THE PPT^2 CONJECTURE THROUGH A QUTRIT: BARRIERS AT $\frac{1}{5}$ AND A SMALLER CONSTANT WHEN THE FIRST FACTOR ACTS ON M_3

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ABSTRACT. The PPT^2 conjecture asserts that composites of PPT (completely positive and completely copositive) maps are entanglement breaking; it holds for composites through a qubit (the qubit theorem of K. Henry) and for maps on M_3 (Christandl, Müller-Hermes and Wolf; Chen, Yang and Tang). For PPT maps $\Phi\colon \mathrm{M}_{d_1} \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$, averaging the qubit theorem over the qubit compressions of the middle qutrit shows, as in the companion paper for $d_1 = d_2$, that $W + \frac{1}{5}\,\Phi^*(I)^\top \otimes \Psi(I)$ is separable, where W is the Choi matrix of $\Psi \circ \Phi$; the conjecture says that W itself is separable. Let $\Lambda(3)$ be the least c for which $\rho + c\,I \otimes I$ is separable for the Choi matrix ρ of every PPT channel $\mathrm{M}_3 \to \mathrm{M}_3$. We prove, without the qubit theorem, that $0.118 \le \Lambda(3) < \frac{1}{5}$, where the lower bound comes from an exact certificate whose arithmetic is checked in Lean 4. With the qubit theorem, $\frac{1}{5}$ can then be replaced by $\Lambda(3)$ whenever the first factor Φ acts on M_3, that is, $d_1 = 3$. Certificates built from the qubit theorem independently of Φ and Ψ cannot lower $\frac{1}{5}$. For $d_1, d_2 \ge 3$ with $\max(d_1, d_2) \ge 4$, neither can averages of certificates that act separately on the input, middle and output spaces, even if PPT^2 through a qutrit were known in all smaller outer dimensions.

1. INTRODUCTION

A linear map $\Theta\colon \mathrm{M}_m \to \mathrm{M}_n$ is *PPT* if both Θ and $\vartheta \circ \Theta$ are completely positive, where ϑ is the transpose. It is *entanglement breaking* (EB) if its Choi matrix $J(\Theta) = \sum_{a,b} E_{ab} \otimes \Theta(E_{ab})$ is separable. The PPT^2 conjecture asserts that the composite of two PPT maps is EB. It was proposed by Christandl in 2012; if true, it limits the use of PPT maps in repeater scenarios, where a map acts more than once [1]. In equivalent forms, it says that two parties who each apply a PPT map to their half of a shared state always end with a separable state, and that a positive map applied after a PPT map gives a decomposable map [1].

The middle dimension limits how entangled a composite can be: the Choi matrix of a composite through M_k has Schmidt number at most k [10]. The conjecture holds for composites through a qubit, $\mathrm{M}_m \to \mathrm{M}_2 \to \mathrm{M}_n$, with arbitrary outer dimensions [3] (the qubit theorem); in this paper, results

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whose proofs use it, directly or through another result, are labelled 'given [3]'. Composites through a qutrit are therefore the next case. For them the conjecture holds when both outer dimensions are at most 3 [1, 2]. In general it is open, but it has been proved for further classes of maps, such as Gaussian channels and maps with certain unitary symmetries, and asymptotic versions are known, for instance for iterates of unital or trace-preserving PPT maps (see [1] and the references there).

This paper concerns composites $\Psi \circ \Phi$ of PPT maps through a qutrit, $\Phi\colon \mathrm{M}_{d_1} \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$, with $W := J(\Psi \circ \Phi)$ and $M := \Phi^*(I)^\top \otimes \Psi(I)$, the Choi matrix of $\Psi \circ \tau \circ \Phi$ with $\tau(X) = \mathrm{Tr}(X)\, I_3$. Applying [3] to the compressions of the middle qutrit to a qubit and averaging shows that $W + \frac{1}{5}M$ is separable (Proposition 2.1; for $d_1 = d_2$ this is [4, Proposition 7.21]), while PPT^2 through a qutrit says that W itself is separable. We ask how far the constant $\frac{1}{5}$ can be lowered, for arbitrary outer dimensions. Since PPT^2 through a qutrit is the statement with the constant 0, every constant below $\frac{1}{5}$ that holds for all pairs in a class narrows, for that class, the gap between what the qubit theorem gives and what the conjecture asserts. The companion paper [4] studies, for $d_1 = d_2$, the twirl of W over local diagonal unitaries.

Main results. Three statements summarize the paper. The first shows that the averaging argument behind $\frac{1}{5}$ cannot be improved without using the pair.

Theorem A (Theorem 3.3, Corollary 3.4 and Theorem 3.5)**.** *For $\lambda < \frac{1}{5}$, no certificate that is independent of the pair and applies the qubit theorem* [3] *after fixed completely positive maps, transposes, conjugations $\mathrm{Ad}_G \circ \cdot \circ \mathrm{Ad}_{G^{-1}}$ with invertible G, or averages of these shows that $W + \lambda M$ is separable for all PPT pairs. For outer dimensions $d_1, d_2 \geq 3$ with $\max(d_1, d_2) \geq 4$, the bound $\frac{1}{5}$ persists for averages of certificates that apply fixed maps separately to the input, middle and output spaces, even when PPT^2 for maps on M_3, the separability of PPT states on $\mathbb{C}^2 \otimes \mathbb{C}^3$ and PPT^2 through a qutrit in all smaller outer dimensions are added.*

So induction on the outer dimension by averaging cannot work. For a first factor on M_3, however, a smaller constant suffices.

Theorem B (Theorems 8.1 and 8.5, Remark 5.7)**.** *Let $\Lambda(3)$ be the least c such that $\rho + c\, I \otimes I$ is separable for the Choi matrix ρ of every PPT channel $\mathrm{M}_3 \to \mathrm{M}_3$. Then $0.118 \leq \Lambda(3) < \frac{1}{5}$.*

Theorem C (given [3]; Corollary 8.2)**.** *For all PPT maps $\Phi\colon \mathrm{M}_3 \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$, the operator $W + \Lambda(3)M$ is separable.*

Theorem B does not use [3].

The route. A certificate below $\frac{1}{5}$ that applies the qubit theorem after operations of the kind in Theorem A must therefore depend on the pair;

the certificates behind Theorems B and C depend on one factor. Certificates adapted to one factor Φ express $\rho + \lambda\,\rho_A \otimes I$, where $\rho = J(\Phi)$ and $\rho_A = \Phi^*(I)^\top$, as a separable operator plus a sum of PPT operators supported on $\mathbb{C}^{d_1} \otimes V$ with planes $V \subseteq \mathbb{C}^3$; then $W + \lambda M$ is separable for every partner Ψ (Lemma 5.1). The least such λ is the one-sided constant $\lambda_{\mathrm{QD}}(\Phi) \le \frac{1}{5}$, and the difficulty is uniformity. For fixed d, the largest one-sided constant $\Lambda(d)$ over all nonzero PPT maps $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$ is attained at an extreme PPT channel $\mathrm{M}_n \to \mathrm{M}_3$ with $n \le d$. Call a PPT channel *rigid* if its one-sided constant is $\frac{1}{5}$. Then $\Lambda(d) < \frac{1}{5}$ unless some extreme PPT channel $\mathrm{M}_n \to \mathrm{M}_3$ with $n \le d$ is rigid, and $W + \Lambda(d)M$ is separable for every pair with $d_1 = d$ (Lemma 5.6(d), (e)).

At $d = 3$ no PPT channel is rigid (Theorem 8.1), so by compactness $\Lambda(3) < \frac{1}{5}$ (Lemma 5.6(b), (d)); the proof uses that the kernels of the Choi matrix of a rigid channel and of its partial transpose meet every $\mathbb{C}^3 \otimes V$ (Theorem 5.9). Since PPT operators on $\mathbb{C}^3 \otimes \mathbb{C}^2$ are separable, this $\Lambda(3)$ is the constant of Theorem B. This proves the upper bound of Theorem B, and Theorem C follows. The lower bound comes from an explicit strictly PPT map and an exact rational entanglement witness, certified by the dual form of the PPT symmetric extension test of level 2 [19] (Theorem 8.5). It also bounds what certificates adapted to one factor can give at $d_1 = 3$, even for pairs with $d_1 = d_2 = 3$, where PPT^2 is known. Numerical searches in Section 8 suggest that $\Lambda(3)$ is about 0.1188, close to the lower bound.

Computations. Computational facts are marked † and listed in Table 1. The exact facts, †1–†3, †8, †10 and †11, use rational or Gaussian-rational arithmetic. They are used in the proofs that cite them and are also proved in Lean 4 on the deposited data (Appendix B). Neither the deductions in the text that derive results from them, such as Lemma 8.4, nor the cited results, such as [3] and the separability of PPT operators on $\mathbb{C}^3 \otimes \mathbb{C}^2$, are formalized. The numerical facts, †4–†7, †9, †12 and †13, use floating point and are not used in any proof. Scripts, data and the Lean code are in the software deposit [8].

Organization. Section 2 fixes notation and restates the bound $\frac{1}{5}$, Sections 3–5 and 8 contain the results described above, and Section 9 poses open problems, among them whether $\Lambda(d) < \frac{1}{5}$ for $d \ge 4$, which Corollary 5.12 reduces to a class of extreme PPT channels. Sections 6 and 7 are not used in Section 8: Section 6 gives exact certificates at explicit strictly PPT factors near points computed in floating point, for instance $W - \frac{1}{20}M$ separable for a pair at $d = 4$ (Theorem 6.1) and $W + \frac{1}{10}M$ separable for every partner of a factor near a residual point (Section 2.1, Theorem 6.2), and Section 7 reports numerical values of pair-adapted and one-sided constants. Appendix A describes the exact certificates, Appendix B their Lean checks, and Appendix C proves Theorem 8.1; the supplement [Supp] lists the deposited computations (S1).

2. Preliminaries

2.1. **Notation.** We write M_n for the complex $n \times n$ matrices, E_{ab} for the matrix units, Θ^* for the adjoint of a linear map Θ with respect to the trace inner product, ϑ for the transpose, and $\mathrm{Ad}_X(Y) := XYX^*$. An operator on $\mathbb{C}^m \otimes \mathbb{C}^n$ is separable if it is a positive combination of products $P \otimes Q$ of positive semidefinite operators. For PPT maps $\Phi\colon \mathrm{M}_{d_1} \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ put $\rho := J(\Phi)$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^3$ and $\sigma := J(\Psi^*)$ on $\mathbb{C}^{d_2} \otimes \mathbb{C}^3$, with the outer factor first, and $\rho_A := \mathrm{Tr}_2\, \rho = \Phi^*(I_3)^\top$. The residual points of [4, †1] are 20 pairs of maps $\mathrm{M}_4 \to \mathrm{M}_3 \to \mathrm{M}_4$ that the random semidefinite searches of [4, Section 9] found in a configuration that the theorems of [4] leave open. They are PPT to within 10^{-7} and known only in floating point, and they are not known to be extreme.

2.2. **The constant $\lambda(\Phi, \Psi)$.** Throughout this paper $\Phi\colon \mathrm{M}_{d_1} \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ are nonzero PPT maps with arbitrary $d_1, d_2 \geq 1$; in [4], $d_1 = d_2 = d$. We write $\tau(X) := \mathrm{Tr}(X)\, I_3$ for $X \in \mathrm{M}_3$, $\Omega := \sum_i e_i \otimes e_i$ (not normalized), and $^\Gamma$ for the partial transpose on the first tensor factor. Put

$$W := J(\Psi \circ \Phi), \qquad M := J(\Psi \circ \tau \circ \Phi) = \Phi^*(I_3)^\top \otimes \Psi(I_3),$$

and, for linear $S\colon \mathrm{M}_3 \to \mathrm{M}_3$, $L_{\Phi,\Psi}(S) := J(\Psi \circ S \circ \Phi)$, so that $W = L_{\Phi,\Psi}(\mathrm{id})$ and $M = L_{\Phi,\Psi}(\tau)$. Since M is a product of positive semidefinite operators, the $\lambda \in \mathbb{R}$ for which $W + \lambda M$ is separable form a closed half-line; let $\lambda(\Phi, \Psi)$ be its left end. Proposition 2.1 gives $\lambda(\Phi, \Psi) \leq \frac{1}{5}$, and PPT^2 through a qutrit is the statement that $\lambda(\Phi, \Psi) \leq 0$ for all pairs. Let $\lambda^*(d)$ be the supremum of $\lambda(\Phi, \Psi)$ over all pairs with $d_1 = d$ and arbitrary d_2.

2.3. **The constant $\frac{1}{5}$.** For $d_1 = d_2$ the following proposition is [4, Proposition 7.21]; its proof applies verbatim to arbitrary d_1, d_2, and we restate it in that form.

Proposition 2.1 (given [3])**.** *Let $d_1, d_2 \geq 1$ and let $\Phi\colon \mathrm{M}_{d_1} \to \mathrm{M}_3$ and $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ be PPT. Then*

$$5\, J(\Psi \circ \Phi) + \Phi^*(I_3)^\top \otimes \Psi(I_3)$$

is separable.

Proof. For a rank-2 projection $P = VV^*$ on $\mathbb{C}^3$ ($V\colon \mathbb{C}^2 \to \mathbb{C}^3$ an isometry), $\Psi \circ \mathrm{Ad}_P \circ \Phi = (\Psi \circ \mathrm{Ad}_V) \circ (\mathrm{Ad}_{V^*} \circ \Phi)$ is a composite of PPT maps through M_2: $\mathrm{Ad}_{V^*} \circ \Phi$ is PPT, and $\Psi \circ \mathrm{Ad}_V$ is completely positive with $\vartheta \circ \Psi \circ \mathrm{Ad}_V = (\vartheta \circ \Psi) \circ \mathrm{Ad}_V$ completely positive. By [3] it is EB. For P Haar distributed, $\mathbb{E}[P \otimes P] = \frac{5}{12} I + \frac{1}{12} F$, F the swap (take traces against I and F: $9a + 3b = (\mathrm{Tr}\, P)^2 = 4$ and $3a + 9b = \mathrm{Tr}\, P^2 = 2$), so $\mathbb{E}[\mathrm{Tr}(PXPY)] = \frac{5}{12} \mathrm{Tr}(XY) + \frac{1}{12} \mathrm{Tr}\, X \,\mathrm{Tr}\, Y$ and $\mathbb{E}[\mathrm{Ad}_P] = \frac{5}{12}\mathrm{id} + \frac{1}{12} \mathrm{Tr}(\cdot) I_3$. Hence $\frac{5}{12} J(\Psi \circ \Phi) + \frac{1}{12} J\big(\mathrm{Tr}(\Phi(\cdot))\Psi(I_3)\big)$ is an average of separable operators, and $J(\mathrm{Tr}(\Phi(\cdot))\Psi(I_3)) = \Phi^*(I_3)^\top \otimes \Psi(I_3)$. $\square$

3. Pair-independent certificates

This section proves Theorem A of the introduction: it treats certificates that apply the qubit theorem [3] after pair-independent operations (Theorem 3.3) and averages of slot-wise certificates (Theorem 3.5).

Let $\mathcal{K}$ be the closed convex cone of linear maps on M_3 generated by the maps Ad_K and $\vartheta \circ \mathrm{Ad}_K$ with $K \in \mathrm{M}_3$ and $\operatorname{rank} K \le 2$.

Lemma 3.1 (given [3])**.** *For every $S \in \mathcal{K}$ and every PPT pair (Φ, Ψ) as above, $\Psi \circ S \circ \Phi$ is EB.*

Proof. Let $\operatorname{rank} K \le 2$. Choose isometries $\iota_1, \iota_2 \colon \mathbb{C}^2 \to \mathbb{C}^3$ with $\operatorname{ran} K \subseteq \operatorname{ran} \iota_1$ and $(\ker K)^\perp \subseteq \operatorname{ran} \iota_2$. Then $K = \iota_1 K_0 \iota_2^*$ with $K_0 := \iota_1^* K \iota_2 \in \mathrm{M}_2$, and $\Psi \circ \mathrm{Ad}_K \circ \Phi = (\Psi \circ \mathrm{Ad}_{\iota_1 K_0}) \circ (\mathrm{Ad}_{\iota_2^*} \circ \Phi)$. A completely positive map after a PPT map, and a PPT map after a completely positive map, are PPT, so this is a composite of PPT maps through M_2, which is EB by [3]. For $\vartheta \circ \mathrm{Ad}_K$, the map $\Psi \circ \vartheta$ is PPT ($J(\Psi \circ \vartheta) = J(\Psi)^\Gamma$), and the first case applies to the pair $(\Phi, \Psi \circ \vartheta)$. Finally, EB maps form a closed convex cone. □

Remark 3.2. The cone $\mathcal{K}$ is closed under the operations that do not depend on the pair: $C_1 \circ \mathcal{K} \circ C_2 \subseteq \mathcal{K}$ for completely positive maps C_1, C_2 on M_3 (since $\mathrm{Ad}_A \circ \mathrm{Ad}_K \circ \mathrm{Ad}_B = \mathrm{Ad}_{AKB}$), $\vartheta \circ \mathcal{K} = \mathcal{K} \circ \vartheta = \mathcal{K}$ (since $\vartheta \circ \mathrm{Ad}_K = \mathrm{Ad}_{\bar{K}} \circ \vartheta$), and $\mathrm{Ad}_G \circ \mathcal{K} \circ \mathrm{Ad}_{G^{-1}} = \mathcal{K}$ for invertible G. Every EB map on M_3 lies in $\mathcal{K}$, and so do the averages of Proposition 2.1 and every average of compressions to qubits in any frame. Moreover, by duality with the PPT cone, a linear map $A \colon \mathrm{M}_3 \to \mathrm{M}_2$ makes $A \circ \Phi$ PPT for every PPT Φ exactly when A is decomposable, and likewise for $B \colon \mathrm{M}_2 \to \mathrm{M}_3$ and $\Psi \circ B$. Every composite $B \circ A$ of decomposable maps through M_2 is a finite sum of generators of $\mathcal{K}$, and every generator is such a composite. So $\mathcal{K}$ is the closed cone of middle maps that the qubit theorem [3] certifies uniformly over all PPT pairs.

