s\|\delta\||^2,$$

and a suitable s gives $\langle x, J(\Phi')^\Gamma x\rangle \ge 0$. The last statement is immediate. □

Corollary 5.15 (twin lemma)**.** *Let $d = 4$, let $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_4$ be PPT, and let $x \neq x'$ be indices with $H_{x'} = \lambda H_x$ and $T_{x'} = \gamma T_x$, where $\lambda \neq 0$ and $\dim\operatorname{col} H_x \ge 2$. Then $\Psi \circ \Phi$ is EB. The same holds with $G_{x'} = \lambda G_x$ and $\dim\operatorname{col} G_x \ge 2$.*

Proof. $\lambda G_{x'}^* H_x = \gamma G_x^* H_x$, so $G(n)^* u = 0$ for $u \in \operatorname{col} H_x$, where $n := \bar\kappa e_x - e_{x'}$ and $\kappa := \gamma/\lambda$. Lemma 5.14 gives $\Phi = \Phi' + \mathrm{Ad}_K$ with $\Phi' = \Phi' \circ \mathrm{Ad}_{P_{n^\perp}}$. Every Kraus operator L_q of Ψ satisfies $L_q^*(e_{x'} - \lambda e_x) = 0$, so $\Psi = \mathrm{Ad}_Q \circ \Psi$ with Q the projection onto $(e_{x'} - \lambda e_x)^\perp$. Hence $\Psi \circ \Phi' = \mathrm{Ad}_Q \circ \Psi \circ \Phi' \circ \mathrm{Ad}_{P_{n^\perp}}$ is a composite of PPT maps through supports of dimensions $(3,3,3)$, which is EB by [1, 2]; and $\Psi \circ \mathrm{Ad}_K$ has Kraus operators of rank at most one, so it is EB. The second statement is the first for the adjoint pair (Ψ^*, Φ^*). □

Lemma 5.16 (near twins)**.** *Let $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_4$ be PPT, and let $a \neq b$ satisfy $T_b = \gamma T_a$ with $\gamma \neq 0$ and $H_b = \lambda H_a + w\zeta^*$ with $\lambda \neq 0$, $w \in \mathbb{C}^3$, $\zeta \in \mathbb{C}^Q$, such that $H_a(\zeta^\perp)$ has dimension at least 2. Then $\mathrm{tw}(W)$ is separable. The same holds with the roles of G and H exchanged.*

For the proof, given in [Supp, S9], we apply Lemma 5.14 with $n = -\bar\gamma e_a + \bar\lambda e_b$ and $U = H_a(\zeta^\perp)$; the remainder Φ' has $G'_b = (\bar\gamma/\bar\lambda)G'_a$ and $T'_b = \gamma T'_a$, and the pair (Φ', Ψ) is then settled by Corollary 5.15, by a peel or by a zero-row compression.

Lemma 5.17 (nested rows)**.** *Let Ψ be PPT and let $a \neq b$ be indices with $\operatorname{col} H_b^* \subseteq \operatorname{col} H_a^*$ and $\dim\operatorname{col} H_a^* \le 2$. Then either the index a peels on the Ψ side, or $H_b = cH_a$ for some $c \in \mathbb{C}$. The mirror statement holds for Φ.*

Proof. Let $F_a := \operatorname{col} H_a^*$. If $\dim F_a \le 1$, every $\tilde{N}_y$ is a scalar and a peels, or $H_a = 0 = H_b$. Let $\dim F_a = 2$. By Lemma 3.1(b) (pivot a, index b), $\tilde{N}_b$ is normal and $\tilde{N}_y \tilde{N}_b^* = \tilde{N}_b^* \tilde{N}_y$ for every y. If $\tilde{N}_b$ is not a scalar, the eigenspaces of the normal operator $\tilde{N}_b^*$ are two orthogonal lines in F_a, every $\tilde{N}_y$ commutes with $\tilde{N}_b^*$ and so preserves them, so all $\tilde{N}_y$ are diagonal in one orthonormal basis, and a peels. If $\tilde{N}_b = cI$, then $H_b P_{F_a} = cH_a$ and $H_b(I - P_{F_a}) = 0$, so $H_b = cH_a$. □

Theorem 5.18. *Let $d = 4$, let the coherence rank be 3 with all $B_{aa} > 0$, and let the relation vanish at exactly two indices. Then* $\operatorname{tw}(W)$ *is separable.*

In [Supp, S9] we reduce the problem to the corner of rank at most 3 (Lemma 5.3 with $|I| = 2$), which is separable. If its outer marginal has full rank and (H) holds, Corollary 5.7 applies; proportional slices are the twin case (Corollary 5.15, or a peel); two parallel outer factors give near twins (Lemma 5.16); and a corner with a planar middle support leads, through the structure of the filtered opposite corner, to near twins or to nested rows (Lemma 5.17).

5.7. **One vanishing index and a corner of rank at most three.** In this subsection and the next, $d = 4$, $\sum_{c\in I} \lambda_c T_c = 0$ with all $\lambda_c \neq 0$ on I, and for $p \in \mathbb{C}^I$ we put $n_\lambda(p) := \sum_{c\in I} \overline{\lambda_c p_c}\, e_c$. The *output support* of Ψ is the smallest subspace containing every range of Ψ, and the *input support* of Φ is the orthogonal complement of the common kernel of its Kraus operators.

Lemma 5.19. *Let Φ, Ψ be completely positive with $\sum_{c\in I} \lambda_c T_c = 0$. If $p \otimes u$ lies in the range of σ_I, then $G(n_\lambda(p))^* u = 0$, and if Φ is PPT, $J(\Phi)^\Gamma(n_\lambda(p) \otimes u) = 0$. The mirror statement holds for the range of ρ_I.*

Proof. The range of $\sigma_I = \mathbf{H}_I \mathbf{H}_I^*$ is $\operatorname{col} \mathbf{H}_I$ with $\mathbf{H}_I = [H_c]_{c\in I}$. If $\mathbf{H}_I q = p \otimes u$, then $H_c q = p_c u$ and $G(n_\lambda(p))^* u = \sum_c \lambda_c p_c G_c^* u = \sum_c \lambda_c T_c q = 0$. The PPT statement is the first step of the proof of Lemma 5.14. □

Lemma 5.20. *Let Φ be PPT, $\sum_{c\in I} \lambda_c T_c = 0$, and let $p\colon \mathbb{C}^2 \to \mathbb{C}^I$ and $u\colon \mathbb{C}^2 \to \mathbb{C}^3$ be polynomial maps (in τ, not $\bar\tau$) with $p(\tau) \otimes u(\tau)$ in the range of σ_I for every τ. Then $J(\Phi)^\Gamma$ vanishes on $n_\lambda(\operatorname{span}\{p(\tau)\}) \otimes \operatorname{span}\{u(\tau)\}$.*

Proof. Write $p(\tau) = \sum_\alpha \tau^\alpha p_\alpha$ and $u(\tau) = \sum_\beta \tau^\beta u_\beta$. By Lemma 5.19, every $n_\lambda(p(\tau)) \otimes u(\tau) = \sum_{\alpha,\beta} \bar\tau^\alpha \tau^\beta\, n_\lambda(p_\alpha) \otimes u_\beta$ lies in $\ker J(\Phi)^\Gamma$. The functions $\tau \mapsto \bar\tau^\alpha \tau^\beta$ are linearly independent on $\mathbb{C}^2$, so every $n_\lambda(p_\alpha) \otimes u_\beta$ lies in the kernel, and these span $n_\lambda(\operatorname{span}\{p(\tau)\}) \otimes \operatorname{span}\{u(\tau)\}$. □

Lemma 5.21. *Let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ be PPT, $U \subseteq \mathbb{C}^3$ with $\dim U \ge 2$, and $N \subseteq \mathbb{C}^d$ a subspace with $J(\Phi)^\Gamma(n \otimes u) = 0$ for all $n \in N$ and $u \in U$. Then $\Phi = \Phi_0 + \sum_{j=1}^{\dim N} \operatorname{Ad}_{K_j}$ with $\operatorname{rank} K_j \le 1$ and Φ_0 PPT with Kraus operators vanishing on N. If $d = 4$ and $\dim N \ge 2$, then Φ is EB.*

Proof. $\|G(n)^* u\|^2 = \langle n \otimes u, J(\Phi)^\Gamma(n \otimes u)\rangle = 0$, so $G(n)^* U = 0$ for $n \in N$. Apply Lemma 5.14 successively with the vectors $n_1, \dots, n_m$ of a basis of

N; by its last statement the hypothesis persists for the remainders, and the earlier n_i stay in the kernels. If $d = 4$ and $\dim N \geq 2$, then Φ_0 factors through an input of dimension at most 2, so its Choi matrix is a PPT operator on $\mathbb{C}^{\leq 2} \otimes \mathbb{C}^3$, which is separable [17, 18, 19]; the terms Ad_{K_j} are EB. □

Proposition 5.22. *Let Φ, Ψ be PPT with $\sum_{c\in I} \lambda_c T_c = 0$. Suppose the range of σ_I contains $p \otimes U$ with $\dim U \geq 2$ and $n_\lambda(p) \neq 0$, and Ψ has output support of dimension at most 3 (for instance, the outer marginal of σ_I is singular). Then $\Psi \circ \Phi$ is EB. The mirror statement holds for ρ_I, with the input support of Φ.*

Proof. By Lemma 5.19, $G(n_\lambda(p))^* U = 0$, and Lemma 5.14 gives $\Phi = \Phi' + \mathrm{Ad}_K$ with $\Phi' = \Phi' \circ \mathrm{Ad}_P$, P the projection onto $n_\lambda(p)^\perp$. With Q the projection onto a 3-dimensional subspace containing the output support of Ψ, $\Psi \circ \Phi' = \mathrm{Ad}_Q \circ \Psi \circ \Phi' \circ \mathrm{Ad}_P$ is EB by [1, 2], and $\Psi \circ \mathrm{Ad}_K$ is EB. If $\sum_c m_c H_c = 0$ with $m \neq 0$, every range of Ψ lies in $(\sum_c m_c e_c)^\perp$; the outer marginal of σ_I is the Gram matrix of $(H_c)_{c\in I}$. □

Theorem 5.23. *Let $d = 4$, let the coherence rank be 3 with all $B_{aa} > 0$, and let the relation vanish at exactly one index, with support I. If ρ_I or σ_I has rank at most 3, or has rank 4 and an outer marginal of rank at most 2, then $\Psi \circ \Phi$ is EB.*

We summarize the proof from [Supp, S10]. Passing to the adjoint pair if necessary, we assume that σ_I satisfies the hypothesis. With S and Y_0 the outer and inner supports of σ_I: if $\dim S = \operatorname{rank} \sigma_I \leq 3$, (H-i) holds and Ψ is EB; if $\dim S = 1$, the slices on I are proportional and Ψ factors through M_2, so it is EB. Otherwise $\dim S = 2$. If $\operatorname{rank} \sigma_I = 3$ and the inner marginal has rank 3, then $\sigma_I = \sum_{k\leq 3} |x_k \otimes y_k\rangle\langle x_k \otimes y_k|$ with independent y_k [10], and either the x_k are pairwise non-parallel, so that (H-ii) holds and Ψ is EB by Corollary 5.7(a), or two of them are parallel and the range contains a product plane, to which Proposition 5.22 applies. In the remaining cases the range of σ_I (spanned by its product vectors, since σ_I is separable) contains either a product plane or a polynomial family of product vectors whose second factors span a plane. In the first case Proposition 5.22 applies, and in the second Φ is EB by Lemmas 5.20 and 5.21. Each branch concludes that Ψ, Φ or $\Psi \circ \Phi$ is EB. No peel is used. The same argument extends to the configurations $(4, 5)$ and $(5, 4)$ in which the rank-5 corner has an outer marginal of rank 2 [Supp, S10].

5.8. **Separable corners of rank four.** A PPT operator of rank 4 on $\mathbb{C}^3 \otimes \mathbb{C}^3$ with local ranks $(3, 3)$ is either separable or entangled; if it is separable, it is the sum of four pure product states, or the sum of a pure product state and a separable operator of rank 3 with local ranks $(2, 2)$ [13, Theorem 3]. A PPT operator of rank 4 with a local rank at most 2 is separable [17, 18, 19].

Theorem 5.24. *In Theorem 5.13 with $|Z| = 1$, suppose σ_I has rank 4, outer marginal of rank 3 and inner marginal of rank 2. Then Φ or Ψ is EB. The same holds for ρ_I.*

Following [Supp, S11], we write the range of σ_I as a 4-dimensional subspace of $\mathbb{C}^I \otimes Y_0$ with $\dim Y_0 = 2$; its product vectors $p \otimes u(\tau)$, $u(\tau) = \tau_1 f_1 + \tau_2 f_2$, are the kernels of a 2×3 pencil $M(\tau)$ that has rank 2 for generic τ. The cross product of its rows factors as $g(\tau)q(\tau)$ with g a binary form of degree 0, 1 or 2. If $\deg g \leq 1$, the vectors $q(\tau)$ span at least a plane and it follows from Lemmas 5.20 and 5.21 that Φ is EB; the case $\deg g = 0$ does not occur, since it would force $T_c = 0$ for $c \in I$. If $\deg g = 2$, the range is spanned by $q_0 \otimes Y_0$ and at most two product vectors $p_i \otimes u(\tau_i)$, and after a maximal subtraction Lemma 5.12 (with $\dim \Pi_0 = 1$) implies that Ψ is EB.

Corollary 5.25. *In Theorem 5.13 with $|Z| = 1$, let $\sigma_I = \sum_{k=1}^{4} |x_k \otimes y_k\rangle\langle x_k \otimes y_k|$ have rank 4 and $\operatorname{span}\{x_k\} = \operatorname{span}\{y_k\} = \mathbb{C}^3$. Then Ψ is EB. The same holds for ρ_I and Φ.*

On each side we sort the configurations of four vectors spanning $\mathbb{C}^3$ into three types (no three dependent, exactly one dependent triple, exactly one parallel pair). The details are in [Supp, S11]. In all but two configurations (I^+) and (R) hold (Lemma 5.10 and three variants for collinear or coplanar triples and a parallel inner pair) or the outer factors have exactly one repeated direction; then Lemma 5.8 or Lemma 5.12 applies. In the two remaining configurations, x_1, x_2, x_3 span a plane Π and pairwise non-parallel y_1, y_2, y_3 span a plane P (or $y_1 \parallel y_2$ and $P := \operatorname{span}\{y_1, y_3\}$), condition (R) fails for all data (its solution space has dimension 5, resp. 6), and Lemma 5.12 applies with $\sigma_0 = \sum_{k \leq 3} |x_k \otimes y_k\rangle\langle x_k \otimes y_k|$ on $\Pi \otimes P$ and $\sigma_1 = |x_4 \otimes y_4\rangle\langle x_4 \otimes y_4|$.

Theorem 5.26. *In Theorem 5.13 with $|Z| = 1$, suppose σ_I or ρ_I has rank 4 and is separable. Then $\Psi \circ \Phi$ is EB. If that corner has an outer marginal of rank 3, then Φ or Ψ is EB.*

Proof. Let (r_o, r_i) be the outer and inner local ranks of the corner, say σ_I; $r_o r_i \geq 4$. If $r_o \leq 2$, Theorem 5.23 applies. If $r_o = 3$ and $r_i = 2$, Theorem 5.24 applies. If $r_o = r_i = 3$, by [13, Theorem 3] either σ_I is a sum of four pure product states, and Corollary 5.25 applies; or $\sigma_I = |x \otimes y\rangle\langle x \otimes y| + \sigma'$ with σ' separable of rank 3 supported on $\Pi \otimes P$ with Π, P planes; then $x \notin \Pi$, because the outer support is $\mathbb{C}^3$, and Lemma 5.12 with $\sigma_0 = \sigma'$ and $\sigma_1 = |x \otimes y\rangle\langle x \otimes y|$ applies. For ρ_I exchange the roles of Φ and Ψ. □

When $r_o \leq 2$, the proof of Theorem 5.23 may conclude through Proposition 5.22, where no factor need be EB.

5.9. **Entangled corners of rank four: rigidity.** This subsection settles the case of Theorem 5.13 in which both corners have rank 4 and are PPT entangled (Theorem 5.31). The key notion is rigidity of a corner, defined before Lemma 5.28: for a rigid corner on three outer indices, the PPT

property forces the Gram data at the fourth index to be, up to an EB remainder, a combination of those on the corner (Lemma 5.28). So Ψ (or Φ) splits into a map supported on three outer indices and an EB part, and two rigid corners reduce the composite to PPT^2 in dimension 3 (Theorem 5.29). The main step is Theorem 5.30: every PPT entangled operator of rank 4 on $\mathbb{C}^3 \otimes \mathbb{C}^3$ is rigid. Its proof reduces to the projector onto the complement of an unextendible product basis and then to a linear system in the coordinates of Lemma 5.27; a nontrivial solution would have to satisfy an identity, (I1), that the orthogonality pattern of the basis rules out.

By [13, Proposition 5(i) and Theorem 3(iii)] (see also [12, 31]), a PPT entangled operator σ of rank 4 on $\mathbb{C}^3 \otimes \mathbb{C}^3$ has local ranks $(3, 3)$, $\operatorname{rank} \sigma^\Gamma = 4$, and $\sigma \propto (A \otimes B)\Pi_\psi (A \otimes B)^*$ with A, B invertible and Π_ψ the projector onto the orthogonal complement of an unextendible product basis (UPB) $\{\psi_k\}$ [14, 15]; its kernel contains exactly six product vectors up to scale.

Lemma 5.27. *Every UPB of $\mathbb{C}^3 \otimes \mathbb{C}^3$ has exactly five members. If $\{\psi_k = a_k \otimes b_k\}_{k \in \mathbb{Z}_5}$ is one, with $\|a_k\| = \|b_k\| = 1$, then any three of the a_k and any three of the b_k are linearly independent, and after relabelling $\langle a_k, a_{k\pm1}\rangle = 0$ and $\langle b_k, b_{k\pm2}\rangle = 0$ for all k, while $\langle a_k, a_{k\pm2}\rangle \neq 0$ and $\langle b_k, b_{k\pm1}\rangle \neq 0$; no two a_k, and no two b_k, are parallel.*

Proof. Four product vectors $a_k \otimes b_k$ ($k \leq 4$) are always extended by $a' \otimes b'$ with $a' \perp a_1, a_2$ and $b' \perp b_3, b_4$. Six to eight orthonormal product vectors leave a complement of dimension 3 to 1 (nine form a complete product basis, which is not a UPB) whose projector Π is PPT, since Π^Γ is the projector onto $\operatorname{span}\{\bar{a}_k \otimes b_k\}^\perp$, these vectors being orthonormal; a PPT operator of rank at most 3 is separable [13, Proposition 5(i)], so the complement contains a product vector, contradicting unextendibility. So a UPB has five members. If three a's lay in a plane, a vector a' orthogonal to it and b' orthogonal to the other two b's would give a product vector orthogonal to all ψ_k; similarly for the b's. Hence each a_k is orthogonal to at most two other a's, and likewise for the b's. Each of the ten pairs $\{k, l\}$ has $\langle a_k, a_l\rangle\langle b_k, b_l\rangle = 0$. The orthogonality graphs on the a's and on the b's have maximal degree 2, hence at most five edges each; so both have exactly five edges, every pair is an edge of exactly one of them, and both are 2-regular on five vertices, i.e. complementary 5-cycles. Relabel so that the a-cycle is 0-1-2-3-4-0; the b-cycle is then 0-2-4-1-3-0. Parallel vectors would make three of them dependent. □

The five-member statement and the orthogonality graph are in [15, Section IV A], and the labelling is that of [12, Section IV A]; the argument above is direct and also gives the general position used below.

For a positive semidefinite σ on $\mathbb{C}^I \otimes \mathbb{C}^3$ with $\sigma^\Gamma \succeq 0$ write $\sigma = VV^*$ with $V\colon \mathbb{C}^r \to \mathbb{C}^I \otimes \mathbb{C}^3$ of full column rank, $V_a := (e_a^* \otimes I)V \in \mathbb{C}^{3\times r}$ and

$V(c) := \sum_a c_a V_a$, and let

$$S(\sigma) := \Big\{ Z \in \mathbb{C}^{3\times r} : \sum_{a\in I} e_a \otimes ZV_a^* v \in \operatorname{range} \sigma^\Gamma \text{ for all } v \in \mathbb{C}^3 \Big\}.$$

It contains every $V(c)$, because $\sum_a e_a \otimes V(c)V_a^* v = \sigma^\Gamma(c \otimes v)$. We call σ *rigid* if $S(\sigma) = \{V(c) : c \in \mathbb{C}^I\}$; this does not depend on the factor V.

Lemma 5.28. *Let $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_4$ be PPT, let j be an index and $I := \{1,2,3,4\} \setminus \{j\}$, and let Ω and $P := \Omega\Omega^*$ be as in step (2) of the proof of Lemma 5.6 for $\sigma_I = VV^*$, so that $H_a = V_a\Omega^*$ and $H_a(I-P) = 0$ for $a \in I$. Then $H_j\Omega \in S(\sigma_I)$. If σ_I is rigid, there is $c \in \mathbb{C}^I$ with $H_jP = \sum_{b\in I} c_b H_b$, and*

$$\Psi = \mathrm{Ad}_M \circ \Psi_I + \Psi_R, \qquad \Psi_I := \mathrm{Ad}_{\Pi_I} \circ \Psi,$$
$$Me_b := e_b + \bar{c}_b e_j \ (b \in I), \qquad \Psi_R(X) := \mathrm{Tr}(M_R X)\, e_j e_j^*,$$

where $M_R := H_j(I-P)H_j^* \succeq 0$; $\Psi_I\colon \mathrm{M}_3 \to \mathrm{M}_I$ is PPT and Ψ_R is EB.

Proof. $K := J(\Psi^*)^\Gamma \succeq 0$ has corner $K_{II} = \sigma_I^\Gamma$, and for $a \in I$, $K_{aj} = H_jH_a^* = H_jPH_a^* = (H_j\Omega)V_a^*$; its block column maps v to $\sum_a e_a \otimes H_j\Omega V_a^* v$, which lies in $\operatorname{range} K_{II}$. So $H_j\Omega \in S(\sigma_I)$. If σ_I is rigid, $H_j\Omega = V(c)$, so $H_jP = V(c)\Omega^* = \sum_b c_b H_b$. Write $\Psi^*(E_{xy}) = H_xPH_y^* + H_x(I-P)H_y^*$; the second term has the single nonzero entry M_R at (j,j), which gives Ψ_R. The first term has Gram data H_b ($b \in I$) and $H_jP = \sum_b c_b H_b$, so its Choi matrix is $(L\otimes I)\sigma_I(L^*\otimes I)$ with $Le_b := e_b + c_b e_j$, which is PPT; its Kraus operators are $M\Pi_I L_q$, where L_q are those of Ψ, so it equals $\mathrm{Ad}_M \circ \Psi_I$. □

The mirror statement for Φ uses $J(\Phi)^\Gamma$, whose block column at j is formed by the blocks $G_jG_a^*$: if ρ_I is rigid, then $\Phi = \Phi_I \circ \mathrm{Ad}_{N^*} + \Phi_R$ with $\Phi_I := \Phi \circ \mathrm{Ad}_{\Pi_I^*}$ PPT, $Ne_b := e_b + \bar{d}_b e_j$ for some $d \in \mathbb{C}^I$, and $\Phi_R(X) = \langle e_j, Xe_j\rangle M'_R$ with $M'_R \succeq 0$. If the outer marginal of the corner has rank 3, c (resp. d) is unique; this is not needed.

