POSITIVE CORRELATION OF BOND ENERGIES IN THE O(3) AND O(4) MODELS

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ABSTRACT. We study the O(N) model on a finite graph and ask whether the scalar products $\sigma_x \cdot \sigma_y$ of two pairs of vertices, such as the bond energies of two edges, are positively correlated, a form of Griffiths' second inequality for scalar products of spins (Simon's Problem 5C). For all couplings we prove the inequality on several classes of graphs, which at $N = 3$ include the class of all graphs that have no minor isomorphic to the wheel W_4 once the two pairs are joined by edges. For $N = 3, 4$ and uniform coupling we prove the inequality at every value of the coupling and for every pair of edges on the strips $P_3 \times P_n$, which have a W_4 minor once $n \geq 3$. The proofs combine two tools. The first consists of expansions of the edge weights into spin networks and, at $N = 3$, into powers of $(1+\sigma_x \cdot \sigma_y)/2$, an expansion in which total positivity of the label laws is carried along graphs without a W_4 minor. The second is a transfer operator on the strips. Together with strong-coupling expansions on the whole graph of each short strip, it reduces positivity to finitely many inequalities, which are verified in rigorous computer arithmetic.

1. INTRODUCTION

1.1. **The problem.** Let $G = (V, E)$ be a finite graph and let $J = (J_e)_{e \in E}$ be couplings with $J_e \geq 0$. The O(N) model on G is the probability measure

$$\mu_{G,J}(d\sigma) = \frac{1}{Z_{G,J}} \exp\Big(\sum_{xy \in E} J_{xy}\, \sigma_x \cdot \sigma_y \Big) \prod_{x \in V} \omega(d\sigma_x)$$

on $(S^{N-1})^V$, where ω is the rotation-invariant probability measure on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. For an edge $e = xy$ we write $s_e = \sigma_x \cdot \sigma_y$ for its bond energy. The question studied here is whether

$$\mathrm{Cov}_{G,J}(\sigma_a \cdot \sigma_b,\ \sigma_c \cdot \sigma_d) \ \geq\ 0 \tag{1}$$

for all vertices a, b, c, d. Since $\partial_{J_e} \langle \sigma_c \cdot \sigma_d \rangle = \mathrm{Cov}(s_e, \sigma_c \cdot \sigma_d)$, inequality (1) for all graphs says that every two-point function $\langle \sigma_c \cdot \sigma_d \rangle$ is nondecreasing in every coupling. It is Griffiths' second inequality for scalar products of spins.

Date: November 2026.

2020 *Mathematics Subject Classification.* Primary 82B20, 60K35; Secondary 60E15, 33C55, 15B48, 05C83, 65G20, 68V20.

Key words and phrases. O(N) model, Griffiths' second inequality, correlation inequalities, bond energies, spin networks, total positivity, transfer operator, computer-assisted proof, formal verification.

It holds for $N = 1$ by Griffiths' theorem [16] and for $N = 2$ by Ginibre's [15]. For $N \geq 3$ it is a well-known open problem [11, (1.6)], [30, Problem 5C], [27, §2.3], [1, §1].

It suffices to treat pairs of edges. If ab is not an edge, add it with coupling J_{ab}. Then $\mathrm{Cov}(\sigma_a \cdot \sigma_b, \sigma_c \cdot \sigma_d)$ is the limit as $J_{ab} \downarrow 0$ of $\partial_{J_{ab}} \partial_{J_{cd}} \log Z$, and the same applies to cd. Thus (1) for all pairs of edges of all graphs in a class closed under adding edges gives (1) for all pairs of vertices. For a fixed graph we state results for pairs of edges $e \neq f$.

1.2. **Prior work.** Dunlop [11] proved Griffiths-type inequalities for products of single spin components at $N = 3$ and $N = 4$, in a nonnegative (possibly zero) field along one component. Kunz, Pfister and Vuillermot announced similar results [20]; the full paper is [21]. These statements concern components, not scalar products, and Dunlop records (1) as open for $N > 2$ [11, (1.6)].

Ginibre's proof for $N = 2$ rests on a positive-definiteness property of functions on the cycle group of a graph. Sylvester [32] showed that this property fails for every $N \geq 3$. His counterexamples are nonplanar, and each needs at least four negative signs. The case with two negative signs, which would give (1), had no counterexample in his survey.

Campbell [5] states (1) for every $N \geq 3$ on every finite graph; the claim is in the published abstract, and it is Theorem 2(ii) of Campbell's preprint [6]. The proof uses the identity of his Lemma 1, which fails already for $N = 3$ on three sites (Appendix C). An erratum was published [7]. Our counterexample in Appendix C applies to Lemma 1 as originally stated. Lees [23] announced Griffiths inequalities for all $N \geq 2$, for single components only; the paper was withdrawn. Peled and Spinka [27, §2.3] describe the monotonicity question for $N \geq 3$ as open. Abdesselam [1] formulates a coefficientwise strengthening (his Conjecture 1.1). He proves large-power asymptotics, and he shows that determinantal substitutes for the moments, which include the asymptotic moment map, satisfy these inequalities; for the moments themselves they remain open.

1.3. **Results.** Our results are of two kinds. The first holds for all couplings on certain classes of graphs. The second holds for uniform coupling on $3 \times n$ grids, where at $N = 3, 4$ we prove the inequality for every pair of edges at every value of the coupling (Theorem 1.4(i)).

Theorem 1.1 (cycles and cacti)**.** *Let $N \geq 2$ and let every block of G be a cycle or a single edge. Then $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of edges $e \neq f$ and all couplings $J \geq 0$. The inequality is strict when e and f lie on a common cycle whose couplings are all positive.*

A 2-*edge cut* of a finite graph G is a set $\{e, f\}$ of two edges such that $G - \{e, f\}$ has more connected components than G, while $G - e$ and $G - f$ do not; equivalently, e and f lie in the same component, neither is a bridge, and removing both disconnects that component. If one of two edges is a bridge,

or they lie in different components, then $\mathrm{Cov}(s_e, s_f) = 0$, by independence or by Lemma 2.3.