Theorem 3.3. *For $\lambda \in \mathbb{R}$, $\mathrm{id} + \lambda\tau \in \mathcal{K}$ if and only if $\lambda \ge \frac{1}{5}$. Equivalently, $|\Omega\rangle\langle\Omega| + \lambda\, I \otimes I$ lies in $\mathrm{SN}_2 + \mathrm{SN}_2^\Gamma$, where SN_2 is the cone of positive semi-definite operators on $\mathbb{C}^3 \otimes \mathbb{C}^3$ of Schmidt number at most 2, if and only if $\lambda \ge \frac{1}{5}$.*

Proof. Let $u_1, \dots, u_{12}$ be the vectors of four mutually unbiased bases of $\mathbb{C}^3$ and $\Pi_k := I - u_k u_k^*$. These vectors form a complex projective 2-design [9], $\sum_k (u_k u_k^*)^{\otimes 2} = I + F$ with F the swap, and a direct computation gives

$$\sum_{k=1}^{12} \mathrm{Ad}_{\Pi_k} = 5\,\mathrm{id} + \tau, \qquad \sum_{k=1}^{12} \Pi_k = 8\, I. \tag{1}$$

Hence $\mathrm{id} + \frac{1}{5}\tau = \frac{1}{5}\sum_k \mathrm{Ad}_{\Pi_k} \in \mathcal{K}$, and since $\tau = \sum_{i,j} \mathrm{Ad}_{E_{ij}} \in \mathcal{K}$, also $\mathrm{id} + \lambda\tau \in \mathcal{K}$ for $\lambda \ge \frac{1}{5}$.

Conversely, let $Z := 2\, I \otimes I - |\Omega\rangle\langle\Omega|$. For $\operatorname{rank} K \le 2$ we have $J(\mathrm{Ad}_K) = |k\rangle\langle k|$ with $k = (I \otimes K)\Omega$, $\langle k|k\rangle = \|K\|_2^2$ and $\langle\Omega|k\rangle = \operatorname{Tr} K$, so

$\operatorname{Tr}(Z\,J(\mathrm{Ad}_K)) = 2\|K\|_2^2 - |\operatorname{Tr} K|^2 \geq 0$, because $|\operatorname{Tr} K| \leq \|K\|_1 \leq \sqrt{2}\,\|K\|_2$ when K has at most two nonzero singular values. For every K, $J(\vartheta \circ \mathrm{Ad}_K)$ is the partial transpose of $|k\rangle\langle k|$ on the second factor, and the partial transpose of Z on that factor is $2I - F \succeq 0$; so $\operatorname{Tr}(Z\,J(\vartheta \circ \mathrm{Ad}_K)) \geq 0$. Hence $\operatorname{Tr}(Z\,J(S)) \geq 0$ for all $S \in \mathcal{K}$. Since $\operatorname{Tr}(Z\,J(\mathrm{id})) = 6 - 9 = -3$ and $\operatorname{Tr}(Z\,J(\tau)) = 18 - 3 = 15$, the inclusion $\mathrm{id} + \lambda\tau \in \mathcal{K}$ forces $\lambda \geq \frac{1}{5}$. The second form is the first under $S \mapsto J(S)$: the operators $J(\mathrm{Ad}_K)$ with $\operatorname{rank} K \leq 2$ are the rank-one elements of SN_2, and the partial transposes on the two factors give the same cone SN_2^{Γ}, because SN_2 is invariant under the full transpose. □

On the maps Ad_K alone, Theorem 3.3 is the Schmidt-number bound for isotropic states used in the remark after [4, Proposition 7.21] [10]; the transposed maps do not help either, because the partial transpose of Z is positive semidefinite.

Corollary 3.4. (a) *For $\lambda < \frac{1}{5}$, no certificate that is independent of the pair and applies the qubit theorem* [3] *after the operations of Remark 3.2 shows that $W + \lambda M$ is separable for all PPT pairs.*

(b) *Suppose that $L_{\Phi,\Psi}$ is injective. This holds if and only if the $\Phi(E_{ij})$ span M_3 and Ψ is injective on M_3, which needs $d_1, d_2 \geq 3$ and holds for generic pairs with $d_1, d_2 \geq 3$. Then for $\lambda < \frac{1}{5}$ there are no $S_1, \ldots, S_m \in \mathcal{K}$, chosen in any way depending on (Φ, Ψ), with $W + \lambda M = \sum_k L_{\Phi,\Psi}(S_k)$.*

Proof. (a) follows from Theorem 3.3, Lemma 3.1 and Remark 3.2. (b) If $W + \lambda M = \sum_k L_{\Phi,\Psi}(S_k)$, then $L_{\Phi,\Psi}(\sum_k S_k - \mathrm{id} - \lambda\tau) = 0$, so $\mathrm{id} + \lambda\tau = \sum_k S_k \in \mathcal{K}$ and $\lambda \geq \frac{1}{5}$. For the criterion, $L_{\Phi,\Psi}(S) = (\Phi^\natural \otimes \Psi)(J(S))$ with $\Phi^\natural(Y) := \Phi^*(Y^\top)^\top$, and $S \mapsto J(S)$ is a linear bijection. □

Corollary 3.4(b) concerns maps in $\mathcal{K}$. Pair-adapted factorizations $S = B \circ A$ with $A\colon \mathrm{M}_3 \to \mathrm{M}_2$ and $B\colon \mathrm{M}_2 \to \mathrm{M}_3$ such that $A \circ \Phi$ and $\Psi \circ B$ are PPT for the given pair are certified by [3] as well, and they are used in Sections 6 and 7; their factors A, B need not be decomposable, or even positive.

A pair-independent argument may also apply maps on the outer spaces. Call a triple (a, b, c) of linear maps on M_{d_1}, M_3 and M_{d_2} *certified* if $c \circ \Psi \circ b \circ \Phi \circ a$ is EB for all PPT pairs. Since $(a, b, c) \mapsto J(c \circ \Psi \circ b \circ \Phi \circ a)$ is trilinear, a representation

$$\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} + \lambda\,\mathrm{id} \otimes \tau \otimes \mathrm{id} = \lim \textstyle\sum_k t_k\, a_k \otimes b_k \otimes c_k \tag{2}$$

with $t_k \geq 0$ and certified triples (a_k, b_k, c_k) shows that $W + \lambda M$ is separable for all pairs. Every average of slot maps (compressions, frames, gauges, transposes, positive pre- and post-processing) is of this form. On M_n let POS_n and DEC_n be the cones of positive and of decomposable maps, and for $r \leq n$ let $Q_r^{\mathrm{in}}(n)$ and $Q_r^{\mathrm{out}}(n)$ be the cones generated by $\mathrm{Ad}_A \circ p$ and by

$p \circ \mathrm{Ad}_A$, respectively, with $\operatorname{rank} A \le r$ and $p \in \mathrm{POS}_n$. The known theorems certify the following product cones of triples:

- $\mathcal{C}_1 = Q_3^{\mathrm{in}}(d_1) \otimes \mathrm{DEC}_3 \otimes Q_3^{\mathrm{out}}(d_2)$, by PPT^2 for maps on M_3 [1, 2];
- $\mathcal{C}_2 = \mathrm{POS}_{d_1} \otimes \mathcal{K} \otimes \mathrm{POS}_{d_2}$, by Lemma 3.1 and because EB maps stay EB after positive maps;
- $\mathcal{C}_3 = Q_2^{\mathrm{in}}(d_1) \otimes \mathrm{DEC}_3 \otimes \mathrm{POS}_{d_2}$ and its mirror $\mathcal{C}_3'$, because PPT operators on $\mathbb{C}^2 \otimes \mathbb{C}^3$ are separable [11, 7, 6].

If PPT^2 through a qutrit were known in all smaller outer dimensions, the cones $P_{r,r'} = Q_r^{\mathrm{in}}(d_1) \otimes \mathrm{DEC}_3 \otimes Q_{r'}^{\mathrm{out}}(d_2)$ with $r \le d_1$, $r' \le d_2$ and $(r, r') \ne (d_1, d_2)$ would be certified as well.

Theorem 3.5. (a) *Let $d_1, d_2 \ge 3$ with $\max(d_1, d_2) \ge 4$. If $\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} + \lambda\, \mathrm{id} \otimes \tau \otimes \mathrm{id}$ lies in the closed convex cone generated by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_3'$, then $\lambda \ge \frac{1}{5}$.*

(b) *The same holds with all cones $P_{r,r'}$ added, for the same d_1, d_2.*

In both cases $\lambda = \frac{1}{5}$ is attained, by $\mathcal{C}_2$ and the compressions of Theorem 3.3.

Proof. For a linear map a on M_n let $P_n(a) := \int \mathrm{Ad}_U \circ a \circ \mathrm{Ad}_{U^*}\, dU$ over the Haar measure of $U(n)$. It commutes with every Ad_U, so by Schur's lemma ($\mathrm{M}_n = \mathbb{C}I \oplus \mathfrak{sl}_n$) $P_n(a) = \alpha\, \mathrm{id} + \beta\, \tau_n$ with $\tau_n(X) = \mathrm{Tr}(X) I_n$; write $v(a) = (\alpha, \beta)$. The twirl preserves $\mathrm{Tr}_{\mathrm{super}}$, the trace of a map as an operator on M_n, and $\mathrm{Tr}(a(I))$. Since $\mathrm{Tr}_{\mathrm{super}}(\alpha\, \mathrm{id} + \beta\tau_n) = \alpha n^2 + \beta n$ and $\mathrm{Tr}\big((\alpha\, \mathrm{id} + \beta\tau_n)(I)\big) = \alpha n + \beta n^2$, the inequality $\beta \ge \mu_r(n)\, \alpha$ with $\mu_r(n) := \frac{n-r}{rn-1}$ is equivalent to $\mathrm{Tr}_{\mathrm{super}}(a) \le r\ \mathrm{Tr}(a(I))$.

For a positive map q on M_r, $\mathrm{Tr}_{\mathrm{super}}(q) \le r\ \mathrm{Tr}(q(I))$: the difference is $\mathrm{Tr}[J(q)(rI - |\Omega_r\rangle\langle\Omega_r|)]$, $J(q)$ is block-positive, and $rI - |\Omega_r\rangle\langle\Omega_r|$ is separable (an isotropic state with singlet fidelity 0). For $a = \mathrm{Ad}_A \circ p$ with p positive and $A = \sum_{k \le r} s_k |u_k\rangle\langle w_k|$ (orthonormal u_k and w_k), put $U = [u_1, \dots, u_r]$, $V = [w_1, \dots, w_r]$, $S = \mathrm{diag}(s)$ and $q(X) := V^* p(UXU^*) V$, a positive map on M_r. By cyclicity $\mathrm{Tr}_{\mathrm{super}}(a) = \mathrm{Tr}_{\mathrm{super}}(\mathrm{Ad}_S \circ q)$, and $\mathrm{Tr}(a(I)) = \sum_k s_k^2 \langle w_k | p(I) | w_k \rangle \ge \mathrm{Tr}\big((\mathrm{Ad}_S \circ q)(I)\big)$ because $UU^* \le I$; so $\mathrm{Tr}_{\mathrm{super}}(a) \le r\ \mathrm{Tr}(a(I))$, and the same holds for $a = p \circ \mathrm{Ad}_A$. Hence $v(a)$ lies in the cone spanned by $(1, 0)$ and $(-1, 1)$ for $a \in \mathrm{POS}_n \cup \mathrm{DEC}_n$ (positivity of $\alpha\, \mathrm{id} + \beta\tau_n$ means $\beta \ge 0$ and $\alpha + \beta \ge 0$), in the cone spanned by $(1, \mu_r(n))$ and $(-1, 1)$ for $a \in Q_r^{\mathrm{in}}(n) \cup Q_r^{\mathrm{out}}(n)$, and in the cone spanned by $(1, \frac{1}{5})$ and $(-1, 1)$ for $a \in \mathcal{K}$ (Theorem 3.3).

On $\mathrm{span}\{\mathrm{id}, \tau\}^{\otimes 3}$, with $s(x) := x_0 + x_1$, let

$$F_a(v \otimes w \otimes u) := (w_0 + w_1)\big[a\, s(v)s(u) + (a + \tfrac{1}{5})(v_1 u_1 - v_0 u_0)\big] + \tfrac{6}{5}\, w_1\, s(v)s(u).$$

Then $F_a(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}) = -\frac{1}{5}$ and $F_a(\mathrm{id} \otimes \tau \otimes \mathrm{id}) = 1$. On products of the rays above: if an outer ray is $(-1, 1)$, then $F_a = (w_0 + w_1)(a + \frac{1}{5})\, s(\cdot) \ge 0$, since $w_0 + w_1 \ge 0$ on every middle ray and $s \ge 0$ on every outer ray. For outer rays $(1, \mu)$ and $(1, \mu')$ with $\mu, \mu' \ge 0$: the middle ray $(1, \frac{1}{5})$ gives $F_a = \frac{6}{5}(a + \frac{1}{5})(\mu + \mu' + 2\mu\mu') \ge 0$; the middle ray $(-1, 1)$ gives $\frac{6}{5}(1 + \mu)(1 +$

$\mu') \geq 0$; and the middle ray $(1,0)$, which occurs only with DEC_3, gives $a(\mu+\mu'+2\mu\mu') - \frac{1}{5}(1-\mu\mu')$. Every generating cone with middle slot DEC_3 has a proper compression in some outer slot, so $\mu+\mu' > 0$ there ($\mu_3(n) > 0$ for $n \geq 4$ and $\mu_r(n) > 0$ for $r < n$), and only finitely many values occur. If a is larger than the finitely many bounds $\frac{1-\mu\mu'}{5(\mu+\mu'+2\mu\mu')}$, then the functional $F_a \circ (P_{d_1} \otimes P_3 \otimes P_{d_2})$ is nonnegative on all generators, hence on the closed cone, and takes the value $\lambda - \frac{1}{5}$ on the left side of (2). □

So, within positive sums of certified products, induction on the outer dimension by averaging is impossible: even with the cones of Theorem 3.5(b), averaging reaches $\frac{1}{5}$ but gives no constant below it. A proof of $\lambda^*(d) < \frac{1}{5}$ must use at least one of the following: the structure of the given pair; a certificate that is not a positive sum of slot-wise products; or a certified type, not listed above and without outer compression, whose middle cone contains $\mathrm{id} + \gamma\tau$ for some $\gamma < \frac{1}{5}$.

The qubit-middle argument of [3] brings a factor to a 2×2 block form with equal diagonal blocks, and a positive semidefinite block matrix $\left(\begin{smallmatrix} I & K \\ K^* & I \end{smallmatrix}\right)$ is separable, since the contraction K is a mean of two unitaries, which are normal. With three blocks this fails, even under PPT.

Proposition 3.6. *There is a real positive definite operator X on $\mathbb{C}^3 \otimes \mathbb{C}^3$ (here the middle factor comes first), written as a 3×3 block matrix over the middle factor, with $\operatorname{Tr} X = 1$, diagonal blocks $X_{aa} = \frac{1}{9} I_3$ and $X^\Gamma \succ 0$, that is entangled. For every $d \geq 3$, embedding the second factor isometrically into $\mathbb{C}^d$ and adding $\frac{1}{9} I_3 \otimes P$, with P the projection onto the complement, gives, after normalization, entangled positive definite PPT operators on $\mathbb{C}^3 \otimes \mathbb{C}^d$ whose three diagonal blocks are equal multiples of I_d.*

Proof. The rational matrix X and a rational matrix O with $I - O^\top O \succ 0$ are listed in [Supp, S1]. Entanglement follows from the realignment criterion [5]: for separable σ of trace one the realignment $R(\sigma)$ has trace norm at most 1, so $\operatorname{Tr}(O^\top R(\sigma)) < 1$, while $\operatorname{Tr}(O^\top R(X)) = \frac{3109624483}{3000000000}$. All properties of X and O are checked in exact arithmetic (†3). The embedded operator is PPT as a direct sum of PPT pieces, and it is entangled because its compression to the original factor is a multiple of X. □

4. Classes reduced to known results

This section collects classes of factors for which [3] or PPT^2 for maps on M_3 already makes the composite EB; Theorem 4.1(a) underlies the certificates adapted to one factor of the route in Section 1.

As before, $\rho := J(\Phi)$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^3$ and $\sigma := J(\Psi^*)$ on $\mathbb{C}^{d_2} \otimes \mathbb{C}^3$, with the outer factor first. For $n \geq 1$ let QD_n (respectively $\mathrm{OR3}_n$) be the closed convex cone generated by the PPT operators on $\mathbb{C}^n \otimes \mathbb{C}^3$ whose middle support, the range of the partial trace over the outer factor, has dimension at most 2 (respectively, whose outer support, the range of the partial trace over $\mathbb{C}^3$, has dimension at most 3), and let SEP_n be the cone of separable

operators. The middle and outer ranks of an operator are the dimensions of its middle and outer supports.