Theorem 5.29. *Let $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_4$ be PPT. If $J(\Psi^*)$ has a rigid corner σ_I and $J(\Phi)$ has a rigid corner $\rho_{I'}$, with $|I| = |I'| = 3$, then $\Psi \circ \Phi$ is EB.*

Proof. By Lemma 5.28 and its mirror, $\Psi = \mathrm{Ad}_M \circ \Psi_I + \Psi_R$ and $\Phi = \Phi_{I'} \circ \mathrm{Ad}_{N^*} + \Phi_R$, so

$$\Psi \circ \Phi = \mathrm{Ad}_M \circ (\Psi_I \circ \Phi_{I'}) \circ \mathrm{Ad}_{N^*} + \mathrm{Ad}_M \circ \Psi_I \circ \Phi_R + \Psi_R \circ \Phi.$$

$\Psi_I \circ \Phi_{I'}\colon \mathrm{M}_{I'} \to \mathrm{M}_3 \to \mathrm{M}_I$ is a composite of PPT maps with all dimensions equal to 3, hence EB [1, 2]; composing with completely positive maps keeps it EB, and the other two terms are EB because Φ_R and Ψ_R are. □

Theorem 5.30. *Every PPT entangled operator of rank 4 on $\mathbb{C}^3 \otimes \mathbb{C}^3$ is rigid.*

Proof. 0. *Reduction.* By [13, Theorem 3], $\sigma \propto (A\otimes B)\Pi(A\otimes B)^*$ with $\Pi := \Pi_\psi$ for a UPB $\{\psi_k = a_k \otimes b_k\}$ in the form of Lemma 5.27. With $V' = (A\otimes B)V$ and $((A\otimes B)X(A\otimes B)^*)^\Gamma = (\bar{A}\otimes B)X^\Gamma(\bar{A}\otimes B)^*$ one checks $S((A\otimes B)\sigma(A\otimes B)^*) = B\cdot S(\sigma)$, and the trivial family maps onto the trivial family. So let $\sigma = \Pi$, $\mathcal{V} := \operatorname{range}\Pi = \operatorname{span}\{\psi_k\}^\perp$, and V an isometry onto $\mathcal{V}$.

A. *Reformulation.* $\Pi^\Gamma = I - \sum_k |\bar{a}_k\otimes b_k\rangle\langle\bar{a}_k\otimes b_k|$ is the projector onto $\operatorname{span}\{\bar{a}_k\otimes b_k\}^\perp$. So $Z\in S$ if and only if $\langle \bar{a}_k\otimes b_k, \sum_a e_a\otimes ZV_a^*v\rangle = b_k^*ZV^*(a_k\otimes v) = 0$ for all k and v. With $\hat{Z} := VZ^*$, a bijection onto $\operatorname{Hom}(\mathbb{C}^3,\mathcal{V})$, this reads $\hat{Z}b_k \perp a_k\otimes\mathbb{C}^3$ for all k, and $V(c)$ corresponds to $\hat{Z}_c\colon u\mapsto \Pi(\bar{c}\otimes u)$.

B. *The slots.* $Q_k := \mathcal{V}\cap(a_k\otimes\mathbb{C}^3)^\perp$ equals $\Pi(\mathbb{C}^3\otimes b_k)$, of dimension 2. Indeed $\langle a_k\otimes u, \Pi(x\otimes b_k)\rangle = \langle a_k,x\rangle\langle u,b_k\rangle - \sum_l\langle a_k,a_l\rangle\langle u,b_l\rangle\langle a_l,x\rangle\langle b_l,b_k\rangle = 0$, because the term $l=k$ cancels the first and $\langle a_k,a_l\rangle\langle b_l,b_k\rangle = 0$ for $l\neq k$. Next, $x\otimes b_k\in\operatorname{span}\{\psi_l\}$ forces $x\in\mathbb{C}a_k$: write $x=\alpha a_k + x_0$ with $x_0\perp a_k$; the coefficients of $x_0\otimes b_k$ in the orthonormal basis (ψ_l) are $\langle a_l,x_0\rangle\langle b_l,b_k\rangle$, which vanish for $l = k, k\pm 2$, so $x_0\otimes b_k = \mu_+\psi_{k+1} + \mu_-\psi_{k-1}$; as a 3×3 matrix the right side has rank equal to the number of nonzero μ's, and one nonzero μ would make b_k parallel to $b_{k\pm1}$; so $x_0 = 0$. In the same way $a_k\otimes u\in\operatorname{span}\{\psi_l\}$ forces $u\in\mathbb{C}b_k$. Hence $x\mapsto\Pi(x\otimes b_k)$ has kernel $\mathbb{C}a_k$, so $\dim\Pi(\mathbb{C}^3\otimes b_k) = 2$; and the map $y\mapsto(a_k^*\otimes I)y$ on $\mathcal{V}$, whose adjoint $u\mapsto\Pi(a_k\otimes u)$ has kernel $\mathbb{C}b_k$, has rank 2, so its kernel Q_k has dimension $4-2=2$.

C. *Coordinates.* By A and B, S corresponds to the linear $\hat{Z}\colon\mathbb{C}^3\to\mathbb{C}^9$ with $\hat{Z}b_k\in\Pi(\mathbb{C}^3\otimes b_k)$ for all k. Write $\hat{Z}b_k = \Pi(x_k\otimes b_k)$ (x_k unique modulo $\mathbb{C}a_k$) and $\Delta := \{\delta\in\mathbb{C}^5 : \sum_k\delta_k b_k = 0\}$, of dimension 2. Then $S\cong\mathcal{X}/\bigoplus_k\mathbb{C}a_k$ with $\mathcal{X} := \{(x_k)\in(\mathbb{C}^3)^5 : \sum_k\delta_k x_k\otimes b_k\in\operatorname{span}\{\psi_l\}\text{ for all }\delta\in\Delta\}$. The trivial solutions $x_k = \bar{c} + \nu_k a_k$ form an 8-dimensional subspace of $\mathcal{X}$, and rigidity means $\dim\mathcal{X} = 8$.

D. *Solving.* Let $\delta^{[0]},\delta^{[1]}\in\Delta$ with $\delta_0^{[0]} = 0 = \delta_1^{[1]}$ and $\delta_3^{[0]} = \delta_3^{[1]} = 1$ (all their other entries are nonzero by Lemma 5.27), and $\rho_k := \delta_k^{[1]}/\delta_k^{[0]}$ for $k = 2,3,4$. For $\delta\in\Delta$, $\sum_k\delta_k x_k\otimes b_k = \sum_k\mu_k a_k\otimes b_k$ for some μ if and only if $\delta_k x_k - \mu_k a_k = \delta_k^{[0]}p + \delta_k^{[1]}q$ for some $p,q\in\mathbb{C}^3$ and all k. For $\delta=\delta^{[0]}$ this gives $q = sa_0$, $x_1\equiv p$ and $x_k\equiv p+\rho_k s a_0$ for $k=2,3,4$ (modulo $\mathbb{C}a_k$); for $\delta=\delta^{[1]}$ it gives $x_0\equiv q'$ and $x_k\equiv q' + \rho_k^{-1}s'a_1$ for $k=2,3,4$. So $\mathcal{X}$ is parametrized by (p,q',s,s',ν), with $x_0 = q'+\nu_0 a_0$, $x_1 = p+\nu_1 a_1$, $x_k = p+\rho_k s a_0 + \nu_k a_k$, subject to

$$(\dagger)\qquad w + \rho_k s a_0 - \rho_k^{-1}s'a_1\in\mathbb{C}a_k\quad(k=2,3,4),\qquad w := p - q'.$$

This parametrization is injective: $x=0$ gives $p = -\nu_1 a_1$ and then, any three a's being independent, $\nu_1 = s = \nu_k = 0$ and $p=0$; also $q' = -\nu_0 a_0$, and $(\dagger)$ for $k=2$ reads $\nu_0 a_0 - \rho_2^{-1}s'a_1\in\mathbb{C}a_2$, so $\nu_0 = s' = 0$. Since q' and ν are free, $\dim\mathcal{X} = 8 + \dim\{(w,s,s')\text{ satisfying }(\dagger)\}$.

E. *The obstruction.* The ρ_k are pairwise distinct and nonzero (two equal ones would give a relation among three b's). Write $(\dagger)$ as $w + \rho_k s a_0 -$

$\rho_k^{-1}s'a_1 = \lambda_k a_k$ and subtract for $k = 2, 3$ and $k = 3, 4$: $(\rho_2 - \rho_3)sa_0 - (\rho_2^{-1} - \rho_3^{-1})s'a_1 - \lambda_2 a_2 + \lambda_3 a_3 = 0$ and $(\rho_3 - \rho_4)sa_0 - (\rho_3^{-1} - \rho_4^{-1})s'a_1 - \lambda_3 a_3 + \lambda_4 a_4 = 0$. Four of the a's satisfy a unique relation, with all coefficients nonzero: $\varepsilon_0 a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3 = 0$ and $\varepsilon'_0 a_0 + \varepsilon'_1 a_1 + \varepsilon'_3 a_3 + \varepsilon'_4 a_4 = 0$. So the two coefficient vectors are $\tau\varepsilon$ and $\tau'\varepsilon'$, and comparing the coefficients of a_3 and a_0 gives $\tau\varepsilon_3 = -\tau'\varepsilon'_3$, $(\rho_2 - \rho_3)s = \tau\varepsilon_0$, $(\rho_3 - \rho_4)s = \tau'\varepsilon'_0$. If $\tau \neq 0$, then

$$\text{(I1)} \qquad \frac{\varepsilon_0}{\rho_2 - \rho_3} = -\frac{\varepsilon_3\varepsilon'_0/\varepsilon'_3}{\rho_3 - \rho_4}.$$

If $\tau = 0$, then $\tau' = 0$, $s = s' = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $w = 0$. So it suffices to show that (I1) fails.

F. *(I1) fails.* Put $c_{kl} := \langle a_k, a_l\rangle$ and $e_{kl} := \langle b_k, b_l\rangle$. Applying $\langle a_2, \cdot\rangle$ and $\langle a_0, \cdot\rangle$ to ε gives $\varepsilon_2 = -\varepsilon_0 c_{20}$ and $\varepsilon_0(1 - |c_{02}|^2) = -\varepsilon_3 c_{03}$; applying $\langle a_0, \cdot\rangle$ to ε' gives $\varepsilon'_0 = -\varepsilon'_3 c_{03}$. Applying $\langle b_1, \cdot\rangle$, $\langle b_4, \cdot\rangle$, $\langle b_2, \cdot\rangle$ to $\delta^{[0]}$ and $\langle b_0, \cdot\rangle$, $\langle b_2, \cdot\rangle$, $\langle b_4, \cdot\rangle$ to $\delta^{[1]}$ gives $\delta_2^{[0]} = -e_{23}/(1 - |e_{12}|^2)$, $\delta_4^{[0]} = -e_{43}$, $\delta_2^{[1]} = -e_{23}$, $\delta_4^{[1]} = -e_{43}/(1 - |e_{04}|^2)$, hence

$$\rho_2 = 1 - |e_{12}|^2, \qquad \rho_3 = 1, \qquad \rho_4 = 1/(1 - |e_{04}|^2).$$

Since $\varepsilon_3, c_{03} \neq 0$, (I1) is equivalent to $\rho_4 - \rho_3 = (1 - |c_{02}|^2)(\rho_2 - \rho_3)$, that is, to $|e_{04}|^2/(1 - |e_{04}|^2) = -(1 - |c_{02}|^2)|e_{12}|^2$. The left side is positive and the right side negative, since $0 < |e_{04}| < 1$, $|c_{02}| < 1$ and $e_{12} \neq 0$ by Lemma 5.27. □

Theorem 5.31. *In Theorem 5.13 with* $|Z| = 1$ *and* $\operatorname{rank}\rho_I = \operatorname{rank}\sigma_I = 4$*, if both corners are PPT entangled, then* $\Psi \circ \Phi$ *is EB.*

Proof. Both corners are rigid by Theorem 5.30, and Theorem 5.29 applies with $I' = I$. □

5.10. **Corner ranks** $(5, 4)$ **and** $(4, 5)$**.** This subsection treats the case of Theorem 5.13 in which one corner is PPT entangled of rank 4 and the other has rank 5 (Theorem 5.40). After a filter, the corner of rank 5 has its range in the kernel of the entangled corner, and the PPT conditions confine it, and the block column next to it, to the span of the six product vectors in that kernel (Lemmas 5.36–5.38); this rests on Theorem 5.30 through a dimension count (Proposition 5.34). A sign analysis then writes $J(\Phi)$ as a sum of rank-one terms of Schmidt rank at most 2 (Lemma 5.39), rigidity of the entangled corner splits Ψ (Lemma 5.28), and a Schmidt-number argument as in [2] (Lemma 5.32) shows that the composite is EB.

The *Schmidt number* of a positive semidefinite operator on $\mathbb{C}^m \otimes \mathbb{C}^n$ is the least k such that it is a sum of rank-one operators $|\psi\rangle\langle\psi|$ with every ψ of Schmidt rank at most k [30].

Lemma 5.32. (a) *Let* $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ *be completely positive,* j *an index,* $I := \{1, \dots, d\} \setminus \{j\}$*, and let* ρ_I *be separable. Then* $J(\Phi) = \sum_k |\psi_k\rangle\langle\psi_k| + E_{jj} \otimes S$ *with* $S \succeq 0$ *and* $\psi_k = x_k \otimes y_k + e_j \otimes z_k$*; in particular* $J(\Phi)$ *has Schmidt number at most* 2*.*

(b) *If $J(\Phi)$ has Schmidt number at most 2 and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_m$ is PPT with $m \le 3$, then $\Psi \circ \Phi$ is EB.*

Proof. (a) Write $\rho_I = \sum_k |x_k \otimes y_k\rangle\langle x_k \otimes y_k|$ and $J(\Phi) = [[\rho_I, J_{Ij}], [J_{jI}, J_{jj}]]$ along $\mathbb{C}^d \otimes \mathbb{C}^3 = (\mathbb{C}^I \otimes \mathbb{C}^3) \oplus (e_j \otimes \mathbb{C}^3)$. Since $J(\Phi) \succeq 0$, range $J_{Ij} \subseteq$ range ρ_I. With $T := J_{jI}\rho_I^+$, the operator $J_A := [I;T]\rho_I[I;T]^*$ has blocks ρ_I, J_{Ij} and $J_{jI}\rho_I^+ J_{Ij}$, so $J(\Phi) - J_A = E_{jj} \otimes S$ with $S := J_{jj} - J_{jI}\rho_I^+ J_{Ij} \succeq 0$, and $J_A = \sum_k |\psi_k\rangle\langle\psi_k|$ with $\psi_k = x_k \otimes y_k + e_j \otimes T(x_k \otimes y_k)$. (b) $J(\Psi \circ \Phi) = (\mathrm{id} \otimes \Psi)(J(\Phi)) = \sum_k (\mathrm{id} \otimes \Psi)(|\psi_k\rangle\langle\psi_k|)$ with $\psi_k \in A_k \otimes \mathbb{C}^3$, $\dim A_k \le 2$. Each term is supported on $A_k \otimes \mathbb{C}^m$, is positive semidefinite, and its partial transpose on the second factor, $(\mathrm{id} \otimes \vartheta \circ \Psi)(|\psi_k\rangle\langle\psi_k|)$, is positive semidefinite. A PPT operator on $\mathbb{C}^2 \otimes \mathbb{C}^m$ with $m \le 3$ is separable [17, 18, 19]. □

Part (b) is [2, Theorem 5] with $\varphi_2 = \mathrm{id}$ and $\rho = J(\Phi)$; it is the argument by which [2] derive PPT² in dimension 3 from the dual form of [16]: two-qutrit PPT states have Schmidt number at most 2. For $d = m = 3$, part (b) also holds when $J(\Psi)$ is only 1-undistillable [4].

Corollary 5.33. *Let $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_4$ be PPT. If $J(\Psi^*)$ has a rigid corner on three outer indices and $J(\Phi)$ has a separable corner on three outer indices, then $\Psi \circ \Phi$ is EB. The same holds with the roles of $J(\Phi)$ and $J(\Psi^*)$ exchanged.*

Proof. By Lemma 5.28, $\Psi = \mathrm{Ad}_M \circ \Psi_I + \Psi_R$ with $\Psi_I\colon \mathrm{M}_3 \to \mathrm{M}_I$ PPT and Ψ_R EB, and Lemma 5.32 shows that $\Psi_I \circ \Phi$ is EB. For the mirror, Lemma 5.28 for Φ gives $\Phi = \Phi_{I'} \circ \mathrm{Ad}_{N^*} + \Phi_R$, and $\Phi_{I'}^* \circ \Psi^*$ is EB by Lemma 5.32, applied to the completely positive map Ψ^* and the PPT map $\Phi_{I'}^*\colon \mathrm{M}_3 \to \mathrm{M}_3$; the adjoint of an EB map is EB. □

For $J \in \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^I \otimes \mathbb{C}^3)$ put $J_a := (e_a^* \otimes I)J$ and $J^\sharp v := \sum_a e_a \otimes J_a^* v$, and for subspaces $R, R' \subseteq \mathbb{C}^I \otimes \mathbb{C}^3$ let $\mathcal{S}(R, R') := \{J \in \mathrm{Hom}(\mathbb{C}^3, R) : J^\sharp \in \mathrm{Hom}(\mathbb{C}^3, R')\}$. The block column of $J(\Phi)^\Gamma$ at j, rows in I, is $J_{Ij}^\sharp$, and for $\sigma = VV^*$ the map $Z \mapsto VZ^*$ is a conjugate-linear bijection from $S(\sigma)$ onto $\mathcal{S}(\mathrm{range}\,\sigma, \mathrm{range}\,\sigma^\Gamma)$.

Proposition 5.34. *For subspaces R, R' of $\mathbb{C}^n \otimes \mathbb{C}^3$,*

$$\dim \mathcal{S}(R^\perp, R'^\perp) = 9n - 3\dim R - 3\dim R' + \dim \mathcal{S}(R, R').$$

Proof. Identify J with $T_J \in \mathbb{C}^n \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, $T_J(a,i,k) := \langle e_a \otimes e_i, Je_k\rangle$. Then $J \in \mathrm{Hom}(\mathbb{C}^3, R)$ if and only if $T_J \in \mathcal{A} := \{T : T(\cdot,\cdot,k) \in R\ \forall k\}$; and since $(J^\sharp v)(a,k) = \sum_i \overline{T_J(a,i,k)} v_i$, $J^\sharp \in \mathrm{Hom}(\mathbb{C}^3, R')$ if and only if $T_J \in \mathcal{B} := \{T : T(\cdot,i,\cdot) \in \overline{R'}\ \forall i\}$. The spaces built in the same way from $R^\perp$ and $\overline{R'^\perp} = \overline{R'}^\perp$ are $\mathcal{A}^\perp$ and $\mathcal{B}^\perp$. So $\mathcal{S}(R^\perp, R'^\perp) \cong (\mathcal{A} + \mathcal{B})^\perp$, with $\dim \mathcal{A} = 3\dim R$, $\dim \mathcal{B} = 3\dim R'$ and $\mathcal{S}(R, R') \cong \mathcal{A} \cap \mathcal{B}$. □

For the rest of this subsection σ is a PPT entangled operator of rank 4 on $\mathbb{C}^3 \otimes \mathbb{C}^3$, $K := \ker\sigma$ and $K' := \ker\sigma^\Gamma$, both of dimension 5.

Corollary 5.35. $\dim \mathcal{S}(K, K') = 6$.

Proof. $\mathcal{S}(\operatorname{range}\sigma, \operatorname{range}\sigma^\Gamma) \cong S(\sigma) = \{V(c)\}$ by Theorem 5.30, and $c \mapsto V(c)$ is injective because $\|V(c)\|^2 = c^\top M \bar{c}$ with M the outer marginal of σ, which has rank 3; so it has dimension 3. Proposition 5.34 gives $27 - 12 - 12 + 3 = 6$. □

Lemma 5.36. *K contains exactly six product vectors up to scale* [13, Theorem 3]*, $p_m = u_m \otimes w_m$ ($m = 1, \dots, 6$). Any five of them are linearly independent, so they satisfy a single relation $\sum_m \mu_m p_m = 0$ with all $\mu_m \neq 0$. Moreover:*

(a) *any three of the u_m, and any three of the w_m, are linearly independent;*
(b) *the operators $w_m w_m^*$ span a space of dimension at least 5;*
(c) *the partial conjugates $p_m^\Gamma := \bar{u}_m \otimes w_m$ are the six product vectors of K', and their relation $\sum_m \mu'_m p_m^\Gamma = 0$ satisfies $|\mu'_m| = |\mu_m|$.*

Proof. For every X, $\langle \bar{x}\otimes y, X^\Gamma(\bar{x}\otimes y)\rangle = \langle x\otimes y, X(x\otimes y)\rangle$; applied to σ and σ^Γ this shows that $x\otimes y \in K$ if and only if $\bar{x}\otimes y \in K'$. By [13, Theorem 3], $\sigma \propto (A\otimes B)\Pi(A\otimes B)^*$ with Π as in Lemma 5.27; then $K = (A\otimes B)^{-*}\operatorname{span}\{\psi_k\}$, $K' = (\bar{A}\otimes B)^{-*}\operatorname{span}\{\bar{a}_k \otimes b_k\}$, and the statements are invariant under these filters, so let $\sigma = \Pi$. Five of the product vectors are the ψ_k; write the sixth as $u \otimes w = \sum_l \nu_l \psi_l$ with $\nu_l = \langle a_l, u\rangle\langle b_l, w\rangle$. All $\nu_k \neq 0$: if $\langle a_k, u\rangle = 0$, apply $a_k^* \otimes I$ and use $\langle a_k, a_{k\pm1}\rangle = 0$: $\nu_{k+2}\langle a_k, a_{k+2}\rangle b_{k+2} + \nu_{k-2}\langle a_k, a_{k-2}\rangle b_{k-2} = 0$, so $\nu_{k\pm2} = 0$; then $u \otimes w = \nu_{k+1}\psi_{k+1} + \nu_{k-1}\psi_{k-1}$ has matrix rank equal to the number of nonzero coefficients and would be parallel to ψ_{k+1} or ψ_{k-1}. If $\langle b_k, w\rangle = 0$, apply $I \otimes b_k^*$ and use $\langle b_k, b_{k\pm2}\rangle = 0$ in the same way. So any five of the six vectors are independent, and in the filtered frame the relation is the same. (a) Three a's are independent by Lemma 5.27. If $u \in \operatorname{span}\{a_k, a_l\}$, take $y \neq 0$ orthogonal to a_k, a_l: then $0 = \langle y, u\rangle w = \sum_m \nu_m \langle y, a_m\rangle b_m$ is a combination of the three independent b_m, $m \notin \{k, l\}$, so y is orthogonal to all five a's, and $y = 0$. The same argument applies to w. (b) If $\sum_k \gamma_k b_k b_k^* = 0$, applying it to b_m gives $\gamma_m b_m + \gamma_{m+1}\langle b_{m+1}, b_m\rangle b_{m+1} + \gamma_{m-1}\langle b_{m-1}, b_m\rangle b_{m-1} = 0$, so $\gamma_m = 0$. (c) $\bar{u} \otimes w \in K' = \operatorname{span}\{\bar{a}_k \otimes b_k\}$, an orthonormal family, so $\bar{u} \otimes w = \sum_k \nu'_k\, \bar{a}_k \otimes b_k$ with $\nu'_k = \overline{\langle a_k, u\rangle}\langle b_k, w\rangle$, and $|\nu'_k| = |\nu_k|$. □

Lemma 5.37. *$\mathcal{S}(K, K') = \operatorname{span}\{|p_m\rangle\langle w_m| : m = 1, \dots, 6\}$, and these six maps are linearly independent.*

Proof. For $J = |p_m\rangle\langle w_m|$, $J_a = u_m(a) w_m w_m^*$ and $J^\sharp v = \langle w_m, v\rangle\, \bar{u}_m \otimes w_m \in K'$. If $\sum_m \zeta_m |p_m\rangle\langle w_m| = 0$, then $(\zeta_m \langle w_m, y\rangle)_m$ is a multiple of μ for every y; for $y \perp w_1$ its first entry vanishes while $\mu_1 \neq 0$, so $\zeta_m\langle w_m, y\rangle = 0$ for all m and all $y \perp w_1$, and $\zeta_m = 0$ for $m \neq 1$, since $w_m \not\perp w_1$; then $\zeta_1 = 0$. Corollary 5.35 completes the proof. □

Lemma 5.38. *Every Hermitian X with $\operatorname{range} X \subseteq K$ and $\operatorname{range} X^\Gamma \subseteq K'$ is a real combination of the six projectors $|p_m\rangle\langle p_m|$.*

Proof. For $b \in \{1,2,3\}$ the map $J^{(b)}\colon v \mapsto X(e_b \otimes v)$ has range in K, blocks $J^{(b)}_a = X_{ab}$, and $(J^{(b)})^\sharp v = \sum_a e_a \otimes X_{ba} v = X^\Gamma(e_b \otimes v) \in K'$, since X is Hermitian and $(X^\Gamma)_{ab} = X_{ba}$. By Lemma 5.37, $X(e_b \otimes v) = \sum_m \zeta^{(b)}_m \langle w_m, v\rangle p_m$, so $X = \sum_m |p_m\rangle\langle f_m \otimes w_m|$ for some $f_m \in \mathbb{C}^3$, and also $X = X^* = \sum_m |f_m \otimes w_m\rangle\langle p_m|$. Comparing, $\sum_m A_m \otimes w_m w_m^* = 0$ with $A_m := |u_m\rangle\langle f_m| - |f_m\rangle\langle u_m|$. By Lemma 5.36(b) the linear relations among the $w_m w_m^*$ form a space of dimension at most 1. If it is zero, all $A_m = 0$. If it is spanned by $\gamma \neq 0$, then $A_m = \gamma_m A$ for a fixed matrix A, and γ has at least three nonzero entries, since two would make two w's parallel. If $A \neq 0$, then $u_m \in \operatorname{range} A_m = \operatorname{range} A$ for each such m, so three u's lie in a space of dimension at most 2, contradicting Lemma 5.36(a). So all $A_m = 0$, i.e. $f_m = \lambda_m u_m$ with λ_m real, and $X = \sum_m \lambda_m |p_m\rangle\langle p_m|$. □

Lemma 5.39 (an elementary identity)**.** *Let $q_0, \dots, q_N$ be vectors with $\sum_m q_m = 0$, and let $\sigma_0, \dots, \sigma_N$ be real and nonzero, exactly one of them negative, $\sigma_n < 0$, with $\sum_m 1/\sigma_m = 0$. Put $P := \{0, \dots, N\} \setminus \{n\}$ and $\chi_{lm} := \sigma_l q_l - \sigma_m q_m$. Then*

$$\sum_m \sigma_m\, q_m q_m^* = \sum_{l<m \text{ in } P} \frac{|\sigma_n|}{\sigma_l \sigma_m}\, |\chi_{lm}\rangle\langle\chi_{lm}|.$$

Proof. $\sum_{l\in P} 1/\sigma_l = 1/|\sigma_n|$. Expanding, the right side is $\sum_{l\in P}(\sigma_l - |\sigma_n|) q_l q_l^* - |\sigma_n| \sum_{l\neq m \text{ in } P} q_l q_m^* = \sum_{l\in P} \sigma_l q_l q_l^* - |\sigma_n| (\sum_{l\in P} q_l)(\sum_{l\in P} q_l)^*$, and $\sum_{l\in P} q_l = -q_n$. □

Theorem 5.40. *In Theorem 5.13 with $|Z| = 1$:*

(a) *if $\operatorname{rank}\rho_I = 5$ and σ_I is PPT entangled of rank 4, then $J(\Phi)$ has Schmidt number at most 2 and $\Psi \circ \Phi$ is EB;*
(b) *if $\operatorname{rank}\sigma_I = 5$ and ρ_I is PPT entangled of rank 4, then $J(\Psi^*)$ has Schmidt number at most 2 and $\Psi \circ \Phi$ is EB.*

Proof. (a) After the filter that makes $\lambda_a = 1$ on I (a local filter of $J(\Phi)$, which preserves Schmidt numbers and the property that $\Psi \circ \Phi$ is EB), Lemma 5.3 gives $\operatorname{range}\rho_I = K := \ker\sigma_I$ and $\operatorname{range}\rho_I^\Gamma \subseteq K' := \ker\sigma_I^\Gamma$. By Lemma 5.38, $\rho_I = \sum_m t_m |p_m\rangle\langle p_m|$ with t_m real; and $J_{Ij} \in \mathcal{S}(K, K')$, since $\operatorname{range} J_{Ij} \subseteq \operatorname{range}\rho_I$ and $\operatorname{range} J_{Ij}^\sharp \subseteq \operatorname{range}\rho_I^\Gamma$ by positivity of $J(\Phi)$ and $J(\Phi)^\Gamma$; so $J_{Ij} = \sum_m \zeta_m |p_m\rangle\langle w_m|$ by Lemma 5.37.