Theorem 1.2 (O(4), all couplings)**.** *Let $N = 4$ and $J \geq 0$.*

(a) *If every block of G is an edge, a generalized theta graph, or a 2-connected outerplanar graph whose weak dual is a path, then $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of edges $e \neq f$.*
(b) *On every finite graph, $\mathrm{Cov}(s_e, s_f) \geq 0$ whenever e and f share a vertex, and whenever $\{e, f\}$ is a 2-edge cut.*

Part (a) contains every $2 \times n$ ladder, every fan and every triangulated strip. Part (b) holds also at $N = 3$: for pairs sharing a vertex by Theorem 3.2, and for 2-edge cuts by Theorem 5.2 with $\Gamma_0 = \{e, f\}$, whose Γ_0^+ (defined there) is a subgraph of a 4-cycle or of a triangle, possibly with parallel edges. At $N = 3$ the law of the common label of a 2-edge cut is moreover nonnegative, and the inequality is strict when e and f lie on a cycle with positive couplings (Theorem 3.7).

Theorem 1.3 (O(3), graphs without a W_4 minor)**.** *Let $N = 3$ and $J \geq 0$. If the graph $G \cup \{ab, cd\}$ has no minor isomorphic to the wheel W_4 (a 4-cycle and a hub joined to its four vertices), then* (1) *holds.*

In Section 5 we prove Theorem 1.3 for all pair weights that are power series in $(1 + t)/2$ with nonnegative coefficients, in a form in which only the part of the graph between the two pairs has to avoid a W_4 minor (Theorem 5.2).

Theorem 1.4 (uniform coupling on $3 \times n$ grids)**.** *Let $N \in \{3, 4\}$ and let all couplings of the grid $P_3 \times P_n$ equal J, where P_n is the path with n vertices; the n copies of P_3 are the columns, their edges are called vertical and the other edges horizontal. The three copies of P_n are the top, middle and bottom rows.*

(i) *For every $n \geq 1$ and every $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of edges $e \neq f$.*
(ii) *For every $n \geq 3$ and every $0 < J \leq 20$, $\mathrm{Cov}(s_e, s_f) > 0$ for every pair of edges $e \neq f$.*

Theorem 6.34 in Section 6.6 states six special cases of part (i) that enter the proof of part (i), among them the case $n = 3$ and the case of the couplings $0 \leq J \leq 300$ ($0 \leq J \leq 350$ at $N = 4$). Part (i) follows from parts (b), (e) and (f) of that theorem together with Theorem 6.22 (Section 6.6), a theorem that treats the pairs with a horizontal edge of the middle row at $n \geq 6$ and $J \geq 300$ ($J \geq 350$ at $N = 4$).
For $0 < J \leq 20$ and $n \geq 3$ the inequality is strict by part (ii). For $J > 20$ it is strict on the pairs named in Remark 6.35 in Section 6.6, among them the pairs of the strict clause of Theorem 6.22.
The grid $P_3 \times P_2$ is the 2×3 ladder, which is covered for all couplings by Theorems 1.2 and 1.3. By contrast, the grid $P_3 \times P_3$ is the smallest grid

with a W_4 minor. It is the wheel W_4 with its four rim edges subdivided. The end of Section 6.6 assembles Theorem 1.4.

For $N = 4$ one step separates Theorem 1.2 from all series-parallel graphs. It is a statement about zeros of Chebyshev expansions (Conjecture 4.11). Formal verification. The deductions behind Theorems 1.1 and 1.4 and behind parts (a), (e) and (f) of Theorem 6.34 have been checked in Lean 4 with Mathlib, and so has the deduction behind part (b) of Theorem 6.34 for $J \leq 300$. The certified computations and some analytic facts enter the check as stated hypotheses, and so do Theorems 1.2(a) and 1.3 for the grid $P_3 \times P_2$. For Theorem 1.4(i) and parts (e) and (f) of Theorem 6.34 the hypotheses also include Theorem 3.2, Lemma 6.1 and Theorem 6.21. These hypotheses include parts (a), (c) and (d) of Theorem 6.34 as well. The hypotheses `hBoxAll5`, `hM1` and `hM2tail` state the inequality for every pair of a class of pairs and stand for the records of Propositions B.14, B.22 and B.24, while each record gives its bound for one pair of the class. The other pairs of the class are the images of that pair under the reflections of $P_3 \times P_n$ in the middle row and in the vertical midline, and the step from that pair to the other pairs uses the invariance of the Gibbs measure at uniform coupling under these two reflections, an invariance that is not formalized in Lean.

1.4. **Methods.** For all couplings we use the expansion of $e^{J\sigma\cdot\sigma'}$ in zonal spherical harmonics. It writes Z as a sum over labellings of the edges, and every covariance of bond energies becomes a covariance of increasing functions of the labels (Section 2). At $N = 4$ the labelling weights are nonnegative on every graph, since $S^3 = SU(2)$. The covariance then has a sign whenever the law of two labels is totally positive. We obtain total positivity from two classical facts: tridiagonal matrices with nonnegative entries generate totally nonnegative semigroups (Karlin and McGregor [19]), and Hadamard products of real-rooted polynomials with one-signed zeros are zero or real-rooted (Malo [25]).