Theorem 4.1 (given [3])**.** (a) *If* $\rho \in \mathrm{QD}_{d_1} + \mathrm{SEP}_{d_1}$ *or* $\sigma \in \mathrm{QD}_{d_2} + \mathrm{SEP}_{d_2}$*, then* $\Psi \circ \Phi$ *is EB.*

(b) *If* $\rho \in \mathrm{QD}_{d_1} + \mathrm{OR3}_{d_1} + \mathrm{SEP}_{d_1}$ *and* $\sigma \in \mathrm{QD}_{d_2} + \mathrm{OR3}_{d_2} + \mathrm{SEP}_{d_2}$*, then* $\Psi \circ \Phi$ *is EB.*

Proof. The map $(\rho, \sigma) \mapsto J(\Psi \circ \Phi)$ is bilinear and continuous, every PPT summand of ρ is $J(\Phi_k)$ for a PPT map Φ_k, and every PPT summand of σ is $J(\Psi_l^*)$ for a PPT map Ψ_l. Since EB maps form a closed convex cone, it suffices to show that $\Psi_l \circ \Phi_k$ is EB for every pair of generators that occurs. If the middle support of $J(\Phi_k)$, the range of $\Phi_k(I)$, lies in a plane with projection Π, then $0 \preceq \Phi_k(X) \preceq c\,\Phi_k(I)$ for $0 \preceq X \preceq cI$ gives $\Phi_k = \mathrm{Ad}_\Pi \circ \Phi_k$, and $\Psi_l \circ \Phi_k = \Psi_l \circ \mathrm{Ad}_\Pi \circ \Phi_k$ is EB by Lemma 3.1. Symmetrically, $\Psi_l = \Psi_l \circ \mathrm{Ad}_{\Pi'}$ when the middle support of $J(\Psi_l^*)$ lies in a plane. A separable generator gives an EB map, and a composite with an EB map is EB. If both outer supports have dimension at most 3, then $(A \otimes I)J(\Phi_k)(A \otimes I)^* = J(\Phi_k \circ \mathrm{Ad}_{A^\top})$ gives $\Phi_k = \Phi_k' \circ \mathrm{Ad}_{\iota^*}$ with an isometry $\iota\colon \mathbb{C}^3 \to \mathbb{C}^{d_1}$ onto a three-dimensional subspace containing the transposed outer support (if $d_1 < 3$, factor through an isometry $\mathbb{C}^{d_1} \to \mathbb{C}^3$ instead) and $\Phi_k' = \Phi_k \circ \mathrm{Ad}_\iota$ PPT on M_3; likewise $\Psi_l = \mathrm{Ad}_{\iota'} \circ \Psi_l'$. Then $\Psi_l \circ \Phi_k = \mathrm{Ad}_{\iota'} \circ (\Psi_l' \circ \Phi_k') \circ \mathrm{Ad}_{\iota^*}$ is EB by PPT^2 for maps on M_3 [1, 2]. Under (a) every pair of generators contains a member of middle support at most 2 or a separable one, and under (b) every pair is of one of the three kinds. □

Each cone enlarges the separable one: $\mathrm{QD}_n + \mathrm{SEP}_n \neq \mathrm{SEP}_n$ for $n \geq 4$ (PPT entangled states on $\mathbb{C}^4 \otimes \mathbb{C}^2$ placed in a two-dimensional middle subspace), and $\mathrm{OR3}_n + \mathrm{SEP}_n \neq \mathrm{SEP}_n$ for $n \geq 3$. The condition defining class (a) does not refer to a basis, and class (a) contains factors that are not EB.

Corollary 4.2 (given [3])**.** *PPT^2 through a qutrit holds for all outer dimensions at most* d *if and only if* $\Psi \circ \Phi$ *is EB for every pair with* $d_1, d_2 \leq d$ *such that:* ρ *and* σ *span extreme rays of the PPT cones; both have middle rank* 3*; both have rank at least* 2*; and at least one of them has outer rank at least* 4*.*

Proof. Every PPT operator is a finite sum of extreme rays of the (closed, pointed) PPT cone, and $J(\Psi \circ \Phi)$ depends bilinearly on ρ and σ. An extreme ray of middle rank at most 2 lies in QD, one of rank 1 is a product, and a pair with both outer ranks at most 3 is covered by Theorem 4.1(b). □

Compressing the outer spaces instead of the middle one gives a bound in terms of the marginals of W; it uses only PPT^2 for maps on M_3.

Proposition 4.3. *Let* $W_A := \mathrm{Tr}_2 W$ *and* $W_B := \mathrm{Tr}_1 W$*, and let* $\mu_i := \max\{0, \frac{d_i - 3}{3d_i - 1}\}$ *for* $i = 1, 2$*. Then the operator*

$$W + \mu_2\, W_A \otimes I + \mu_1\, I \otimes W_B + \mu_1 \mu_2 \,\mathrm{Tr}(W)\, I \otimes I$$

is separable. For $d_1 = d_2 = 4$ the coefficients are $\mu_1 = \mu_2 = \frac{1}{11}$.

Proof. A side with $d_i \le 3$ is not compressed: embed $\mathbb{C}^{d_i}$ isometrically into $\mathbb{C}^3$ there ($\mu_i = 0$), and if $d_1, d_2 \le 3$ the claim is PPT^2 for maps on M_3. For rank-3 projections A on $\mathbb{C}^{d_1}$ and B on $\mathbb{C}^{d_2}$, $(\bar{A}\otimes B)W(\bar{A}\otimes B) = J(\mathrm{Ad}_B \circ \Psi \circ \Phi \circ \mathrm{Ad}_A)$ is, up to isometries, the Choi matrix of a composite of two PPT maps on M_3, hence separable [1, 2]. For Haar-random rank-3 projections on $\mathbb{C}^n$, $\mathbb{E}[\mathrm{Ad}_A] = a_n \,\mathrm{id} + b_n\, \tau_n$ with $a_n = \frac{3(3n-1)}{n(n^2-1)}$ and $b_n = \frac{3(n-3)}{n(n^2-1)}$ (from $a_n + nb_n = \frac{3}{n}$ and $n^2 a_n + nb_n = 9$, as in the proof of Proposition 2.1), and $\bar{A}$ is Haar-distributed with A. Averaging gives the separable operator $a_{d_1}a_{d_2}W + a_{d_1}b_{d_2}W_A \otimes I + b_{d_1}a_{d_2} I \otimes W_B + b_{d_1}b_{d_2}\,\mathrm{Tr}(W)\, I\otimes I$, and dividing by $a_{d_1}a_{d_2}$ gives the claim. □

The coefficient μ_i is optimal for each side among pair-independent averages of compressions of rank at most 3 and their transposes, by the argument of Theorem 3.3 with the witness $3I - |\Omega\rangle\langle\Omega|$ on $\mathbb{C}^n \otimes \mathbb{C}^n$. The remainder is built from the marginals of W, so Proposition 4.3 is not comparable with Proposition 2.1.

5. Certificates adapted to one factor

This section carries out the route of Section 1 up to rigidity: it studies certificates adapted to one factor and the one-sided constants λ_{QD} and $\Lambda(d)$ they define.

Let $\rho_A := \mathrm{Tr}_2\,\rho = \Phi^*(I_3)^\top$ and $R := \rho_A \otimes I_3$, so that $\rho + \lambda R = J(\Phi + \lambda\,\tau\circ\Phi)$, and define

$$\lambda_{\mathrm{QD}}(\Phi) := \min\{\lambda \in \mathbb{R} : \rho + \lambda R \in \mathrm{QD}_{d_1}\}.$$

Lemma 5.1. (a) *$\lambda_{\mathrm{QD}}(\Phi)$ is well defined, and $-\frac{1}{3} \le \lambda_{\mathrm{QD}}(\Phi) \le \frac{1}{5}$.*

(b) *(given [3]) If $\lambda \ge \lambda_{\mathrm{QD}}(\Phi)$, then $W + \lambda M$ is separable for every PPT map $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ and every d_2. In particular $\lambda(\Phi,\Psi) \le \lambda_{\mathrm{QD}}(\Phi)$ for every partner Ψ.*

Proof. (a) QD_{d_1} is the cone over the compact set of trace-one PPT operators with middle support in some plane (the Grassmannian is compact and support containment is a closed condition), so it is closed. The set of feasible λ is closed, it is upward closed because $R \in \mathrm{SEP} \subseteq \mathrm{QD}$, and it is bounded below by $-\frac{1}{3}$ because QD consists of positive semidefinite operators and $\mathrm{Tr}(\rho + \lambda R) = (1+3\lambda)\,\mathrm{Tr}\,\rho$. By (1), $\rho + \frac{1}{5}R = \frac{1}{5}\sum_k (I\otimes\Pi_k)\rho(I\otimes\Pi_k)$, and each summand is PPT (the congruence by $I\otimes\Pi_k$ commutes with Γ) with middle support in $\mathrm{ran}\,\Pi_k$. (b) The operator $\rho + \lambda R \in \mathrm{QD} \subseteq \mathrm{PPT}$ is the Choi matrix of the PPT map $\Phi + \lambda\,\tau\circ\Phi$. By Theorem 4.1(a), $\Psi\circ(\Phi + \lambda\,\tau\circ\Phi)$ is EB, and its Choi matrix is $W + \lambda M$. □

For a plane $V \subseteq \mathbb{C}^3$ with an isometry $\iota_V\colon \mathbb{C}^2 \to \mathbb{C}^3$ onto V, let $\rho_V := (I\otimes \iota_V^*)\rho(I\otimes\iota_V) = J(\mathrm{Ad}_{\iota_V^*}\circ\Phi)$ be the Choi matrix of the compression of Φ to a qubit.

Theorem 5.2 (given [3])**.** *If for some plane V both ρ_V and ρ_V^Γ are positive definite, then $\lambda_{\mathrm{QD}}(\Phi) < \frac{1}{5}$. Hence there is $\varepsilon(\Phi) > 0$ such that $W + (\frac{1}{5} - \varepsilon)M$ is separable for every PPT map $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ and every d_2. The same holds with the roles of the factors exchanged, applied to Ψ^*.*

Proof. The hypothesis implies $\rho_A \succeq \operatorname{Tr}_2 \rho_V \succ 0$, so $R = (\rho_A^{1/2} \otimes I)(I \otimes I)(\rho_A^{1/2} \otimes I)$ is an interior point of SEP $\subseteq$ QD: the identity is interior by the separable ball of [12], and invertible local congruences preserve SEP. So $\rho + \lambda R$ is interior to QD for $\lambda > \frac{1}{5}$, the value is finite by Lemma 5.1(a), and conic duality [13] gives $\lambda_{\mathrm{QD}}(\Phi) = \max\{-\operatorname{Tr}(Z\rho) : Z \in \mathrm{QD}^*,\ \operatorname{Tr}(ZR) = 1\}$ with attainment. Suppose $\lambda_{\mathrm{QD}}(\Phi) = \frac{1}{5}$ and let Z be optimal. Then $\operatorname{Tr}(Z(\rho + \frac{1}{5}R)) = 0$. By the Haar identity in the proof of Proposition 2.1, this is an integral over rank-2 projections Π of $\operatorname{Tr}(Z(I \otimes \Pi)\rho(I \otimes \Pi))$, which is nonnegative (the compressions lie in QD) and continuous in Π; since the Haar measure on the Grassmannian has full support, it vanishes for every Π. Now $Z \in \mathrm{QD}^*$ means that $Z_V := (I \otimes \iota_V^*)Z(I \otimes \iota_V)$ lies in $\mathrm{PPT}^* = \mathrm{PSD} + \mathrm{PSD}^\Gamma$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^2$ for every plane V. Writing $Z_V = P + Q^\Gamma$ with $P, Q \succeq 0$ gives $0 = \operatorname{Tr}(Z_V \rho_V) = \operatorname{Tr}(P\rho_V) + \operatorname{Tr}(Q\rho_V^\Gamma)$ with both terms nonnegative. On the open set of planes where $\rho_V \succ 0$ and $\rho_V^\Gamma \succ 0$, which is nonempty by hypothesis (note $\rho_V^\Gamma = (\rho^\Gamma)_V$, as Γ acts on the outer factor), this forces $P = Q = 0$, so $Z_V = 0$. The real and imaginary parts of the entries of $(I \otimes \iota^*)Z(I \otimes \iota)$ are real polynomials in $(\operatorname{Re}\iota, \operatorname{Im}\iota)$ on the connected real-analytic manifold of isometries $\mathbb{C}^2 \to \mathbb{C}^3$, and they vanish on an open set, so they vanish identically: $Z_V = 0$ for every V. Any two vectors of $\mathbb{C}^3$ lie in a common plane, so $Z = 0$, which contradicts $\operatorname{Tr}(ZR) = 1$. The last sentence follows from $J(\Theta^*) = FJ(\Theta)^\top F$ for Hermiticity-preserving linear maps Θ (F the swap), since the swap and the full transpose preserve separability. □

Remark 5.3. The hypothesis needs $\operatorname{rank}\rho \geq 2d_1$ and $\operatorname{rank}\rho^\Gamma \geq 2d_1$, but these rank conditions are not sufficient: if $\ker\rho$ contains $u \otimes S$ with $u \in \mathbb{C}^{d_1}$ and S a plane in $\mathbb{C}^3$, every compression ρ_V is singular, since every plane meets S. For example, $\rho = (P_{u^\perp} \otimes I)\sigma_1(P_{u^\perp} \otimes I) + (I \otimes P_{S^\perp})\sigma_2(I \otimes P_{S^\perp})$ with $\sigma_i \succ 0$ and $\sigma_i^\Gamma \succ 0$ is PPT with $\rho_A \succ 0$ and ranks $(10, 10)$ at $d_1 = 4$. The hypothesis holds for every ρ in the interior of the PPT cone. We do not know whether it holds generically on each boundary stratum of fixed ranks. The two conditions may be met at different planes: if $\rho_{V_1} \succ 0$ and $\rho_{V_2}^\Gamma \succ 0$, some V satisfies both, since $\det\rho_V$ and $\det\rho_V^\Gamma$ are nonzero real-analytic functions on a connected manifold. On the set where $\Phi^*(I) \succ 0$, λ_{QD} is continuous (upper semicontinuous by the interior point above, lower semicontinuous because QD is closed), so $\varepsilon(\Phi)$ can be chosen uniformly on compact subsets of the open set where the hypothesis holds.

The hypothesis of Theorem 5.2 fails at rank-deficient factors such as the residual points of [4, †1], where every compression ρ_V has a kernel. A finite version of the proof needs only a linear condition. Fix a unitary U, let

$V_k := (Uu_k)^\perp$ for the twelve vectors u_k of (1), and let $\Pi_k^U := U\Pi_k U^*$ be the projection onto V_k; the identities (1) hold for the Π_k^U as well. Put $S_k := \mathbb{C}^{d_1} \otimes V_k$, $K_k := \ker\rho \cap S_k$ and $K_k' := \ker\rho^\Gamma \cap S_k$, and let

$$\mathcal{L}_k := \{g \in \mathrm{Herm}(S_k) : P_{K_k} g P_{K_k} = 0,\ P_{K_k'} g^\Gamma P_{K_k'} = 0\},$$

where $\mathrm{Herm}(S)$ is the space of Hermitian operators supported on S and P_K is the orthogonal projection onto K (the partial transpose maps $\mathrm{Herm}(S_k)$ to itself).

Theorem 5.4 (given [3])**.** *If the linear map* $\Sigma\colon \bigoplus_k \mathcal{L}_k \to \mathrm{Herm}(\mathbb{C}^{d_1}\otimes\mathbb{C}^3)$, $(g_k) \mapsto \sum_k g_k$, *is surjective, then* $\lambda_{\mathrm{QD}}(\Phi) < \frac{1}{5}$. *Hence there is* $\varepsilon > 0$ *such that* $W + (\frac{1}{5} - \varepsilon)M$ *is separable for every PPT map* $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ *and every* d_2.

Proof. By (1) for the Π_k^U, $\rho + \frac{1}{5}R = \sum_k \sigma_k(0)$ with $\sigma_k(0) := \frac{1}{5}(I\otimes\Pi_k^U)\rho(I\otimes\Pi_k^U)$, which is PPT and supported on S_k; inside S_k its kernel is K_k and the kernel of its partial transpose is K_k', because $\rho \succeq 0$ and $\rho^\Gamma \succeq 0$. For each k, the projection $X_k := I\otimes\Pi_k^U \in \mathrm{Herm}(S_k)$ onto S_k satisfies $X_k^\Gamma = X_k$, so $P_{K_k}X_kP_{K_k}$ and $P_{K_k'}X_k^\Gamma P_{K_k'}$ are the identity on K_k and on K_k'. By surjectivity choose $g_k \in \mathcal{L}_k$ with $\sum_k g_k = \sum_k X_k$, and for $c > 0$ put $e_k := c(X_k - g_k)$; then $\sum_k e_k = 0$, $P_{K_k}e_kP_{K_k} = cP_{K_k}X_kP_{K_k}$ and $P_{K_k'}e_k^\Gamma P_{K_k'} = cP_{K_k'}X_k^\Gamma P_{K_k'}$. Let $\delta_k := -\frac{1}{8}(I\otimes\Pi_k^U)R(I\otimes\Pi_k^U) + e_k$; since $\sum_k \Pi_k^U = 8I$, $\sum_k \delta_k = -R$, and for c large δ_k is positive definite on K_k and δ_k^Γ on K_k'. In the decomposition $S_k = \mathrm{ran}\,\sigma_k(0) \oplus K_k$, the Schur complement shows that $\sigma_k(\varepsilon) := \sigma_k(0) + \varepsilon\delta_k \succeq 0$ for small $\varepsilon > 0$, and likewise $\sigma_k(\varepsilon)^\Gamma \succeq 0$. One ε serves all twelve planes, and $\rho + (\frac{1}{5} - \varepsilon)R = \sum_k \sigma_k(\varepsilon) \in \mathrm{QD}_{d_1}$. The rest is Lemma 5.1(b). □

Remark 5.5. If all K_k and K_k' vanish, then $\mathcal{L}_k = \mathrm{Herm}(S_k)$ and Σ is onto; if the hypothesis of Theorem 5.2 holds at one plane, it holds on an open dense set of planes, so this happens for almost every U. In general Σ fails to be onto if and only if some nonzero Z has every compression Z_k in $\mathrm{Herm}(K_k) + \mathrm{Herm}(K_k')^\Gamma$. On each locus where $\dim K_k$, $\dim K_k'$ and $\dim(\mathrm{Herm}(K_k) + \mathrm{Herm}(K_k')^\Gamma)$ are constant, failure is a closed real-algebraic condition (the vanishing of all maximal minors of Σ); across loci it need not be closed, because the kernels jump when the ranks change. We have not proved that surjectivity holds generically on the relevant strata. Numerically, the map Σ has full rank 144, with a wide margin, at floating-point models of three rank-deficient factors with $d_1 = 4$ (†4). Two of them are residual points of [4, †1], with kernel dimensions $(2, 2)$ at all twelve planes; the third is a factor of ranks $(7, 7)$ from a solve flagged as possibly inaccurate, with kernel dimensions $(1, 1)$. The kernels are decided at a tolerance, and the exact extreme rays are not known.

The constant ε in Theorems 5.2 and 5.4 depends on Φ. A uniform constant needs control of the degenerate factors; after a normalization it reduces to

a pointwise statement. For $n \geq 1$ let T_n be the set of operators $\rho \succeq 0$ on $\mathbb{C}^n \otimes \mathbb{C}^3$ with $\rho^\Gamma \succeq 0$ and $\operatorname{Tr}_2 \rho = I_n$, the Choi matrices of PPT channels $\mathrm{M}_n \to \mathrm{M}_3$ (a compact convex set), and let $\Lambda(d)$ be the supremum of $\lambda_{\mathrm{QD}}(\Phi)$ over nonzero PPT maps $\Phi\colon \mathrm{M}_d \to \mathrm{M}_3$. We call $\rho \in T_n$ *rigid* if $\lambda_{\mathrm{QD}}(\rho) = \frac{1}{5}$; this is unrelated to the rigidity of corners in [4, Section 5.9], which every PPT entangled operator of rank 4 on $\mathbb{C}^3 \otimes \mathbb{C}^3$ has ([4, Theorem 5.30]).