Signs. Put $q_m := \mu_m p_m$ (so $\sum_m q_m = 0$, and any five form a basis of K) and $s_m := t_m/|\mu_m|^2$, so $\rho_I = \sum_m s_m q_m q_m^*$. If $s_l, s_m < 0$ for some $l \neq m$, take $z \in K$ orthogonal to the other four q's; then $\langle z, q_l\rangle = -\langle z, q_m\rangle \neq 0$ and $\langle z, \rho_I z\rangle < 0$. The same test excludes a negative s_n together with a zero s_l. If $s_n < 0$ and all other $s_m > 0$, then in the basis $(q_m)_{m\neq n}$ the corner is $D + s_n \mathbf{1}\mathbf{1}^\top$ with $D = \operatorname{diag}(s_m)_{m\neq n}$, which is positive semidefinite of rank 5 if and only if $H := \sum_m 1/s_m = \sum_m |\mu_m|^2/t_m < 0$. So either all $t_m \geq 0$, or exactly one $t_n < 0$, the others are positive, and $H < 0$.

If all $t_m \geq 0$ and some $t_m = 0$, then ρ_I is separable and Lemma 5.32(a) applies. Otherwise all $t_m \neq 0$ and $H \neq 0$ ($H > 0$ when all $t_m > 0$). Put

$\eta_m := \bar{\zeta}_m/t_m$, $\hat{x}_m := u_m + \eta_m e_j \in \mathbb{C}^4$, $\varphi_m := \hat{x}_m \otimes w_m$, $v := \sum_m \mu_m \eta_m w_m$ and $\varphi_0 := e_j \otimes v$; then $\sum_m \mu_m \varphi_m = \varphi_0$. Let

$$J_A := \sum_{m=1}^{6} t_m |\varphi_m\rangle\langle\varphi_m| - \frac{1}{H}|\varphi_0\rangle\langle\varphi_0|.$$

With $q_0 := \varphi_0$, $q_m := -\mu_m \varphi_m$, $\sigma_0 := -1/H$ and $\sigma_m := s_m$ we have $\sum_{m=0}^{6} q_m = 0$, $J_A = \sum_m \sigma_m q_m q_m^*$, $\sum_m 1/\sigma_m = -H + H = 0$, and exactly one σ_m is negative (σ_0 if all $t_m > 0$, σ_n otherwise). By Lemma 5.39, J_A is a positive combination of the operators $|\chi_{lm}\rangle\langle\chi_{lm}|$, and each χ_{lm} is a combination of two product vectors, so it has Schmidt rank at most 2. The corner of J_A on I is $\sum_m t_m |p_m\rangle\langle p_m| = \rho_I$, and its block (I, j) is $\sum_m t_m \bar{\eta}_m |p_m\rangle\langle w_m| = J_{Ij}$. Its rank is at least $\operatorname{rank}\rho_I = 5$, and at most 5 because its range is spanned by the χ_{lm}, differences of six vectors. A positive semidefinite completion of ρ_I and J_{Ij} of rank $\operatorname{rank}\rho_I$ has (j, j) block $J_{jI}\rho_I^+ J_{Ij}$, so $J(\Phi) - J_A = E_{jj} \otimes S$ with $S \succeq 0$. Hence $J(\Phi)$ has Schmidt number at most 2.

By Theorem 5.30 and Lemma 5.28, $\Psi = \mathrm{Ad}_M \circ \Psi_I + \Psi_R$ with $\Psi_I \colon \mathrm{M}_3 \to \mathrm{M}_I$ PPT and Ψ_R EB, and Lemma 5.32(b) guarantees that $\Psi_I \circ \Phi$ is EB. So $\Psi \circ \Phi$ is EB.

(b) After the same filter, ρ_I is still PPT entangled of rank 4, and $\rho_I \sigma_I = 0 = \rho_I^\Gamma \sigma_I^\Gamma$. The argument of (a), with the roles of the two corners exchanged and the Gram data of Ψ^*, shows that $J(\Psi^*)$ has Schmidt number at most 2. By Theorem 5.30 and the mirror of Lemma 5.28, $\Phi = \Phi_I \circ \mathrm{Ad}_{N^*} + \Phi_R$; by Lemma 5.32(b) for Ψ^* and the PPT map $\Phi_I^* \colon \mathrm{M}_3 \to \mathrm{M}_3$, the map $\Phi_I^* \circ \Psi^*$ is EB, hence its adjoint $\Psi \circ \Phi_I$ is EB, and so is $\Psi \circ \Phi$. □

Remark 5.41. The proof of Theorem 5.40 uses the relation only through $\rho_I \sigma_I = 0 = \rho_I^\Gamma \sigma_I^\Gamma$. So, for any PPT maps $\Phi \colon \mathrm{M}_4 \to \mathrm{M}_3$ and $\Psi \colon \mathrm{M}_3 \to \mathrm{M}_4$: if σ_I is PPT entangled of rank 4 and, after a diagonal filter of the inputs I of Φ, $\rho_I \sigma_I = 0$ and $\rho_I^\Gamma \sigma_I^\Gamma = 0$, then $\Psi \circ \Phi$ is EB. Indeed $\rho_I = \sum_m t_m |p_m\rangle\langle p_m|$ by Lemma 5.38; if all $t_m \geq 0$, ρ_I is separable and Corollary 5.33 applies; if $\operatorname{rank}\rho_I = 5$, the proof above applies; and if one $t_n < 0$ and $\operatorname{rank}\rho_I \leq 4$, then $H = 0$ and ρ_I is PPT entangled of rank 4 (its expansion in the six independent projectors $|p_m\rangle\langle p_m|$ is unique, and a separable decomposition would use only product vectors of K), so Theorems 5.30 and 5.29 apply. The entangled branch of the proof of Theorem 5.40 occurs: the rank-5 corner is PPT entangled, with one negative t_n, in an exact PPT operator of this form (†12) and, numerically, at one of the two points of †8.

5.11. **Full-support relations.** In this subsection $d = 4$, $\operatorname{rank} B = 3$ and $B_{aa} > 0$ for all a. Then the relation $\sum_a \lambda_a T_a = 0$ is unique up to a factor. Theorem 5.13 treats relations with some $\lambda_a = 0$; here all $\lambda_a \neq 0$, and we call the pair an *FS pair*. The diagonal filter that makes $\lambda = \mathbf{1}$ preserves the PPT property and the separability of the twirl, so we assume $\lambda = \mathbf{1}$. Then

$$\textstyle\sum_a T_a = 0, \qquad \rho\sigma = 0, \qquad B\mathbf{1} = 0$$

(Lemma 5.3 with $I = \{1, \dots, 4\}$). We write $\Omega := \sum_a |aa\rangle$, $\rho_{ac} := G_a G_c^*$ and $\sigma_{ac} := H_a H_c^*$ for the blocks of ρ and σ, $\Pi_a := E_{aa} \otimes I_3$, and $D_e := \mathrm{diag}(e)$ for $e \in \mathbb{C}^4$. Let

$$\mathfrak{M}(e) := \Big\{ X \in \mathrm{M}_3 : \sum_a e_a G_a^* X H_a = 0 \Big\} = \{ X \in \mathrm{M}_3 : \ \rho(D_e \otimes X)\sigma = 0 \}.$$

The two descriptions agree for all Gram data: with the block columns $\mathbf{G} = [G_a]_a$ and $\mathbf{H} = [H_a]_a$, $\sum_a e_a G_a^* X H_a = \mathbf{G}^*(D_e \otimes X)\mathbf{H}$, and $\mathbf{G}^* Y \mathbf{H} = 0$ if and only if $\mathbf{G}\mathbf{G}^* Y \mathbf{H}\mathbf{H}^* = 0$. So $\mathfrak{M}(e)$ depends only on Φ and Ψ, and $I \in \mathfrak{M}(\mathbf{1})$. We say that an FS pair satisfies condition (G) if $\mathfrak{M}(e) = \{0\}$ for every $e \notin \mathbb{C}\mathbf{1}$ and $\mathfrak{M}(t\mathbf{1}) = \mathbb{C}I$ for every $t \neq 0$. Condition (G) says that these kernels are as small as the relation allows. We do not know whether it holds for generic FS pairs; numerically it holds at the 20 residual points of †1 (†24).

This subsection does not settle FS pairs, which remain open (Section 11); it explains why the methods of this paper do not reach them. Theorem 5.42 gives equivalent forms of the relation and of the twirl. The main result, Theorem 5.43, shows under condition (G) that in a decomposition of $\mathrm{tw}(W)$ into twirls of composites with diagonal outer filters and completely positive middle maps, every term with nonzero coherence matrix is a multiple of a diagonal filter of $\mathrm{tw}(W)$ itself, and that no such decomposition exists when these middle maps are PPT or have Kraus operators of rank at most 2; so these operations cannot reduce FS pairs to [1, 2] or to [3]. Proposition 5.44 shows, for data (A, B), that the PPT property and separable corner twirls do not suffice either. Propositions 5.45–5.47 and Lemmas 5.48 and 5.49 treat special cases and constraints on supports and certificates.

Theorem 5.42. *Let K_p and L_q be Kraus operators of Φ and Ψ with $K_p e_a = G_a e_p$ and $L_q^* e_a = H_a e_q$.*

(a) *The following are equivalent: $\sum_a T_a = 0$; $\rho\sigma = 0$; $\langle\Omega|W|\Omega\rangle = 0$; $\mathrm{Tr}(L_q K_p) = 0$ for all p, q.*

(b) *(given [3]) Suppose that $d = 4$, $\mathrm{rank}\, B = 3$, all $B_{aa} > 0$, and that the conditions of (a) hold. Let U be a unitary 4×4 matrix. The pair $(\Phi \circ \mathrm{Ad}_U, \mathrm{Ad}_{U^*} \circ \Psi)$ is PPT, its composite has Choi matrix $W_U = (U^\top \otimes U^*)W(\bar{U} \otimes U)$, and $\langle\Omega|W_U|\Omega\rangle = 0$. So its coherence matrix B_U satisfies $B_U\mathbf{1} = 0$, and either $\mathrm{tw}(W_U)$ is separable, by Theorem 1.1 or by Lemma 2.3 and [1, 2], or the rotated pair is again an FS pair with relation $\mathbf{1}$.*

(c) *$\mathrm{tw}(W) = X(A, B)$ with $A_{ab} = \mathrm{Tr}(\rho_{aa}\sigma_{bb})$ for $a \neq b$ and $B_{ac} = \mathrm{Tr}(\Pi_a \rho \Pi_c \sigma)$.*

(d) *If $\sum_a T_a = 0$ and $X(A, B) = \sum_n \mathrm{tw}(|x_n \otimes y_n\rangle\langle x_n \otimes y_n|) + \sum_{a \neq b} c_{ab}|ab\rangle\langle ab|$ with $c_{ab} \geq 0$, then $\sum_a x_n(a) y_n(a) = 0$ for every n.*

Proof. (a) The equivalence of the first two conditions is Lemma 5.3 with $I = \{1, \dots, 4\}$. By Lemma 2.1, $\langle\Omega|W|\Omega\rangle = \sum_{a,c} B_{ac} = \| \sum_a T_a \|^2$. Finally $G_a[m, p] = K_p[m, a]$ and $H_a[m, q] = \overline{L_q[a, m]}$, so $(\sum_a T_a)_{pq} = \overline{\mathrm{Tr}(L_q K_p)}$.

(b) For a linear map Λ on M_4, comparing entries gives $J(\mathrm{Ad}_{U^*}\circ\Lambda\circ\mathrm{Ad}_U) = (U^\top\otimes U^*)J(\Lambda)(\bar U\otimes U)$, and $(\bar U\otimes U)\Omega = (I\otimes UU^*)\Omega = \Omega$. So $\mathbf{1}^*B_U\mathbf{1} = \langle\Omega|W_U|\Omega\rangle = 0$, and $B_U\succeq 0$ gives $B_U\mathbf{1} = 0$. If $\operatorname{rank} B_U\le 2$, Theorem 1.1 applies; if some $(B_U)_{aa} = 0$, zero-row compression reduces to outer dimension at most 3; otherwise the relation of the rotated pair is unique and equals $\mathbf{1}$.

(c) Lemma 2.1 gives $\langle ab|W|cd\rangle = \mathrm{Tr}(\rho_{ac}\sigma_{db})$, and $\Pi_a\rho\Pi_c = E_{ac}\otimes\rho_{ac}$.

(d) By Lemma 6.1, $B = \sum_n z_n z_n^*$ with $z_n = x_n\odot y_n$, so $0 = \mathbf{1}^*B\mathbf{1} = \sum_n|\mathbf{1}^*z_n|^2$. □

The invariance in (b) concerns the normalized relation. For a full-support relation λ the condition is $\langle\Omega_\lambda|W|\Omega_\lambda\rangle = 0$ with $\Omega_\lambda := \sum_a\lambda_a|aa\rangle$, and a change of the outer basis does not preserve it in general. By (d), the product vectors of a separable decomposition of $\mathrm{tw}(W)$, written as matrices $x_ny_n^\top$, are nilpotent.

For $s, s'\in\mathbb{C}^4$ and a completely positive map $\mathcal{N}$ on M_3 put

$$W_{s,s',\mathcal{N}} := J(\mathrm{Ad}_{D_{s'}}\circ\Psi\circ\mathcal{N}\circ\Phi\circ\mathrm{Ad}_{D_s}).$$

For $s = s' = \mathbf{1}_S$ ($S\subseteq\{1,\dots,4\}$) and $\mathcal{N} = \mathrm{id}$ this is the coordinate compression of W to S, as in Theorem 5.1; for $s = s' = \mathbf{1}$ and $\mathcal{N} = \mathrm{Ad}_X$ with $\operatorname{rank} X\le 2$ it is a composite through a qubit, as in Proposition 7.21.

Theorem 5.43 (a barrier for coordinate compressions, diagonal filters and middle maps)**.** *Let (Φ,Ψ) be an FS pair with relation $\mathbf{1}$, and suppose that*

$$\mathrm{tw}(W) = \sum_{j=1}^{J}\mathrm{tw}(W_{s_j,s_j',\mathcal{N}_j}) + \sum_{a\neq b}c_{ab}|ab\rangle\langle ab| \tag{4}$$

with $s_j, s_j'\in\mathbb{C}^4$, completely positive maps $\mathcal{N}_j$ on M_3 and $c_{ab}\ge 0$. Put $e_j := \overline{s_j\odot s_j'}$. Then every Kraus operator Y of $\mathcal{N}_j$, in any Kraus decomposition, satisfies $Y^*\in\mathfrak{M}(e_j)$. Consequently, if the pair satisfies condition (G):

(a) *if $e_j\notin\mathbb{C}\mathbf{1}$, then $\mathcal{N}_j = 0$; if $e_j = 0$, the j-th term has zero coherence matrix;*

(b) *if $e_j\in\mathbb{C}\mathbf{1}$ and $e_j\neq 0$, then $\mathcal{N}_j = c_j\,\mathrm{id}$ with $c_j\ge 0$, and the j-th term is $c_j(D_{s_j}\otimes D_{s_j'})\,\mathrm{tw}(W)(D_{s_j}\otimes D_{s_j'})^*$;*

(c) *a decomposition (4) does not exist if every $\mathcal{N}_j$ with $e_j\in\mathbb{C}\mathbf{1}\setminus\{0\}$ is PPT or has a Kraus decomposition by operators of rank at most 2.*

Proof. Apply $E^*\cdot E$ to (4). It annihilates the terms $|ab\rangle\langle ab|$ with $a\neq b$ and commutes with the twirl, so $B = \sum_j B^{(j)}$ with $B^{(j)} := E^*W_{s_j,s_j',\mathcal{N}_j}E\succeq 0$. Then $0 = \mathbf{1}^*B\mathbf{1} = \sum_j\mathbf{1}^*B^{(j)}\mathbf{1}$, and each summand is nonnegative, so $\mathbf{1}^*B^{(j)}\mathbf{1} = 0$ for every j. Fix j, write $s = s_j$, $s' = s_j'$, $e = e_j$, and let $\mathcal{N}_j = \sum_k\mathrm{Ad}_{Y_k}$. The map $\mathcal{N}_j\circ\Phi\circ\mathrm{Ad}_{D_s}$ has Gram data $[s_aY_1G_a\,|\,s_aY_2G_a\,|\,\cdots]$, and the adjoint $\Psi^*\circ\mathrm{Ad}_{D_{s'}^*}$ of $\mathrm{Ad}_{D_{s'}}\circ\Psi$ has Gram data $\bar s_a'H_a$. So the j-th pair has the coherent matrices $T_a^{(j)} = [e_aG_a^*Y_k^*H_a]_k$, and

$$\mathbf{1}^*B^{(j)}\mathbf{1} = \|\textstyle\sum_a T_a^{(j)}\|^2 = \sum_k\|\sum_a e_aG_a^*Y_k^*H_a\|^2.$$

So every Y_k^* lies in $\mathfrak{M}(e)$. Under (G): if $e \notin \mathbb{C}\mathbf{1}$, all Y_k vanish; if $e = 0$, all $T_a^{(j)}$ vanish; if $e = t\mathbf{1}$ with $t \neq 0$, then $Y_k \in \mathbb{C}I$, so $\mathcal{N}_j = c_j \,\mathrm{id}$, and $J(\mathrm{Ad}_{D'} \circ \Lambda \circ \mathrm{Ad}_D) = (D \otimes D')J(\Lambda)(D \otimes D')^*$ for diagonal D, D'. For (c): by (b) such a $\mathcal{N}_j$ equals $c_j \,\mathrm{id}$. If it is PPT, then $c_j = 0$, since $J(\mathrm{id})^\Gamma$ is the flip, which has the eigenvalue -1; if it has a Kraus decomposition by operators of rank at most 2, these lie in $\mathbb{C}I$ and vanish. So every term has zero coherence matrix, and $B = 0$, which contradicts $\operatorname{rank} B = 3$. □

For $J = 1$ and $s = s' = \mathbf{1}$ the proof gives: if $\mathfrak{M}(\mathbf{1}) = \mathbb{C}I$, the identity is the only completely positive map $\mathcal{N}$ on M_3 for which the pair $(\mathcal{N} \circ \Phi, \Psi)$ has coherence matrix B. Theorem 5.43 concerns decompositions of the form (4), with the same maps Φ and Ψ in every term, diagonal outer filters, completely positive middle maps and a diagonal remainder. Within this family it excludes, under (G), coordinate compressions to at most three indices with any completely positive middle maps (so the overshoot $\frac{1}{2}D_B$ of Theorem 5.1 cannot be removed this way), filters with non-constant $s \odot s'$, and middle maps not proportional to the identity, in particular the qubit compressions averaged in Proposition 7.21 and PPT middle maps. Such terms are separable by [1, 2], [3] and, for PPT middle maps, [16] with [3]. Theorem 5.43 does not cover non-diagonal outer maps, middle maps that are completely copositive, splits of Φ or Ψ (peels, Lemma 5.51, the decomposition of Lemma 7.1), the coherent sandwich of Lemma 5.50 with a general separable S, or direct constructions of product terms as in Lemma 6.3 and Corollary 7.3. For non-diagonal outer maps with the identity middle map, Proposition 5.47 below shows, when $\ker W = \mathbb{C}\Omega$, that a piece with a singular outer map X or Y must satisfy $X^*\bar{Y} = 0$; the complementary pieces defined before (5) satisfy this condition and are not excluded. If moreover Φ and Ψ span extreme rays and are not EB, Lemma 7.2 excludes peels at every index; condition (G) for a pair does not pass to its extreme summands, and the residual points of †1 are not known to be extreme.