For Theorem 1.3 we use the Hopf map $S^3 \to S^2$, under which $(1 + \sigma \cdot \sigma')/2 = |\langle\psi|\psi'\rangle|^2$ for unit spinors $\psi, \psi' \in \mathbb{C}^2$. We show that the moment array $U_G(n) = \int \prod_e \big((1+s_e)/2\big)^{n_e} \prod_x \omega(d\sigma_x)$ is multivariate totally positive of order two when G has no W_4 minor; a total covariance decomposition and the FKG inequality then give the inequality for all pair weights that are power series in $(1+t)/2$ with nonnegative coefficients. Gaussian integration over spinors shows that every two-terminal kernel is such a series. Cycles and K_4 have totally positive transfer densities, for K_4 by an $SU(2)$ argument with stretched Clebsch–Gordan coefficients and by a positivity statement for configurations of bits, and Karlin's composition formula carries total positivity along the tree-like structure of graphs without a W_4 minor.

For uniform coupling on $3 \times n$ grids we use a transfer operator on the rotation-invariant functions of a column. Every covariance of two bond energies has an exact spectral expansion. A finite list of inequalities on the leading eigen-data, together with a direct evaluation of the covariance

at small separations, implies positivity for all n at once. We certify these conditions on a cover of $J \in [0.0099, 20]$ with ball arithmetic and Taylor models in J, and a perturbative argument with explicit constants covers $0 < J \le 1/100$. For $P_3 \times P_3$ at $J > 20$, symmetry and three identities reduce all pairs to one covariance. A trace formula over the hub covers $J \in [20, 100]$, and a strong-coupling expansion with explicit remainders covers large J. On $P_3 \times P_n$ the covariance of two bond energies is a 2×2 determinant of matrix elements of the transfer operator. For $n \ge 4$ and $20 \le J \le 300$ (350 at $N = 4$), we read it as a matrix element of the second exterior power of this operator, written in the frame of the middle spin of each column, and we certify the leading modes of that power on a cover of this range. Above this range, expansions in $1/J$ with explicit remainders give the vertical pairs and the pairs with a top-row or bottom-row edge (Appendix B), and on the whole graph they give $P_3 \times P_4$ and $P_3 \times P_5$, as well as those rows of $P_3 \times P_6$ and $P_3 \times P_7$ that Proposition B.1 excludes. For a pair with a horizontal edge of the middle row the leading term at strong coupling comes from the second of the even modes just below the vacuum, since the middle-row insertion meets the first of them only at order $1/J$; one inequality for each class of pairs gives every n at once, by monotonicity in the distance to the ends and in the separation (Appendix B). Steps that rest on a computer calculation are marked † and are collected in Section 7.

1.5. **Organization.** We set up the spin-network expansion in Section 2 and treat cycles, cacti, 2-edge cuts and pairs sharing a vertex in Section 3. We prove Theorem 1.2 and state Conjecture 4.11 in Section 4, and prove Theorem 1.3 in a more general form in Section 5 (Theorems 5.1 and 5.2). The proof of Theorem 1.4 occupies Section 6, and the computations are described in Section 7. In Appendix A we prove Theorem 6.5, which is the case $0 < J \le 1/100$ of Theorem 1.4(ii). In Appendix B we prove Theorem 6.22 and the certified steps of parts (e) and (f) of Theorem 6.34, and in Appendix C we treat the identity in Campbell's Lemma 1.

2. The spin-network expansion

2.1. **Zonal harmonics.** Let $N \ge 2$ and $\lambda = N/2 - 1$. Let μ_N be the law of $\sigma \cdot e$ for σ distributed by ω, so $\mu_N(dt) \propto (1 - t^2)^{(N-3)/2} dt$ on $[-1, 1]$. Let $\widetilde{C}_\ell$ be the Gegenbauer polynomial C_ℓ^λ normalized by $\widetilde{C}_\ell(1) = 1$ (for $N = 2$, $\widetilde{C}_\ell(\cos\theta) = \cos \ell\theta$). Let H_ℓ be the space of spherical harmonics of degree ℓ on S^{N-1}, of dimension d_ℓ. The orthogonal projection onto H_ℓ in $L^2(\omega)$ is the integral operator with kernel $\Pi_\ell(\sigma, \sigma') = d_\ell \, \widetilde{C}_\ell(\sigma \cdot \sigma')$, and

$$\int \Pi_a(\sigma, \tau) \, \Pi_b(\tau, \sigma') \, \omega(d\tau) = \delta_{ab} \, \Pi_a(\sigma, \sigma'). \tag{2}$$

By the Funk–Hecke formula and Gegenbauer's expansion of e^{Jt} [35],

$$\begin{aligned} e^{J\sigma\cdot\sigma'} &= \sum_{\ell\ge 0} k_\ell(J)\,\Pi_\ell(\sigma,\sigma'),\\ k_\ell(J) &= \int e^{Jt}\widetilde{C}_\ell(t)\,\mu_N(dt) = \Gamma(N/2)\Big(\frac{2}{J}\Big)^{\lambda} I_{\ell+\lambda}(J), \end{aligned} \tag{3}$$

where I_ν is the modified Bessel function. For $J>0$ every $k_\ell(J)$ is positive, and $k_\ell(J)\le C(J)(J/2)^\ell/\ell!$. More precisely, $k_\ell(J)=\Gamma(N/2)\sum_{m\ge0}(J/2)^{\ell+2m}/(m!\,\Gamma(\ell+m+N/2))$, so on every interval $[a,b]\subset(0,\infty)$ there is a constant c with $|k_\ell^{(j)}(J)|\le c\,(\ell+1)^j(J/2)^\ell/\ell!$ for $j=0,1,2$ and every ℓ, and on every interval $[-R,R]$ the same holds with $(R/2)^\ell$ in place of $(J/2)^\ell$. Moreover $d_\ell\le(\ell+1)^{N-1}$.