Lemma 5.6. (a) *If $\rho_A \succ 0$, then $(\rho_A^{-1/2} \otimes I)\rho(\rho_A^{-1/2} \otimes I) \in T_{d_1}$ has the same value of λ_{QD}. If ρ_A is singular with support Y of dimension $n \geq 1$, then ρ is supported on $Y \otimes \mathbb{C}^3$, and $\lambda_{\mathrm{QD}}(\rho)$ equals the value computed on $Y \otimes \mathbb{C}^3 \cong \mathbb{C}^n \otimes \mathbb{C}^3$.*

(b) *For $\rho, \rho' \in T_n$, $|\lambda_{\mathrm{QD}}(\rho) - \lambda_{\mathrm{QD}}(\rho')| \leq \|\rho - \rho'\|_2$.*

(c) *For PPT ρ, ρ' and $t \in [0,1]$, the value $\lambda_{\mathrm{QD}}(t\rho + (1-t)\rho')$ is at most $\max(\lambda_{\mathrm{QD}}(\rho), \lambda_{\mathrm{QD}}(\rho'))$.*

(d) *$\Lambda(d) = \max_{n \leq d} \max\{\lambda_{\mathrm{QD}}(\rho) : \rho$ an extreme point of $T_n\}$, the maximum is attained, and Λ is nondecreasing in d. Hence $\Lambda(d) < \frac{1}{5}$ if and only if no extreme point of T_n, $n \leq d$, is rigid.*

(e) *(given [3]) $\lambda^*(d) \leq \Lambda(d)$. In particular, by (d), if no extreme point of T_n, $n \leq d$, is rigid, then $\lambda^*(d) < \frac{1}{5}$.*

(f) *For $\rho \in T_n$ and a plane V, let K_V and K'_V be the kernels of ρ_V and of ρ_V^Γ in $\mathbb{C}^n \otimes \mathbb{C}^2$, and for an operator Z on $\mathbb{C}^n \otimes \mathbb{C}^3$ let $Z_V := (I \otimes \iota_V^*)Z(I \otimes \iota_V)$. The following are equivalent:*

- (i) *ρ is rigid;*
- (ii) *there is $Z \neq 0$ with $Z_V \in \mathrm{PSD}(K_V) + \mathrm{PSD}(K'_V)^\Gamma$ for every plane V, where $\mathrm{PSD}(K)$ is the set of positive semidefinite operators supported on K;*
- (iii) *there are $A, B \succeq 0$, not both zero, with $A\rho = 0$, $B\rho^\Gamma = 0$ and $\mathcal{T}^{-1}(A + B^\Gamma) \in \mathrm{QD}_n^*$, where $\mathcal{T}(X) := \frac{5}{12}X + \frac{1}{12}\operatorname{Tr}_2(X) \otimes I_3$ and $\mathcal{T}^{-1}(Y) = \frac{12}{5}\big(Y - \frac{1}{8}\operatorname{Tr}_2(Y) \otimes I_3\big)$.*

We do not claim the converse of (e): membership in QD is only sufficient for the separability of the composites, so $\lambda^*(d) < \frac{1}{5}$ could hold even if $\Lambda(d) = \frac{1}{5}$.

Proof. (a) The congruence $\sigma \mapsto (G \otimes I)\sigma(G \otimes I)^*$ with G invertible preserves PPT operators and middle supports, hence QD, and it maps $\rho + \lambda\rho_A \otimes I$ to $\tilde{\rho} + \lambda\tilde{\rho}_A \otimes I$ with $\tilde{\rho}_A = G\rho_A G^*$; take $G = \rho_A^{-1/2}$. If $\rho_A x = 0$, then $\sum_j \langle x \otimes e_j | \rho | x \otimes e_j \rangle = 0$, so $\rho(x \otimes y) = 0$ for all y. Compressing a QD decomposition by $P_Y \otimes I$ keeps each summand PPT with the same middle support, so the minimum is the same on $Y \otimes \mathbb{C}^3$, where the marginal is invertible. (b) On T_n, $R = I_n \otimes I_3$. The ball $\{I + X : \|X\|_2 \leq 1\}$ consists of separable operators [12], so it lies in QD. If $\rho + \lambda R \in \mathrm{QD}$ and $\delta := \|\rho' - \rho\|_2 > 0$, then $\rho' + (\lambda + \delta)R = (\rho + \lambda R) + \delta\big(I + (\rho' - \rho)/\delta\big) \in \mathrm{QD}$. (c) R depends linearly on ρ, so for each λ the set of PPT ρ with $\rho + \lambda R \in \mathrm{QD}$ is a convex cone. (d) By (a), every value $\lambda_{\mathrm{QD}}(\Phi)$ with $\Phi \neq 0$ is a value on some T_n with $n = \operatorname{rank}\Phi^*(I) \leq d$, and every element of T_n, padded

by zeros through an isometry $\mathbb{C}^n \to \mathbb{C}^d$, is the Choi matrix of a PPT map $\mathrm{M}_d \to \mathrm{M}_3$ with the same value. By (b) the maximum over the compact set T_n is attained; writing a maximizer as a convex combination of extreme points [13], (c) shows that one of them is a maximizer. (e) Lemma 5.1(b). (f) By duality, QD_n^* is the set of Z with $Z_V \in \mathrm{PSD} + \mathrm{PSD}^\Gamma$ on $\mathbb{C}^n \otimes \mathbb{C}^2$ for every plane V, and since $R = I$ is interior, $\lambda_{\mathrm{QD}}(\rho) = \max\{-\operatorname{Tr}(Z\rho) : Z \in \mathrm{QD}_n^*,\ \operatorname{Tr}(ZR) = 1\}$ with attainment. (i)⇒(ii): for an optimal Z, $\operatorname{Tr}(Z(\rho + \frac{1}{5}R)) = 0$. Every plane belongs to a rotated design as in (1), and for each such design $\rho + \frac{1}{5}R$ is $\frac{1}{5}$ times the sum of the compressions of ρ to its planes, all of which pair nonnegatively with Z; so $\operatorname{Tr}(Z_V\rho_V) = 0$ for every V. Writing $Z_V = a + b^\Gamma$ with $a, b \succeq 0$ gives $\operatorname{Tr}(a\rho_V) = \operatorname{Tr}(b\rho_V^\Gamma) = 0$, so $a \in \mathrm{PSD}(K_V)$ and $b \in \mathrm{PSD}(K_V')$. (ii)⇒(i): such a Z lies in QD_n^* and $\operatorname{Tr}(Z_V\rho_V) = 0$ for every V, so $\operatorname{Tr}(Z(\rho + \frac{1}{5}R)) = 0$ by (1). As $Z \neq 0$ lies in SEP^* and R is interior to SEP, $\operatorname{Tr}(ZR) > 0$, so $\operatorname{Tr}(Z(\rho + \lambda R)) < 0$ for $\lambda < \frac{1}{5}$. (ii)⇒(iii): $\mathcal{T}(Z)$ is the average of the compressions of Z over the planes of one design, hence of the form $A + B^\Gamma$ with $A, B \succeq 0$, $\operatorname{Tr}(A\rho) = 0 = \operatorname{Tr}(B\rho^\Gamma)$, and $Z = \mathcal{T}^{-1}(A + B^\Gamma)$. (iii)⇒(i): $Z := \mathcal{T}^{-1}(A + B^\Gamma)$ lies in QD_n^* and is nonzero, since $A + B^\Gamma = 0$ would give $\operatorname{Tr} A + \operatorname{Tr} B = 0$; and $\operatorname{Tr}(Z(\rho + \frac{1}{5}R)) = \frac{12}{5}\operatorname{Tr}(\mathcal{T}(Z)\rho) = \frac{12}{5}\big(\operatorname{Tr}(A\rho) + \operatorname{Tr}(B\rho^\Gamma)\big) = 0$, since $\mathcal{T}$ is self-adjoint and $\mathcal{T}(\rho) = \frac{5}{12}(\rho + \frac{1}{5}R)$. Conclude as in (ii)⇒(i). □

Remark 5.7. For $d \leq 2$, PPT operators on $\mathbb{C}^d \otimes \mathbb{C}^3$ are separable, so $\Lambda(2) \leq 0$. For $d = 3$, PPT operators on $\mathbb{C}^3 \otimes \mathbb{C}^2$ are separable, so $\mathrm{QD}_3 = \mathrm{SEP}_3$ and $\Lambda(3)$ is the largest value of $\min\{\lambda : \rho + \lambda\, I \otimes I \text{ separable}\}$ over PPT channels $\mathrm{M}_3 \to \mathrm{M}_3$, a question about two-qutrit PPT states only. Rigid channels are rare in the following sense. By Lemma 5.6(f)(ii), $Z_V = 0$ wherever $K_V = K_V' = 0$; if this happens at one plane, it happens on an open dense set of planes, hence $Z_V = 0$ for all V by continuity, and $Z = 0$ as in the proof of Theorem 5.2. So a rigid ρ has $K_V \neq 0$ or $K_V' \neq 0$ at every plane, and by Theorem 5.4 it fails the surjectivity condition for every rotation U. Both kernels are needed: by Theorem 5.9 below, a rigid $\rho \in T_n$ has both $\ker\rho$ and $\ker\rho^\Gamma$ meeting $\mathbb{C}^n \otimes V$ for every plane V. At $d = 3$, heuristically, without special structure this needs both ranks to be at most 5; a kernel containing $u \otimes S$ as in Remark 5.3 meets every $\mathbb{C}^3 \otimes V$ with $\dim\ker\rho = 2$.

The operators Z of Lemma 5.6(f)(ii) are controlled by linear algebra once one of the two kernels misses some $\mathbb{C}^n \otimes V$. For a plane V let P_V be the orthogonal projection onto V, and for $\rho \in T_n$, with K_V, K_V' and Z_V as in Lemma 5.6(f), let $\mathcal{Z}(\rho)$ be the set of Hermitian operators Z on $\mathbb{C}^n \otimes \mathbb{C}^3$ such that

$$Z_V \in \mathrm{Herm}(K_V) + \mathrm{Herm}(K_V')^\Gamma \quad \text{for every plane } V.$$

If ρ is rigid, the operator $Z \neq 0$ of Lemma 5.6(f)(ii) lies in $\mathcal{Z}(\rho)$: it is Hermitian, because every Z_V is and any two vectors of $\mathbb{C}^3$ lie in a common plane, and $\mathrm{PSD}(K) \subseteq \mathrm{Herm}(K)$. Since $\rho \succeq 0$, $\rho_V x = 0$ if and only if

$\rho(I\otimes\iota_V)x=0$, so $(I\otimes\iota_V)K_V=\ker\rho\cap(\mathbb{C}^n\otimes V)$, and likewise $(I\otimes\iota_V)K'_V=\ker\rho^\Gamma\cap(\mathbb{C}^n\otimes V)$.

Lemma 5.8. *Let $n\geq 1$ and let $N\subseteq\mathbb{C}^n\otimes\mathbb{C}^3$ be a subspace that contains no subspace $y\otimes S$ with $y\in\mathbb{C}^n$, $y\neq 0$, and S a plane. If Z is a Hermitian operator on $\mathbb{C}^n\otimes\mathbb{C}^3$ such that the range of $(I\otimes P_V)Z(I\otimes P_V)$ lies in N for every plane V, then $Z=0$.*

Proof. For $b=\sum_k b_k\otimes e_k\in\mathbb{C}^n\otimes\mathbb{C}^3$ and $u\in\mathbb{C}^3$ put $b[u]:=(I\otimes u^*)b=\sum_k\bar{u}_k b_k\in\mathbb{C}^n$, so that $\langle y,b[u]\rangle=\langle y\otimes u,b\rangle$ for $y\in\mathbb{C}^n$. Fix $b\in N^\perp$. For a unit vector m and $V=m^\perp$, the Hermitian operator $(I\otimes P_V)Z(I\otimes P_V)$ has range in N, so it vanishes on $N^\perp$; since $(I\otimes P_V)b=b-b[m]\otimes m$, this gives $Z(b-b[m]\otimes m)\in\ker(I\otimes P_V)=\mathbb{C}^n\otimes m$. Hence

$$F(q,m):=(I\otimes q^*)Z(b[m]\otimes m)-\langle m,m\rangle(I\otimes q^*)Zb\in\mathbb{C}^n$$

vanishes whenever $\langle q,m\rangle=0$, first for unit m and then for all m, as $F(q,cm)=|c|^2F(q,m)$. For fixed $m\neq 0$ the maps $q\mapsto F(q,m)$ and $q\mapsto\langle q,m\rangle$ are conjugate-linear and the first vanishes on the kernel of the second, so $F(q,m)=\langle q,m\rangle h(m)$ for some $h(m)\in\mathbb{C}^n$; in particular $m_2F(e_1,m)=m_1F(e_2,m)$ for all m. Both sides are polynomials in the independent variables m_k, $\bar{m}_k$, and they agree as polynomials, since a polynomial in m and $\bar{m}$ that vanishes on $\mathbb{C}^3$ is zero; as m_1 and m_2 are coprime, $F(e_1,m)=m_1g(m)$ for a polynomial g; as $F(e_1,m)$ is linear in m and linear in $\bar{m}$, g is linear in $\bar{m}$ and does not depend on m, so $g(m)=H\bar{m}$ for a linear map $H\colon\mathbb{C}^3\to\mathbb{C}^n$. Then $h(m)=H\bar{m}$ where $m_1\neq 0$, and $F(q,m)=\langle q,m\rangle H\bar{m}$ for all q,m by continuity. Both sides of

$$(I\otimes q^*)Z(b[u]\otimes w)=\langle u,w\rangle(I\otimes q^*)Zb+\langle q,w\rangle H\bar{u}$$

are conjugate-linear in u and linear in w, and they agree for $u=w$; by polarization they agree for all $u,w\in\mathbb{C}^3$. Letting q run through a basis gives $Z(b[u]\otimes w)=\langle u,w\rangle Zb+(H\bar{u})\otimes w$, so $Z(b[u]\otimes w)\in\mathbb{C}^n\otimes w$ whenever $\langle u,w\rangle=0$.

Fix $w\neq 0$. If $y\in\mathbb{C}^n$ is orthogonal to every $b[u]$ with $b\in N^\perp$ and $u\perp w$, then $\langle y\otimes u,b\rangle=0$ for all such b and u, so $y\otimes w^\perp\subseteq N$, and $y=0$ by hypothesis. So these vectors $b[u]$ span $\mathbb{C}^n$, and $Z(x\otimes w)\in\mathbb{C}^n\otimes w$ for every $x\in\mathbb{C}^n$, that is, $Z(x\otimes w)=(L_wx)\otimes w$ with L_w linear. For linearly independent w_1,w_2, comparing $(L_{w_1+w_2}x)\otimes(w_1+w_2)$ with $Z(x\otimes w_1)+Z(x\otimes w_2)=(L_{w_1}x)\otimes w_1+(L_{w_2}x)\otimes w_2$ gives $L_{w_1}=L_{w_1+w_2}=L_{w_2}$, and $L_{cw}=L_w$ for $c\neq 0$. So $L_w=L$ for all $w\neq 0$, and $Z=L\otimes I_3$ with L Hermitian. Now $(I\otimes P_V)Z(I\otimes P_V)=L\otimes P_V$ has range $\operatorname{ran}L\otimes V\subseteq N$ for every plane V; two planes span $\mathbb{C}^3$, so $\operatorname{ran}L\otimes\mathbb{C}^3\subseteq N$, and $L=0$ by hypothesis. □

Theorem 5.9. *Let $n\geq 1$ and $\rho\in T_n$. If $\ker\rho^\Gamma\cap(\mathbb{C}^n\otimes V_0)=\{0\}$ or $\ker\rho\cap(\mathbb{C}^n\otimes V_0)=\{0\}$ for some plane V_0, then $\mathcal{Z}(\rho)=\{0\}$. In particular ρ is not rigid.*

Proof. Suppose first that $\ker\rho^\Gamma \cap (\mathbb{C}^n \otimes V_0) = \{0\}$, that is, $(\rho^\Gamma)_{V_0} \succ 0$. The function $\iota \mapsto \det\big((I\otimes\iota^*)\rho^\Gamma(I\otimes\iota)\big)$ is a real polynomial in $(\operatorname{Re}\iota, \operatorname{Im}\iota)$ on the connected real-analytic manifold of isometries $\iota\colon \mathbb{C}^2 \to \mathbb{C}^3$, and it is nonzero at an isometry onto V_0, so it is nonzero on an open dense set $\mathcal{U}$ of isometries. Let $Z \in \mathcal{Z}(\rho)$ and $N := \ker\rho$. For $\iota \in \mathcal{U}$ and $V = \operatorname{ran}\iota$ we have $K_V' = 0$, so $Z_V \in \operatorname{Herm}(K_V)$, and $(I\otimes P_V)Z(I\otimes P_V) = (I\otimes\iota)Z_V(I\otimes\iota)^*$ has range in $(I\otimes\iota)K_V \subseteq N$. The entries of $P_{N^\perp}(I\otimes\iota\iota^*)Z(I\otimes\iota\iota^*)$ are polynomials in $(\operatorname{Re}\iota, \operatorname{Im}\iota)$ that vanish on $\mathcal{U}$, hence everywhere: the range of $(I\otimes P_V)Z(I\otimes P_V)$ lies in N for every plane V. Moreover, N contains no subspace $y\otimes S$ with $y \neq 0$ and S a plane. Otherwise, for $s \in S$,

$$\langle\bar y\otimes s|\rho^\Gamma|\bar y\otimes s\rangle = \operatorname{Tr}\big((|\bar y\rangle\langle\bar y|\otimes|s\rangle\langle s|)^\Gamma\rho\big) = \langle y\otimes s|\rho|y\otimes s\rangle = 0,$$

so $\bar y\otimes S \subseteq \ker\rho^\Gamma$, as $\rho^\Gamma \succeq 0$; since S meets every plane, $\ker\rho^\Gamma$ would meet $\mathbb{C}^n\otimes V_0$. Lemma 5.8 gives $Z = 0$. If instead $\ker\rho\cap(\mathbb{C}^n\otimes V_0) = \{0\}$, the first case applies to ρ^Γ, which lies in T_n because $\operatorname{Tr}_2\rho^\Gamma = (\operatorname{Tr}_2\rho)^\top$: the roles of K_V and K_V' are exchanged, and $Z \in \mathcal{Z}(\rho)$ if and only if $Z^\Gamma \in \mathcal{Z}(\rho^\Gamma)$, since $(Z^\Gamma)_V = (Z_V)^\Gamma$ and Γ is an involution. So $\mathcal{Z}(\rho) = \{0\}$, and ρ is not rigid, since for a rigid ρ the operator $Z \neq 0$ of Lemma 5.6(f)(ii) lies in $\mathcal{Z}(\rho)$. □

Remark 5.10. The hypothesis of Theorem 5.9 is invariant under the outer congruences of Lemma 5.6(a). For invertible G, the operator $\rho' = (G\otimes I)\rho(G\otimes I)^*$ has $\rho'^\Gamma = (\bar G\otimes I)\rho^\Gamma(\bar G\otimes I)^*$, so $\ker\rho' = ((G^*)^{-1}\otimes I)\ker\rho$ and $\ker\rho'^\Gamma = ((G^\top)^{-1}\otimes I)\ker\rho^\Gamma$, and invertible maps of the form $A\otimes I$ preserve each $\mathbb{C}^n\otimes V$. Hence, by Lemma 5.1(a), Lemma 5.6(a) and Theorem 5.9, $\lambda_{\mathrm{QD}}(\rho) < \frac15$ for every PPT operator ρ on $\mathbb{C}^n\otimes\mathbb{C}^3$ with $\rho_A \succ 0$ such that $\ker\rho$ or $\ker\rho^\Gamma$ misses some $\mathbb{C}^n\otimes V$.