Proposition 5.44. *Let* $\varepsilon = \frac{1}{4}$,

$$B^\star = \begin{pmatrix} 1 & i\varepsilon & -i\varepsilon & -1 \\ -i\varepsilon & 1 & i\varepsilon & -1 \\ i\varepsilon & -i\varepsilon & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}, \qquad A^\star_{ab} := |B^\star_{ab}| \quad (a \neq b).$$

Then $B^\star \succeq 0$ *has rank* 3, $B^\star\mathbf{1} = 0$ *and no zero entry, and* $X(A^\star, B^\star)$ *is PPT. For every set* S *of three indices the pair* $(A^\star_S, B^\star_S)$ *is pairwise completely positive, so its twirl is separable. But* $X(A^\star, B^\star)$ *is not separable.*

Proof. $B^\star = M^* B^\star_{123} M$ with $M = [I_3 \,|\, -\mathbf{1}]$, and $B^\star_{123}$ has leading principal minors 1, $\frac{15}{16}$, $\frac{13}{16}$; so $B^\star \succeq 0$, $\operatorname{rank} B^\star = 3$ and $B^\star\mathbf{1} = 0$. $X(A^\star, B^\star)$ is PPT because $A^\star_{ab}A^\star_{ba} = |B^\star_{ab}|^2$. The triangle products $B^\star_{ij}B^\star_{jk}B^\star_{ki}$ are $-\frac{i}{64}$, $\frac{i}{4}$, $-\frac{i}{4}$, $\frac{i}{4}$ for $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, none of them a positive real, and the comparison matrix $\mathcal{M}(B^\star)$ (diagonal entries $B^\star_{aa}$, off-diagonal entries

$-|B^\star_{ab}|$) satisfies $\mathbf{1}^\top \mathcal{M}(B^\star)\mathbf{1} = -\frac{3}{2} < 0$. By Proposition 7.16, $X(A^\star, B^\star)$ is not separable. Corners: take $v = (1,1,1)$ for $S = \{1,2,3\}$ and $v = (1,1,\frac{3}{4})$ for $S \ni 4$ (with 4 last). Then $\mathcal{M}(B^\star_S)v \geq 0$ entrywise. For $i < j$ in S let Z_{ij} be the rank-one positive semidefinite matrix on $\{i,j\}$ with diagonal $(|B^\star_{ij}|v_j/v_i, |B^\star_{ij}|v_i/v_j)$ and off-diagonal entry $B^\star_{ij}$. Then $B^\star_S - \sum_{i<j} Z_{ij}$ is diagonal with entries $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, respectively $(0, 0, \frac{1}{3})$. Each $Z_{ij} = zz^*$ with $z = x \odot y$ supported on $\{i,j\}$, and x, y can be scaled so that $|x_i|^2|y_j|^2 = |x_j|^2|y_i|^2 = |B^\star_{ij}|$; each diagonal entry r_i is the term $\sqrt{r_i}\, e_i$ with no off-diagonal cost. So each pair (i,j) carries the cost $|B^\star_{ij}| = A^\star_{ij}$, and Lemma 6.1 applies. □

So the PPT property of $\mathrm{tw}(W)$ and the separability of the four corner twirls, which [1, 2] provide, do not by themselves settle the full-support case. Proposition 5.44 is a statement about data (A, B): we do not claim that $(A^\star, B^\star)$ is the twirl of a PPT composite.

Proposition 5.45. *Let Φ and Ψ be PPT with $\rho\sigma = 0$, at any coherence rank. Suppose that σ spans an extreme ray of the cone of PPT operators on $\mathbb{C}^4 \otimes \mathbb{C}^3$ and that Ψ is not EB. If $p \otimes u \in \operatorname{range}\sigma$ with $p \in \mathbb{C}^4$ and $u \in \mathbb{C}^3$ nonzero, then $\bar{p} \otimes u \in \ker\rho^\Gamma$ and $\bar{p} \otimes u \notin \operatorname{range}\sigma^\Gamma$; in particular $\operatorname{rank}\rho^\Gamma + \operatorname{rank}\sigma^\Gamma \leq 11$. The same holds with the roles of ρ and σ exchanged, and with $(\rho^\Gamma, \sigma^\Gamma)$ in place of (ρ, σ), with the conclusion $\operatorname{rank}\rho + \operatorname{rank}\sigma \leq 11$. Consequently:*

(a) *at a pair of extreme rays with both factors not EB and complementary supports ($\operatorname{rank}\rho + \operatorname{rank}\sigma = \operatorname{rank}\rho^\Gamma + \operatorname{rank}\sigma^\Gamma = 12$), none of the ranges of ρ, σ, ρ^Γ, σ^Γ contains a product vector, and neither do their kernels, which are these ranges;*

(b) *at a pair of extreme rays with both factors not EB, if one of the four ranks is at least 7, then the rank sum of the other pair is at most 11.*

Proof. Let $p \otimes u \in \operatorname{range}\sigma = \operatorname{range}\mathbf{H}$, and pick q with $H_a q = p_a u$ for all a. Then $\sum_a p_a G_a^* u = (\sum_a T_a)q = 0$ by Theorem 5.42(a). Since the (a,c) block of ρ^Γ is $G_c G_a^*$, $\langle \bar{p} \otimes u, \rho^\Gamma(\bar{p} \otimes u)\rangle = \|\sum_a p_a G_a^* u\|^2 = 0$, so $\bar{p} \otimes u \in \ker\rho^\Gamma$. Suppose $\bar{p} \otimes u \in \operatorname{range}\sigma^\Gamma$. With $v = p \otimes u$ we have $(vv^*)^\Gamma = |\bar{p} \otimes u\rangle\langle \bar{p} \otimes u|$, so $\sigma - tvv^*$ and its partial transpose are positive semidefinite for small $t > 0$. Then $\sigma = (\sigma - tvv^*) + tvv^*$ is a sum of two PPT operators, and extremality makes σ proportional to the product state vv^*, so Ψ is EB, a contradiction. Since $\operatorname{range}\sigma^\Gamma \subseteq \ker\rho^\Gamma$ (Lemma 5.3), the rank bound follows. For the mirror statement, $G_a g = p_a u$ gives $\sum_a p_a H_a^* u = (\sum_a T_a)^* g = 0$. For $(\rho^\Gamma, \sigma^\Gamma)$: these are the Choi matrices of the PPT pair obtained by composing with the transpose, which again satisfies $\rho^\Gamma\sigma^\Gamma = 0$, and the partial transpose preserves the cone of PPT operators, its extreme rays and the EB property. For (a) and (b): a subspace of $\mathbb{C}^4 \otimes \mathbb{C}^3$ of dimension at least 7 contains a product vector, since the Segre variety $\mathbb{P}^3 \times \mathbb{P}^2 \subset \mathbb{P}^{11}$ has dimension 5. □

Proposition 5.46. *Suppose that, after the filter, $\rho = \alpha P$ and $\sigma = \beta(I - P)$ for an orthogonal projection P on $\mathbb{C}^4 \otimes \mathbb{C}^3$ and $\alpha, \beta > 0$. Then $B_{ac} =$*

$-\alpha\beta\|\Pi_a P\Pi_c\|_2^2 \le 0$ *for* $a \neq c$, *so* $\mathrm{tw}(W)$ *is separable (Remark 6.2). Moreover* P^Γ *is a projection.*

Proof. By Theorem 5.42(c),

$$B_{ac} = \alpha\beta\,\mathrm{Tr}(\Pi_a P\Pi_c(I-P)) = \alpha\beta\big[\delta_{ac}\,\mathrm{Tr}(\Pi_a P) - \mathrm{Tr}((\Pi_a P\Pi_c)(\Pi_c P\Pi_a))\big].$$

The twirl is PPT because W is. Lemma 5.3 gives $\rho^\Gamma\sigma^\Gamma = 0$, that is, $P^\Gamma(I - P^\Gamma) = 0$. □

Spans of orthonormal product vectors give such projections, with Φ EB; we do not know whether Proposition 5.46 applies to pairs with both factors not EB.

Finally, we record three facts on compressions for any outer dimension d. For linear maps X, Y on $\mathbb{C}^d$ the *compression* of W is $W_{X,Y} := (X\otimes Y)W(X\otimes Y)^*$. For an orthogonal projection P with $0 < \mathrm{rank}\,P < d$ and $P^\perp := I - P$, the *complementary piece* $W_P := W_{\bar P, P^\perp} = J(\mathrm{Ad}_{P^\perp}\circ\Psi\circ\Phi\circ\mathrm{Ad}_P)$ is the composite with input restricted to $\mathrm{range}\,P$ and output restricted to its orthogonal complement. A *complementary certificate* for W is an identity

$$\mathrm{tw}(W) = \sum_j c_j\,\mathrm{tw}(W_{P_j}) + \sum_{a\neq b} d_{ab}|ab\rangle\langle ab|, \qquad c_j, d_{ab} \ge 0, \tag{5}$$

that is, $\sum_j c_j B^{(j)} = B$ and $\sum_j c_j A^{(j)}_{ab} \le A_{ab}$ for $a \neq b$, where $(A^{(j)}, B^{(j)})$ are the matrices of $\mathrm{tw}(W_{P_j})$. Its *slack* is the largest t with $\sum_j c_j A^{(j)}_{ab} \le (1-t)A_{ab}$ for all $a \neq b$, and $t^*(W)$ is the supremum of the slack over all complementary certificates.

Proposition 5.47. *Let* $W \succeq 0$ *on* $\mathbb{C}^d\otimes\mathbb{C}^d$ *with* $\ker W = \mathbb{C}\Omega$. *Suppose that* $\mathrm{tw}(W) = \sum_j c_j\,\mathrm{tw}(W_{X_j,Y_j}) + \sum_{a\neq b} d_{ab}|ab\rangle\langle ab|$ *with* $c_j > 0$, $d_{ab} \ge 0$, *and* X_j *or* Y_j *singular for each* j. *Then* $X_j^*\bar Y_j = 0$ *for every* j. *For* $X = \bar P$ *and* $Y = Q$ *with orthogonal projections* P, Q *this says* $QP = 0$.

Proof. Apply $E^*\cdot E$; it annihilates the terms $|ab\rangle\langle ab|$ with $a \neq b$, and $E^*\,\mathrm{tw}(Z)E = E^*ZE$ for every Z. So $B = \sum_j c_j E^*W_{X_j,Y_j}E$ with positive semidefinite summands, and $0 = \mathbf{1}^*B\mathbf{1} = \langle\Omega|W|\Omega\rangle$ forces $0 = \langle\Omega|W_{X_j,Y_j}|\Omega\rangle = \langle\omega_j|W|\omega_j\rangle$ with $\omega_j := (X_j^*\otimes Y_j^*)\Omega$ for every j. Hence $\omega_j \in \ker W = \mathbb{C}\Omega$. With $\mathrm{vec}(M) := \sum_{x,y} M_{xy}|xy\rangle$ we have $\omega_j = \mathrm{vec}(X_j^*\bar Y_j)$ and $\Omega = \mathrm{vec}(I)$, so $X_j^*\bar Y_j = \mu_j I$. Its rank is at most $\min(\mathrm{rank}\,X_j, \mathrm{rank}\,Y_j) < d$, so $\mu_j = 0$. For $X = \bar P$ and $Y = Q$, $X^*\bar Y = \overline{PQ}$, and $PQ = 0$ if and only if $QP = 0$. □

At the residual points of coherence rank 3 examined in †3, W has rank 15 (numerically), so $\ker W = \mathbb{C}\Omega$. For the identity middle map and diagonal $X = D_s$, $Y = D_{s'}$ one has $\langle\omega|W|\omega\rangle = \|\sum_a e_a T_a\|^2$ with $e = \overline{s\odot s'}$, which is the quantity of Theorem 5.43. Proposition 5.47 extends this case from diagonal to arbitrary outer maps with X or Y singular, under $\ker W = \mathbb{C}\Omega$ instead of condition (G). It does not concern middle maps other than the identity, nor pairs with X and Y both invertible.

Lemma 5.48. *Let $d \le 5$ and let $W \succeq 0$ on $\mathbb{C}^d \otimes \mathbb{C}^d$ with $W^\Gamma \succeq 0$. If W has a complementary certificate* (5)*, then* $\mathrm{tw}(W)$ *is separable.*

Proof. Let $P = P_j$ have rank k. Then W_P is positive semidefinite and supported on range $\bar{P} \otimes$ range $P^\perp \cong \mathbb{C}^k \otimes \mathbb{C}^{d-k}$, and it is PPT, since $(W_{X,Y})^\Gamma = (\bar{X} \otimes Y) W^\Gamma (\bar{X} \otimes Y)^*$. For $d \le 5$, either $\min(k, d-k) = 1$, and every positive operator on $\mathbb{C}^1 \otimes \mathbb{C}^m$ is separable, or $\{k, d-k\} \subseteq \{2, 3\}$, and PPT operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ are separable [17, 18, 19]. So W_P and $\mathrm{tw}(W_P)$ are separable, and so is the right side of (5). □

For composites the pieces are separable without the dimension count when $k \le 2$ or $d - k \le 2$: then $\Phi \circ \mathrm{Ad}_P$ is a PPT map from M_k into M_3, respectively $\mathrm{Ad}_{P^\perp} \circ \Psi$ is a PPT map from M_3 into M_{d-k}, and such maps are EB [17, 18, 19].

Lemma 5.49 (twirling does not create certificates)**.** *For every $W \succeq 0$, $t^*(W) \ge t^*(\mathrm{tw}(W))$. In particular, if the operator* $\mathrm{tw}(W)$ *has a complementary certificate, so does W, with the same slack.*

Proof. Let D be a diagonal unitary and $Q := DPD^*$. Since $D^* = \bar{D}$ and $D^\top = D$, $(\bar{P} \otimes P^\perp)(D \otimes \bar{D}) = (D \otimes \bar{D})(\bar{Q} \otimes Q^\perp)$, so $(\mathrm{tw}(W))_P$ is the average over D (Haar measure on the torus) of $(D \otimes \bar{D}) W_{DPD^*} (D \otimes \bar{D})^*$. The matrices (A, B) of a twirl are linear in the operator and invariant under conjugation by $D \otimes \bar{D}$, so those of $\mathrm{tw}((\mathrm{tw}(W))_P)$ are the average of those of $\mathrm{tw}(W_{DPD^*})$. This average lies in the convex hull of the compact set formed by the matrices of $\mathrm{tw}(W_Q)$, Q in the torus orbit of P, so by Carathéodory's theorem it is a finite convex combination of them. Substituting in a certificate for $\mathrm{tw}(W)$ gives a certificate for W with the same equation for B and the same costs, hence the same slack, since $\mathrm{tw}(W)$ and W have the same matrices (A, B). □

5.12. **Three further tools.**

Lemma 5.50 (coherent sandwich)**.** *If S is separable, $S \preceq W$ and $E^*SE = E^*WE$, then* $\mathrm{tw}(W)$ *is separable. Moreover $WEB^+E^*W \preceq S$ for every such S.*

Proof. $W - S \succeq 0$ has zero coherence matrix, so $(W-S)E = 0$ and $\mathrm{tw}(W-S)$ is diagonal and nonnegative, while $\mathrm{tw}(S)$ is separable. For the inequality, $SE = WE$ and $S \succeq SE(E^*SE)^+E^*S$. □

Lemma 5.51 (coordinate decompositions)**.** *Let $d = 4$ and suppose $\rho = \sum_j \rho_j$, where each ρ_j is PPT and either separable or has a zero block row at some outer coordinate. Then* $\mathrm{tw}(W)$ *is separable. The same holds on the Ψ side.*

Proof. Each $\rho_j = J(\Phi_j)$ with Φ_j PPT, and $W = \sum_j J(\Psi \circ \Phi_j)$. If ρ_j is separable, then $\Psi \circ \Phi_j$ is EB. A zero block row at a_j gives $T^{(j)}_{a_j} = 0$; then Lemma 2.3 reduces the pair to three outer indices, where [1, 2] apply. □

Lemma 5.52 (middle-block form)**.** *For $a, b \in \{1,2,3\}$ put $R_{ab} := \sum_{i,j} \Phi(E_{ij})_{ab} E_{ij} \in \mathrm{M}_d$ and $C_{ab} := \Psi(E_{ab}) \in \mathrm{M}_d$. Then $W = \sum_{a,b} R_{ab} \otimes C_{ab}$ and $\mathrm{tw}(W) = \sum_{a,b} \Theta(R_{ab}, C_{ab})$, where*

$$\Theta(X,Y) := \sum_{i \neq j} X_{ii} Y_{jj} |ij\rangle\langle ij| + \sum_{i,k} X_{ik} Y_{ik} |ii\rangle\langle kk|.$$

The terms $\Theta(R_{aa}, C_{aa}) = \mathrm{tw}(R_{aa} \otimes C_{aa})$ are separable.

Proof. $W = \sum_{i,j} E_{ij} \otimes \Psi(\Phi(E_{ij}))$ and $\Phi(E_{ij}) = \sum_{a,b} \Phi(E_{ij})_{ab} E_{ab}$; and $\mathrm{tw}(X \otimes Y) = \Theta(X,Y)$ because $\langle ij|X \otimes Y|kl\rangle = X_{ik} Y_{jl}$. The operators R_{aa} and C_{aa} are compressions of $J(\Phi)$ and $J(\Psi)$, hence positive semidefinite. □

6. Frame certificates

For a matrix A with nonnegative off-diagonal entries and $A_{aa} = B_{aa}$, and $B \succeq 0$, the operator $X(A,B)$ of (1) is invariant under $D \otimes \bar{D}$. For vectors $x, y \in \mathbb{C}^d$ write $x \odot y$ for the entrywise product.

Lemma 6.1. $\mathrm{tw}(|x \otimes y\rangle\langle x \otimes y|) = X\big([|x_a|^2 |y_b|^2]_{a,b},\ (x \odot y)(x \odot y)^*\big)$. *Consequently $X(A,B)$ is separable if and only if there are $x_n, y_n \in \mathbb{C}^d$ with*

$$B = \sum_n (x_n \odot y_n)(x_n \odot y_n)^* \quad \text{and} \quad \sum_n |x_n(a)|^2 |y_n(b)|^2 \leq A_{ab} \ (a \neq b).$$

Proof. The first identity follows from $\langle ab|x \otimes y\rangle = x_a y_b$ and the twirl formula. If the conditions hold, $X(A,B)$ is a sum of twirled product states and of nonnegative multiples of $|ab\rangle\langle ab|$, $a \neq b$. Conversely, if $X(A,B) = \sum_n |x_n \otimes y_n\rangle\langle x_n \otimes y_n|$, apply tw, which fixes $X(A,B)$, and compare entries. □

Lemma 6.1 is [23, Theorem 6(c)], with [23, Lemma 3] for the inequalities; we include the proof for completeness. Pairs (A,B) satisfying its condition are called pairwise completely positive [23].

Remark 6.2. If $B = D_\theta M D_\theta^*$ for a diagonal unitary D_θ and a matrix M with $M_{ab} \leq 0$ for $a \neq b$, then $X(A,B)$ is separable as soon as it is PPT, that is, as soon as $A_{ab} A_{ba} \geq |B_{ab}|^2$ for all $a \neq b$. Indeed $M \succeq 0$ is a singular or nonsingular M-matrix on each connected component of its support graph, so there is v with positive entries and $Mv \geq 0$; then $\mathrm{diag}(v) M \,\mathrm{diag}(v)$ is diagonally dominant and decomposes into a nonnegative diagonal and positive semidefinite 2×2 blocks with off-diagonal entries $v_a v_b M_{ab}$. Each block is the coherence matrix of one product term on $\{a,b\}$ whose costs $|x_a|^2 |y_b|^2$ and $|x_b|^2 |y_a|^2$ can be chosen to be A_{ab} and $|B_{ab}|^2 / A_{ab} \leq A_{ba}$ after undoing the scaling by v. The local unitary $D_\theta \otimes I$ restores B. Here $\mathcal{M}(B) = M$ (diagonal entries B_{aa}, off-diagonal entries $-|B_{ab}|$) is positive semidefinite, so this case is also contained in [23, Theorem 5].

Lemma 6.3 (frame certificates)**.** *Let $B \succeq 0$ and $B = FF^*$ with $F \in \mathbb{C}^{d \times r}$, and put $K_{ab} := \sum_n |F_{an}||F_{bn}|$. Suppose that $A_{ab} > 0$ whenever $K_{ab} > 0$, and*

that every oriented cycle $(a_1, \dots, a_L)$ of distinct indices, $2 \le L \le d$, whose edges all have $K > 0$ satisfies

$$\prod_l A_{a_l a_{l+1}} \;\ge\; \prod_l K_{a_l a_{l+1}} \qquad (a_{L+1} := a_1).$$

Then $X(A,B) = \sum_{n=1}^{r} \operatorname{tw}(|x_n \otimes y_n\rangle\langle x_n \otimes y_n|) + \sum_{a \ne b} c_{ab} |ab\rangle\langle ab|$ with $c_{ab} \ge 0$; in particular it is separable.

Proof. The difference constraints $u_a - u_b \le \log(A_{ab}/K_{ab})$, one for each oriented edge with $K_{ab} > 0$, have no negative cycle, since a closed walk splits into simple oriented cycles; so they have a real solution u (Bellman–Ford). Put $x_n(a) = |F_{an}|^{1/2} e^{u_a/2} e^{i \arg F_{an}}$ and $y_n(a) = |F_{an}|^{1/2} e^{-u_a/2}$. Then $x_n \odot y_n$ is the n-th column of F, so $\sum_n (x_n \odot y_n)(x_n \odot y_n)^* = B$, and $\sum_n |x_n(a)|^2 |y_n(b)|^2 = K_{ab} e^{u_a - u_b} \le A_{ab}$. Lemma 6.1 gives the claim with $c_{ab} = A_{ab} - K_{ab} e^{u_a - u_b}$. □

At coherence rank one ($F = b$, $B = bb^*$) this is the cycle condition $A_{ab} \ge |b_a||b_b|\, w_a/w_b$ of the proof of Theorem 4.1. Two choices of F are useful.

- For any F, Cauchy–Schwarz gives $K_{ab} \le \sqrt{B_{aa} B_{bb}}$, so the cycle condition for $A_{ab}/\sqrt{B_{aa} B_{bb}}$ suffices; it requires $A_{ab} A_{ba} \ge B_{aa} B_{bb}$, which is stronger than PPT.
- For a pivot order, the pivoted Cholesky factor of B gives $r = \operatorname{rank} B$ terms. If $B\mathbf{1} = 0$ and $\operatorname{rank} B = 3$ at $d = 4$, every Schur complement keeps the all-ones vector of its index set in its kernel, so the three columns are generically supported on 4, 3 and 2 indices. Compare the recursive (Cholesky-type) construction of [23, Theorem 4], whose success also depends on the order of the indices [23, Example 3].

The certificate of Lemma 6.3 uses r product terms. For F the pivoted Cholesky factor, $r = \operatorname{rank} B$; Section 7 shows that $\operatorname{rank} B$ terms do not always suffice for separable twirls of PPT composites, so the condition of Lemma 6.3 is sufficient but not necessary. Numerically, at every residual point (a PPT pair from the searches of Section 9 at which no index peels on either side; the precise conditions are listed there) some pivot order satisfies the hypothesis of Lemma 6.3 with slack (†7). Random separable twirls pass the same test, so the test reflects the form of the certificate rather than a property of composites.

7. Extreme pairs, Kraus frames and the number of product terms

This section asks how far explicit product-state certificates reach. It reduces the question to extreme pairs (Lemma 7.1), shows that the certificates of Lemma 6.3 built from Kraus operators fail at some extreme pairs although the twirl is separable there (Theorem 7.10), and shows that the number of product terms can exceed the coherence rank (Section 7.4). It also contains

the constant $\frac{1}{5}$ (Proposition 7.21) and a factor-width obstruction (Proposition 7.22). Nothing in Sections 2–6 depends on the results of this section.

7.1. **Extreme pairs.**

Lemma 7.1. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the cones of PPT maps $\mathrm{M}_d \to \mathrm{M}_3$ and $\mathrm{M}_3 \to \mathrm{M}_d$. If $\mathrm{tw}(J(\Psi \circ \Phi))$ is separable whenever Φ and Ψ span extreme rays of $\mathcal{K}_1$ and $\mathcal{K}_2$, then it is separable for every PPT pair. The same holds for $J(\Psi \circ \Phi)$ itself, and for the pairs with $\rho\sigma = 0$: these reduce to pairs of extreme rays with $\rho\sigma = 0$.*

Proof. Under the Choi isomorphism $\mathcal{K}_1 = \{X : X \succeq 0,\ X^\Gamma \succeq 0\}$ is a closed convex cone containing no line, hence the set of finite sums of elements of its extreme rays [28]. The same holds for $\mathcal{K}_2$. Write $\Phi = \sum_k \Phi_k$ and $\Psi = \sum_l \Psi_l$ with extreme summands. Then $J(\Psi \circ \Phi) = \sum_{k,l} J(\Psi_l \circ \Phi_k)$, tw is linear, and sums of separable operators are separable. For the last claim, $\{\rho \in \mathcal{K}_1 : \mathrm{Tr}(\rho\sigma) = 0\}$ is a face of $\mathcal{K}_1$ (the values $\mathrm{Tr}(\rho_k\sigma) \geq 0$ of the summands add up to 0), so the summands of ρ can be taken in it; then decompose σ inside the face determined by each summand. □

Lemma 7.2. *Let Ψ span an extreme ray of $\mathcal{K}_2$ and not be EB. Then no index c with $B_{cc} > 0$ peels on the Ψ side. The mirror statement holds for Φ.*

Proof. A peel gives $\Psi = \Psi_{F_c} + \Psi_{(c)}$ with Ψ_{F_c} EB and $\Psi_{(c)}$ PPT (Lemma 3.1), and $\Psi^*_{F_c}(E_{cc}) = H_c H_c^* \neq 0$ because $B_{cc} > 0$. Extremality gives $\Psi_{F_c} = \lambda\Psi$ with $\lambda > 0$, so Ψ would be EB. □

7.2. **Kraus-frame certificates.** Write $M_{pq} := L_q K_p$ for the composite Kraus operators. Then

$$A_{ij} = \sum_{p,q} |M_{pq}[j,i]|^2, \qquad B_{ij} = \sum_{p,q} M_{pq}[i,i]\,\overline{M_{pq}[j,j]},$$

and we put

$$N_{ij} := \sum_{p,q} |T_i[p,q]|\,|T_j[p,q]| = \sum_{p,q} |M_{pq}[i,i]|\,|M_{pq}[j,j]|.$$

Corollary 7.3 (Kraus-frame condition)**.** *Let Φ and Ψ be completely positive, with any Gram data. If $\prod_{e\in C} A_e \geq \prod_{e\in C} N_e$ for every oriented cycle C of distinct indices (edges with $N_e = 0$ impose nothing; an edge with $N_e > 0 = A_e$ makes the condition fail), then $\mathrm{tw}(W)$ is separable, with an explicit certificate of PQ product terms.*

Proof. We apply Lemma 6.3 with $F := [\overline{T_a[p,q]}]_{a,(p,q)}$, for which $FF^* = B$ and $K = N$. □

Since W, A and B depend only on Φ and Ψ, any Gram data may be used, and the condition depends on that choice. The 2-cycle part, $A_{ij}A_{ji} \geq N_{ij}^2$, strengthens the partial-transpose inequality $A_{ij}A_{ji} \geq |B_{ij}|^2$ by replacing $|\sum_{p,q} \cdot|$ with $\sum_{p,q} |\cdot|$. Numerically, it binds at most of the residual points

of Section 9, but not at all of them (†7), so it does not by itself give the condition.