2.2. **Labels and the covariance identity.** Insert (3) on every edge. For a labelling $\ell\in\mathbb{N}^E$ put

$$W(\ell)=\int\prod_{xy\in E}\Pi_{\ell_{xy}}(\sigma_x,\sigma_y)\prod_{x\in V}\omega(d\sigma_x).$$

Then

$$Z_{G,J}=\sum_{\ell\in\mathbb{N}^E}W(\ell)\prod_{e\in E}k_{\ell_e}(J_e), \tag{4}$$

and the series converges absolutely and locally uniformly in J, because $|W(\ell)|\le\prod_e d_{\ell_e}$. Define the label weight $p(\ell)=W(\ell)\prod_e k_{\ell_e}(J_e)/Z_{G,J}$, a signed measure on $\mathbb{N}^E$ of total mass one, and

$$r_\ell(J)=\partial_J\log k_\ell(J).$$

Proposition 2.1. *For edges $e\neq f$ with $J_e,J_f>0$,*

$$\operatorname{Cov}(s_e,s_f)=\partial_{J_e}\partial_{J_f}\log Z_{G,J}=\operatorname{Cov}_p\big(r_{\ell_e}(J_e),\,r_{\ell_f}(J_f)\big).$$

Proof. Differentiate (4) term by term. This is allowed by the local uniform convergence of the differentiated series. Then $\partial_{J_e}\log Z=\mathbb{E}_p[r_{\ell_e}(J_e)]$, and differentiating once more in J_f gives the claim. □

Lemma 2.2. *For every $N\ge2$ and $J>0$, $\ell\mapsto r_\ell(J)$ is strictly increasing.*

Proof. By (3) and $I_\nu'(x)=I_{\nu+1}(x)+\frac{\nu}{x}I_\nu(x)$, we have $r_\ell(J)=\ell/J+R_{\ell+\lambda}(J)$ with $R_\nu=I_{\nu+1}/I_\nu$. Since $k_{\ell+1}/k_\ell=R_\nu$ with $\nu=\ell+\lambda\ge0$, we have $r_{\ell+1}-r_\ell=\partial_J\log R_\nu(J)$. The ratio satisfies the Riccati equation $R_\nu'=1-(2\nu+1)R_\nu/x-R_\nu^2$. Therefore $R_\nu'>0$ if and only if

$$R_\nu(x)<\frac{x}{\nu+\frac12+\sqrt{(\nu+\frac12)^2+x^2}},$$

which holds for all $x>0$ and $\nu\ge-\frac12$ by [3, (16)] and [29, Theorem 2]. □

Equivalently, $k_\ell(J)$ is totally positive of order two in (ℓ, J). For $N = 2$ the weights $W(\ell)$ count divergence-free orientations and are nonnegative. For $N = 3$, however, they take both signs, already on the complete graph K_4. The case $N = 4$ is treated in Section 4.

2.3. **Blocks.**

Lemma 2.3. *Let $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$. Then every function of the bond energies of G_1 is independent of every function of the bond energies of G_2. In particular $\mathrm{Cov}(s_e, s_f) = 0$ when e and f lie in different blocks, and for e, f in the same block B the covariance equals the covariance computed on B alone.*

Proof. Given σ_v, the spins of G_1 and of G_2 are conditionally independent. By rotation invariance, the conditional law of the bond energies of G_i does not depend on σ_v. Removing the parts attached at cut vertices multiplies Z by a constant and leaves the law of the bond energies of B unchanged. □

2.4. **Two-edge cuts.**

Lemma 2.4. *Let $\{e, f\}$ be a 2-edge cut of G, as defined before Theorem 1.2, so that neither edge is a bridge. Then $W(\ell) = 0$ unless $\ell_e = \ell_f$. Let $J_e, J_f > 0$. If the law of the common label $c = \ell_e = \ell_f$ under p is nonnegative, then $\mathrm{Cov}(s_e, s_f) \geq 0$. If the law is nonnegative for all small positive values of J_e and J_f, the inequality holds also when $J_e = 0$ or $J_f = 0$.*

Proof. By the definition, e and f lie in the same component, and removing both splits it into two sides, each joined to the other by e and by f; the other components only multiply Z by a constant. Integrating out one side, with the endpoints u of e and w of f on the other side held fixed, gives a function of (u, w) that is invariant under the diagonal action of $\mathrm{O}(N)$ and lies in $H_{\ell_e} \otimes H_{\ell_f}$. The spaces H_ℓ are irreducible and pairwise inequivalent, so this function vanishes unless $\ell_e = \ell_f$ (Schur's lemma). For $J_e, J_f > 0$, by Proposition 2.1 the covariance is the covariance of two functions of the single variable c. Both are increasing by Lemma 2.2, so Chebyshev's sum inequality gives $\mathrm{Cov}(s_e, s_f) \geq 0$. The covariance is continuous in J, which gives the last claim. □

A bridge forces its label to vanish, so its bond energy is independent of all others, as Lemma 2.3 also shows.

3. Results on every graph

3.1. **Cycles and cacti.** On a cycle C with couplings J_g, the projections (2) force all labels to be equal, and $Z_C = \sum_{c \geq 0} d_c \prod_{g \in C} k_c(J_g)$. This is the classical solution of the ring [31].

Proof of Theorem 1.1. By Lemma 2.3 it suffices to consider two edges $e \neq f$ of the same cycle C. They form a 2-edge cut of C, and the law of the common label is $p(c) = d_c \prod_{g \in C} k_c(J_g)/Z_C \geq 0$. Lemma 2.4 gives $\mathrm{Cov}(s_e, s_f) \geq 0$

when $J_e, J_f > 0$, and continuity in J gives the remaining cases. If all couplings of C are positive, then $p(c) > 0$ for every c, and the two functions of c in Lemma 2.4 are strictly increasing. Chebyshev's inequality is then strict. □

Remark 3.1. Lemma 2.4 applies on every graph when the law of the common label is nonnegative. This nonnegativity holds for $N = 2$ and $N = 4$ on every graph, because $W \geq 0$ there (Section 2.2 and Lemma 4.1), and for $N = 3$ on every graph by Theorem 3.7. It holds for every N when both sides of the cut are two-terminal series-parallel networks, by the nonnegativity of the linearization coefficients of Gegenbauer polynomials.