Remark 5.11. Some hypothesis on N is needed in Lemma 5.8: if $N = y\otimes\mathbb{C}^3$, every $Z = |y\rangle\langle y|\otimes X$ with X Hermitian satisfies the range condition. For $n \geq 3$ it is not enough that N contains no subspace $y\otimes\mathbb{C}^3$ with $y \neq 0$. Let $\Omega_3 := \sum_{k=1}^3 e_k\otimes e_k \in \mathbb{C}^n\otimes\mathbb{C}^3$, let P_3 be the projection onto the span of e_1, e_2, e_3 in $\mathbb{C}^n$, and let N be the orthogonal complement of Ω_3 in $\operatorname{ran}P_3\otimes\mathbb{C}^3$. Then N contains no $y\otimes\mathbb{C}^3$ with $y \neq 0$, since $\langle\Omega_3, y\otimes t\rangle = \sum_{k\leq3} y_k t_k$. But $Z := P_3\otimes I - \frac12|\Omega_3\rangle\langle\Omega_3| \neq 0$, and for every plane V the Hermitian operator $(I\otimes P_V)Z(I\otimes P_V)$ is supported on $\operatorname{ran}P_3\otimes\mathbb{C}^3$ and annihilates Ω_3, because $\langle\Omega_3|(I\otimes P_V)|\Omega_3\rangle = \operatorname{Tr}P_V = 2$; so its range lies in N.

Corollary 5.12. *Let $d \geq 4$. Then $\Lambda(d) < \frac15$ if and only if no extreme point ρ of T_n with $4 \leq n \leq d$ such that both $\ker\rho$ and $\ker\rho^\Gamma$ meet $\mathbb{C}^n\otimes V$ for every plane V is rigid.*

Proof. By Lemma 5.6(d), $\Lambda(d) < \frac15$ if and only if no extreme point of T_n with $n \leq d$ is rigid. For $n \leq 3$ no element of T_n is rigid (Theorem 8.1), and for $4 \leq n \leq d$ an extreme point outside the stated class is not rigid by Theorem 5.9. □

Remark 5.13. Let $\rho \in T_n$. If $\dim\ker\rho \geq n+1$, then $\ker\rho$ meets every $\mathbb{C}^n \otimes V$, since $(n+1)+2n > 3n$; so every extreme point of T_4 with both ranks at most 7 belongs to the class of Corollary 5.12. If $\dim\ker\rho = r \leq n$ and $k_1, \ldots, k_r$ is a basis of $\ker\rho$, then $\ker\rho$ meets $\mathbb{C}^n \otimes m^\perp$ if and only if the $n \times r$ matrix $\mathcal{M}(m) := [(I \otimes m^*)k_1, \ldots, (I \otimes m^*)k_r]$ has rank less than r; so $\ker\rho$ meets every $\mathbb{C}^n \otimes V$ if and only if the space $\{\mathcal{M}(m) : m \in \mathbb{C}^3\}$, which is at most 3-dimensional, consists of matrices of rank less than r. Extreme points of T_n satisfy $\operatorname{rank}(\rho)^2 + \operatorname{rank}(\rho^\Gamma)^2 \leq 10n^2$: the Hermitian X with $\operatorname{ran} X \subseteq \operatorname{ran}\rho$, $\operatorname{ran} X^\Gamma \subseteq \operatorname{ran}\rho^\Gamma$ and $\operatorname{Tr}_2 X = 0$ form a real space of dimension at least $\operatorname{rank}(\rho)^2 + \operatorname{rank}(\rho^\Gamma)^2 - 10n^2$, and $\rho \pm \varepsilon X \in T_n$ for such X and small ε.

6. Exact certificates beyond the constant

This section gives exact certificates at explicit factors, built from factorizations through M_2 adapted to the pair or to one factor; their data are listed in [Supp, S1], and Appendix A describes the checks.

Theorem 6.1. *Let $\Phi_Q\colon \mathrm{M}_4 \to \mathrm{M}_3$ and $\Psi_Q\colon \mathrm{M}_3 \to \mathrm{M}_4$ be the maps with the Gaussian-rational Choi matrices given in [Supp, S1], and put $W_Q := J(\Psi_Q \circ \Phi_Q)$ and $M_Q := \Phi_Q^*(I)^\top \otimes \Psi_Q(I)$.*

(a) *Φ_Q and Ψ_Q are PPT: their Choi matrices and the partial transposes of these are positive definite.*
(b) *(given [3]) $W_Q - \frac{1}{20}M_Q$ is separable. In particular W_Q is separable.*

Proof. The data contain Gaussian-rational linear maps $A_k\colon \mathrm{M}_3 \to \mathrm{M}_2$ and $B_k\colon \mathrm{M}_2 \to \mathrm{M}_3$, $k = 1, \ldots, 12$, with $\sum_k B_k \circ A_k = \mathrm{id} - \frac{1}{20}\tau$ exactly, such that every $A_k \circ \Phi_Q$ and every $\Psi_Q \circ B_k$ is PPT with positive definite Choi matrix and partial transpose; together with (a) this is checked in exact arithmetic (†1). By [3] each $\Psi_Q \circ B_k \circ A_k \circ \Phi_Q$ is EB, so $W_Q - \frac{1}{20}M_Q = \sum_k J(\Psi_Q \circ B_k \circ A_k \circ \Phi_Q)$ is separable. Since $\Phi_Q^*(I)$ and $\Psi_Q(I)$ are positive definite, M_Q is a product of positive operators, and W_Q is separable. □

The factors are strictly PPT perturbations, of size about 10^{-6}, of numerically extreme points of rank 7 of the PPT cones on $\mathbb{C}^4 \otimes \mathbb{C}^3$, obtained by minimizing random linear objectives over PPT states with marginal floors. The maps A_k and B_k are far from completely positive (their Choi matrices have eigenvalues down to -5.2 and -3.9), and since $\sum_k B_k \circ A_k = \mathrm{id} - \frac{1}{20}\tau$ is an identity of maps, Theorem 3.3 shows that not all $B_k \circ A_k$ lie in $\mathcal{K}$.

For a single factor, Lemma 5.1 turns a decomposition $\mathrm{id} + \lambda\tau = \sum_k \mathrm{Ad}_{\iota_k} \circ A_k$ with isometries $\iota_k\colon \mathbb{C}^2 \to \mathbb{C}^3$ and $A_k \circ \Phi$ PPT into a certificate for every partner, since each $\Psi \circ \mathrm{Ad}_{\iota_k}$ is PPT.

Theorem 6.2. *Let $\Phi_Q\colon \mathrm{M}_4 \to \mathrm{M}_3$ now be the map with the Gaussian-rational Choi matrix given in [Supp, S1] for this theorem; it is a strictly PPT perturbation, of size about 10^{-6}, of a residual point of [4, †1], the one obtained from start 1, which we call s_1 (likewise s_3, s_{17} below).*

(a) $J(\Phi_Q)$ *and* $J(\Phi_Q)^\Gamma$ *are positive definite.*
(b) *(given* [3]*) For every PPT map* $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$ *and every* d_2, $J(\Psi \circ \Phi_Q) + \frac{1}{10}\Phi_Q^*(I)^\top \otimes \Psi(I)$ *is separable; also* $\lambda_{\mathrm{QD}}(\Phi_Q) \le \frac{1}{10}$.
(c) *(given* [3]*) The same holds with* Φ_Q *replaced by any Hermiticity-preserving linear map* $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_3$ *with* $\|J(\Phi) - J(\Phi_Q)\|_2 \le 1791/10^8$.

Proof. The data contain isometries $\iota_k\colon \mathbb{C}^2 \to \mathbb{C}^3$ with entries in $\mathbb{Q}(i)$ and Hermiticity-preserving linear maps $A_k\colon \mathrm{M}_3 \to \mathrm{M}_2$ with entries in $\mathbb{Q}(i)$, $k = 1, \dots, 16$, such that $\sum_k \mathrm{Ad}_{\iota_k} \circ A_k = \mathrm{id} + \frac{1}{10}\tau$ exactly and, for every k, $X_k := J(A_k \circ \Phi_Q)$ satisfies $X_k - \frac{1}{6250} I \succ 0$ and $X_k^\Gamma - \frac{1}{6250} I \succ 0$; moreover $(1791/10^8)^2 \max_k \|J(A_k)\|_2^2 \le (1/6250)^2$, with $\max_k \|J(A_k)\|_2^2 = 79.750126\ldots$. All of this, and (a), is checked in exact arithmetic (†2). For (b), every $A_k \circ \Phi_Q$ is PPT and every $\Psi \circ \mathrm{Ad}_{\iota_k}$ is PPT, so each $\Psi \circ \mathrm{Ad}_{\iota_k} \circ A_k \circ \Phi_Q$ is EB by [3], and summing gives the claim; moreover $\rho + \frac{1}{10}R = \sum_k (I \otimes \iota_k) X_k (I \otimes \iota_k)^* \in \mathrm{QD}_4$. For (c), let $\Delta := J(\Phi) - J(\Phi_Q)$, which is Hermitian. Then $J(A_k \circ \Phi) - X_k = (\mathrm{id} \otimes A_k)(\Delta)$, and

$$\|(\mathrm{id} \otimes A_k)(\Delta)\|_\infty \le \|(\mathrm{id} \otimes A_k)(\Delta)\|_2 \le \|J(A_k)\|_2 \, \|\Delta\|_2,$$

because the operator norm of $\mathrm{id} \otimes A_k$ on Hilbert–Schmidt operators equals that of A_k, which is at most the Frobenius norm of its matrix, and that equals $\|J(A_k)\|_2$. The partial transpose preserves $\|\cdot\|_2$, so the same bound holds for the partial transposes. By Weyl's inequality, $J(A_k \circ \Phi)$ and its partial transpose have least eigenvalues greater than $\frac{1}{6250} - \|J(A_k)\|_2 \|\Delta\|_2 \ge 0$, and the proof of (b) applies. □

Numerically, with $J(s_1)$ the stored point rescaled to trace 3, the eigenvalues of $J(\Phi_Q) - J(s_1)$ are all about 9.79×10^{-7}. The stored floating-point point, made Hermitian and multiplied by 3, lies at distance $3.397 \times 10^{-6} \le 341/10^8$ from $J(\Phi_Q)$ (†2), so Theorem 6.2(c) covers it and every Hermitian point within 1.45×10^{-5} of it. We have not certified that the exact extreme ray near s_1 lies in this ball. Since Φ_Q is strictly PPT of full rank, Theorem 5.2 already gives $\lambda_{\mathrm{QD}}(\Phi_Q) < \frac{1}{5}$; Theorem 6.2 adds the explicit constant $\frac{1}{10}$ and a ball of explicit radius.

7. Numerical observations

This section defines the pair-adapted constant λ_{PF} and reports floating-point values of it and of one-sided constants at residual and numerically extreme points; these values are not used in any proof. For a pair (Φ, Ψ) let $\mathcal{A}_\Phi$ be the set of linear maps $A\colon \mathrm{M}_3 \to \mathrm{M}_2$ with $A \circ \Phi$ PPT, $\mathcal{B}_\Psi$ the set of linear maps $B\colon \mathrm{M}_2 \to \mathrm{M}_3$ with $\Psi \circ B$ PPT, and

$$\lambda_{\mathrm{PF}}(\Phi, \Psi) := \inf\big\{\lambda : \mathrm{id} + \lambda\tau \in \overline{\mathrm{cone}}\{B \circ A : A \in \mathcal{A}_\Phi,\ B \in \mathcal{B}_\Psi\}\big\}.$$

By [3], $\lambda(\Phi, \Psi) \le \lambda_{\mathrm{PF}}(\Phi, \Psi)$, and $\lambda_{\mathrm{PF}} \le \frac{1}{5}$ because $\mathcal{K}$ is contained in the cone. These are the factorizations of Theorem 6.1. When $d_1 d_2 > 9$ the

map $L_{\Phi,\Psi}$ is not onto, since $81 < d_1^2 d_2^2$, so these factorizations are a genuine restriction. At a residual point of [4, †1] one has $\lambda(\Phi,\Psi) \geq 0$: there $W\sum_i |ii\rangle = 0$ ([4, Section 9]) while $M \succ 0$.

- †5 Alternating semidefinite programs over twelve factorizations, with the factors bounded in Frobenius norm by 10, reach $\lambda_{\mathrm{PF}} \leq 0.0030$, 0.0018 and 0.0207 at the residual points s_1, s_3 and s_{17} of [4, †1]. These values are approximate: the solver flagged every iterate as possibly inaccurate, and the PPT constraints and the identity hold up to residuals of at most about 3×10^{-5} ([Supp, S1] lists them). The factors are far from completely positive and several sit at or near the norm bound, so the infimum, if it is 0, may require unbounded factors. At the pair $\Phi = \Psi = \mathrm{id} + 2\tau$ on M_3, where $W + \lambda M = |\Omega\rangle\langle\Omega| + (16+49\lambda)\, I \otimes I$ is isotropic, so that $\lambda(\Phi,\Psi) = -\frac{15}{49}$, the same method (without the iterate check of [Supp, S1]) reaches -0.30604.
- †6 One-sided certificates with twelve fixed planes: the least λ with $\mathrm{id} + \lambda\tau = \sum_k \mathrm{Ad}_{\iota_k} \circ A_k$ and $A_k \circ \Phi$ PPT, an upper bound on $\lambda_{\mathrm{QD}}(\Phi)$, lies between 0.043 and 0.085 at eleven factors with $d_1 = 4$ (five residual points of [4, †1] and six numerically extreme points of rank 6 or 7); the solver flagged every solve as possibly inaccurate, and the five values whose verified PPT violations exceed 10^{-5} are approximate.
- †7 Decompositions into PPT pieces on fixed sets of planes bound λ_{QD} from above, and the bound depends on how well the planes cover $\mathbb{C}^3$: for a pure product state, whose true value is 0, the bounds are 0.046, 0.035 and 0.0063 with 12, 36 and 112 planes; at s_1 they are 0.0435, 0.0303 and 0.0148. So the values of †6 mainly measure the plane sets, and at s_1 the one-sided constant appears to be about 0.01.

These observations are consistent with $\Lambda(4)$ being well below $\frac{1}{5}$, but they do not bound it: Lemma 5.6 reduces a uniform bound to the extreme channels, and the tested factors are few. Searches over T_4, the Choi matrices of PPT channels $\mathrm{M}_4 \to \mathrm{M}_3$, are reported at the end of Section 8.

8. The constant at $d = 3$

This section shows that no PPT channel $\mathrm{M}_n \to \mathrm{M}_3$ with $n \leq 3$ is rigid and bounds $\Lambda(3)$ from below.

Theorem 8.1. *No element of T_n with $n \leq 3$ is rigid. Hence $\Lambda(3) < \frac{1}{5}$.*

Corollary 8.2 (given [3])**.** *$\lambda^*(3) \leq \Lambda(3) < \frac{1}{5}$. In particular, for every PPT map $\Phi\colon \mathrm{M}_3 \to \mathrm{M}_3$, every d_2 and every PPT map $\Psi\colon \mathrm{M}_3 \to \mathrm{M}_{d_2}$, the operator $W + \Lambda(3)\, M$ is separable.*

Proof. Lemma 5.1(b) and Theorem 8.1, since $\lambda_{\mathrm{QD}}(\Phi) \leq \Lambda(3)$ for every nonzero PPT map $\Phi\colon \mathrm{M}_3 \to \mathrm{M}_3$. □

The proof of Theorem 8.1 is in Appendix C; it uses no computation, and its constant is not explicit. By Lemma 5.6(a)–(c) and the decomposition

TABLE 1. The computational facts. Exact facts are proved in Lean (Appendix B); numerical facts are not used in any proof. For each fact, [Supp, S1] gives its deposit folder, the commands that rerun it and the expected outputs.