Lemma 7.4. *If (A, B) is pairwise completely positive and* $\operatorname{rank} B = 1$*, one product term suffices: $B = zz^*$ with $z = x \odot y$ and $|x_a|^2|y_b|^2 \le A_{ab}$ for $a \neq b$.*

Proof. A decomposition $B = \sum_n z_n z_n^*$ of a rank-one $B = zz^*$ has $z_n = c_n z$ with $\sum_n |c_n|^2 = 1$. For a in the support of z and $c_n \neq 0$ write $|x_n(a)|^2 = |c_n||z_a|e^{v_n(a)}$, so that $|y_n(a)|^2 = |c_n||z_a|e^{-v_n(a)}$. For $a \neq b$ in the support, Jensen's inequality with weights $|c_n|^2$ gives

$$A_{ab} \ge |z_a||z_b| \sum_n |c_n|^2 e^{v_n(a)-v_n(b)} \ge |z_a||z_b|\, e^{v(a)-v(b)}, \qquad v := \sum_n |c_n|^2 v_n.$$

On the support we set $|x_a|^2 = |z_a|e^{v(a)}$ and $|y_a|^2 = |z_a|e^{-v(a)}$, with the phases taken from z, and elsewhere we put $x = y = 0$. □

Lemma 7.5. *For every PPT pair (any d, any middle dimension) with coherence rank at most* 1*, the condition of Corollary 7.3 holds in every choice of Kraus bases.*

Proof. Here $T_a = \lambda_a T$ for one T, so $N_{ij} = |\lambda_i||\lambda_j|\|T\|_2^2 = |B_{ij}|$. The twirl is separable for every middle dimension [Supp, S1], so Lemma 7.4 gives one term $x \odot y = b$ with $B = bb^*$. Put $t_a := |x_a|^2/|b_a|$ on the support of b; then $N_{ij}t_i/t_j = |x_i|^2|y_j|^2 \le A_{ij}$, and indices outside the support carry $N = 0$. □

Lemma 7.6. *If $J(\Phi)$ has rank one, the condition of Corollary 7.3 holds in every Kraus basis of Ψ; similarly with the roles of Φ and Ψ exchanged.*

Proof. Being PPT, $J(\Phi) = |\alpha\rangle\langle\alpha| \otimes |g\rangle\langle g|$ is a product, so $P = 1$, $G_a = \alpha_a g$ and $T_a = \bar{\alpha}_a g^* H_a$. By Cauchy–Schwarz $N_{ij} \le |\alpha_i||\alpha_j|\|H_i^* g\|\|H_j^* g\|$, while $A_{ij} = |\alpha_i|^2\|H_j^* g\|^2$. With $t_i := |\alpha_i|/\|H_i^* g\|$ (indices with $T_i = 0$ carry $N = 0$) we get $N_{ij}t_i/t_j \le A_{ij}$. □

7.3. **The Kraus-frame condition at extreme pairs.** By Lemma 7.1, if the condition of Corollary 7.3 held at all pairs spanning extreme rays, then every twirl at outer dimension 4 would be separable. This route is closed: there are extreme pairs at which the condition fails for all Gram data, although the twirl is separable.

Lemma 7.7. *For every factorization $B = FF^*$, the matrix K of Lemma 6.3 satisfies $K_{ab} \ge |B_{ab}|$. In particular $N_{ab} \ge |B_{ab}|$ for all Gram data of Φ and Ψ, with any numbers of Kraus operators. Hence, if some oriented cycle C has $\prod_{e\in C} A_e < \prod_{e\in C} |B_e|$, the hypothesis of Lemma 6.3 fails for every F, and the condition of Corollary 7.3 fails for all Gram data.*

Proof. $K_{ab} = \sum_n |F_{an}||F_{bn}| \ge |\sum_n F_{an}\overline{F_{bn}}| = |B_{ab}|$, and the Kraus frame of any Gram data satisfies $FF^* = B$ and $K = N$. □

Let $\mathcal{C}_{m,n}$ be the cone of PPT operators on $\mathbb{C}^m \otimes \mathbb{C}^n$. The map $\Phi \mapsto J(\Phi)$ identifies the PPT maps $\mathrm{M}_m \to \mathrm{M}_n$ with $\mathcal{C}_{m,n}$, and $\Psi \mapsto J(\Psi^*)$ identifies

the PPT maps $\mathrm{M}_n \to \mathrm{M}_m$ with $\mathcal{C}_{m,n}$. In this notation $J(\Phi) = [\rho_{ab}]$ and $J(\Psi^*) = [\sigma_{ab}]$ with 3×3 blocks $\rho_{ab} = G_a G_b^*$ and $\sigma_{ab} = H_a H_b^*$, so that, by Lemma 2.1,

$$A_{ij} = \mathrm{Tr}(\rho_{ii}\sigma_{jj}), \qquad B_{ij} = \mathrm{Tr}(\rho_{ij}\sigma_{ji}). \tag{6}$$

Lemma 7.8. *An operator $\rho \in \mathcal{C}_{m,n}$ spans an extreme ray if and only if the real space $\mathcal{F}(\rho) := \{X = X^* : \mathrm{range}\, X \subseteq \mathrm{range}\, \rho,\ \mathrm{range}\, X^\Gamma \subseteq \mathrm{range}\, \rho^\Gamma\}$ equals $\mathbb{R}\rho$.*

Proof. If $X \in \mathcal{F}(\rho)$, then $\rho \pm \varepsilon X \in \mathcal{C}_{m,n}$ for small $\varepsilon > 0$, because ρ and ρ^Γ are positive definite on their ranges; so extremality forces $X \in \mathbb{R}\rho$. Conversely, if $\rho = \rho_1 + \rho_2$ with $\rho_i \in \mathcal{C}_{m,n}$, then $0 \preceq \rho_i \preceq \rho$ and $0 \preceq \rho_i^\Gamma \preceq \rho^\Gamma$, so $\rho_i \in \mathcal{F}(\rho)$; if $\mathcal{F}(\rho) = \mathbb{R}\rho$, both ρ_i are multiples of ρ. □

This is the criterion of [29].

Lemma 7.9. *Let $X \in \mathbb{C}^{d\times m}$ have rank m and $Y \in GL_n$. Then $L(\rho) := (X \otimes Y)\rho(X \otimes Y)^*$ maps $\mathcal{C}_{m,n}$ into $\mathcal{C}_{d,n}$, extreme rays to extreme rays, and entangled operators to entangled operators.*

Proof. $L(\rho)^\Gamma = (\bar{X} \otimes Y)\rho^\Gamma(\bar{X} \otimes Y)^*$. If $L(\rho) = \tau_1 + \tau_2$ with $\tau_i \in \mathcal{C}_{d,n}$, then $\mathrm{range}\, \tau_i \subseteq \mathrm{range}\, X \otimes \mathbb{C}^n$, and $\rho_i := (X^+ \otimes Y^{-1})\tau_i(X^+ \otimes Y^{-1})^* \in \mathcal{C}_{m,n}$ satisfy $\rho_1 + \rho_2 = \rho$ and $L(\rho_i) = \tau_i$. Extremality of ρ gives $\rho_i = c_i\rho$, so $\tau_i = c_i L(\rho)$. If $L(\rho)$ were separable, so would be $\rho = (X^+ \otimes Y^{-1})L(\rho)(X^+ \otimes Y^{-1})^*$. □

In map form, $(X \otimes Y)J(\Phi_0)(X \otimes Y)^* = J(\mathrm{Ad}_Y \circ \Phi_0 \circ \mathrm{Ad}_{X^\top})$, where $\mathrm{Ad}_M(Z) = MZM^*$. If Φ and Ψ span extreme rays and neither is EB, then $A_{ij} > 0$ whenever $i \neq j$ and $B_{ii}, B_{jj} > 0$: by Corollary 1.3(c) and Lemma 7.2, $\mathrm{rank}\, G_i \geq 2$ and $\mathrm{rank}\, H_j \geq 2$, and two subspaces of $\mathbb{C}^3$ of dimension at least 2 are not orthogonal. So at such pairs no failure of the condition comes from a zero of A.

Let $\rho_T := \frac{1}{4}\big(I - \sum_{k=1}^5 |\psi_k\rangle\langle\psi_k|\big)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ for the Tiles unextendible product basis $\psi_1 = e_1 \otimes (e_1 - e_2)/\sqrt{2}$, $\psi_2 = (e_1 - e_2) \otimes e_3/\sqrt{2}$, $\psi_3 = e_3 \otimes (e_2 - e_3)/\sqrt{2}$, $\psi_4 = (e_2 - e_3) \otimes e_1/\sqrt{2}$, $\psi_5 = (e_1 + e_2 + e_3) \otimes (e_1 + e_2 + e_3)/3$ [14], and let

$$X = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -2 \\ 2 & -1 & 2 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & 2 & 1 \end{pmatrix},$$

$$Z = \begin{pmatrix} 2 & 0 & 1 \\ 2 & -1 & 0 \\ 2 & 2 & 2 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} -2 & -2 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

For $d = 3$ put $J(\Phi) := (X \otimes Y)\rho_T(X \otimes Y)^\top$ and $J(\Psi^*) := (Z \otimes Y_2)\rho_T(Z \otimes Y_2)^\top$; for $d = 4$ append the row $(1, 1, 1)$ to X and the row $(1, 0, 1)$ to Z.

Theorem 7.10. *For $d = 3$ and for $d = 4$:*

(a) *Φ and Ψ are PPT, span extreme rays and are not EB; all $B_{aa} > 0$, B has no zero entry, the coherence rank is d, and no index peels;*

(b) $A_{13}A_{32}A_{21} = \frac{4586699}{27648} < \frac{1159140625}{2985984} = |B_{13}B_{32}B_{21}|$, *so the hypothesis of Lemma 6.3 fails for every factorization of* B*, and the condition of Corollary 7.3 fails for all Gram data of* Φ *and* Ψ*;*

(c) $\Psi \circ \Phi$ *is EB, by PPT*^{2} *in dimension* 3 [1, 2]*; in particular* W *and* $\mathrm{tw}(W)$ *are separable.*

Proof. (a) $(4\rho_T)^2 = 4\rho_T$ has rank 4 and $\rho_T^\Gamma = \rho_T$, so $\rho_T \in \mathcal{C}_{3,3}$; the space $\mathcal{F}(\rho_T)$ has dimension 1 (an exact computation, †13), so ρ_T spans an extreme ray by Lemma 7.8, and it is entangled [14]. The matrices X and Z have rank 3, and $\det Y = -2$, $\det Y_2 = -4$; by Lemma 7.9, $J(\Phi)$ and $J(\Psi^*)$ are PPT, span extreme rays and are entangled. The entries of A and B are exact rationals by (6) (†13); they give $B_{aa} > 0$, no zero entry and $\operatorname{rank} B = d$. No index peels by Lemma 7.2. (b) follows from the exact computation together with Lemma 7.7. (c) With $J(\Phi_T) = \rho_T$ we have $\Phi = \mathrm{Ad}_Y \circ \Phi_T \circ \mathrm{Ad}_{X^\top}$ and $\Psi = \mathrm{Ad}_Z \circ \Phi_T^* \circ \mathrm{Ad}_{Y_2^\top}$, so

$$\Psi \circ \Phi = \mathrm{Ad}_Z \circ \big[\Phi_T^* \circ \mathrm{Ad}_{Y_2^\top Y} \circ \Phi_T\big] \circ \mathrm{Ad}_{X^\top}\,.$$

The bracket is a composite of two PPT maps $\mathrm{M}_3 \to \mathrm{M}_3 \to \mathrm{M}_3$, hence EB [1, 2], and composing an EB map with completely positive maps keeps it EB. □

At $d = 3$, where PPT^2 holds, the condition of Corollary 7.3 thus fails at some extreme pairs for all Gram data. At $d = 4$ both factors of Theorem 7.10 have outer local rank 3 ($J(\Phi)$ and $J(\Psi^*)$ live on $S \otimes \mathbb{C}^3$ and $S' \otimes \mathbb{C}^3$ for 3-dimensional subspaces $S, S' \subset \mathbb{C}^4$, not coordinate subspaces), which is why PPT^2 on M_3 applies. An exact example of this kind at outer dimension 4 in which a factor has full outer local rank is not known; numerically, there are extreme pairs with full local ranks at which $\prod_{e\in C} A_e < \prod_{e\in C} |B_e|$ for a cycle C (†16).

Theorem 7.11. *In the spectral Kraus bases (eigenvectors of* $J(\Phi)$ *and* $J(\Psi^*)$*, unique up to phases here), the pairs of Theorem 7.10 violate the* 2*-cycle inequality* $A_{ij}A_{ji} \geq N_{ij}^2$*:*

	$N_{13}/\sqrt{A_{13}A_{31}}$	$N_{23}/\sqrt{A_{23}A_{32}}$
$d = 3$:	$1.2880251913\ldots$	$1.3796978034\ldots$
$d = 4$:	$1.2927712568\ldots$	$1.4323694730\ldots$

Moreover, at a pair $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$ *spanning extreme rays, with Gaussian-rational Choi matrices and full local ranks* $(4, 3)$ *for both factors, the spectral Kraus bases give* $N_{12}/\sqrt{A_{12}A_{21}} = 21.270550181\ldots$.

Proof (computer-assisted). The Choi matrices are exact, of exact rank 4 (respectively 6). Approximate eigenpairs from a high-precision eigen-solve are certified in interval arithmetic: the residuals and the gaps between the enclosures of the nonzero eigenvalues bound the angles between the computed and the true unit eigenvectors (the enclosures are disjoint, so the nonzero eigenvalues are simple), which gives enclosures of $|T_i[p, q]|\,|T_j[p, q]|$ and of

N_{ij}; these are compared with the exact $A_{ij}A_{ji}$ (†14). For the second pair we define $\rho_U := \frac{1}{6}(I-P)$ on $\mathbb{C}^4\otimes\mathbb{C}^3$, where P is the projector onto six pairwise orthogonal product vectors $a_k\otimes b_k$ with Gaussian-integer entries such that any four a_k span $\mathbb{C}^4$ and any three b_k span $\mathbb{C}^3$. So these vectors form an unextendible product basis, $(6\rho_U)^\Gamma$ is the projector onto the orthogonal complement of the orthogonal family $\bar{a}_k\otimes b_k$, and $\rho_U\in\mathcal{C}_{4,3}$. The operator ρ_U is entangled, since its range contains no product vector [14]. The face system of Lemma 7.8 for ρ_U has rank 35 modulo three primes; the rank over $\mathbb{Q}$ is at least that and at most 35, since $\rho_U\in\mathcal{F}(\rho_U)$, so $\dim\mathcal{F}(\rho_U)=1$. The factors are $(X\otimes Y)\rho_U(X\otimes Y)^*$ and $(Z\otimes Y_2)\rho_U(Z\otimes Y_2)^*$ for invertible Gaussian-rational matrices X,Y,Z,Y_2 (not those above); they span extreme rays by Lemma 7.9, and the certificate is the same (†14). □

Theorem 7.11 sharpens the remark after Corollary 7.3: in the spectral bases the 2-cycle part can fail outright, also at pairs with full local ranks. At the second pair the twirl is far from the boundary of the PPT set, and Lemma 6.3 with a pivoted Cholesky frame certifies it numerically (†16).

By Lemma 7.7, the matrix $|B| := [|B_{ij}|]$ bounds the matrix K of every frame from below.

Proposition 7.12. (a) *(Standard.) If $d\le 3$, then $|B|\succeq 0$ for every $B\succeq 0$.*

(b) *There is a pair $\mathrm{M}_4\to\mathrm{M}_3\to\mathrm{M}_4$ of the form of Theorem 7.10, with other real filters, at which B is rational, all $B_{aa}>0$, $A_{ij}A_{ji}\ge B_{ij}^2$ for all $i\ne j$, and $|B|$ is not positive semidefinite. Both factors span extreme rays, and $\Psi\circ\Phi$ is EB by PPT² in dimension 3 [1, 2], as in Theorem 7.10(c).*

Proof. (a) The principal minors of $|B|$ of order at most 2 equal those of B, and for $d=3$, $\det|B|-\det B = 2(|B_{12}B_{23}B_{31}|-\mathrm{Re}(B_{12}B_{23}B_{31}))\ge 0$. (b) An explicit rational vector v has $v^\top|B|v<0$ (an exact computation, †15); extremality follows from Lemma 7.9, and $\Psi\circ\Phi$ is EB by the argument of Theorem 7.10(c). □

Together with Theorem 7.10(b), the proposition shows that neither the cycle condition for $(A,|B|)$ nor $|B|\succeq 0$ holds at all extreme pairs at outer dimension 4.

One potential per product term does not repair the 2-cycle failures of a fixed frame. By Lemma 6.1, every separable decomposition of $X(A,B)$ has the form in the following lemma.

Lemma 7.13. *Let $x_n,y_n\in\mathbb{C}^d$ satisfy $\sum_n|x_n(a)|^2|y_n(b)|^2\le A_{ab}$ for $a\ne b$, and let $B=\sum_n z_nz_n^*$ with $z_n := x_n\odot y_n$. Then for every oriented cycle $C=(a_1,\dots,a_L)$ of distinct indices,*

$$\prod_{l=1}^{L}A_{a_la_{l+1}}\ \ge\ \Big(\sum_n\prod_{l=1}^{L}|z_n(a_l)|^{2/L}\Big)^L\qquad(a_{L+1}:=a_1);$$

for $L = 2$ this reads $A_{ij}A_{ji} \geq (\sum_n |z_n(i)||z_n(j)|)^2$.

Proof. Put $c_n(a,b) := |x_n(a)|^2|y_n(b)|^2$. Around C, $\prod_l c_n(a_l, a_{l+1}) = \prod_l |z_n(a_l)|^2$, and the generalized Hölder inequality gives

$$\sum_n \prod_l c_n(a_l,a_{l+1})^{1/L} \leq \prod_l \Big(\sum_n c_n(a_l,a_{l+1})\Big)^{1/L} \leq \prod_l A_{a_l a_{l+1}}^{1/L}.$$

□

So a decomposition whose vectors z_n are the columns of a fixed frame F needs $A_{ij}A_{ji} \geq K_{ij}^2$ for all pairs, whatever the potentials; at the pairs of Theorem 7.11 the spectral Kraus frame admits no certificate. The lemma gives necessary conditions. At the pairs of Theorem 7.10, whose twirls are separable by Theorem 7.10(c), Kraus frames of other Gram data with one potential per product term give numerical certificates (†16). This leads to Question 11.1.

7.4. **Tight pairs and the number of product terms.**

Lemma 7.14 (tight pairs)**.** *Let*

$$B = \sum_n (x_n \odot y_n)(x_n \odot y_n)^* \quad \textit{with} \quad \sum_n |x_n(a)|^2|y_n(b)|^2 \leq A_{ab} \ (a \neq b),$$

and let $i \neq j$ with $A_{ij}A_{ji} = |B_{ij}|^2 > 0$. Then for every n the number $z_n(i)\overline{z_n(j)}$, $z_n := x_n \odot y_n$, is a nonnegative multiple of B_{ij}.

Proof. Since $|z_n(i)||z_n(j)| = |x_n(i)y_n(j)|\cdot|x_n(j)y_n(i)|$, Cauchy–Schwarz gives

$$\begin{aligned} |B_{ij}| &\leq \sum_n |z_n(i)||z_n(j)| \leq \Big(\sum_n |x_n(i)|^2|y_n(j)|^2\Big)^{1/2}\Big(\sum_n |x_n(j)|^2|y_n(i)|^2\Big)^{1/2} \\ &\leq (A_{ij}A_{ji})^{1/2} = |B_{ij}|. \end{aligned}$$

Equality in $|\sum_n w_n| \leq \sum_n |w_n|$ with $w_n = z_n(i)\overline{z_n(j)}$ and $\sum_n w_n = B_{ij} \neq 0$ forces every nonzero w_n to have the phase of B_{ij}. □

The same equality step appears in the proof of [23, Theorem 2]. The proof uses only the off-diagonal entries of B, so Lemma 7.14 and Corollary 7.15 below also hold when $B - D' = \sum_n z_n z_n^*$ for a nonnegative diagonal D'.

Corollary 7.15. *Let S be a set of at least 3 indices such that, for all distinct $i,j,l \in S$, $B_{ij} \neq 0$, $A_{ij}A_{ji} = |B_{ij}|^2$ and $B_{ij}B_{jl}B_{li} < 0$. Then in every decomposition as in Lemma 7.14 each z_n has at most two nonzero entries in S, and the number of terms is at least $|S|(|S|-1)/2$.*

Proof. If $z_n(i), z_n(j), z_n(l) \neq 0$ for distinct $i,j,l \in S$, Lemma 7.14 gives positive c_1, c_2, c_3 with $z_n(i)\overline{z_n(j)} = c_1 B_{ij}$, $z_n(j)\overline{z_n(l)} = c_2 B_{jl}$ and $z_n(l)\overline{z_n(i)} = c_3 B_{li}$; the product of the left sides is $|z_n(i)z_n(j)z_n(l)|^2 > 0$ and that of the right sides is negative. Each pair $\{i,j\} \subseteq S$ has $B_{ij} \neq 0$, so some term is nonzero at i and j; it meets S in exactly $\{i,j\}$, and distinct pairs need distinct terms. □

Proposition 7.16. *Let $B \succeq 0$ have all off-diagonal entries nonzero and no triangle product $B_{ij}B_{jk}B_{ki}$ (i,j,k distinct) a positive real, and put $A_{ij} := |B_{ij}|$ for $i \neq j$. Then $X(A,B)$ is PPT, and it is separable if and only if the comparison matrix $\mathcal{M}(B)$ (diagonal B_{ii}, off-diagonal entries $-|B_{ij}|$) is positive semidefinite.*

Proof. Here $X(A,B)$ is PPT because $A_{ij}A_{ji} = |B_{ij}|^2$ for all $i \neq j$. If $X(A,B)$ is separable, Lemma 7.14 applies to every pair, and as in Corollary 7.15 (with the inequality $B_{ij}B_{jk}B_{ki} \notin (0,\infty)$ in place of negativity) no term has three nonzero entries. So $B = D' + \sum_{i<j} M^{(ij)}$ with $D' \succeq 0$ diagonal and $M^{(ij)} \succeq 0$ supported on $\{i,j\}$ with (i,j) entry B_{ij}; since $\mathcal{M}$ of a positive semidefinite 2×2 matrix is positive semidefinite, $\mathcal{M}(B) = D' + \sum \mathcal{M}(M^{(ij)}) \succeq 0$. Conversely, if $\mathcal{M}(B) \succeq 0$, the irreducible M-matrix $\mathcal{M}(B)$ has a vector v with positive entries and $\mathcal{M}(B)v \geq 0$. For $i<j$ let $z^{(ij)}$ be supported on $\{i,j\}$ with $|z_i^{(ij)}|^2 = |B_{ij}|v_j/v_i$, $|z_j^{(ij)}|^2 = |B_{ij}|v_i/v_j$ and $z_i^{(ij)}\overline{z_j^{(ij)}} = B_{ij}$, realized by $x = y = |z^{(ij)}|^{1/2}$ up to phases. Its only off-diagonal costs are $|z_i^{(ij)}||z_j^{(ij)}| = |B_{ij}|$, and $B - \sum_{i<j} z^{(ij)}z^{(ij)*}$ is diagonal with entries $B_{ii} - \sum_{j\neq i}|B_{ij}|v_j/v_i \geq 0$. □

The "if" direction of Proposition 7.16 is contained in [23, Theorem 5]; the "only if" direction uses Lemma 7.14.