3.2. **Pairs sharing a vertex.**

Theorem 3.2. *Let $N \in \{3, 4\}$, $J \geq 0$, $b \in V$, and let A and B be finite multisets of vertices other than b. Then*

$$\operatorname{Cov}\Big(\prod_{x\in A} \sigma_x \cdot \sigma_b,\ \prod_{y\in B} \sigma_y \cdot \sigma_b\Big) \geq 0.$$

In particular $\operatorname{Cov}(s_e, s_f) \geq 0$ *for every two edges e, f with a common vertex.*

Proof. Write F and H for the two products. Both are invariant under simultaneous rotation of all spins. It follows that $\mathbb{E}[F \mid \sigma_b]$ and $\mathbb{E}[H \mid \sigma_b]$ are constant, and $\operatorname{Cov}(F, H \mid \sigma_b)$ does not depend on σ_b; hence $\operatorname{Cov}(F, H)$ equals the covariance under the conditional law given $\sigma_b = u$, for a fixed unit vector u. That law is the $\mathrm{O}(N)$ model on $G-b$ with couplings J and the field $J_{xb}u$ at every neighbour x of b. All fields point along u and have nonnegative strength, and $F = \prod_{x\in A} \sigma_x \cdot u$, $H = \prod_{y\in B} \sigma_y \cdot u$ are products of components along the field. Choose u along the third axis. For $N = 3$, the inequality $\langle FH\rangle \geq \langle F\rangle\langle H\rangle$ is [11, Theorem 2, (3.1)] with $a = 3$. For $N = 4$, in the coordinates of Lemma 3.3, the couplings contribute $J_{xy}\varrho_x\varrho_y\cos(\varphi_x - \varphi_y)$ to h_1 and $J_{xy}\tau_x\tau_y\cos(\psi_x - \psi_y)$ to h_2, the fields contribute $J_{xb}\tau_x\cos\psi_x$ to h_2, and F, H are products of the components $\tau_x\cos\psi_x$. Thus $h_1 \in Q_1$ and $h_2, F, H \in Q_2$ in the notation of [11], and the inequality is [11, Theorem 1, (2.14)], which applies by Lemma 3.3. □

Lemma 3.3. *The uniform probability measure on S^3 satisfies the hypothesis of negative correlation* [11, (2.4)].

Proof. In the coordinates of [11, (2.1)–(2.3)], a point of $\mathbb{R}^4$ is $S = (\varrho\cos\varphi, \varrho\sin\varphi, \tau\cos\psi, \tau\sin\psi)$ and a single-site measure has the form $d\mu(S) = d\nu(\varrho, \tau)\, d\varphi\, d\psi$. The hypothesis (2.4) is

$$\iint \prod_{p\in P} \big(f_p(\varrho) \pm f_p(\varrho')\big) \prod_{q\in Q} \big(f_q(\tau') \pm f_q(\tau)\big)\, d\nu(\varrho, \tau)\, d\nu(\varrho', \tau') \ \geq\ 0$$

for every finite family $\{f_n : n \in P \cup Q\}$ of positive increasing bounded functions on $[0, \infty)$ and every choice of signs. For the uniform measure on S^3 we have $\varphi, \psi \in [0, 2\pi)$ and $d\nu(\varrho, \tau) = \delta(\varrho^2 + \tau^2 - 1)\, d\varrho^2\, d\tau^2$, the

analogue of [11, Example 3]. On the support, $\tau = \sqrt{1-\varrho^2}$ decreases in ϱ. Where $\varrho \geq \varrho'$ we therefore have $\tau \leq \tau'$, so every factor with a minus sign is nonnegative and every factor with a plus sign is positive. Exchanging the two copies multiplies the integrand by $(-1)^s$, where s is the number of minus signs. If s is even, the integrand is nonnegative everywhere. If s is odd, the integral vanishes. □

3.3. **Two-edge cuts at $N = 3$.** We use 2-edge cuts as defined before Theorem 1.2. If e and f are parallel, then $s_e = s_f$ and $\mathrm{Cov}(s_e, s_f)$ is a variance; other parallel edges can be merged by adding their couplings. Graphs in this subsection are therefore simple. Theorem 5.2 with $\Gamma_0 = \{e, f\}$, whose Γ_0^+ is a subgraph of a 4-cycle or of a triangle, possibly with parallel edges, gives $\mathrm{Cov}(s_e, s_f) \geq 0$ for every 2-edge cut at $N = 3$, for all weights in the class $\mathcal{P}$ of Section 5 (power series in $(1+t)/2$ with nonnegative coefficients). What follows adds that the law of the common label in Lemma 2.4 is nonnegative at $N = 3$, and that the inequality is strict on cycles with positive couplings. The corner steps in the proofs of Theorems 6.4 and 6.5 use exactly this.

For $N = 3$, $\widetilde{C}_\ell = P_\ell$ is the Legendre polynomial, $d_\ell = 2\ell + 1$ and $\mu_3(dt) = dt/2$. For bounded measurable κ on $[-1, 1]$ put $\rho_c(\kappa) = \frac{1}{2}\int_{-1}^{1} \kappa(t) P_c(t)\, dt$. By the Funk–Hecke formula, the integral operator with kernel $\kappa(\sigma \cdot \tau)$ acts on H_c as multiplication by $\rho_c(\kappa)$.