Fact	Content	Kind	Deposit folder
†1	certificate of Theorem 6.1	exact	`sec10/cert_10_17`
†2	certificate of Theorem 6.2	exact	`sec10/cert_10_18`
†3	the operators X and O of Proposition 3.6	exact	`sec10/prop_10_6`
†4	rank of Σ at three factors (Remark 5.5)	numerical	`sec10/thm_10_13`
†5	λ_{PF} at residual points (Section 7)	numerical	`sec10/d29`
†6	one-sided bounds with twelve planes (Section 7)	numerical	`sec10/d30`
†7	dependence on the plane sets (Section 7)	numerical	`sec10/d31`
†8	certificates of Remark 8.3	exact	`sec10/cert_d3`
†9	search over T_3 with a weaker relaxation (Section 8)	numerical	`sec10/d33`
†10	certificate of Theorem 8.5	exact	`sec10/cert_lam3`
†11	certificate of Remark 8.6	exact	`sec10/cert_tiles_lb`
†12	search over T_3 for $\Lambda(3)$ (Section 8)	numerical	`sec10/num_lam3`
†13	searches over T_4 and T_5 (Section 8)	numerical	`sec10/num_lam4`

of PPT operators on $\mathbb{C}^3 \otimes \mathbb{C}^3$ into extreme rays (which does not use [3]), it suffices to show that no PPT entangled extreme ray ρ with $\rho_A \succ 0$ is rigid; such a ρ has $(\operatorname{rank}\rho, \operatorname{rank}\rho^\Gamma)$ among $(4,4)$, $(5,5)$, $(5,6)$, $(6,5)$, $(5,7)$, $(7,5)$, $(6,6)$ (Proposition C.2). For a rigid ρ, Lemma 5.6(f) gives a block-positive $Z \neq 0$ with $\operatorname{Tr}(Z\rho) < 0$ whose compressions to the subspaces $\mathbb{C}^3 \otimes V$ are supported on the kernels of ρ and ρ^Γ. The classification of rank-4 PPT entangled states [15] and of linear spaces of matrices of bounded rank [16, 17] limits the possible kernels, and each case yields a vanishing slice of Z (so Z would be decomposable), a product subspace in a kernel (so ρ would be separable), or a contradiction with extremality or with the fact that both kernels meet every $\mathbb{C}^3 \otimes V$.

Remark 8.3. At explicit rank-deficient points the constant is well below $\frac{1}{5}$. Let $\rho_T = \frac{1}{4}(I - P)$ be the state defined before [4, Theorem 7.10], where P is the projection onto the span of the Tiles basis [18]; ρ_T is a PPT entangled state of rank 4. Let $\rho_H = (I \otimes H)\rho_T(I \otimes H)^*$ with $H = \left(\begin{smallmatrix} 5 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{smallmatrix}\right)$. For $\rho = \rho_T$ and for $\rho = \rho_H$ there are isometries $\iota_k \colon \mathbb{C}^2 \to \mathbb{C}^3$ and 6×6 matrices X_k,

$k = 1, \dots, 18$, all with entries in $\mathbb{Q}(i)$, such that $X_k \succ 0$, $X_k^\Gamma \succ 0$ and

$$\textstyle\sum_k (I \otimes \iota_k) X_k (I \otimes \iota_k)^* = \rho + \frac{1}{10}\,\rho_A \otimes I$$

exactly (†8). Each summand is PPT with middle support in a plane, so $\lambda_{\mathrm{QD}}(\rho) \le \frac{1}{10}$ at both points and, by Lemma 5.6(a), at their normalizations in T_3. Appendix A.1 describes the data and the checks.

Lower bounds on $\Lambda(3)$ come from the dual form of the PPT symmetric extension test of level 2 [19], the level-2 test for short. Let $f_{11}, f_{22}, f_{33}, f_{12}, f_{13}, f_{23}$ be the standard basis of $\mathbb{C}^6$, in this order, let $\xi\colon \mathbb{C}^6 \to \mathbb{C}^3 \otimes \mathbb{C}^3$ be the real linear map with $\xi(f_{ii}) = e_i \otimes e_i$ and $\xi(f_{ij}) = e_i \otimes e_j + e_j \otimes e_i$ for $i < j$, whose range is the symmetric subspace $\mathrm{Sym}^2(\mathbb{C}^3)$, and let $\Xi := I_3 \otimes \xi\colon \mathbb{C}^3 \otimes \mathbb{C}^6 \to \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. As before, ${}^\Gamma$ is the partial transpose of the first factor, here on $\mathbb{C}^3 \otimes \mathbb{C}^6$ and on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$; let ${}^{\Gamma_3}$ be that of the third factor of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Lemma 8.4. *Let Z be a Hermitian operator on $\mathbb{C}^3 \otimes \mathbb{C}^3$, and let $Q_0, Q_1 \succeq 0$ on $\mathbb{C}^3 \otimes \mathbb{C}^6$ and $Q_2 \succeq 0$ on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ satisfy*

$$\Xi^*(Z \otimes I_3)\Xi = Q_0 + Q_1^\Gamma + \Xi^* Q_2^{\Gamma_3} \Xi.$$

Then $\mathrm{Tr}(Z\sigma) \ge 0$ for every separable σ on $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Proof. Let $\sigma = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$ with $p_i \ge 0$ and $\|b_i\| = 1$, and put $S' := \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i| \otimes |b_i\rangle\langle b_i|$. Then $\mathrm{Tr}_3 S' = \sigma$, and S' is supported on $\operatorname{ran}\Xi$, so $S' = \Xi S \Xi^*$ with $S := \Xi^+ S' (\Xi^+)^* \succeq 0$, where $\Xi^+ := (\Xi^*\Xi)^{-1}\Xi^* = I_3 \otimes \xi^+$ with $\xi^+ := (\xi^*\xi)^{-1}\xi^*$. Using $\mathrm{Tr}(XY^\Gamma) = \mathrm{Tr}(X^\Gamma Y)$, the same for ${}^{\Gamma_3}$, and $\Xi S \Xi^* = S'$, we get

$$\mathrm{Tr}(Z\sigma) = \mathrm{Tr}\big(S'(Z \otimes I_3)\big) = \mathrm{Tr}(SQ_0) + \mathrm{Tr}(S^\Gamma Q_1) + \mathrm{Tr}(S'^{\Gamma_3} Q_2).$$

Congruence by $I_3 \otimes \xi^+$ commutes with the partial transpose of the first factor, so $S^\Gamma = \Xi^+ S'^\Gamma (\Xi^+)^*$ with $S'^\Gamma = \sum_i p_i |\bar a_i\rangle\langle \bar a_i| \otimes |b_i\rangle\langle b_i| \otimes |b_i\rangle\langle b_i| \succeq 0$; and $S'^{\Gamma_3} = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i| \otimes |\bar b_i\rangle\langle \bar b_i| \succeq 0$. So all three terms are nonnegative. □

Theorem 8.5. *Let ρ_q be the real symmetric 9×9 matrix with entries in $2^{-40}\mathbb{Z}$ of Appendix A.2. Then $\rho_q \succ 0$ and $\rho_q^\Gamma \succ 0$, so ρ_q is the Choi matrix of a PPT map $\mathrm{M}_3 \to \mathrm{M}_3$, and $\lambda_{\mathrm{QD}}(\rho_q) \ge \frac{59}{500}$. Hence*

$$0.118 = \tfrac{59}{500} \le \Lambda(3) < \tfrac{1}{5}.$$

Proof. The exact checks of Appendix A.2 (†10) give $\rho_q \succ 0$, $\rho_q^\Gamma \succ 0$ and $\rho_A = \mathrm{Tr}_2 \rho_q \succ 0$, and real symmetric rational matrices Z, Q_0, Q_1, Q_2 with $Q_0, Q_1, Q_2 \succ 0$ that satisfy the identity of Lemma 8.4 entrywise, together with $\mathrm{Tr}(ZR) > 0$ and $-\mathrm{Tr}(Z\rho_q) \ge \frac{59}{500}\mathrm{Tr}(ZR)$, where $R = \rho_A \otimes I$. If $\rho_q + \lambda R \in \mathrm{QD}_3$, then $\rho_q + \lambda R$ is separable by Remark 5.7, and Lemma 8.4 gives $0 \le \mathrm{Tr}(Z\rho_q) + \lambda\,\mathrm{Tr}(ZR)$, so $\lambda \ge \frac{59}{500}$. As ρ_q is the Choi matrix of a nonzero PPT map $\mathrm{M}_3 \to \mathrm{M}_3$, $\Lambda(3) \ge \lambda_{\mathrm{QD}}(\rho_q)$. The upper bound is Theorem 8.1. □

By Lemma 5.6(a), the PPT channel with Choi matrix $\tilde{\rho}_q := (\rho_A^{-1/2} \otimes I)\rho_q(\rho_A^{-1/2} \otimes I) \in T_3$ has the same value, so $\tilde{\rho}_q + \lambda\, I \otimes I$ is not separable for $\lambda < \frac{59}{500}$; thus this channel alone gives the lower bound $\Lambda(3) \geq 0.118$ of Theorem B. In particular, certificates adapted to one factor cannot give a constant below 0.118 at $d_1 = 3$, although PPT^2 for maps on M_3 [1, 2] gives $\lambda(\Phi, \Psi) \leq 0$ for every pair with $d_1 = d_2 = 3$.

Remark 8.6. At the Tiles state of Remark 8.3, $\frac{1}{20} \leq \lambda_{\mathrm{QD}}(\rho_T) \leq \frac{1}{10}$. The upper bound is Remark 8.3. The lower bound follows as in the proof of Theorem 8.5 from rational matrices Z, Q_0, Q_1, Q_2 for ρ_T with $Q_0, Q_1, Q_2 \succ 0$, the identity of Lemma 8.4, $\mathrm{Tr}(ZR) > 0$ and $-\mathrm{Tr}(Z\rho_T) \geq \frac{1}{20}\mathrm{Tr}(ZR)$, checked exactly together with $\mathrm{Tr}_2\, \rho_T \succ 0$ (†11).

All values in this paragraph are numerical. They suggest that the bound of Theorem 8.5 is close to $\Lambda(3)$. An alternating search over T_3 that uses the level-2 test of Lemma 8.4 reaches 0.1188 from four of five starts, at points of ranks $(5, 5)$. The matrix ρ_q is such a point, shifted by $2^{-20}I$ and rounded to the grid 2^{-40}, and at ρ_q the level-2 lower bound and an upper bound from PPT pieces on 60 random planes both round to 0.11877 (†12). A weaker relaxation uses a swap-invariant extension that need not be supported on $\mathbb{C}^3 \otimes \mathrm{Sym}^2(\mathbb{C}^3)$ and is PPT only for the partial transpose of the first factor. With it, the search stops at points of ranks $(4, 4)$ where $\lambda_{\mathrm{QD}} \geq 0.0505$, and at such a point 60 random planes give $\lambda_{\mathrm{QD}} \leq 0.093$ (†9). For $d = 4$, where $\mathrm{QD}_4 \neq \mathrm{SEP}_4$, searches over T_4 bound λ_{QD} by PPT pieces on planes and find no larger value: five starts end at points where these bounds are 0.1186 to 0.1192, and a longer run at one of them gives $\lambda_{\mathrm{QD}} \approx 0.1176$. So at $d = 4$ the searches find, within their accuracy, no one-sided constant above $\Lambda(3) \approx 0.1188$, consistent with $\Lambda(4) = \Lambda(3)$; the inequality $\Lambda(4) \geq \Lambda(3)$ is Lemma 5.6(d). At one point of T_5 the same method gives estimates between 0.1188 and 0.1197 (†13).

9. Open problems

The value of $\Lambda(3)$. What is the exact value of $\Lambda(3)$, which lies in $[0.118, \frac{1}{5})$ by Theorems 8.1 and 8.5, and is it attained at points of ranks $(5, 5)$, where the searches of Section 8 end? The proof of Theorem 8.1 gives no explicit upper bound below $\frac{1}{5}$.

Rigid channels and the constants $\Lambda(d)$ *for* $d \geq 4$. Is there a rigid PPT channel $\mathrm{M}_d \to \mathrm{M}_3$ for some $d \geq 4$? If not, then $\Lambda(d) < \frac{1}{5}$ for every d, and $\lambda^*(d) < \frac{1}{5}$ given [3] (Lemma 5.6(d), (e) and Theorem 8.1). By Corollary 5.12 only the extreme points ρ of T_n, $4 \leq n \leq d$, at which both $\ker \rho$ and $\ker \rho^\Gamma$ meet every $\mathbb{C}^n \otimes V$ need to be examined, and at $n = 4$ these include every extreme point with both ranks at most 7 (Remark 5.13). For $d \geq 4$, QD_d is larger than SEP_d, and besides the equality at $d = 3$ (Remark 5.7) the proof of Theorem 8.1 uses the classification of PPT entangled states of rank 4 on

$\mathbb{C}^3 \otimes \mathbb{C}^3$ and the size of the kernels at $d = 3$. In view of the searches at $d = 4$ in Section 8 (†13): is $\sup_d \Lambda(d) < \frac{1}{5}$, or even $\Lambda(d) = \Lambda(3)$ for every $d \geq 3$?

Pair-adapted certificates. Is $\lambda_{\mathrm{PF}}(\Phi, \Psi) \leq 0$ for every PPT pair? Since $\lambda(\Phi, \Psi) \leq \lambda_{\mathrm{PF}}(\Phi, \Psi)$ by [3] (Section 7), a positive answer would prove PPT² through a qutrit. At three residual points of [4, †1], where $\lambda(\Phi, \Psi) \geq 0$, the searches of Section 7 (†5) find λ_{PF} close to 0; is $\lambda_{\mathrm{PF}} = 0$ there?

Appendix A. The exact certificates

The checkers of this appendix read only the deposited data files of their certificate and work in exact rational or Gaussian-rational arithmetic, so their checks do not depend on how the data were produced. The checkers also run on altered copies of certificate data, and every such control run fails as it must; [Supp, S1] lists them.

The data of Theorems 6.1 and 6.2 are Choi matrices $J(S) = \sum_{a,b} E_{ab} \otimes S(E_{ab})$, input factor first, with entries in $\mathbb{Q}(i)$ written as exact fractions. For Theorem 6.1 they are $J(\Phi_Q)$ and $J(\Psi_Q)$ (12×12) and $J(A_k)$, $J(B_k)$ (6×6), $k = 1, \ldots, 12$; for Theorem 6.2 they are $J(\Phi_Q)$, the $J(A_k)$ and the isometries ι_k (3×2), $k = 1, \ldots, 16$. The factors are printed in [Supp, S1]; all data files are in the software deposit [8], and [Supp, S1] names their folders and the SHA256SUMS files that pin them. The factors Φ_Q (and Ψ_Q) are obtained from numerically extreme floating-point points, made strictly PPT by a shift of about $10^{-6} I$ and rounded to a dyadic grid, and the maps A_k, B_k come from a semidefinite program, rounded and then corrected exactly; [Supp, S1] and the deposit give the details.

The checkers (†1, †2) verify:

(a) the positive definiteness of $J(\Phi_Q)$ (and $J(\Psi_Q)$) and of the partial transposes;
(b) for Theorem 6.2, $\iota_k^* \iota_k = I_2$;
(c) the identity of maps, entrywise on Choi matrices: $\sum_k J(B_k \circ A_k) = J(\mathrm{id}) - \frac{1}{20} J(\tau)$, respectively $\sum_k (I \otimes \iota_k) J(A_k) (I \otimes \iota_k)^* = J(\mathrm{id}) + \frac{1}{10} J(\tau)$, where $J(\mathrm{id}) = |\Omega\rangle\langle\Omega|$ and $J(\tau) = I \otimes I$;
(d) the positive definiteness of $J(A_k \circ \Phi_Q) = (\mathrm{id} \otimes A_k)(J(\Phi_Q))$ and of $J(\Psi_Q \circ B_k)$, and of their partial transposes;
(e) for Theorem 6.2, the margins $J(A_k \circ \Phi_Q) - \frac{1}{6250} I \succ 0$ and $J(A_k \circ \Phi_Q)^\Gamma - \frac{1}{6250} I \succ 0$, the Hermiticity of every $J(A_k)$, and $(1791/10^8)^2 \max_k \|J(A_k)\|_2^2 \leq (1/6250)^2$.

Positive definiteness is decided by first testing exact Hermiticity and then applying Sylvester's criterion, with fraction-free Bareiss elimination, to the real symmetric matrix $\left(\begin{smallmatrix} \mathrm{Re}\, H & -\mathrm{Im}\, H \\ \mathrm{Im}\, H & \mathrm{Re}\, H \end{smallmatrix}\right)$, which is positive definite if and only if H is. The control runs alter the constant, one entry, the margin, the radius or a data file. The distance between the stored floating-point point s_1 and $J(\Phi_Q)$ is computed exactly by reading the binary64 entries as rationals.

A.1. **The certificates of Remark 8.3.** For each of the two points the data are the integer matrix H (the identity for ρ_T), the constant $\frac{1}{10}$, the isometries ι_k (3×2) and the matrices X_k (6×6, outer factor first), $k = 1, \dots, 18$, with entries in $\mathbb{Q}(i)$ written as exact fractions, in the format of the other certificates of this appendix. The checker (†8) rebuilds ρ_T from the five Tiles vectors and verifies:

(a) $P^2 = P$, $(P^\Gamma)^2 = P^\Gamma$ and $\operatorname{Tr} P = 5$, so that ρ_T and ρ_T^Γ are positive semidefinite;
(b) $\det H \neq 0$ and the positive definiteness of $\rho_A = \operatorname{Tr}_2 \rho_H$;
(c) $\iota_k^* \iota_k = I_2$ for every k;
(d) the Hermiticity of every X_k and the positive definiteness of X_k and X_k^Γ, decided by Sylvester's criterion on the real matrix above, with exact rational elimination without pivoting;
(e) the identity $\sum_k (I \otimes \iota_k) X_k (I \otimes \iota_k)^* = \rho_H + \frac{1}{10} \rho_A \otimes I$, entrywise, and $\frac{1}{10} < \frac{1}{5}$.

It also reports the exact ranks of ρ_H and ρ_H^Γ, both equal to 4; they follow from (a) and (b), because ρ_H and ρ_H^Γ are congruent, by the invertible $I \otimes H$, to $\rho_T = \frac{1}{4}(I - P)$ and $\rho_T^\Gamma = \frac{1}{4}(I - P^\Gamma)$, and $I - P$ and $I - P^\Gamma$ are idempotents of trace 4. The data come from a semidefinite program at $\lambda = \frac{1}{10}$, rounded and then corrected exactly; the control runs alter the constant, one entry or a data file. The facts (a)–(e) and the ranks are also proved in Lean (Appendix B).