Example 7.17. Let $\{|m\rangle\}$ be an orthonormal basis of $\mathbb{C}^3$ and, for the three pairs $m = \{a,b\}$ of $\{1,2,3\}$, put $\varphi_m = e_a + e_b$ and $r_m = e_a - e_b$. The maps $\Phi(X) = \sum_m \langle\varphi_m|X|\varphi_m\rangle |m\rangle\langle m|$ and $\Psi(Y) = \sum_m \langle m|Y|m\rangle\, r_m r_m^{\top}$ are EB, hence PPT, and $W = \sum_m (\varphi_m \otimes r_m)(\varphi_m \otimes r_m)^{\top}$. Its twirl has $A_{ij} = 1$ for $i \neq j$ and $B = 3I - J$ (the Laplacian of K_3; rank 2, $B\mathbf{1} = 0$, all $B_{aa} > 0$, irreducible). All pairs are tight and the triangle product is -1, so by Corollary 7.15 every certificate needs at least $3 > \operatorname{rank} B$ product terms. Embedding φ_m, r_m in $\mathbb{C}^4$ gives a PPT pair $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$ with the same property.

Example 7.18. Let $t > 0$ and $z_1 = (1, -1-t, 0, t)$, $z_2 = (1, 0, -1-t, t)$, $z_3 = (0, 1, -1, 0)$; let φ_m be the indicator vector of $\operatorname{supp} z_m$ and $r_m = z_m$, and define Φ, Ψ as in Example 7.17 (now $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$). Then $B = \sum_m z_m z_m^{\top}$ has rank 2 ($z_1 - z_2 = -(1+t)z_3$), $B\mathbf{1} = 0$, all $B_{aa} > 0$, and B is irreducible ($B_{14} = 2t$). On $S = \{1,2,3\}$ each pair has exactly one contributing term, in A as well as in B, because φ_m and z_m have the same support; so S is tight, and $B_{12}B_{23}B_{31} = -(1+t)^2 < 0$. By Corollary 7.15 every certificate needs at least 3 terms.

Example 7.19. For $A_{ij} = 1$ ($i \neq j$) and $B = 4I - J$ (the Laplacian of K_4, rank 3), $X(A,B) = \sum_{a<b} \mathrm{tw}(|x_{ab} \otimes y_{ab}\rangle\langle x_{ab} \otimes y_{ab}|)$ with $x_{ab} = e_a + e_b$, $y_{ab} = e_a - e_b$ is separable, and by Corollary 7.15 every certificate needs at least 6 terms.

So for separable twirls, even of PPT composites through a qutrit, the minimal number of product terms can exceed the coherence rank. Corollary 7.15 excludes certificates with rank B terms only when $|S|(|S|-1)/2 > \operatorname{rank} B$. A tight triangle with negative product and $\operatorname{rank} B = 3$ is no obstruction: for the EB pair at $d = 3$ built as in Example 7.17 from $x_m = (1,1,0),(1,0,1),(0,1,1)$ and $y_m = (1,-1,0),(1,0,-2),(0,1,-3)$, all pairs are tight, $B_{12}B_{23}B_{31} = -6$, $\operatorname{rank} B = 3$, and the three product terms of W form a certificate. In Examples 7.17 and 7.18 the marginal $\Psi(I)$ is singular, and so is $\Phi^*(I)$ in Example 7.18 and in the embedding of Example 7.17; whether $\operatorname{rank} B$ terms suffice for pairs with nonsingular marginals is open. Two further examples of the same construction show that the cycle condition for $(A,|B|)$ and positivity of $|B|$ can fail even for EB pairs (in Theorem 7.10(b) and Proposition 7.12(b) the factors are extreme and entangled): (i) $x_1 = (s,1,0)$, $y_1 = (1/s,1,0)$ and its two cyclic shifts, $0 < s < 1$, give $A_{12}A_{23}A_{31} = s^6 < 1 = |B_{12}B_{23}B_{31}|$; (ii) $z_1 = (1,c,0,-c)$, $z_2 = (0,c,1,c)$, $c = 2^{-1/2}$, with $x_n = \operatorname{sign}(z_n)|z_n|^{1/2}$ and $y_n = |z_n|^{1/2}$ through a qubit, give $|B|$ with eigenvalue $1-\sqrt{2}$. The composite examples are sums of extreme pairs (their Choi matrices are sums of rank-one products).

7.5. **Kraus-adapted pure terms.**

Lemma 7.20. *Let r be the coherence rank, and let $X_1,\dots,X_{PQ}$ be an orthonormal basis of $\mathbb{C}^{P\times Q}$ whose first r elements span $\mathcal{T}$. Put $M_{ij}^{(k)} := \langle X_k, G_i^* H_j\rangle$ and $n_k := \sum_{i,j} \overline{M_{ij}^{(k)}}\,|ij\rangle$. Then $W = \sum_k n_k n_k^*$ and*

$$\operatorname{tw}(W) = \sum_{k\le r} \operatorname{tw}(n_k n_k^*) + \sum_{k>r}\sum_{i\neq j} |M_{ij}^{(k)}|^2 |ij\rangle\langle ij|.$$

Proof. By Lemma 2.1 and Parseval, $\langle ab|W|cd\rangle = \sum_k \overline{M_{ab}^{(k)}} M_{cd}^{(k)}$. For $k > r$, $X_k \perp \mathcal{T}$ gives $M_{aa}^{(k)} = \langle X_k, T_a\rangle = 0$, so the coherence part of $\operatorname{tw}(n_k n_k^*)$ vanishes. □

So $\operatorname{tw}(W)$ is always a sum of $\operatorname{rank} B$ twirled pure states and a nonnegative diagonal. The proof of Theorem 4.1 replaces the single pure state by a product state within the budget A.

7.6. **An average over qubit compressions.**

Proposition 7.21 (given [3])**.** *Let $d \ge 1$ and let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_d$ be PPT. Then*

$$5\,J(\Psi\circ\Phi) + \Phi^*(I_3)^\top \otimes \Psi(I_3)$$

is separable. In particular $\operatorname{tw}(W) + \frac{1}{5}\operatorname{tw}(\rho_A \otimes \sigma_A^\top)$ is separable, where $\rho_A = \operatorname{Tr}_2 \rho = \Phi^*(I)^\top$ and $\sigma_A = \operatorname{Tr}_2 \sigma = \Psi(I)^\top$.

Proof. For a rank-2 projection $P = VV^*$ on $\mathbb{C}^3$ ($V\colon \mathbb{C}^2 \to \mathbb{C}^3$ an isometry), $\Psi\circ\mathrm{Ad}_P\circ\Phi = (\Psi\circ\mathrm{Ad}_V)\circ(\mathrm{Ad}_{V^*}\circ\Phi)$ is a composite of PPT maps through M_2: $\mathrm{Ad}_{V^*}\circ\Phi$ is PPT, and $\Psi\circ\mathrm{Ad}_V$ is completely positive with $\vartheta\circ\Psi\circ\mathrm{Ad}_V = (\vartheta\circ$

$\Psi)\circ \mathrm{Ad}_V$ completely positive. By [3] it is EB. For P Haar distributed, $\mathbb{E}[P\otimes P] = \frac{5}{12}I + \frac{1}{12}F$, F the swap (take traces against I and F: $9a+3b = (\operatorname{Tr} P)^2 = 4$ and $3a+9b = \operatorname{Tr} P^2 = 2$), so $\mathbb{E}[\operatorname{Tr}(PXPY)] = \frac{5}{12}\operatorname{Tr}(XY) + \frac{1}{12}\operatorname{Tr} X \operatorname{Tr} Y$ and $\mathbb{E}[\mathrm{Ad}_P] = \frac{5}{12}\mathrm{id} + \frac{1}{12}\operatorname{Tr}(\cdot)I_3$. Hence $\frac{5}{12}J(\Psi\circ\Phi) + \frac{1}{12}J\big(\operatorname{Tr}(\Phi(\cdot))\Psi(I_3)\big)$ is an average of separable operators, and $J(\operatorname{Tr}(\Phi(\cdot))\Psi(I_3)) = \Phi^*(I_3)^\top \otimes \Psi(I_3)$. □

The Haar average may be replaced by the average over the 216 elements, up to phases, of the single-qutrit Clifford group, which is a unitary 2-design; the separable operator is then a finite sum of 216 separable operators. Among $U(3)$-invariant averages of compressions of rank at most 2 the constant $\frac{1}{5}$ is optimal: the averaged map has an isotropic Choi matrix of Schmidt number at most 2, so its fidelity is at most $\frac{2}{3}$ [30], and the Haar average attains $\frac{2}{3}$. This does not bound the best constant c for which $W + c\,\rho_A \otimes \sigma_A^\top$ is separable; the companion paper [34] studies this constant (Section 10).

7.7. **Factor width.** Call $B \succeq 0$ 3-*sparse* if $B = \sum_{|S|=3} \iota_S(P_S)$ with $P_S \succeq 0$ supported on S.

Proposition 7.22. *Let $d = 4$. If B is not 3-sparse, then in every separable decomposition $\operatorname{tw}(W) = \sum_n \operatorname{tw}(|x_n \otimes y_n\rangle\langle x_n \otimes y_n|) + (\text{nonnegative diagonal})$ some $x_n \odot y_n$ has four nonzero entries.*

Proof. If every $z_n = x_n \odot y_n$ vanished at some index l_n, then

$$B = \sum_{l=1}^{4} \sum_{n:\, l_n = l} z_n z_n^*$$

would be 3-sparse. □

This complements Proposition 5.2, which excludes one corner form for every frustrated B with $B\mathbf{1} = 0$; frustrated B can be 3-sparse, for example $B = \sum_k z_k z_k^\top$ with $z_1 = (0,1,1,-2)$, $z_2 = (1,0,1,-2)$, $z_3 = (1,1,0,-2)$. So when B is not 3-sparse, no construction whose product terms all live on three coordinates (coordinate compressions to three indices, zero-row compressions, the corner averages of Theorem 5.1 with any weights) can produce a separable decomposition of $\operatorname{tw}(W)$, whatever the budget A. At $d = 4$ and coherence rank 3 with $B\mathbf{1} = 0$, $B = RR^*$ with $R \in \mathbb{C}^{4\times 3}$ is 3-sparse if and only if $I_3 = \sum_l Q_l$ for some $Q_l \succeq 0$ with $Q_l R^* e_l = 0$. Numerically, this fails at 11 of the 42 residual points, 7 of coherence rank 3 and 4 of coherence rank 4 (†9).

8. Ferromagnetic coherence and self-composites

This section gives criteria for the separability of $X(A,B)$ at $d \le 4$ that use neither peels nor corners, and applies them to a class of composites at every coherence rank. The *comparison matrix* $\mathcal{M}(R)$ of a Hermitian matrix R has diagonal entries R_{aa} and off-diagonal entries $-|R_{ab}|$.

8.1. **Nonnegative frames.** A matrix $B \succeq 0$ is *ferromagnetic* if D^*BD is entrywise nonnegative for some diagonal unitary D (equivalently, by a choice of phases along a spanning forest of the support graph, every product $B_{a_1a_2}B_{a_2a_3}\cdots B_{a_La_1}$ over a cycle of distinct indices is a nonnegative real). A real matrix N is *doubly nonnegative* if $N \succeq 0$ and N is entrywise nonnegative, and *completely positive* if $N = VV^\top$ with V entrywise nonnegative. Every doubly nonnegative matrix of order at most 4 is completely positive [24] (for an elementary proof see [25]). The statement fails at order 5 (Example 8.4 below).

Lemma 8.1. *Let $d \le 4$, let $B \succeq 0$ and let D be a diagonal unitary with $N := D^*BD$ entrywise nonnegative. Let A be nonnegative off the diagonal, with $A_{ab} > 0$ whenever $N_{ab} > 0$, and suppose that every oriented cycle $(a_1, \dots, a_L)$ of distinct indices, $2 \le L \le d$, whose edges all have $N > 0$ satisfies $\prod_l A_{a_la_{l+1}} \ge \prod_l N_{a_la_{l+1}}$ ($a_{L+1} := a_1$). Then $X(A, B)$ is a sum of m twirled product states and a nonnegative diagonal operator, where m is the number of columns of a nonnegative factor of N; in particular it is separable.*

Proof. N is doubly nonnegative, so $N = VV^\top$ with $V \in \mathbb{R}^{d\times m}_{\ge 0}$ [24]. Put $F := DV$. Then $FF^* = DND^* = B$, and the matrix of Lemma 6.3 is $K_{ab} = \sum_n V_{an}V_{bn} = N_{ab}$, so Lemma 6.3 applies. □

With the potential of Lemma 6.3 realized by a local diagonal filter, Lemma 8.1 follows from [23, Corollary 1] combined with [24]. We say that A is *cycle-balanced on B* if $\prod_{e\in C} A_e = \prod_{e\in C} A_{\bar e}$ for every oriented cycle C of at least 3 distinct indices whose edges e all have $B_e \ne 0$, where $\bar e$ is the reversed edge. Symmetric matrices are cycle-balanced, and so is $A_{ab} = S_{ab}s_a/s_b$ with S symmetric and $s_a > 0$.

Theorem 8.2. *Let $d \le 4$ and let $X(A, B)$ be PPT. If B is ferromagnetic and A is cycle-balanced on B, then $X(A, B)$ is separable.*

Proof. Let $N := D^*BD \ge 0$, so $N_{ab} = |B_{ab}|$. If $N_{ab} > 0$, PPT gives $A_{ab}A_{ba} \ge N_{ab}^2 > 0$. For a 2-cycle the condition of Lemma 8.1 is the PPT inequality. For an oriented cycle C of length $L \ge 3$ with all $N_e > 0$, the product of the PPT inequalities $N_e^2 \le A_eA_{\bar e}$ over its edges is $(\prod_C N)^2 \le \prod_C A \prod_C A_{\bar e} = (\prod_C A)^2$, by balance. Lemma 8.1 applies. □

For symmetric A, Theorem 8.2 follows, as Lemma 8.1 does, from [23, Corollary 1] combined with [24], after the local diagonal unitary $D \otimes I$ that makes B nonnegative. The cycle-balanced case reduces to it: by balance, $\log(A_{ab}/A_{ba})$ is a gradient $\phi_a - \phi_b$ on the support graph of B, so $A_{ab} = S_{ab}s_a/s_b$ there with S symmetric and $s = e^{\phi/2}$, and the local filter $\mathrm{diag}(s)^{-1/2} \otimes \mathrm{diag}(s)^{1/2}$ fixes B, preserves PPT and maps A to S on the support of B. Theorem 8.2 is the ferromagnetic counterpart of Remark 6.2. Each hypothesis is needed:

- *ferromagnetic:* for $d = 3$, $B = \left(\begin{smallmatrix} 1 & c & \bar{c} \\ \bar{c} & 1 & c \\ c & \bar{c} & 1 \end{smallmatrix}\right)$ with $c = \frac{1}{2} + \frac{i}{4}$ and $A_{ab} = |B_{ab}| = \frac{\sqrt{5}}{4}$ $(a \neq b)$, the leading minors of B are $1, \frac{11}{16}, \frac{1}{8}$, the triangle product $c^3 = \frac{1}{32} + \frac{11}{64}i$ is not a positive real, and $\mathcal{M}(B)$ has the eigenvalue $1 - \frac{\sqrt{5}}{2} < 0$; so the PPT operator $X(A, B)$, with A symmetric, is entangled by Proposition 7.16;
- *balanced:* for $d = 3$, $B = J$ (the all-ones matrix), $A_{12} = A_{23} = A_{31} = \frac{1}{2}$ and $A_{21} = A_{32} = A_{13} = 2$, $X(A, B)$ is PPT, and the cycle $(1, 2, 3)$ has $\prod A = \frac{1}{8} < 1$, so $X(A, B)$ is entangled by Lemma CYC [Supp, S1];
- $d \leq 4$*:* Example 8.4.

Embedding with a zero row gives the first two examples at $d = 4$.

Lemma 8.3 ([23])**.** *Let N be doubly nonnegative of order d, and let $X(N, N)$ have $A_{ab} = N_{ab}$ $(a \neq b)$ and $B = N$. Then $X(N, N)$ is PPT, and it is separable if and only if N is completely positive.*

The lemma is taken from [23, Theorems 2 and 6(c)]. After partial transposition, $X(N, N)$ becomes a mixture of Dicke-type states, for which the statement can be found in [26, 27]. For completeness we include the argument. The operator $X(N, N)$ is PPT because $A_{ab}A_{ba} = N_{ab}^2$ for all $a \neq b$. If $N = \sum_n v_n v_n^\top$ with $v_n \geq 0$, the vectors $x_n = y_n = v_n^{1/2}$ (entrywise) give $X(N, N) = \sum_n \mathrm{tw}(|x_n \otimes x_n\rangle\langle x_n \otimes x_n|)$ by Lemma 6.1. Conversely, let x_n, y_n be as in Lemma 6.1 and $z_n := x_n \odot y_n$. For $a \neq b$, by Cauchy–Schwarz,

$$N_{ab} \leq \sum_n |z_n(a)z_n(b)| = \sum_n |x_n(a)y_n(b)|\,|x_n(b)y_n(a)| \leq \sqrt{N_{ab}N_{ba}} = N_{ab};$$

equality forces $z_n(a)\overline{z_n(b)} = |z_n(a)||z_n(b)|$ for all n, so $N = \sum_n |z_n||z_n|^\top$.

Example 8.4. Let $N := \frac{9}{5}I + \mathrm{Adj}(C_5)$, with $\mathrm{Adj}(C_5)$ the adjacency matrix of the 5-cycle. The eigenvalues of $\mathrm{Adj}(C_5)$ are at least $-2\cos\frac{\pi}{5} > -\frac{9}{5}$, so N is doubly nonnegative. It is not completely positive: in a factorization $N = \sum_n v_n v_n^\top$ with $v_n \geq 0$, each $\operatorname{supp} v_n$ is a vertex or an edge of C_5, since N vanishes on the non-edges and C_5 has no triangles; then $\mathcal{M}(N) = \sum_n \mathcal{M}(v_n v_n^\top) \succeq 0$, whereas $\mathbf{1}^\top \mathcal{M}(N)\mathbf{1} = 9 - 10 < 0$. By Lemma 8.3, $X(N, N)$ is PPT and entangled, with $B = N$ ferromagnetic and $A = N$ symmetric.

8.2. **A combined criterion.** A Hermitian matrix R is 2*-sparse positive semidefinite* if $R = \Lambda + \sum_{a<b} R^{(ab)}$ with $\Lambda \succeq 0$ diagonal and each $R^{(ab)} \succeq 0$ supported on $\{a, b\}$. As in Remark 6.2, this holds if and only if $\mathcal{M}(R) \succeq 0$.

Proposition 8.5. *Let $d \leq 4$, $B \succeq 0$, and A nonnegative off the diagonal. Suppose that $B = DND^* + R$ with D a diagonal unitary, N doubly nonnegative and R 2-sparse positive semidefinite, and that there is $t \in \mathbb{R}^d_{>0}$ with*

$$\alpha_{ab} := A_{ab} - N_{ab}t_a/t_b \geq 0 \quad \textit{and} \quad \alpha_{ab}\alpha_{ba} \geq |R_{ab}|^2 \qquad (a \neq b).$$

Then $X(A,B)$ is separable.

Proof. Write $N = VV^\top$ with $V \geq 0$ [24]. For a column v of V put $x := D(vt)^{1/2}$ and $y := (v/t)^{1/2}$ (entrywise operations); then $x \odot y = Dv$, and the costs $|x_a|^2|y_b|^2 = v_a v_b t_a/t_b$ add up to $N_{ab}t_a/t_b$ over the columns. For $a < b$ with $R_{ab} \neq 0$ write $R^{(ab)} = ww^* +$ (nonnegative diagonal) with w supported on $\{a,b\}$, $|w_a|^2 = R^{(ab)}_{aa}$ and $w_a\overline{w_b} = R_{ab}$, and choose x, y supported on $\{a,b\}$ with $x \odot y = w$ and $|x_a|^2|y_b|^2 = \alpha_{ab}$; then $|x_b|^2|y_a|^2 = |R_{ab}|^2/\alpha_{ab} \leq \alpha_{ba}$, and the other costs vanish. The remaining diagonal (Λ, the diagonal leftovers, and the blocks $R^{(ab)}$ with $R_{ab} = 0$) is a sum of nonnegative multiples of $|aa\rangle\langle aa|$. All terms together have coherence B and costs at most A_{ab}, and Lemma 6.1 applies. □

For $N = 0$ the proposition reduces to the criterion "PPT and $\mathcal{M}(B) \succeq 0$" of [23, Theorem 5], valid for every d, which contains Remark 6.2. For $R = 0$ it reduces to Lemma 8.1 (the cycle conditions are equivalent to the existence of t, as in the proof of Lemma 6.3), and so it contains Theorem 8.2. For fixed D and t the hypotheses are a semidefinite feasibility problem.

8.3. **Self-composites.**

Corollary 8.6. *Let $d \leq 4$ and $k \geq 1$, let $\Phi\colon \mathrm{M}_d \to \mathrm{M}_k$ be PPT, and let $\Xi(Z) = \sum_{r=1}^{R} Y_r Z Y_r$ with $Y_r \succeq 0$. Put $\Psi := \Phi^* \circ \Xi\colon \mathrm{M}_k \to \mathrm{M}_d$. Then:*

(a) *Ψ is PPT;*

(b) *with Gram data G_a of Φ (as in Section 2, with 3 replaced by k) and $\rho_{ab} = G_a G_b^*$, the matrices $H_a := [Y_1 G_a \ \cdots \ Y_R G_a]$ are Gram data of Ψ, and*

$$A_{ij} = \sum_r \mathrm{Tr}(Y_r \rho_{ii} Y_r \rho_{jj}) = A_{ji}, \qquad B_{ij} = \sum_r \|Y_r^{1/2} \rho_{ij} Y_r^{1/2}\|_2^2 \geq 0;$$

(c) *$A_{ij} \geq B_{ij}$ for $i \neq j$, and B is doubly nonnegative;*

(d) *$\mathrm{tw}(J(\Psi \circ \Phi)) = \sum_n \mathrm{tw}(|x_n \otimes x_n\rangle\langle x_n \otimes x_n|) + \sum_{i \neq j} (A_{ij} - B_{ij}) |ij\rangle\langle ij|$, where $B = \sum_n v_n v_n^\top$ with $v_n \geq 0$ and $x_n = v_n^{1/2}$ entrywise. In particular $\mathrm{tw}(J(\Psi \circ \Phi))$ is separable, whatever the coherence rank.*

Proof. (a) Φ^* is PPT because Φ is, so Ψ and $\vartheta \circ \Psi = (\vartheta \circ \Phi^*) \circ \Xi$ are completely positive. (b) Ξ is self-adjoint, so $\Psi^*(E_{ab}) = \Xi(\Phi(E_{ab})) = \sum_r Y_r G_a G_b^* Y_r = H_a H_b^*$. Lemma 2.1 gives $A_{ij} = \sum_r \|G_i^* Y_r G_j\|_2^2 = \sum_r \mathrm{Tr}(Y_r \rho_{ii} Y_r \rho_{jj})$, which is symmetric, and $B_{ij} = \sum_r \mathrm{Tr}(G_i^* Y_r G_i G_j^* Y_r G_j) = \sum_r \mathrm{Tr}(Y_r \rho_{ij} Y_r \rho_{ij}^*)$. (c) W is PPT, hence so is $\mathrm{tw}(W)$, and $A_{ij}A_{ji} \geq B_{ij}^2$ with $A_{ij} = A_{ji}$ and $B_{ij} \geq 0$ gives $A_{ij} \geq B_{ij}$; B is a Gram matrix. (d) By [24], $B = \sum_n v_n v_n^\top$ with $v_n \geq 0$, and Lemma 6.1 gives $\mathrm{tw}(|x_n \otimes x_n\rangle\langle x_n \otimes x_n|) = X([v_n(a) v_n(b)]_{a,b}, v_n v_n^\top)$; these terms have coherence B and costs $B_{ab} \leq A_{ab}$. □

The last step follows from [23, Corollary 1] combined with [24]. Corollary 8.6 is the special case of Theorem 8.2 in which A is symmetric, and the case $\Xi = \mathrm{id}$ corresponds to $\Psi = \Phi^*$. A diagonal filter $\Psi \mapsto \mathrm{Ad}_\Delta \circ \Psi$ keeps

the twirl separable, so the conclusion extends to $\mathrm{Ad}_\Delta \circ \Phi^* \circ \Xi$. For $d \geq 5$ the proof works whenever B is completely positive.