Lemma 3.4. *For $m, c \in \mathbb{N}$, $\gamma(m, c) := \frac{1}{2}\int_{-1}^{1} \left(\frac{1+t}{2}\right)^m P_c(t)\, dt$ equals $\frac{(m!)^2}{(m-c)!\,(m+c+1)!}$ if $c \leq m$, and 0 if $c > m$. Hence $\rho_c(\kappa) = \sum_m \kappa_m \gamma(m, c) \geq 0$ for every $\kappa = \sum_m \kappa_m \left(\frac{1+t}{2}\right)^m$ in $\mathcal{P}$, and $\rho_c(\kappa) > 0$ when $\kappa_m > 0$ for some $m \geq c$.*

Proof. By Rodrigues' formula $P_c = (2^c c!)^{-1} (d/dt)^c (t^2 - 1)^c$ and c integrations by parts, $\int_{-1}^{1} f P_c = (-1)^c (2^c c!)^{-1} \int_{-1}^{1} f^{(c)}(t)(t^2 - 1)^c\, dt$. For $f = ((1+t)/2)^m$ the right side vanishes if $c > m$. For $c \leq m$, the substitution $t = 2s - 1$ and a Beta integral give $\int_{-1}^{1} f P_c = 2(m!)^2/((m-c)!(m+c+1)!)$. The series for κ converges uniformly on $[-1, 1]$, so ρ_c can be taken term by term. □

Lemma 3.5. *Let H be a finite graph with distinct vertices x, y and couplings $J_g \geq 0$, and let $\kappa_H(\sigma_x \cdot \sigma_y)$ be its kernel, written K_H in Section 5.1. At $N = 3$, $\rho_c(\kappa_H) \geq 0$ for every c and $\rho_0(\kappa_H) > 0$. If H contains a path from x to y with positive couplings, then $\rho_1(\kappa_H) > 0$.*

Proof. By Proposition 5.4, $\kappa_H \in \mathcal{P}$, so Lemma 3.4 gives $\rho_c(\kappa_H) \geq 0$, and $\rho_0(\kappa_H) = \int \kappa_H\, d\mu_3 > 0$. In the proof of Proposition 5.4, $\kappa_H = \sum_n w(n) K_H(n; \cdot)$ with $w(n) \geq 0$ and polynomials $K_H(n; \cdot)$ in $(1+t)/2$ with nonnegative coefficients. This series converges uniformly on $[-1, 1]$, so $\rho_1(\kappa_H) = \sum_n w(n) \rho_1(K_H(n; \cdot))$ with every term ≥ 0. Let n_P be 1 on the edges of the path P, of length r, and 0 elsewhere. Then $w(n_P) > 0$, and $K_H(n_P; \cdot)$ is the kernel of P with weight $(1+t)/2$ on each edge. Since $(1 + \sigma \cdot \tau)/2 = \frac{1}{2}\Pi_0 + \frac{1}{6}\Pi_1$, (2) gives $\rho_1(K_H(n_P; \cdot)) = 6^{-r} > 0$. □

Proposition 3.6. *Let $V(G) = V_A \sqcup V_B$, with $e = xy$ and $f = zw$ the only edges between V_A and V_B, where $x, z \in V_A$ and $y, w \in V_B$, and let A, B be the graphs induced on V_A, V_B. If $x \neq z$, put $\alpha_c = \rho_c(\kappa_A)$ for the kernel of A with terminals x, z; if $x = z$, put $\alpha_c = K_A$ for every c, where $K_A > 0$ is the integral of the weight of A over the spins of $V_A \setminus \{x\}$. Define β_c from B and (y, w) in the same way. Then, with the couplings of A and B fixed,*

$$Z_G = \sum_{c \geq 0} (2c+1)\, \alpha_c\, \beta_c\, k_c(J_e)\, k_c(J_f)$$

for all real J_e, J_f, and the series and its derivatives up to order two converge absolutely and locally uniformly.

Proof. Integrate first over the spins of $V_A \setminus \{x, z\}$ and of $V_B \setminus \{y, w\}$, and insert (3) for e and f. If $y \neq w$, the Funk–Hecke formula and (2) give

$$\iint e^{J_e \sigma_x \cdot \sigma_y} \kappa_B(\sigma_y \cdot \sigma_w) e^{J_f \sigma_w \cdot \sigma_z}\, \omega(d\sigma_y)\, \omega(d\sigma_w) = \sum_c k_c(J_e) k_c(J_f) \beta_c \Pi_c(\sigma_x, \sigma_z),$$

and the case $y = w$ follows from (2) alone. Integrating against the side A gives the factor $(2c+1)\alpha_c$. The exchanges of sums and integrals, and the convergence of the derivatives, follow from $|\Pi_c| \leq 2c+1$, the boundedness of α_c and β_c, and the bounds on $k_c^{(j)}$ on $[-R, R]$ in Section 2.1. □

Theorem 3.7. *Let $N = 3$ and $J \geq 0$, and let $\{e, f\}$ be a 2-edge cut of a finite graph G. Then the law of the common label in Lemma 2.4 is nonnegative, and $\mathrm{Cov}(s_e, s_f) \geq 0$. If $J_e, J_f > 0$ and e, f lie on a cycle of G whose couplings are positive, then $\mathrm{Cov}(s_e, s_f) > 0$.*