A.2. **The certificates of Theorem 8.5 and Remark 8.6.** For Theorem 8.5 the data are ρ_q (9×9, entries in $2^{-40}\mathbb{Z}$), the constant $\frac{59}{500}$, and the matrices Z (9×9), Q_0, Q_1 (18×18) and Q_2 (27×27), all real and written as exact fractions; Z, Q_1 and Q_2 lie on the grid 2^{-30}, and Q_0 is computed exactly from the identity of Lemma 8.4. For Remark 8.6 the data are the constant $\frac{1}{20}$ and matrices Z, Q_0, Q_1, Q_2 of the same shapes, and ρ_T is rebuilt from the five Tiles vectors as in Appendix A.1. All these files are in the software deposit [8], and [Supp, S1] names their folders and the SHA256SUMS files that pin them. Indices are ordered lexicographically, with the basis $e_a \otimes f_s$ of $\mathbb{C}^3 \otimes \mathbb{C}^6$ in the order of the definition of Ξ. Two independent checkers (†10, †11) verify:

(a) for Theorem 8.5, the symmetry of ρ_q and the positive definiteness of ρ_q, ρ_q^Γ and $\operatorname{Tr}_2 \rho_q$; for Remark 8.6, the facts (a) of Appendix A.1 and the positive definiteness of $\operatorname{Tr}_2 \rho_T$;
(b) the symmetry of Z, Q_0, Q_1, Q_2 and the identity of Lemma 8.4, entrywise;
(c) the positive definiteness of Q_0, Q_1 and Q_2;
(d) with $R = \operatorname{Tr}_2 \rho \otimes I$ and the stated constant c_0: $\operatorname{Tr}(ZR) > 0$, $-\operatorname{Tr}(Z\rho) \geq c_0 \operatorname{Tr}(ZR)$ and $c_0 < \frac{1}{5}$.

The first checker decides positive definiteness by an $LDL^\top$ factorization without pivoting, cross-checked by an LU factorization; the second, which

does not form Ξ as a matrix, uses Sylvester's criterion with fraction-free Bareiss elimination. As printed by the checkers, the decimal expansions of the exact ratios $-\operatorname{Tr}(Z\rho)/\operatorname{Tr}(ZR)$ begin 0.11820000 and 0.05199999. The control runs for Theorem 8.5 alter the constant, the data or the state, among them the constant $59100001/(5 \cdot 10^8)$ just above the ratio and a positive definite ρ that is not PPT. Those for Remark 8.6 alter a certificate of $\frac{3}{40}$ at $(I \otimes H')\rho_T(I \otimes H')^*$ for another matrix H'; the same checkers verify this certificate, which this paper does not use. The facts (a)–(d) are also proved in Lean, with corresponding Lean controls (Appendix B); for Remark 8.6 these include the constant $\frac{13}{250}$ just above the ratio. The data were found by the floating-point search of †12.

Appendix B. The Lean checks

The exact facts †1–†3, †8, †10 and †11 are proved in Lean 4 with Mathlib, on the deposited data. The data modules are generated from the deposited files and checked against them by an independent parser; the generators and the parser are in the folder `lean/gen` of the Lean archive. The integer copies used in kernel-only proofs are tied to the data modules by Lean theorems that evaluate compiled code, and decimal values are checked through rational brackets. For each theorem the deposit records whether the Lean kernel alone checks it or whether the proof also evaluates compiled code. Of the 507 theorems in the modules for the facts of this paper (statements, auxiliary and generated lemmas, and negative controls; the modules also check a third certificate that this paper does not use), the kernel alone checks 424, among them every fact of †8, †10 and †11 and, for the integer copies of the data, every fact of †1–†3; the proofs of the other 83 also evaluate compiled code. The facts of Appendix A.1 are in target TC3 and the ranks there in targets TC3R and TC4; the facts of Appendix A.2 are in target TC4, with corresponding Lean controls. The Lean kernel alone checks every theorem of these three targets. Two kinds of steps are not formalized: the deductions in the text that derive results from the facts, such as the dual form of the level-2 test (Lemma 8.4), which turns the facts of †10 and †11 into the bounds $\lambda_{\mathrm{QD}}(\rho_q) \geq \frac{59}{500}$ and $\lambda_{\mathrm{QD}}(\rho_T) \geq \frac{1}{20}$, and cited results, such as [3], the realignment criterion [5], PPT^2 in dimension 3 [1, 2] and the separability of PPT operators on $\mathbb{C}^3 \otimes \mathbb{C}^2$ [6, 7], which turns the facts of †8 into the separability of $\rho + \frac{1}{10}\,\rho_A \otimes I$ and underlies Remark 5.7. The Lean code is in the software deposit [8].

Appendix C. Proof of Theorem 8.1

This appendix proves Theorem 8.1, following the roadmap given after Corollary 8.2; the proof is completed at its end, from Proposition C.1 and Theorem C.17. We use the notation of Section 5 (T_n, λ_{QD}, $\Lambda(d)$ and rigidity), together with Lemma 5.6 and Remark 5.7. Here U and S' denote planes

of the outer space and V a plane of the middle space, introduced where they occur.

Notation. ρ acts on $\mathbb{C}^3\otimes\mathbb{C}^3$ (outer factor first, then middle). Γ denotes the partial transpose on the outer factor; for product vectors $\langle x\otimes y|B^\Gamma|x\otimes y\rangle = \langle\bar{x}\otimes y|B|\bar{x}\otimes y\rangle$. $T_3 = \{\rho \succeq 0,\ \rho^\Gamma \succeq 0,\ \mathrm{Tr}_2\,\rho = I\}$; $R = \rho_A\otimes I$ with $\rho_A = \mathrm{Tr}_2\,\rho$. $\lambda_{\mathrm{QD}}(\rho) = \min\{\lambda : \rho+\lambda R \text{ separable}\}$ (Remark 5.7). $K = \ker\rho$, $K' = \ker\rho^\Gamma$. A plane is a 2-dimensional subspace V of the middle space; n denotes a normal vector, $V = n^\perp$. $S_V = \mathbb{C}^3\otimes V$, $K_V = K\cap S_V$, $K'_V = K'\cap S_V$ (the kernels of the compressions ρ_V, ρ_V^Γ). Vectors of $\mathbb{C}^3\otimes\mathbb{C}^3$ are identified with 3×3 matrices k (rows outer, columns middle): $x\otimes y \leftrightarrow xy^\top$. Then $k\in S_{n^\perp}$ iff $k\bar{n} = 0$, $\langle k|x\otimes y\rangle = \langle k\bar{y}|x\rangle$, and $(I\otimes\langle y|)k = k\bar{y}$. For an operator Z: $z(x,y) = \langle x\otimes y|Z|x\otimes y\rangle$, $Z_x = (\langle x|\otimes I)Z(|x\rangle\otimes I)$ (middle 3×3), $Z^y = (I\otimes\langle y|)Z(I\otimes|y\rangle)$ (outer 3×3). $\pi_y\colon K\to\mathbb{C}^3$, $k\mapsto k\bar{y}$. "Generic" means outside a proper real-algebraic subset. $A(c)$ is the antisymmetric tensor whose matrix is the cross-product matrix $[c]_\times$; $A_3 = \{A(c)\}$ is the antisymmetric subspace.

Reduction to extreme rays of the PPT cone.

Proposition C.1. $\Lambda(3) = \max\lambda_{\mathrm{QD}}$ *over extreme rays* ρ *of the PPT cone of* $\mathbb{C}^3\otimes\mathbb{C}^3$. *If no extreme ray with* $\rho_A\succ 0$ *is rigid, then* $\Lambda(3) < \frac{1}{5}$.

Proof. A maximizer $\rho^*\in T_3$ exists (Lemma 5.6(b), compactness). It is a finite sum of extreme rays ρ_k. If every $\rho_k+\lambda R_k$ is separable, then so is $\rho^*+\lambda R^*$, by linearity of R in ρ. Therefore $\lambda_{\mathrm{QD}}(\rho^*)\le\max_k\lambda_{\mathrm{QD}}(\rho_k)$, and each $\lambda_{\mathrm{QD}}(\rho_k)\le\Lambda(3)$, since Lemma 5.6(a) shows that values of PPT operators with singular marginal are values on some T_n, $n\le 2$. Rays with singular ρ_A live on a 2×3 system and are separable (value ≤ 0); separable rays have value ≤ 0. Rigidity is invariant under outer congruences (Lemma 5.6(a)). □

Proposition C.2. *Let* ρ *be a PPT entangled extreme ray with* $\rho_A\succ 0$, $r = \mathrm{rank}\,\rho$, $s = \mathrm{rank}\,\rho^\Gamma$. *Then*

$$(r,s)\in\{(4,4),(5,5),(5,6),(6,5),(5,7),(7,5),(6,6)\}.$$

Proof. We have $r,s\ge 4$ because a PPT operator of rank ≤ 3 whose outer marginal has rank 3 is separable [14] and $\mathrm{Tr}_2\,\rho^\Gamma = \rho_A^\top$. The real span of the face of ρ is $\{X = X^* : \mathrm{ran}\,X \text{ in } \mathrm{ran}\,\rho,\ \mathrm{ran}\,X^\Gamma \text{ in } \mathrm{ran}\,\rho^\Gamma\}$, of dimension $\ge r^2+s^2-81$; extremality makes it 1, so $r^2+s^2\le 82$. By [15, Corollary 28], $r = 4$ forces $s = 4$, and symmetrically $s = 4$ forces $r = 4$. This follows from [15, Theorem 25], by which every rank-4 PPT entangled two-qutrit state is $(A\otimes B)\Pi_\psi(A\otimes B)^*$ with Π_ψ proportional to the projection onto the orthocomplement of an orthogonal UPB ψ, so ρ^Γ is the analogous state for the UPB $\{\bar{\varphi}_k\otimes\chi_k\}$. □

Fact C.3 (Remark 5.7)**.** *If* $\rho\in T_3$ *(or a PPT* ρ *with* $\rho_A\succ 0$*) is rigid, then* K *and* K' *meet* S_V *for every plane* V. *This is Theorem 5.9 with* $n = 3$, *applied with Remark 5.10 when* $\rho_A\neq I$; *its proof uses only linear algebra.*

Fact C.4. *Rigidity of ρ and of ρ^Γ are equivalent (Lemma 5.6(f)(ii) is symmetric under $\rho \leftrightarrow \rho^\Gamma$, $Z \leftrightarrow Z^\Gamma$), and ρ^Γ is again a PPT extreme ray with outer marginal $\rho_A^\top \succ 0$ and kernels (K', K).*

The dual witness: zeros and slices.

Lemma C.5 (zeros)**.** *Let ρ (PPT, $\rho_A \succ 0$) be rigid. After an outer congruence assume $\rho \in T_3$. There is $Z \neq 0$ with*

(a) *Z block-positive;*
(b) *$\operatorname{Tr}(Z\rho) = -\operatorname{Tr}(Z)/5 < 0$;*
(c) *for every plane V: $Z_V = a_V + b_V^\Gamma$ with $a_V \succeq 0$ supported on K_V and $b_V \succeq 0$ supported on K_V' (Z_V the compression of Z to S_V);*
(d) *$z(x, y) = 0$ whenever, for some plane V containing y, $x \otimes y$ is orthogonal to K_V and $\bar{x} \otimes y$ is orthogonal to K_V'.*

Proof. Part (c) is the implication (i) $\Longrightarrow$ (ii) of Lemma 5.6(f). For (a), given x and y, pick V containing y. Then $z(x, y) = \langle xy|a_V|xy\rangle + \langle \bar{x}y|b_V|\bar{x}y\rangle \geq 0$. In the situation of (d), both terms vanish. For (b), $\operatorname{Tr}(Z_V\rho_V) = \operatorname{Tr}(a_V\rho_V) + \operatorname{Tr}(b_V\rho_V^\Gamma) = 0$ for every V. Summing over the twelve planes of (1) ($\sum_k \mathrm{Ad}_{\Pi_k} = 5\,\mathrm{id} + \tau$) gives $5\operatorname{Tr}(Z\rho) + \operatorname{Tr} Z = 0$, and $\operatorname{Tr} Z > 0$ as $Z \neq 0$ is block-positive and $I \otimes I$ is interior to SEP. The witness transforms covariantly under outer congruences, so (a), (c), (d) hold for any PPT ρ with $\rho_A \succ 0$ in its own coordinates, and (b) reads $\operatorname{Tr}(Z\rho) = -\operatorname{Tr}(Z(\rho_A \otimes I))/5 < 0$. Later arguments use (b) only through the inequality $\operatorname{Tr}(Z\rho) < 0$. □

Lemma C.6 (slices)**.** *For Z as in Lemma C.5: $Z_x \neq 0$ for all $x \neq 0$ and $Z^y \neq 0$ for all $y \neq 0$.*

Proof. If $Z_x = 0$, then $z(x, y) = 0$ for all y. Since $Z^y \succeq 0$ by (a), this gives $Z^y x = 0$, that is, $(I \otimes \langle y|)Z(x \otimes y) = 0$ for all y. The expression $(I \otimes \langle y'|)Z(x \otimes y)$ is sesquilinear in (y', y) and vanishes on the diagonal, so by polarization $Z(x \otimes y) = 0$ for all y. So Z lives on $x^\perp \otimes \mathbb{C}^3$, a 2×3 system, where block-positive operators are decomposable [6]. Transporting the decomposition with an isometry ι onto $x^\perp$, using $(\iota \otimes I)Q^\Gamma(\iota \otimes I)^* = ((\bar{\iota} \otimes I)Q(\bar{\iota} \otimes I)^*)^\Gamma$ with Γ on the $\mathbb{C}^2$ factor in Q^Γ, gives $Z = Q_1 + Q_2^\Gamma$ with $Q_i \succeq 0$, and then $\operatorname{Tr}(Z\rho) \geq 0$, contradicting (b). $Z^y = 0$ is the same with the factors exchanged (3×2 system). □

Lemma C.7 (slice of rank ≤ 1)**.** *A PPT σ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ with an outer slice $\sigma_p = (\langle p| \otimes I)\sigma(|p\rangle \otimes I)$ of rank ≤ 1 ($p \neq 0$) is separable; likewise for a middle slice.*

Proof. Apply an outer unitary O with $Op = \|p\|e_1$, and note that the partial transpose of $(O \otimes I)\sigma(O \otimes I)^*$ is $(\bar{O} \otimes I)\sigma^\Gamma(\bar{O} \otimes I)^*$. Denote the blocks of σ by s_{kl}. If $s_{11} = 0$, the state lives on the 2×3 system $\operatorname{span}(e_2, e_3) \otimes \mathbb{C}^3$ and is separable. Otherwise $s_{11} = c\,tt^*$ with $c > 0$. Positivity gives $s_{1l} = tv_l^*$; positivity of σ^Γ, whose $(1, l)$ block is $s_{l1} = v_l t^*$, gives $v_l = \gamma_l t$. With

$v = e_1 \otimes t$ and $g = ce_1 + \gamma_2 e_2 + \gamma_3 e_3$: $\sigma v = \|t\|^2 g \otimes t$, $\sigma^\Gamma v = \|t\|^2 \bar{g} \otimes t$, $\langle v|\sigma|v\rangle = \langle v|\sigma^\Gamma|v\rangle = c\|t\|^4$. The Schur complements $\sigma'' = \sigma - (1/c)|gt\rangle\langle gt|$ and $\sigma''^\Gamma = \sigma^\Gamma - (1/c)|\bar{g}t\rangle\langle\bar{g}t|$ are positive semidefinite, and $\sigma''(e_1 \otimes y) = 0$, so σ'' is PPT on $\mathrm{span}(e_2, e_3) \otimes \mathbb{C}^3$, hence separable. The case of a middle slice follows by exchanging the factors. □

Corollary C.8. *If ρ is PPT entangled with $\rho_A \succ 0$, neither K nor K' contains a subspace $u \otimes S$ or $S' \otimes q$ (S, S' planes).*

Proof. If $u \otimes S$ lies in K, then ρ_u kills S and so has rank ≤ 1, and if $u \otimes S$ lies in K', then $(\rho^\Gamma)_u$ has rank ≤ 1, so ρ^Γ, and hence ρ, is separable. The case $S' \otimes q$ is the same with middle slices. Also $u \otimes \mathbb{C}^3$ is not in K or K', as it would make ρ_A singular. □

Lemma C.9 (middle slices of rank ≤ 1)**.** *If Z is block-positive and $\operatorname{rank} Z^y \leq 1$ for every y, then $Z = c|w\rangle\langle w|$, $Z = c(|w\rangle\langle w|)^\Gamma$ or $Z = |c_0\rangle\langle c_0| \otimes Y$ with $c \geq 0$, $Y \succeq 0$; in every case Z is decomposable, so under Lemma C.5 this is impossible.*

Proof. $M(a,b) := (I \otimes \langle a|)Z(I \otimes |b\rangle)$ is antilinear in a, linear in b, $M(a,b)^* = M(b,a)$, $M(y,y) = Z^y$. The 2×2 minors of M are polynomials in $(\bar{a}, b)$ vanishing on $\{a = b\}$, a maximal totally real set, hence everywhere, so $\operatorname{rank} M(a,b) \leq 1$ for all a, b. A bilinear map N with values of rank ≤ 1 has one of the forms $c(\alpha)r(b)^\top$, $c(b)r(\alpha)^\top$, $c_0 r(\alpha, b)^\top$, $c(\alpha, b)r_0^\top$, where $\alpha = \bar{a}$. For fixed α, the values form a linear space of rank-one matrices, so they share a column or a row. If the values share a column for generic α and a row for generic b, then $N = c(\alpha)r(b)^\top$ after rescaling, since bilinearity makes the scalar constant. If both share columns, the column is constant. The other cases are the transposes. Hermiticity then gives $M(a,b) = \lambda C\bar{a}b^\top C^*$ (so $Z = \lambda|w\rangle\langle w|$, $w = \sum_s Ce_s \otimes e_s$), $M(a,b) = \mu Cba^*C^*$ (so $Z = \mu(|w'\rangle\langle w'|)^\Gamma$, $w' = \sum_t \bar{C}e_t \otimes e_t$), or $M(a,b) = m(a,b)c_0c_0^*$ (so $Z = |c_0\rangle\langle c_0| \otimes Y$); positivity of $M(y,y)$ fixes the signs. □

Lemma C.10 (product surfaces)**.** *Let L be a subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ with $\dim L \leq 4$ that contains $u \otimes c$ for u in a nonempty open subset of $\mathbb{P}^2$ (outer), or for c in a nonempty open subset of $\mathbb{P}^2$ (middle). Then L contains $u_0 \otimes S$ or $S' \otimes c_0$ for some planes S, S'.*