Example 8.7. Let ρ_U be the operator on $\mathbb{C}^4 \otimes \mathbb{C}^3$ in the proof of Theorem 7.11, let $X := I_4 + J_4$ (J_4 the all-ones matrix, $\det X = 5$), and let Φ_X be the map with $J(\Phi_X) = (X \otimes I_3)\rho_U(X \otimes I_3)^*$. Take $\Psi = \Phi_X^*$, or $\Psi = \Phi_X^* \circ \mathrm{Ad}_Y$ with $Y = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right) \succ 0$. Since ρ_U spans an extreme ray and is entangled, both factors span extreme rays and are entangled (Lemma 7.9), and both have full local ranks $(4, 3)$. In both cases A is symmetric, B is real, entrywise positive and positive definite (coherence rank 4), and $\mathbf{1}^\top \mathcal{M}(B)\mathbf{1} < 0$. In both cases no index peels (Lemma 7.2), and neither Remark 6.2 nor [23, Theorem 5] applies. With minimal Gram data ($P = Q = 6$) every T_a has rank 3, and the column spaces of the T_a span $\mathbb{C}^6$ (exact computations, †17). By Corollary 8.6 the twirl is separable.

The composite of the pair of Theorem 7.10 for $d = 4$ is EB by Theorem 7.10(c), and no frame certificate with a common potential exists there (Lemma 7.7). At this pair the hypotheses of Proposition 8.5 hold numerically for $X(\kappa A, B)$ down to $\kappa \approx 0.886$, while $X(\kappa A, B)$ is PPT down to $\kappa_{\mathrm{PPT}} \approx 0.838$, where $\kappa_{\mathrm{PPT}} := \max_{i \neq j} |B_{ij}| / \sqrt{A_{ij} A_{ji}}$ is the least κ for which $X(\kappa A, B)$ is PPT (†18). At two of the residual points of †2 these hypotheses hold down to the PPT threshold (†18).

8.4. **Real coherence, self-adjoint composites and stability.** We first record that diagonal-covariant processing before Φ or after Ψ preserves the separability of the twirl.

Proposition 8.8. *Let $D_1, D_2 \colon \mathrm{M}_d \to \mathrm{M}_d$ be completely positive with $D_i(UXU^*) = UD_i(X)U^*$ for every diagonal unitary U, and let $\Phi \colon \mathrm{M}_d \to \mathrm{M}_k$ and $\Psi \colon \mathrm{M}_k \to \mathrm{M}_d$ be completely positive. Then*

$$\mathrm{tw}\big(J(D_2 \circ \Psi \circ \Phi \circ D_1)\big) = (D_1^\# \otimes D_2)\big(\mathrm{tw}(J(\Psi \circ \Phi))\big), \qquad D_1^\# := \vartheta \circ D_1^* \circ \vartheta,$$

so $\mathrm{tw}(J(D_2 \circ \Psi \circ \Phi \circ D_1))$ is separable whenever $\mathrm{tw}(J(\Psi \circ \Phi))$ is. If (Φ, Ψ) is a PPT pair, so is $(\Phi \circ D_1, D_2 \circ \Psi)$.

Proof. For a map Λ, $J(\Lambda \circ D_1) = \sum_{c,e} \big(\sum_{a,b} D_1(E_{ab})_{ce} E_{ab}\big) \otimes \Lambda(E_{ce})$, and $\sum_{a,b} D_1(E_{ab})_{ce} E_{ab} = D_1^*(E_{ec})^\top = D_1^\#(E_{ce})$; so $J(\Lambda \circ D_1) = (D_1^\# \otimes \mathrm{id})J(\Lambda)$, and $J(D_2 \circ \Lambda) = (\mathrm{id} \otimes D_2)J(\Lambda)$. The map $D_1^\#$ is completely positive and covariant under diagonal unitaries, since the diagonal unitaries are closed under complex conjugation. The twirl is the average of $\mathrm{Ad}_{U \otimes \bar{U}}$ over the diagonal unitaries U, and $D_1^\# \otimes D_2$ commutes with each $\mathrm{Ad}_{U \otimes \bar{U}}$, hence with tw; a local completely positive map preserves separability. Finally $\vartheta \circ \Phi \circ D_1$ and $\vartheta \circ D_2 \circ \Psi = (\vartheta \circ D_2 \circ \vartheta) \circ (\vartheta \circ \Psi)$ are completely positive when Φ and Ψ are PPT. □

Proposition 8.8 is routine. It contains the diagonal scalings $(X, Y) \mapsto (DXD^*, DYD^*)$, $D \succeq 0$ diagonal, which preserve pairwise complete positivity [23, Section 3]. For a covariant inner map the formula is the composition rule of covariant maps [7, Proposition 4.4], and the twirl is the projection of [6, Proposition 2.3]. The Schur multipliers $S_M(X) = M \circ X$ with $M \succeq 0$ are completely positive and covariant, and they are not PPT unless M is diagonal. Applied to Corollary 8.6: for every PPT map $\Phi\colon \mathrm{M}_d \to \mathrm{M}_k$ with $d \leq 4$, every Ξ as there and all $M_1, M_2 \succeq 0$, the pair $(\Phi \circ S_{M_1}, S_{M_2} \circ \Phi^* \circ \Xi)$ is PPT and its twirl is separable. Its coherence matrix carries the phases of the Schur factors and is not ferromagnetic in general, while A remains cycle-balanced. For composites the ferromagnetic hypothesis of Theorem 8.2 is therefore sufficient but not necessary.

For a Hermitian matrix B let $|B|$ be its entrywise absolute value. Let $d = 4$ and let $B \succeq 0$ be real with all off-diagonal entries nonzero. The local unitaries $S \otimes I$ with $S = \mathrm{diag}(\pm 1)$ map $X(A, B)$ to $X(A, SBS)$. The number of triangles $\{a, b, c\}$ with $B_{ab}B_{bc}B_{ca} < 0$ is 0, 2 or 4. If it is 0, B is ferromagnetic; if it is 4, $B = SMS$ with $M_{ab} < 0$ for all $a \neq b$ (we call B antiferromagnetic); if it is 2, B is *mixed*, and after a sign switching exactly one off-diagonal entry B_{pq} is negative.

Lemma 8.9. *Let $d = 4$, and let $B \succeq 0$ be real and mixed, with all off-diagonal entries nonzero, $B_{pq} < 0$ and $B_{ab} > 0$ for the other pairs $a \neq b$; let o be the set of the other two indices.*

(a) *If $X(|B|, B)$ is separable, then $|B|$ is positive semidefinite, and indeed completely positive.*
(b) *If $|B| \succeq 0$, then $X(A, B)$ is separable for every A with $A_{ab} \geq |B_{ab}|$ for all $a \neq b$.*
(c) *Let $B \succ 0$, and let $S := B_{\{p,q\}} - B_{\{p,q\},o}B_o^{-1}B_{o,\{p,q\}}$ be the Schur complement on $\{p, q\}$. Then $|B| \succeq 0$ if and only if $2|B_{pq}| \leq \sqrt{s_{pp}s_{qq}} - s_{pq}$, if and only if $B - ww^\top \succeq 0$ for some real w supported on $\{p, q\}$ with $w_p w_q = B_{pq}$.*

In particular $X(|B|, B)$ is separable if and only if $|B| \succeq 0$.

Proof. (a) By Lemma 6.1, $B = \sum_n z_n z_n^*$ with $z_n = x_n \odot y_n$ and $\sum_n |x_n(a)|^2 |y_n(b)|^2 \leq |B_{ab}|$ for $a \neq b$. As in the proof of Lemma 8.3, $|B_{ab}| \leq \sum_n |z_n(a)||z_n(b)| = \sum_n |x_n(a)y_n(b)|\,|x_n(b)y_n(a)| \leq |B_{ab}|$, so $|B| = \sum_n |z_n||z_n|^\top$.

(c) The matrices $|B|$ and B differ only in the entries (p, q) and (q, p), and $|B|_o = B_o \succ 0$, $|B|_{o,\{p,q\}} = B_{o,\{p,q\}}$. So the Schur complement of $|B|$ on $\{p, q\}$ is $S + 2|B_{pq}|(E_{pq} + E_{qp})$, which is positive semidefinite if and only if $|s_{pq} + 2|B_{pq}|| \leq \sqrt{s_{pp}s_{qq}}$; since $S \succ 0$ gives $s_{pq} > -\sqrt{s_{pp}s_{qq}}$, this is the first condition. For the second, $B - ww^\top \succeq 0$ if and only if $w^\top B^{-1} w \leq 1$, the $\{p, q\}$ block of B^{-1} is S^{-1}, and the minimum of $w^\top S^{-1} w$ over real w supported on $\{p, q\}$ with $w_p w_q = -|B_{pq}|$ is $2|B_{pq}|/(\sqrt{s_{pp}s_{qq}} - s_{pq})$.

(b) Let first $B \succ 0$. By (c) there is w with $N := B - ww^\top \succeq 0$. Then $N_{pq} = 0$ and $N_{ab} = B_{ab} > 0$ for the other pairs, so N is doubly nonnegative and, since $d = 4$, completely positive [24]: $N = \sum_n v_n v_n^\top$ with $v_n \geq 0$. The terms with $x = y = v_n^{1/2}$ (entrywise) have costs $v_n(a)v_n(b)$, which add up to N_{ab}, and one term supported on $\{p, q\}$ with $x \odot y = w$ and $|x_p| = |y_p|$, $|x_q| = |y_q|$ costs $|w_p w_q| = |B_{pq}|$ in both directions. The coherence of these terms is $N + ww^\top = B$, and their cost at (a, b) is $|B_{ab}| \leq A_{ab}$, so Lemma 6.1 gives separability. If B is singular, $B + \epsilon I \succ 0$ has the same sign pattern and $|B + \epsilon I| = |B| + \epsilon I \succeq 0$, so $X(A, B + \epsilon I)$ is separable for every $\epsilon > 0$, and so is its limit $X(A, B)$. □

Part (a), which holds at every d, is also the realignment criterion [22] in the form of [23, Theorem 1(e)]. Part (b) also follows from [23, Corollary 1] for N, [23, Theorem 3] for the term on $\{p, q\}$, and additivity. The new statement is the equivalence in the mixed class.

Corollary 8.10. *Let $d = 4$ and let $B \succeq 0$ be real with all off-diagonal entries nonzero. Then $X(|B|, B)$ is separable if and only if $|B| \succeq 0$, and in that case $X(A, B)$ is separable whenever $A_{ab} \geq |B_{ab}|$ for all $a \neq b$. If B is ferromagnetic or antiferromagnetic, then $|B| \succeq 0$.*

Proof. If B is ferromagnetic, then $|B| = SBS$ is doubly nonnegative, hence completely positive [24], and $X(|B|, B)$, the image of $X(|B|, |B|)$ under $S \otimes I$, is separable by Lemma 8.3. If B is antiferromagnetic, $X(|B|, B)$ is PPT and separable by Remark 6.2, and $|B| \succeq 0$ by Lemma 8.9(a), whose proof uses no hypothesis on the signs. The mixed class is Lemma 8.9. Finally $X(A, B) = X(|B|, B) + \sum_{a \neq b}(A_{ab} - |B_{ab}|)|ab\rangle\langle ab|$. □

For symmetric A the condition $A_{ab} \geq |B_{ab}|$ is the PPT condition. In the mixed class $|B|$ need not be positive semidefinite: for the rational matrix B of †20 (mixed, positive definite) $X(|B|, B)$ is PPT, with a symmetric A, and entangled. This operator is also detected by the realignment criterion.

Corollary 8.11. *Let $d = 4$ and let (Φ, Ψ) be a PPT pair whose composite $\Lambda := \Psi \circ \Phi$ is self-adjoint, $\Lambda = \Lambda^*$. Then A is symmetric and B is real. If all off-diagonal entries of B are nonzero and $|B| \succeq 0$, in particular if B is ferromagnetic or antiferromagnetic, then* $\mathrm{tw}(J(\Lambda))$ *is separable.*

Proof. $J(\Lambda^*) = FJ(\Lambda)^\top F$ with F the flip, so the twirl data of Λ^* are $(A^\top, \bar{B})$, and $\Lambda = \Lambda^*$ gives $A = A^\top$ and $B = \bar{B}$. The twirl is PPT and A is symmetric, so $A_{ab} \geq |B_{ab}|$, and Corollary 8.10 applies. □

Consider the composites $\Psi \circ \Phi$ with $\Psi = \Phi^* \circ \Xi$, where $\Xi = \Xi^*$ is completely positive or, equivalently, has a Kraus decomposition by Hermitian operators. These composites are self-adjoint, and Ψ is PPT as in Corollary 8.6(a). Corollary 8.11 does not cover all of them: there are Φ and Hermitian H for which $\Psi = \Phi^* \circ \mathrm{Ad}_H$ has B mixed and $|B|$ not positive semidefinite, also with Φ spanning an extreme ray of the cone of PPT maps (†21). Their

twirls are separable: in the first example Φ is EB, and in the second, every pivoted Cholesky frame of B satisfies the hypothesis of Lemma 6.3 exactly (†21). By Lemma 8.9(a), a proof that the twirls of all such composites are separable has to use the slack of A over $|B|$.

Finally, the ferromagnetic case of Proposition 8.5 is stable under perturbation.

Corollary 8.12. *Let $d \le 4$. Let $B_0 \succ 0$ be ferromagnetic, with $N_0 := D^*B_0D$ entrywise nonnegative for a diagonal unitary D, let $\lambda := \lambda_{\min}(B_0)$, and let $t \in \mathbb{R}^d_{>0}$ and $\eta > 0$ satisfy $(A_0)_{ab} - (N_0)_{ab}t_a/t_b \ge \eta$ for all $a \ne b$. Put $\epsilon := \min(\lambda/(2d), \eta/2)$. Then $X(A, B)$ is separable for all $B \succeq 0$ and all A, nonnegative off the diagonal, with $|B_{ab} - (B_0)_{ab}| \le \epsilon$ for all a, b and $|A_{ab} - (A_0)_{ab}| \le \epsilon$ for all $a \ne b$.*

Proof. We apply Proposition 8.5 with the same D and t, $N := N_0 - \frac{\lambda}{2}I$ and $R := B - DND^* = (B - B_0) + \frac{\lambda}{2}I$. The matrix N is doubly nonnegative: its off-diagonal entries are those of N_0, and $N \succeq \frac{\lambda}{2}I$. Moreover $R_{aa} \ge \frac{\lambda}{2} - \epsilon \ge (d-1)\epsilon \ge \sum_{b\ne a}|R_{ab}|$, because $\epsilon \le \lambda/(2d)$, and a diagonally dominant Hermitian matrix is 2-sparse positive semidefinite. Finally $\alpha_{ab} = A_{ab} - (N_0)_{ab}t_a/t_b \ge \eta - \epsilon \ge \epsilon$ and $\alpha_{ab}\alpha_{ba} \ge \epsilon^2 \ge |R_{ab}|^2$. □

A strict instance of Theorem 8.2 (A_0 cycle-balanced, strict PPT inequalities, all $(A_0)_{ab} > 0$) has such a slack η, with t from the balance reduction after Theorem 8.2. At Example 8.7 with $\Psi = \Phi_X^*$, $D = I$ and $t = \mathbf{1}$, one has $\lambda = 0.159613$, $\eta = 0.162867$ and $\epsilon = 0.019952$, and with $\Phi_X^* \circ \mathrm{Ad}_Y$ one has $\epsilon = 0.0792$ (†22). The twirl data depend continuously on $(J(\Phi), J(\Psi^*))$, so the PPT pairs whose data lie in the open box $\max|B - B_0| < \epsilon$, $\max_{a\ne b}|A_{ab} - (A_0)_{ab}| < \epsilon$ form a relatively open set that contains the pair of Example 8.7. At that pair every slice has rank 3 and no index peels on either side (Lemma 7.2). Since rank 3 is the largest possible, the slices keep rank 3 nearby. Then the subspaces $F_c = \mathrm{col}\, H_c^*$ (and $E_c = \mathrm{col}\, G_c^*$) have dimension 3 and depend continuously on the pair, and so do the operators $\tilde{N}_y$ of Lemma 3.1. The existence of an orthonormal basis (f_i) of F_c with $H_y f_i \in \mathbb{C}H_c f_i$ for all y is then a closed condition. Hence no index peels on a neighbourhood. Coherence rank 4 and $\mathcal{M}(B) \not\succeq 0$ are open conditions. So the twirl is separable on a nonempty open set of PPT pairs $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$ with coherence rank 4, all slices of rank 3, no peel and $\mathcal{M}(B)$ not positive semidefinite, outside Remark 6.2 and [23, Theorem 5]. This set contains pairs that are not self-composites. For instance, take $J(\Phi) = \rho_X + \delta I$ and $J(\Psi^*) = \rho_X + \delta' I$ with $\rho_X = J(\Phi_X)$ and $(\delta, \delta') = (\frac{1}{400}, \frac{1}{200})$. The blocks ρ_{ab} then span a 7-dimensional subspace of M_3, while the pairs (ρ_{ab}, σ_{ab}) span an 8-dimensional subspace (†22), so no linear map L satisfies $\sigma_{ab} = L(\rho_{ab})$ for all a, b. This pair lies in the open box, and no index peels on either side (†22). Numerically, the open set also contains pairs with B not ferromagnetic and A not cycle-balanced, which need the 2-sparse part of Proposition 8.5 (†22).

At two residual points the threshold of Proposition 8.5 coincides numerically with κ_{PPT} (†18). Along the rays $X(\kappa A, B)$, the separability threshold also equals κ_{PPT}, to numerical accuracy, for random entanglement-breaking composites and for random mixtures of product states with nonsingular B (†23). It differs from κ_{PPT} on a nonempty open set of separable operators with directed imbalance. For $d = 4$ let $B = J + \delta I$ with $0 < \delta < \frac{1}{4}$, and let $A_{12} = A_{23} = A_{31} = 1$, $A_{21} = A_{32} = A_{13} = 4$ and $A_{a4} = A_{4a} = 2$ ($a \le 3$). Since all ratios $A_{ij}A_{ji}/B_{ij}^2$ equal 4, we get $\kappa_{\mathrm{PPT}} = \frac{1}{2}$. Every cycle product of A is at least 1, so $X(A, J)$ is separable by Lemma CYC [Supp, S1], and so is $X(A, B) = X(A, J) + \delta \sum_a |aa\rangle\langle aa|$. The map $\tau_{3,1}(Y) := 2\varepsilon(Y) + \varepsilon(SYS^*) - Y$ on M_3 is positive [32, Theorem 1], where ε is the diagonal part and $Se_i = e_{i+1}$ (indices mod 3). Placed on the indices $1, 2, 3$ in the order of the cycle $1 \to 2 \to 3 \to 1$, the Choi matrix of $\tau_{3,1}$ is a block-positive operator Z with $\mathrm{Tr}(ZX(\kappa A, B)) = 3(\kappa - 1 + \delta)$, so $X(\kappa A, B)$ is entangled for $\kappa < 1 - \delta$ (†23). By continuity the two thresholds also differ at all (A', B') near (A, B). For $\mu > 1$ the data $(\mu A, B)$ lie on the same ray, and all operators near them are separable. To see this, write $B'' = J + E$, with E close to δI, and $X(A'', B'') = X(A'' - C, J) + X(C, E)$ with $C_{ab} = |E_{ab}|$ for $a \ne b$ and $C_{aa} = E_{aa}$. The first term is separable by Lemma CYC, since every cycle product of μA exceeds 1, and the second by [23, Theorem 5], since E is diagonally dominant.

9. Computations

Every computational fact is marked †. Table 2 gives, for each, its kind, where it is used and its folder in the software deposit [33], which lists its programs and outputs with their SHA-256. No numerical result is used in the proofs; the exact computations and the interval-arithmetic certificates are used in the proofs that cite them.

The exact facts (†11–†15, †17, †19–†21 and the exact parts of †22–†23) and the outer local ranks stated after Theorem 7.10 are also proved in Lean 4 with Mathlib, on the deposited data, except for these parts: the face-system rank of †14 (Lean proves $\dim \mathcal{F} = 1$ there through another linear system); the enclosure widths and the uniqueness up to phases in †14; the PPT property and the entanglement of the blocks in †19(C); the extremality of Φ in †21 (Lemma 7.9); the identification of the maps of †17 and †21 with the Choi matrices used in Lean; the entanglement of $X(|B|, B)$ in †20 (Lemma 8.9(a)); the randomness of the instances in †11 and †20; and cited results, such as [13], [3] and PPT^2 in dimension 3 [1, 2]. The same Lean development serves this paper and [34]: of its 2850 theorems, the kernel alone checks 1995, and the proofs of the other 855 also evaluate compiled code; 2225 of them, 1470 checked by the kernel alone, are in the modules for the facts of this paper. The Lean code is in the software deposit [33].

9.1. **Construction.** Write $J(\Phi) = \sum_p k_p k_p^*$ and $J(\Psi^*) = \sum_q s_q s_q^*$, and take the Gram data $(G_a)_{ip} = k_p[a, i]$ and $(H_a)_{iq} = s_q[a, i]$. Then for every $\mu \in \mathbb{C}^d$

$$(\textstyle\sum_a \mu_a T_a)_{pq} = \langle k_p, (D_\mu \otimes I) s_q \rangle.$$

Hence $\sum_a T_a = 0$ if and only if $J(\Phi)J(\Psi^*) = 0$, and

$$\textstyle\sum_a \mu_a T_a = 0 \iff \operatorname{supp} J(\Psi^*) \perp (D_{\bar{\mu}} \otimes I) \operatorname{supp} J(\Phi).$$

A relation with all $\mu_a \neq 0$ is normalized to $\mu = (1, \dots, 1)$ by a diagonal filter $G_a \mapsto \alpha_a G_a$, which preserves PPT and the separability of the twirl. We construct PPT pairs by alternating semidefinite programs over PPT states with marginal floors equal to 4% of the maximally mixed marginals. The supports are then made exactly orthogonal, and the Gram data are read off from eigendecompositions. A point is *residual* if:

- both factors are PPT to 10^{-7} and satisfy the floors;
- the coherence rank is exact;
- $B_{aa} \geq 0.01 \operatorname{Tr} B$ for all a;
- every G_a and H_a has rank 3 with singular-value ratios $\sigma_2/\sigma_1 \geq 0.05$ and $\sigma_3/\sigma_1 \geq 0.02$, and $P, Q \geq 4$;
- no index peels on either side: the commutator defect $\max_{x,y} \|\tilde{N}_y \tilde{N}_x^* - \tilde{N}_x^* \tilde{N}_y\| / \max_y \|\tilde{N}_y\|^2$ is at least 0.05 at every pivot.

By Corollary 1.3(c), slices of rank one always peel, which is why the slice ranks are bounded below.

9.2. **Results.**

†1 $d = 4$, coherence rank 3, one relation: 20 residual points, all with $(P, Q) = (6, 6)$. At each, the twirl has a separable fit with relative residual $\leq 7.0 \times 10^{-18}$; at 2 of them both factors are entangled by the realignment criterion.

†2 $d = 4$, coherence rank 4 (independent factors): 22 residual points. Their twirls have separable fits with relative residual $\leq 4.3 \times 10^{-25}$; at 17 of them both factors are entangled by the realignment criterion.

†3 $d = 4$, the matrix W itself, at 37 of the 42 residual points (15 of rank 3, all 22 of rank 4): the least white-noise weight p for which $(1 - p)W + p\, I/16$ has a Bose-symmetric PPT extension of level 2 [21] lies in $[-1.3 \times 10^{-11}, 1.6 \times 10^{-12}]$, the realignment value [22] is ≤ 0.943, and product-state fits reach relative residual $\leq 1.9 \times 10^{-20}$ at coherence rank 4 and $\leq 5.3 \times 10^{-12}$ at coherence rank 3, where they converge slowly.