Proof. By the definition of a 2-edge cut, neither e nor f is a bridge, and e and f lie in the same component G' of G, since otherwise removing both would not disconnect anything. Then f is a bridge of $G' - e$, so $G' - \{e, f\}$ has two components with vertex sets V_A, V_B, and e joins them as well, since otherwise f would be a bridge of G. Proposition 3.6 applies, with $\alpha_c, \beta_c \geq 0$ by Lemma 3.5, so $Z = \sum_c u_c k_c(J_e) k_c(J_f)$ with $u_c = (2c+1)\alpha_c\beta_c \geq 0$ independent of (J_e, J_f). Grouping the labellings of Section 2.2 by the common label gives $Z = \sum_c v_c k_c(J_e) k_c(J_f)$, and this series converges absolutely near $(0, 0)$, since $|W(\ell)| \leq \prod_g d_{\ell_g}$ gives $|v_c| \leq C(2c+1)^2$. As $k_c(a) = \frac{\sqrt{\pi}}{2}(a/2)^c / \Gamma(c + \frac{3}{2}) + O(a^{c+2})$, comparing coefficients gives $u_c = v_c$. Therefore $q_c = u_c k_c(J_e) k_c(J_f)/Z \geq 0$ is the law of the common label. For $J_e, J_f > 0$, Proposition 2.1 and term-by-term differentiation give $\mathrm{Cov}(s_e, s_f) = \mathrm{Cov}_q(r_c(J_e), r_c(J_f))$, which is nonnegative by Lemma 2.2 and Chebyshev's inequality, and positive as soon as q charges two labels. Zero couplings follow by continuity. A cycle through e and f with positive couplings consists of e, f, a path from x to z in A (empty if

$x = z$) and a path from y to w in B (empty if $y = w$), so Lemma 3.5 gives $q_0, q_1 > 0$. □

Corollary 3.8. *Let v be a vertex of degree two of a finite graph G, with edges $e = vx$ and $f = vy$, let all couplings be ≥ 0 with $J_e, J_f > 0$, and suppose that $G - v$ contains a path from x to y with positive couplings. Then, at $N = 3$ and at $N = 4$, the law of the common label of e and f is nonnegative and charges the labels 0 and 1, and $\mathrm{Cov}(s_e, s_f) = \mathrm{Cov}_q(r_c(J_e), r_c(J_f)) > 0$.*

Proof. At $N = 3$ this is Theorem 3.7, with $A = \{v\}$, $\alpha_c = 1$, and the path closing a cycle through e and f. At $N = 4$ every label weight is nonnegative (Lemma 4.1), so the law is nonnegative. The labelling with label 0 everywhere has weight 1, and the labelling with label 1 on a cycle through e and f with positive couplings and 0 elsewhere has weight $d_1 > 0$ by (2); both have positive k-factors. So $q_0, q_1 > 0$, and Lemma 2.2 with Chebyshev's inequality gives a positive covariance. □

4. The O(4) model at arbitrary couplings

Throughout this section $N = 4$, so $\lambda = 1$, $d_\ell = (\ell+1)^2$ and $\widetilde{C}_\ell(\cos\alpha) = \sin((\ell+1)\alpha)/((\ell+1)\sin\alpha)$.

4.1. **Positivity of the label weights.**

Lemma 4.1. *For $N = 4$, $W(\ell) \geq 0$ for every graph and every labelling ℓ.*

Proof. Identify S^3 with $SU(2)$, so that $\sigma \cdot \sigma' = \frac{1}{2}\operatorname{tr}(\sigma^{-1}\sigma')$ and ω is Haar measure. Let D^j be the irreducible unitary representation of spin j. Then $\Pi_\ell(\sigma, \sigma') = (\ell+1)\,\chi_{\ell/2}(\sigma^{-1}\sigma') = (\ell+1)\sum_{m,n} \overline{D^{\ell/2}_{mn}(\sigma)}\, D^{\ell/2}_{mn}(\sigma')$. Insert this on every edge and integrate each vertex variable. At a vertex the integral of the tensor product of the incident representations is the orthogonal projection onto its invariant subspace, $\sum_k |\iota_k\rangle\langle\iota_k|$ for an orthonormal basis of intertwiners. The row indices and the column indices of the matrix entries contract separately. The two contractions are complex conjugate spin-network evaluations $A_\kappa(\ell)$ and $\overline{A_\kappa(\ell)}$, indexed by the choices κ of intertwiners. Hence $W(\ell) = \prod_e(\ell_e+1)\sum_\kappa |A_\kappa(\ell)|^2 \geq 0$. □

It follows that p is a probability measure on $\mathbb{N}^E$. But on the planar graph K_4 the analogous weights at $N = 3$ take negative values, so this positivity is special to $N = 4$ (and $N = 2$).

4.2. **Two-terminal networks.**

A two-terminal network H with terminals x, y has the kernel $K_H(\sigma_x \cdot \sigma_y) = \int \prod_{e\in E(H)} e^{J_e s_e} \prod_{z\neq x,y} \omega(d\sigma_z)$. Write $K_H(t) = \sum_a \rho_a^H\, d_a \widetilde{C}_a(t)$. Series composition multiplies the coefficients, $\rho_a^{H\circ H'} = \rho_a^H \rho_a^{H'}$, by (2). Parallel composition multiplies the kernels. For a path P, $\rho_a^P = \prod_{g\in P} k_a(J_g)$.

Let $p_a = \sqrt{d_a}\,\widetilde{C}_a$, an orthonormal basis of $L^2(\mu_N)$, and let X be the matrix of multiplication by t in this basis. It is tridiagonal with zero diagonal and

positive off-diagonal entries, and its spectrum is $[-1,1]$. For a function K on $[-1,1]$, $K(\mathsf{X})$ is the matrix of multiplication by K.

Lemma 4.2. *Let G be the parallel composition, between terminals x and y, of networks $H_1 \ni e$, $H_2 \ni f$ and R. Then*

$$\operatorname{Cov}(s_e, s_f) = \operatorname{Cov}_q\big(\phi_1(a), \phi_2(b)\big), \quad q(a,b) \propto \rho_a^{H_1}\sqrt{d_a}\;\rho_b^{H_2}\sqrt{d_b}\;[K_R(\mathsf{X})]_{ab},$$

with $\phi_1(a) = \partial_{J_e}\log\rho_a^{H_1}$ and $\phi_2(b) = \partial_{J_f}\log\rho_b^{H_2}$. If $q \geq 0$ is totally positive of order two on $\mathbb{N}^2$ and ϕ_1, ϕ_2 are nondecreasing, then $\operatorname{Cov}(s_e, s_f) \geq 0$.