Proof. Suppose not. Then for each u the set $\{c : u \otimes c \in L\}$ and for each c the set $\{u : u \otimes c \in L\}$ are at most lines, so $\mathcal{S} = \mathbb{P}(L) \cap \text{Segre}$ has a 2-dimensional irreducible component $\mathcal{S}_0$ dominating both factors. If $\dim L \leq 3$, $\mathcal{S}_0 = \mathbb{P}(L)$ would be a linear space of rank-one matrices ($u_0 \otimes \mathbb{C}^3$ or $\mathbb{C}^3 \otimes c_0$), contrary to our assumption. Let $\dim L = 4$. Rank-one matrices are singular points of $\{\det = 0\}$ ($\mathrm{adj} = 0$). If det is not identically 0 on L, $D_L = \mathbb{P}(L) \cap \{\det = 0\}$ is a cubic surface whose singular locus contains $\mathcal{S}_0$; a reduced cubic surface has singular locus of dimension ≤ 1, so $\det|_L = l^2 l'$ or l^3 and $\mathcal{S}_0$ is the plane $\{l = 0\}$, a $\mathbb{P}^2$ of rank-one matrices, which is excluded. If $\det = 0$ on L, L is a space of singular matrices of dimension 4, hence a compression space,

as Skew_3 has dimension 3. A common kernel vector v_0 forces $c^\top v_0 = 0$, a common cokernel vector forces u into a plane, and the $(2,1)$-type (a plane A mapped into a line B) forces c onto a line or u into B ($\mathcal{S}_0$ is irreducible); each contradicts dominance. □

Lemma C.11 (kernels that meet every S_V)**.** *Let L be a subspace with $L \cap S_V \neq 0$ for every plane V, containing no $u \otimes S$, $S' \otimes q$, $u \otimes \mathbb{C}^3$.*

(a) $\dim L = 2$ *is impossible.*
(b) *If* $\dim L = 3$*, then L is contained in $U \otimes \mathbb{C}^3$ for an outer plane U ("type U"), or $L = (G \otimes H)A_3$ with G, H invertible ("type T"); in both cases* $\dim L_V = 1$ *for generic V.*
(c) *If* $\dim L = 4$ *and* $\dim L_V \geq 2$ *for every V, then L is in $U \otimes \mathbb{C}^3$.*

Proof. With a basis k_m of L, L meets $S_{n^\perp}$ iff the matrix $N(l) = [k_1 l \,|\, \dots]$ ($l = \bar{n}$) is singular. (a) The 3×2 matrices $N(l)$ all have rank ≤ 1, so they share a column space, since a common row space would give a linear relation among the k_m. Then $L = u \otimes S$, which the hypothesis rules out. (b) $\{N(l)\}$ is a linear space of singular 3×3 matrices, so by [16, 17] it lies in a compression space or is equivalent to Skew_3. A common kernel vector gives a linear relation among the k_m. A common cokernel vector u gives $u^\top k_m = 0$, i.e. L in $U \otimes \mathbb{C}^3$. The $(2,1)$-type gives $u \otimes S$ in L for an outer vector u and a middle plane S, contrary to the hypothesis. Skew-type gives $N(l) = X[L_0 l]_\times Y$, i.e. $L = (X \otimes L_0^\top)A_3$. If $\dim L_V \geq 2$ for every V, the maps $k \mapsto kl$ have rank ≤ 1, which forces L into $u \otimes \mathbb{C}^3$ or a common 2-dimensional kernel, and both are excluded.

(c) The 3×4 maps $k \mapsto kl$ have rank ≤ 2 for all l. By [16], spaces of bounded rank 2 lie in compression spaces or are of Skew_3-type. Skew-type and the types $(2,0)$, $(3,1)$ are excluded, because a common kernel vector of all $k \mapsto kl$ is 0 and a 3-dimensional subspace mapped into a line is $u \otimes \mathbb{C}^3$. This leaves L in $U \otimes \mathbb{C}^3$. □

Proposition C.12 (rank 4)**.** *No PPT entangled $\rho \in T_3$ with* $\operatorname{rank}\rho = 4$ *is rigid.*

Proof. By [15, Theorem 25] ρ is proportional to $(A \otimes B)\Pi(A \otimes B)^*$ with A, B invertible and Π the projection onto the orthocomplement of an orthogonal UPB $\{\varphi_k \otimes \chi_k\}$, $k = 1, \dots, 5$. So $K = \operatorname{span}\{a_k \otimes b_k\}$ with $a_k = A^{-*}\varphi_k$, $b_k = B^{-*}\chi_k$. Any three a_k are linearly independent: if φ_1, φ_2, φ_3 spanned at most a plane, a vector x orthogonal to them and y orthogonal to χ_4, χ_5 would give a product vector $x \otimes y$ orthogonal to the UPB. The same holds for the b_k. Hence the $a_k \otimes b_k$, and the $\bar{a}_k \otimes b_k$, are linearly independent, as contracting with a vector orthogonal to two of the b's shows. Each $\bar{a}_k \otimes b_k$ lies in K', as $\langle \bar{a}b|\rho^\Gamma|\bar{a}b\rangle = \langle ab|\rho|ab\rangle = 0$. By [14], a PPT operator of rank ≤ 3 with rank-3 marginal is separable, so $\operatorname{rank}\rho^\Gamma \geq 4$. Hence $K' = \operatorname{span}\{\bar{a}_k \otimes b_k\}$.

(1) For $i \neq j$ let $V_{ij} = \operatorname{span}(b_i, b_j)$. Then $K_{V_{ij}} = \operatorname{span}(a_i b_i, a_j b_j)$. If $\sum_k c_k a_k \otimes b_k$ lies in $S_{V_{ij}}$, contracting with n orthogonal to V_{ij} gives

$\sum_{k\notin\{i,j\}} c_k\langle n|b_k\rangle a_k = 0$ with $\langle n|b_k\rangle \neq 0$, so those c_k vanish. Likewise $K'_{V_{ij}} = \operatorname{span}(\bar{a}_i b_i, \bar{a}_j b_j)$.

(2) Fix $i \neq l$ and $x \neq 0$ orthogonal to a_i and a_l. For $y \in V_{il}$, $x \otimes y$ is orthogonal to $K_{V_{il}}$ and $\bar{x} \otimes y$ to $K'_{V_{il}}$, so $z(x, y) = 0$ (Lemma C.5(d)).

(3) The three b_j with $j \notin \{i, l\}$ span $\mathbb{C}^3$, so $\langle b_j|b_i\rangle \neq 0$ for one of them. Put $w = b_i - (\langle b_j|b_i\rangle/\langle b_j|b_j\rangle)b_j \in V_{ij}$, orthogonal to b_j. Then $x \otimes w$ is orthogonal to $K_{V_{ij}}$ (as $\langle a_i|x\rangle = 0$ and $\langle b_j|w\rangle = 0$) and $\bar{x} \otimes w$ to $K'_{V_{ij}}$, so $z(x, w) = 0$. And w is not in V_{il} (otherwise b_j would lie in $\operatorname{span}(b_i, b_l)$).

(4) The positive semidefinite matrix Z_x vanishes, as a quadratic form, on V_{il} and at w, hence on $V_{il} + \mathbb{C}w = \mathbb{C}^3$, so $Z_x = 0$, contradicting Lemma C.6. □

Four lemmas on rigid rays. In this subsection ρ is PPT and rigid with $\rho_A \succ 0$, and Z is as in Lemma C.5.

Lemma C.13 (type U)**.** *There is no $x_0 \neq 0$ such that $x_0 \otimes y$ is orthogonal to K_V for every $y \in V$, for generic V, and such that $M'_V := (\langle \bar{x}_0| \otimes I)K'_V$ (a subspace of V) has dimension ≤ 1 for generic V.*

Proof. Suppose that such an x_0 exists. For generic V let $Y_V = \{y \in V : \langle m|y\rangle = 0 \text{ for } m \in M'_V\}$. Then $Y_V \neq 0$. For $y \in Y_V$, $x_0 \otimes y$ is orthogonal to K_V and $\bar{x}_0 \otimes y$ to K'_V ($\langle k'|\bar{x}_0 \otimes y\rangle = \langle(\langle \bar{x}_0| \otimes I)k'|y\rangle$), so $z(x_0, y) = 0$ by Lemma C.5(d), and $Z_{x_0} y = 0$. By Lemma C.6, $Z_{x_0} \neq 0$, so all Y_V lie in a plane $g^\perp$. If $M'_V = 0$ for generic V, then $Y_V = V$, which is impossible. Hence $M'_V = \mathbb{C}m'_V$ for generic V, and $Y_V = V \cap g^\perp$, i.e. m'_V is parallel to $\Pi_V g$ (the vectors of V orthogonal to $V \cap g^\perp$). K'_V, hence M'_V, depends algebraically on $\nu = \bar{n}$; take a polynomial generator $m'(\nu)$. Then

$$p(\nu, n) := m'(\nu) \wedge \big((\nu^\top n)g - (\nu^\top g)n\big)$$

vanishes for $\nu = \bar{n}$, n in an open set, hence identically. For $\nu^\top g \neq 0$ the matrix $g\nu^\top - (\nu^\top g)I$ has rank 2, so $m'(\nu)$ is parallel to every vector of a plane. Therefore $m'(\nu) = 0$ for generic ν, a contradiction. □

Lemma C.14 (rank-2 middle slices)**.** *Suppose* $\dim K_V = \dim K'_V = 1$ *for generic V, $K_V = \mathbb{C}k_V$, $K'_V = \mathbb{C}k'_V$. For $y \in V$ put $u_V(y) = k_V \bar{y}$ and $u'_V(y) = \operatorname{conj}(k'_V \bar{y})$. Then on an open dense set of y,* $\operatorname{rank} Z^y = 2$ *and, for generic V containing y, $u_V(y)$ and $u'_V(y)$ are linearly independent and span $\Pi(y) := \operatorname{ran} Z^y$.*

Proof. For $y \in V$, $Z^y = (I \otimes \langle y|)Z_V(I \otimes |y\rangle) = s_V u_V u_V^* + t_V u'_V {u'_V}^*$ with $s_V, t_V \geq 0$ by Lemma C.5(c). Here the Γ-part contributes the transpose of an outer block, and $(vv^*)^\top = \bar{v}\bar{v}^*$. So $\operatorname{rank} Z^y \leq 2$ for all y. By Lemma C.9, $\operatorname{rank} Z^y \leq 1$ cannot hold for all y; that condition is algebraic, so $\operatorname{rank} Z^y = 2$ on an open dense set, and there the formula forces $s_V, t_V > 0$ and the stated spans. □

Lemma C.15 (type T)**.** *It cannot be that, after an outer congruence,* $K_V = \mathbb{C}A(\bar{n})$ *for generic* $V = n^\perp$*,* $\dim K'_V = 1$ *for generic* V*, and* $\dim K' \le 4$*.*

Proof. Suppose that these three conditions hold after an outer congruence. Here $u_V(y) = A(\bar{n})\bar{y} = \pm\,\mathrm{conj}(n \times y)$. As V runs through the planes containing y, n runs through $y^\perp$ and $\mathrm{conj}(n \times y)$ through the whole outer plane $y^\perp$. By Lemma C.14, $\Pi(y) = y^\perp$ for generic y, hence $u'_V(y)$ lies in $y^\perp$, that is, $y^\top k'_V \bar{y} = 0$. For generic V this holds on an open subset of V, hence for all $y \in V$. Put $C = \mathrm{conj}(k'_V)$. Then $Cn = 0$ because k'_V lies in S_V, and $y^* C y = 0$ on V, so $\Pi_V C \Pi_V = 0$ and $C = nc^*$ with $c \in V$, i.e. $k'_V = \bar{n} \otimes c$. Thus K' contains product vectors whose outer factors $\bar{n}$ fill an open set, and Lemma C.10 gives $u_0 \otimes S$ or $S' \otimes c_0$ inside K', contradicting Corollary C.8. □

Lemma C.16 (4-dimensional kernels of generic dimension 1)**.** *Suppose* $\dim K = 4$ *and* $\dim K_V = \dim K'_V = 1$ *for generic* V*. Let* $k(v)$ *be a polynomial generator of* K_V *as a function of* $v = \bar{n}$ *(so* $k(v)v = 0$*), and* $K_0 = \mathrm{span}\{k(v)\}$*. Then* $\dim K_0 = 3$*,* $K_V = K_0 \cap S_V$ *for generic* V*, and* K_0 *meets every* S_V*.*

Proof. For y let $E_y = \{v : v^\top y = 0\}$ (the planes through y) and $S_y = \mathrm{span}\, k(E_y)$, of constant dimension for generic y. By Lemma C.14, $\pi_y(S_y)$ lies in the plane $\Pi(y)$. The kernel of π_y on K is $K_{y^\perp} = \mathbb{C}k(\bar{y})$ ($y^\perp$ is a generic plane). Hence $\dim S_y - [k(\bar{y}) \in S_y] \le 2$.

- $\dim S_y = 4$ is impossible. $\dim S_y = 1$: then $k(E_y)$ is a single vector killing the plane E_y, i.e. $a(y) \otimes y \in K$ for generic y; Lemma C.10 (middle factors fill an open set) and Corollary C.8 exclude this.
- $\dim S_y = 3$: then $k(\bar{y})$ is in S_y for generic y. With $E_y = \mathrm{span}(y \times e_1, y \times e_2)$ the coefficients $c_j(y)$ of k on E_y are polynomial in y; $F(y, w) = \det[c_{j1}(y), c_{j2}(y), c_{j3}(y), k(w)]$ vanishes for $w = \bar{y}$, y in an open set, hence identically, so $k(w)$ is in S_y for every w. Hence $K_0 = S_y$, so $\dim K_0 = 3$.
- $\dim S_y = 2$: the image of every line of $\mathbb{P}^2$ under k lies in a line of $\mathbb{P}(K)$, so the closure of the image contains the line through any two general points of it; it is linear of dimension ≤ 2, and $\dim K_0 \le 3$.

In the last two cases K_0 contains K_V for generic V, and meeting S_V is a closed condition, so K_0 meets every S_V. By Lemma C.11(a), K_0 cannot be 2-dimensional, so $\dim K_0 = 3$. □

No rigid extreme ray.

Theorem C.17. *No PPT entangled extreme ray* ρ *with* $\rho_A \succ 0$ *is rigid.*

Proof. Suppose that ρ is a rigid PPT entangled extreme ray with $\rho_A \succ 0$, and let r, s be as in Proposition C.2. By Fact C.3, K and K' meet every S_V. By Fact C.4 we may swap ρ and ρ^Γ. The case $(r, s) = (4, 4)$ is excluded by Proposition C.12. By Lemma C.11(a), a 2-dimensional kernel cannot meet

every S_V, so the cases $(r,s) = (5,7)$ and $(7,5)$ are excluded. Otherwise both kernels have dimension 3 or 4 ((r,s) among $(5,5)$, $(5,6)$, $(6,5)$, $(6,6)$). By Lemma C.11, a 3-dimensional kernel is of type U or type T and has generic dimension 1; a 4-dimensional kernel has generic dimension 1 unless it lies in some $U \otimes \mathbb{C}^3$.

(i) One kernel is contained in $U \otimes \mathbb{C}^3$ for an outer plane U. Swapping ρ and ρ^Γ if necessary, we may assume that this kernel is K. Take x_0 orthogonal to U.

- If $\dim K'_V = 1$ generically, Lemma C.13 applies (M'_V has dimension ≤ 1).
- Otherwise K' is 4-dimensional and lies in $U' \otimes \mathbb{C}^3$ (Lemma C.11(c)). If $\dim K = 3$, then $\dim K_V = 1$ generically and Lemma C.13 applies to ρ^Γ. If $\dim K = 4$, then both kernels are 4-dimensional and of type U, and for every $a \in U$ the two linear conditions that $a \otimes b$ be orthogonal to the 2-dimensional orthocomplement of K in $U \otimes \mathbb{C}^3$ have a solution $b \neq 0$, so $a \otimes b$ is in K and $\bar{a} \otimes b$ in K' (as $\langle \bar{a}b|\rho^\Gamma|\bar{a}b\rangle = \langle ab|\rho|ab\rangle = 0$), whence $\bar{a}$ is in U' for all $a \in U$. Thus $U' = \bar{U}$, so $\bar{x}_0$ is orthogonal to U'. Then $x_0 \otimes y$ lies in $\operatorname{ran}\rho$ and $\bar{x}_0 \otimes y$ in $\operatorname{ran}\rho^\Gamma$, so $\rho - \varepsilon|x_0 y\rangle\langle x_0 y|$ is PPT for small $\varepsilon > 0$ and ρ is not extreme, a contradiction.

(ii) No kernel lies in any $U \otimes \mathbb{C}^3$. Then $\dim K_V = \dim K'_V = 1$ generically.

- If K is 3-dimensional, it is of type T, $K = (G \otimes H)A_3$. Write it with an outer map:

  $$(G \otimes H)A_3 = (GH^{-1} \otimes I)(H \otimes H)A_3 = (GH^{-1} \otimes I)A_3,$$

  since $(M \otimes M)A_3 = A_3$. The outer congruence by $F = (GH^{-1})^*$ gives the kernel $(F^{-*} \otimes I)K = A_3$, so $K_V = \mathbb{C}A(\bar{n})$, and Lemma C.15 applies ($\dim K' \leq 4$). If K' is 3-dimensional, the same argument applies to ρ^Γ.
- If both are 4-dimensional ($(r,s) = (5,5)$), Lemma C.16 gives a 3-dimensional K_0 inside K with $K_V = K_0 \cap S_V$ generically, meeting every S_V and containing no $u \otimes S$ (Corollary C.8). By Lemma C.11(b), K_0 is of type U or of type T. If K_0 is of type U, Lemma C.13 applies with x_0 orthogonal to U, since K_V is in K_0. If K_0 is of type T, then $K_0 = (G \otimes H)A_3 = (GH^{-1} \otimes I)A_3$, and Lemma C.15 applies after the outer congruence by $F = (GH^{-1})^*$, which makes $K_0 = A_3$ and $K_V = \mathbb{C}A(\bar{n})$ for generic V, with $\dim K' = 4$.

Each case leads to a contradiction. □

Proof of Theorem 8.1. By Proposition C.1 and Theorem C.17, $\Lambda(3) < \frac{1}{5}$, so $\lambda_{\mathrm{QD}}(\rho) \leq \Lambda(3) < \frac{1}{5}$ for every $\rho \in T_3$, and no $\rho \in T_n$, $n \leq 3$, is rigid (for $n \leq 2$, every element of T_n is separable). □

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