†4 $d = 5$, coherence rank 3 (two relations 1 and μ with distinct entries): no valid point, because the space left for $\operatorname{supp} J(\Psi^*)$ is too small or the semidefinite program is infeasible (see the remark below).

†5 $d = 5$, coherence rank 4, one relation: 6 residual points, all with $(P, Q) = (7, 8)$. Their twirls have separable fits with relative residual $\leq 7.7 \times 10^{-18}$; for W, the level-2 extension test gives $|p| \leq 2.2 \times 10^{-11}$, the realignment value is ≤ 0.855, and product-state fits reach $\leq$

5.1×10^{-11}. The realignment values of the factors are below 1, so these points may have entanglement-breaking factors.

†6 Controls with the same fitters: a PPT entangled state of Horodecki type on $\mathbb{C}^3 \otimes \mathbb{C}^3$, also embedded in $\mathbb{C}^4 \otimes \mathbb{C}^4$ and $\mathbb{C}^5 \otimes \mathbb{C}^5$, gives twirl-fit residuals 6.0×10^{-3}, product-fit residual 2.7×10^{-3} and extension weights 0.164 and 0.228; separable controls (random mixtures of product states, and the Horodecki-type state in its separable range) give residuals below 10^{-16} and extension weights below 10^{-12}.

†7 Frame certificates at the 42 residual points of †1–†2. With F the pivoted Cholesky factor of B, the hypothesis of Lemma 6.3 holds for 18 to 24 of the 24 pivot orders at each point of coherence rank 3 and for all 24 at each point of coherence rank 4; at the best order the smallest cycle geometric mean of A/K is at least 1.061. In the spectral Kraus bases (eigenvectors of $J(\Phi)$ and $J(\Psi^*)$), the Kraus-frame condition of Corollary 7.3 holds at every point: a potential t gives $\max_{i \neq j} N_{ij} t_i / (t_j A_{ij})$ between 0.757 and 0.9915 (confirmed in exact rational arithmetic), and a 2-cycle binds at 38 of the 42 points, a 3- or 4-cycle at the other four. The Kraus-frame condition also holds at 484 random composites built from extreme points of the PPT cones (extremality checked for the 12 with the smallest margins), and fails at 8 of 40 random non-extreme composites, whose twirls are nevertheless separable. Random separable twirls with $B\mathbf{1} = 0$ at the boundary of separability also pass the Cholesky test.

†8 Relations with zeros at $d = 4$: among 16 valid pairs with a relation vanishing at one index, two have B irreducible, no peel and corner ranks $(5, 4)$, the case of Theorem 5.40; their twirls have separable fits with 3 terms and pass the Cholesky test of †7 at all 24 pivot orders.

†9 3-sparsity. At 7 of the 20 residual points of coherence rank 3 and at 4 of the 22 of coherence rank 4, B is not 3-sparse: a semidefinite program gives Hermitian Y whose 3×3 principal submatrices are positive semidefinite, with $\langle Y, B \rangle / \operatorname{Tr} B$ between -7.6×10^{-2} and -2.3×10^{-3}. At each of the 20 points of coherence rank 3 there is a certificate with 3 product terms, at most one of which has four nonzero entries, with relative slack at least 0.08 in A.

†10 Certificates with $\operatorname{rank} B$ terms. For Examples 7.17 and 7.18, fits with 2 terms stall at relative residual at least 5.8×10^{-3}, while fits with 3 terms reach 10^{-21}. At one residual point of coherence rank 3, along the ray $(\kappa A, B)$ fits with 10 terms reach relative residual at most 1.7×10^{-17} from $\kappa = 0.4874$ on (the partial-transpose margin is 0.4872; residuals of this size are counted as fits here), while fits with 3 terms reach 10^{-25} only for $\kappa \geq 0.55$ and stay at 2×10^{-4} or more for $\kappa \leq 0.52$. An alternating search for a PPT pair $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$ with $B = 4I - J$ and small $\sum_{i \neq j} A_{ij}$ (Example 7.19 would need 12) stops at values between 18.76 and 20.94.

The empty search †4 has a simple explanation.

Remark 9.1. Let $d = 5$, and let the two relations be $(1, \dots, 1)$ and μ with distinct entries. Suppose the supports are in general position: $\dim(S_\Phi + (D_{\bar{\mu}} \otimes I)S_\Phi) = \min(2P, 15)$ and $\dim(S_\Psi + (D_\mu \otimes I)S_\Psi) = \min(2Q, 15)$, where $S_\Phi = \operatorname{supp} J(\Phi)$ and $S_\Psi = \operatorname{supp} J(\Psi^*)$ have dimensions P and Q. Then $Q \le 15 - 2P$ and $P \le 15 - 2Q$, so $\min(P, Q) \le 5$. If that factor has an input marginal of rank 5, its Choi matrix is a PPT state of rank at most its larger local rank, hence separable by [10], and $\mathrm{tw}(W)$ is separable. So a residual point of coherence rank 3 at $d = 5$ with two non-EB factors needs supports in special position.

9.3. **Computations cited in Sections 5, 7 and 8.** As in Section 8, $\kappa_{\mathrm{PPT}} = \max_{i \ne j} |B_{ij}| / \sqrt{A_{ij} A_{ji}}$ is the least κ such that $X(\kappa A, B)$ is PPT.

- †11 For the two configurations of Remark 5.11, any three x_k and any three y_k are linearly independent, (I^+) holds, and the solution space of (R) has dimension 6. Random Gaussian-integer configurations in general position satisfying (I^+), with $r = 3, 4, 5$, give dimension r.
- †12 For the Tiles basis: the six product vectors of the kernel, the dimension 6 in Corollary 5.35 and Lemmas 5.37 and 5.38, and $\dim S(\Pi_\psi) = 3$ (Theorem 5.30); and an exact operator $J(\Phi) \succeq 0$ with $J(\Phi)^\Gamma \succeq 0$ on $\mathbb{C}^4 \otimes \mathbb{C}^3$ whose rank-5 corner is $\rho_I = \sum_m t_m |p_m\rangle\langle p_m|$ with $t = (1, 2, 1, \frac{3}{2}, 1, -\frac{1}{59})$, so that $H = -\frac{59}{2}$. Its minimal completion is a positive combination of 15 rank-one operators of Schmidt rank at most 2.
- †13 Theorem 7.10: $\dim \mathcal{F}(\rho_T) = 1$ (the face system has rank 15 in 16 real unknowns), the ranks and determinants of the filters, A and B for $d = 3, 4$, and the cycle values. Independently, A and B are recomputed from rational Kraus vectors with the formulas of Section 7.
- †14 Theorem 7.11: exact ranks, and enclosures in interval arithmetic with outward rounding of the eigenpairs, of the N_{ij} and of the ratios (widths at most 7.2×10^{-78} and 1.3×10^{-76}). For the second pair: the six Gaussian-integer product vectors, the projector identities, the rank 35 of the face system (computed modulo three primes), and the Gaussian-rational filters, with nonzero determinants.
- †15 Proposition 7.12(b): rational filters with $\operatorname{rank} X = \operatorname{rank} Z = 3$, $\det Y = \frac{35897}{125000}$ and $\det Y_2 = \frac{36451}{40000}$; B rational with $B_{aa} > 0$; $A_{ij} A_{ji} \ge B_{ij}^2$; and $v^\top |B| v = -0.7453\ldots$ for an explicit rational v with $\|v\| \approx 1$.
- †16 Section 7. Face walks started near an exact pair of the form of Theorem 7.10 at $d = 4$ with other filters end at extreme rays of ranks $(8, 9)$ and $(9, 8)$ with full local ranks, at which a cycle has $(\prod_C |B| / \prod_C A)^{1/|C|} = 1.44$ to 1.45. At the second pair of Theorem 7.11, $\kappa_{\mathrm{PPT}} = 0.238$, and a pivoted Cholesky frame satisfies the hypothesis of Lemma 6.3 with worst cycle ratio 0.238. Kraus frames with one potential per product term, optimized over Gram data, have

worst cost ratio 0.853 ($d = 3$) and 0.849 ($d = 4$) at the pairs of Theorem 7.10. At five extreme pairs on filtered Tiles orbits, obtained by maximizing the violation of the cycle condition for $(A, |B|)$, the worst ratio is between 0.99689 and 0.99915, below 1 in floating point, at four of them, so these frames certify the twirls numerically, and it is 1.00001 at a pair with $\kappa_{\mathrm{PPT}} = 0.99995$ (inconclusive). At these pairs both factors have outer local rank 3, so their composites are EB by PPT^2 in dimension 3, and the computations test the form of the certificates.

†17 Example 8.7, in arithmetic over $\mathbb{Q}(i)$: the projector identities for ρ_U; A symmetric; B real and entrywise positive with positive leading principal minors; $A_{ij} \geq B_{ij}$; $\mathbf{1}^\top \mathcal{M}(B)\mathbf{1} = -5.3537\ldots$ (for $\Psi = \Phi_X^*$) and $-28.5469\ldots$ (for $\Phi_X^* \circ \mathrm{Ad}_Y$); $\operatorname{rank} J(\Phi_X) = 6$ with all diagonal blocks of rank 3; and the ranks and column spaces of the T_a.

†18 Proposition 8.5. For fixed D and t the hypotheses are a semidefinite feasibility problem; a local search over (D, t) finds them satisfied, to solver tolerance, for $X(\kappa A, B)$ with $\kappa \geq 0.8865$ at the pair of Theorem 7.10 for $d = 4$, and with $\kappa \geq \kappa_{\mathrm{PPT}}$ (0.4047 and 0.9265) at two residual points of †2.

†19 Remark 3.5. (A), (B): at $d = 6$ with coherence rank 6 and at $d = 5$ with coherence rank 5, every chain of peels without compressions stops at coherence rank 4 with four nonzero rows, while the chain peel, compression, peel, compression (the second peel is available only after the compression) ends at outer dimension 3; in these two examples Φ is EB by construction, so they serve to test the chains. (C): at $d = 6$ with coherence rank 6, B irreducible and $J(\Psi^*)$ with PPT entangled blocks, the peel-only chains stop at coherence rank 5 with five nonzero rows, while a Ψ-peel, a compression, a Φ-peel and a compression reach outer dimension 2.

†20 Lemma 8.9 and Corollary 8.10. The matrix

$$B = \begin{pmatrix} 1 & \frac{1}{10} & \frac{4}{5} & \frac{1}{2} \\ \frac{1}{10} & 1 & \frac{1}{2} & -\frac{2}{5} \\ \frac{4}{5} & \frac{1}{2} & 1 & \frac{1}{10} \\ \frac{1}{2} & -\frac{2}{5} & \frac{1}{10} & 1 \end{pmatrix}$$

is positive definite and mixed. The eigenvalues of $|B|$ are $(8 \pm \sqrt{10})/5$ and $(2 \pm \sqrt{5})/5$, so $X(|B|, B)$ is PPT and entangled; moreover $\||B|\|_{\mathrm{tr}} - \operatorname{Tr}|B| = (2\sqrt{5} - 4)/5 > 0$. For the matrix

$$B = \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{2} & \frac{1}{10} \\ \frac{1}{10} & 1 & \frac{1}{10} & -\frac{1}{5} \\ \frac{1}{2} & \frac{1}{10} & 1 & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{5} & \frac{1}{10} & 1 \end{pmatrix}$$

the eigenvalues of $|B|$ are $\frac{1}{2}$, $\frac{4}{5}$, $\frac{11}{10}$ and $\frac{8}{5}$. The two conditions of Lemma 8.9(c) agree on 1980 random integer mixed matrices $B \succ 0$.

†21 Corollary 8.11: two composites $\Psi = \Phi^* \circ \mathrm{Ad}_H$ with H Hermitian, B real, positive definite and mixed, and $|B|$ not positive semidefinite. In the first, $G_a = \mathrm{diag}(\varphi_a)$ with $\varphi_1 = (\frac{7}{4}, -\frac{3}{2}, 1)$, $\varphi_2 = (\frac{1}{2}, 0, -1)$, $\varphi_3 = (\frac{5}{4}, -\frac{5}{4}, \frac{1}{2})$ and $\varphi_4 = (\frac{5}{4}, -\frac{3}{2}, -\frac{1}{4})$, so that Φ is EB, and

$$H = \begin{pmatrix} -\frac{5}{4} & \frac{5}{4} & 1 \\ \frac{5}{4} & -\frac{7}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix};$$

the vector $v = (\frac{313}{1000}, -\frac{67}{200}, -\frac{423}{500}, \frac{273}{1000})$ has $v^\top |B| v = -\frac{414813}{5120000}$. In the second, $J(\Phi) = (X \otimes I)\rho_U(X \otimes I)^*$ with ρ_U as in Example 8.7 and

$$X = \begin{pmatrix} -56 - 7i & -1 - 32i & 25 - 67i & -5 + 12i \\ 69 - 73i & -5 + 34i & -15 + 32i & -18 - 7i \\ 26 - 10i & -27 - 59i & 46 - 11i & 1 + 9i \\ 25 - 49i & -76 + 11i & 9 - 30i & 8 - 7i \end{pmatrix}$$

and

$$H = \begin{pmatrix} 20 & 8 + 2i & -3 + 100i \\ 8 - 2i & 6 & 23 \\ -3 - 100i & 23 & 15 \end{pmatrix}.$$

Here $\det X = 2632047 + 379515i$, so Φ spans an extreme ray of the cone of PPT maps (Lemma 7.9). The matrices A and B are rational; for each of the 24 pivot orders the exact factorization $PBP^\top = LDL^\top$ has $D \succ 0$, and the frame $F = LD^{1/2}$ satisfies all 20 cycle conditions of Lemma 6.3. The binding cycle is the 2-cycle $(3, 4)$, with ratio $\kappa_{\mathrm{PPT}} = 0.894150$.

†22 Corollary 8.12 (λ, η and ϵ at Example 8.7 are given in Section 8). Exact: for $(\delta, \delta') = (\frac{1}{400}, \frac{1}{200})$ the 16 blocks ρ_{ab} span a space of dimension 7 and the pairs (ρ_{ab}, σ_{ab}) one of dimension 8 (for $\delta = \delta'$ both dimensions are 7); at this pair no index peels on either side, $\max |B - B_0| = 0.010757$ and $\max_{a \neq b} |A_{ab} - (A_0)_{ab}| = 0.010617$ are smaller than ϵ, and $\mathbf{1}^\top \mathcal{M}(B)\mathbf{1} = -5.3134$. Numerical: five pairs $J(\Phi') = \rho_X + sR_1$, $J(\Psi'^*) = \rho_X + sR_2$ with random separable R_1, R_2 and $5 \cdot 10^{-3} \leq s \leq 10^{-2}$ lie in the open box and satisfy the hypotheses of Proposition 8.5 with $N = N_0 - \frac{\lambda}{2} I$ and $t = \mathbf{1}$. Their triangle phases of B are of order $10^{-3}\pi$ and the 3-cycle imbalances of A of order 10^{-3}; $\mathcal{M}(B)$ is not positive semidefinite, $\mathrm{rank}\, B = 4$, all slices have rank 3, and no index peels.

†23 Thresholds along rays. For 8 random entanglement-breaking composites of coherence rank 4, the separability threshold along the ray $(\kappa A, B)$ equals κ_{PPT} within $3 \cdot 10^{-4}$ (relative), and for 18 random mixtures of product states with nonsingular B within $5 \cdot 10^{-4}$ (the bisection resolution is about $4 \cdot 10^{-4}$ there). For the example of Section 8.4, the ratios $A_{ij}A_{ji}/B_{ij}^2$, the cycle products of A (their

minimum is 1) and the value of $\mathrm{Tr}(ZX(\kappa A, B))$ are computed exactly.

†24 Condition (G) of Section 5.11. At each of the 20 residual points of †1 the map $X \mapsto \sum_a G_a^* X H_a$ on M_3 has kernel $\mathbb{C}I$, and the ratio of its second smallest to its largest singular value is between 8.9×10^{-3} and 1.05×10^{-1}. A local search of the minimum over $e \in \mathbb{C}^4 \setminus \mathbb{C}\mathbf{1}$ of $s_9(M(e))/(s_1(M(\mathbf{1}))\,\mathrm{dist}(e, \mathbb{C}\mathbf{1}))$, where $M(e)$ is the matrix of $X \mapsto \sum_a e_a G_a^* X H_a$, gives values between 3.0×10^{-3} and 3.1×10^{-2}.

TABLE 2. The computational facts of Section 9, their kind, where they are used, and their folders in the software deposit [33] (under `sec9/` unless given in full). The files `MAP.md` and `MAP.tsv` of the deposit list the programs and outputs of each dagger with their SHA-256, and each folder lists its files with their SHA-256 in its `SHA256SUMS`.

Fact	Kind	Used in	Folder
†1, †2 residual points at $d = 4$	numerical	Sections 5.11, 8, 11	`d01`
†3 W at the residual points	numerical	Section 5.11	`d03`
†4, †5 searches at $d = 5$	numerical	Section 9	`d04`
†6 controls of the fits	numerical	Section 9	`d01`, `d03`, `d04`
†7 frame certificates	numerical	Sections 6, 7, 11	`d07a`, `d07b`
†8 relations with a zero at $d = 4$	numerical	Remark 5.41	`d07a`
†9 3-sparsity	numerical	Sections 7, 11	`d09a`–`d09d`
†10 certificates with rank B terms	numerical	Section 11	`d09c`, `d09d`, `d10`
†11 condition (R)	exact	Remark 5.11	`d11`
†12 the Tiles basis; a corner of rank 5	exact	Remark 5.41	`d12`
†13 the pairs of Theorem 7.10	exact	Theorem 7.10	`d13`
†14 the pairs of Theorem 7.11	interval, exact	Theorem 7.11	`d13`, `d14`
†15 filters with $\lvert B\rvert \not\succeq 0$	exact	Proposition 7.12(b)	`d13`
†16 face walks, Kraus frames	numerical	Sections 7.3, 11	`d13`
†17 Example 8.7	exact	Example 8.7	`d17`
†18 hypotheses of Proposition 8.5	numerical	Section 8	`d17`, `d18`
†19 chains with compressions	exact	Remark 3.5	`d19a`, `d19b`
†20 mixed coherence matrices	exact	Section 8.4	`d20a`, `d20b`
†21 self-adjoint composites	exact	Section 8.4	`d20b`, `d21`
†22 pairs without peels	exact, numerical	Section 8.4	`d20a`, `d20b`, `d22`
†23 thresholds along rays	numerical, exact	Section 8.4	`d20a`, `d20b`
†24 condition (G)	numerical	Section 5.11	`d24`
the outer local ranks after Theorem 7.10	exact	the text after Theorem 7.10	`sec7/thm_7_10`

10. The constant $\frac{1}{5}$

With $M = \Phi^*(I)^\top \otimes \Psi(I)$ as in the introduction, Proposition 7.21 shows, given [3], that $W + \frac{1}{5}M$ is separable for every PPT pair. The companion paper [34] studies how far the constant $\frac{1}{5}$ can be lowered; we state its main results. The constant $\frac{1}{5}$ is optimal for every certificate that does not depend on the pair and applies [3] after fixed completely positive maps, transposes, conjugations $\mathrm{Ad}_G \circ \cdot \circ \mathrm{Ad}_{G^{-1}}$ with invertible G, or averages [34, Theorem 3.3 and Corollary 3.4], and, when both outer dimensions are at least 3 and one of them is at least 4, for averages of certificates that apply fixed maps separately to the input, middle and output spaces and are certified by the known theorems [34, Theorem 3.5]. Certificates adapted to one factor $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ hold, given [3], for every partner, and the least constant they give is at most $\frac{1}{5}$; its supremum $\Lambda(d)$ over all nonzero PPT maps $\mathrm{M}_d \to \mathrm{M}_3$ is below $\frac{1}{5}$ unless some extreme PPT channel is rigid, that is, has least constant $\frac{1}{5}$, and, given [3], $W + \Lambda(d)M$ is separable for all PPT maps $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_n$ [34, Lemmas 5.1 and 5.6]. This notion of rigidity is unrelated to the rigidity of corners in Theorem 5.30. No PPT channel $\mathrm{M}_3 \to \mathrm{M}_3$ is rigid, and $0.118 \le \Lambda(3) < \frac{1}{5}$, the lower bound by an exact certificate; neither result uses the qubit theorem [34, Theorems 8.1 and 8.5]. Hence, given [3], $W + \Lambda(3)M$ is separable for all PPT maps $\Phi\colon \mathrm{M}_3 \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_n$ and every n [34, Corollary 8.2]. Given [3], exact certificates also give constants below $\frac{1}{5}$ at explicit factors with Gaussian-rational Choi matrices near points computed in floating point [34, Theorems 6.1 and 6.2].

11. Open problems

(a) *Coherence rank at least* 3. Let $d \ge 5$. Is the twirl separable when, after zero-row compression to outer dimension at least 5, no sequence of peels and zero-row compressions reaches coherence rank at most 2 or outer dimension at most 3? Apart from pairs that another separability criterion settles at a reachable pair, these are the open configurations.

(b) *Outer dimension* 4. At $d = 4$, are the twirls separable at coherence rank 3 with a relation of full support (after a filter, $J(\Phi)J(\Psi^*) = 0$) and no peel, and at coherence rank 4 without peels and outside the classes of Section 8? Relations with a zero coefficient are settled by Theorem 5.13. Both open configurations occur numerically, and at every residual point of †1–†2 the twirl satisfies the frame criterion of Lemma 6.3 to numerical accuracy (†7). For the first configuration, Theorem 5.43 (under condition (G)) and Proposition 5.44 show which reductions cannot suffice. The classes of Section 8 contain extreme pairs at which every T_a has rank 3 (Example 8.7) and a nonempty open set of pairs without peels (Corollary 8.12). A related question: are the twirls of all composites $\Phi^* \circ \Xi \circ \Phi$ with $\Xi = \Xi^*$

completely positive separable? Corollary 8.11 does not cover all self-adjoint composites (Section 8.4).

(c) *Kraus frames with one potential per term.* The Kraus-frame condition of Corollary 7.3 fails at some extreme pairs for all Gram data (Theorem 7.10), so Lemma 7.1 combined with that condition cannot settle outer dimension 4. The following weakening remains open.

Question 11.1. Does every pair of PPT maps $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$ spanning extreme rays have Gram data whose Kraus frame F admits vectors x_n, y_n with $x_n \odot y_n$ equal to the n-th column of F and $\sum_n |x_n(a)|^2 |y_n(b)|^2 \le A_{ab}$ for all $a \neq b$?

By Lemma 6.1 such vectors certify $\mathrm{tw}(W)$, so a positive answer, combined with Lemma 7.1, would imply that every twirl at outer dimension 4 is separable. By Lemma 7.13 the Gram data must satisfy $A_{ij}A_{ji} \ge N_{ij}^2$ for all pairs, which the spectral Kraus bases need not do (Theorem 7.11). Numerically, such Gram data exist at the pairs of Theorem 7.10 and at all but one of the tested extreme pairs on filtered Tiles orbits; at the remaining one the search is inconclusive (†16). At the tested extreme pairs on filtered Tiles orbits both factors have outer local rank 3, so their twirls are already separable by PPT^2 in dimension 3.

(d) *Number of product terms.* Examples 7.17 and 7.18 show that separable twirls of PPT composites can need more product terms than the coherence rank, and Example 7.19 gives data (A, B) that need more, although no PPT pair with these data is known. Is $\operatorname{rank} B$ enough for pairs with nonsingular marginals, or on the residual configurations of item (b), where all tested points have strict partial-transpose slack (†9, †10)?

(e) *Entanglement breaking.* Theorem 1.1 concerns the twirl: is $\Psi \circ \Phi$ itself EB at coherence rank 2? Several cases of Table 1 show that a factor or the composite is EB, and so does Theorem 5.13 when $|Z| = 1$; for $|Z| = 2$ its proof gives the separability of the twirl but not more.

(f) *The constant* $\frac{1}{5}$. The open problems on the constant of Proposition 7.21 are stated in [34, Section 9]; among them are the value of $\Lambda(3)$ and whether some PPT channel $\mathrm{M}_d \to \mathrm{M}_3$ with $d \ge 4$ is rigid in the sense of Section 10.

The supplement also states two conjectures on its case analysis, Conjectures J4 and SW2 [Supp, S7], and a statement OP1 that some results of [Supp, S8] assume; no result of this paper uses them.

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