Proof. Integrate everything except σ_x, σ_y. Then

$$Z = \int K_{H_1}K_{H_2}K_R\,d\mu_N = \sum_{a,b}\rho_a^{H_1}\rho_b^{H_2}\,d_a d_b \int \widetilde{C}_a\widetilde{C}_b K_R\,d\mu_N,$$

and $\int \widetilde{C}_a\widetilde{C}_b K_R\,d\mu_N = (d_a d_b)^{-1/2}[K_R(\mathsf{X})]_{ab}$. Differentiating in J_e and J_f gives the covariance formula. A probability measure on $\mathbb{N}^2$ that is totally positive of order two is positively associated (FKG), which gives the sign. □

Lemma 4.3. *Let H be built from single edges with positive couplings by series compositions and by parallel compositions with networks S for which $K_S(\mathsf{X})$ is totally nonnegative. Then for every edge e of H, $(J_e, a) \mapsto \rho_a^H$ is totally positive of order two, so $a \mapsto \partial_{J_e}\log\rho_a^H$ is nondecreasing.*

Proof. For an edge this is Lemma 2.2. Series composition multiplies ρ_a^H by a positive function of a alone. Parallel composition with S gives $\rho_c^H = \sum_a \rho_a^{H'}\sqrt{d_a}\,[K_S(\mathsf{X})]_{ac}/\sqrt{d_c}$, and the basic composition formula [18] preserves total positivity of order two. □

4.3. **The class $\mathcal{K}$.**

Definition 4.4. $\mathcal{K}$ is the class of entire functions $K(t) = Ce^{\sigma t}\prod_n(1+t/\zeta_n)$ with $C > 0$, $\sigma \geq 0$, $\zeta_n \geq 1$ and $\sum_n 1/\zeta_n < \infty$: the Laguerre–Pólya functions whose zeros are all real and at most -1.

Five facts are used.

- (K1) If $K \in \mathcal{K}$ then $K(\mathsf{X})$ is totally nonnegative. Indeed $I + \mathsf{X}/\zeta$ ($\zeta \geq 1$) is tridiagonal with nonnegative entries and positive semidefinite, hence totally nonnegative [12]. The same holds for $e^{\sigma\mathsf{X}} = \lim_n(I + \sigma\mathsf{X}/n)^n$ and for the norm-convergent product. This is the semigroup argument of Karlin and McGregor [19].
- (K2) $\mathcal{K}$ is closed under products.
- (K3) $\mathcal{K}$ is closed under d/dt for nonconstant K. For a nonconstant polynomial in $\mathcal{K}$ this is Rolle's theorem, by which the zeros of the derivative lie between the smallest and the largest zero; in general K' is the locally uniform limit of the derivatives of the polynomials of (K5) below, and it lies in $\mathcal{K}$ by (K5). This is Laguerre's theorem [24, Ch. VIII].

(K4) For a polynomial $K = \kappa_0 + 2\sum_{n\ge1}\kappa_n T_n$, put $F(z) = K((z + z^{-1})/2) = \sum_{n\in\mathbb{Z}}\kappa_{|n|}z^n$. Then $K \in \mathcal{K}$ if and only if $K(1) > 0$ and F has only negative real zeros in $\mathbb{C}^*$, because $z + z^{-1} = 2t_0$ has negative real roots exactly when $t_0 \le -1$. For $K \in \mathcal{K}$ of degree m, $z^m F$ then has a positive leading coefficient and only negative zeros, so $\kappa_n > 0$ for $0 \le n \le m$.

(K5) Every $K \in \mathcal{K}$ is a locally uniform limit in $\mathbb{C}$ of polynomials in $\mathcal{K}$: replace $e^{\sigma t}$ by $(1+\sigma t/n)^n$, which lies in $\mathcal{K}$ for $n \ge \sigma$, and truncate the product. Conversely, a locally uniform limit in $\mathbb{C}$ of polynomials in $\mathcal{K}$ that is not identically zero lies in $\mathcal{K}$. After the substitution $t = s - 1$ this is the theorem of Laguerre and Pólya on locally uniform limits of polynomials with only real nonpositive zeros [24, Ch. VIII], whose proof rests on Hurwitz's theorem. The constant C of the limit is positive, since the limit is nonnegative at $t = 1$ and does not vanish on $(-1, \infty)$.

Lemma 4.5 (Malo)**.** *If $f = \sum a_k x^k$ has only real zeros and $g = \sum b_k x^k$ has only real zeros, all in $(-\infty, 0]$ or all in $[0, \infty)$, then $\sum a_k b_k x^k$ is zero or has only real zeros.*

See [25] and the survey [8].

Lemma 4.6. *If $K_1, K_2 \in \mathcal{K}$, then their O(2) series composition $K_1 \star_2 K_2$, whose Chebyshev coefficients are $\kappa_n = \kappa_n^1\kappa_n^2$, lies in $\mathcal{K}$.*

Proof. For polynomials of degree at most m, $z^m F_1$ and $z^m F_2$ have only real nonpositive zeros and nonnegative coefficients by (K4), and their Hadamard product is $z^m F$. It is not zero, since $\kappa_0 = \kappa_0^1\kappa_0^2 > 0$, so it is real-rooted by Lemma 4.5; its coefficients are nonnegative, so F has only negative zeros in $\mathbb{C}^*$. As $K(1) = F(1) \ge \kappa_0 > 0$, (K4) gives $K \in \mathcal{K}$. In general, approximate K_1 and K_2 by polynomials in $\mathcal{K}$ as in (K5). The approximants are bounded on every ellipse with foci ±1, uniformly in the approximation, so their Chebyshev coefficients satisfy Cauchy bounds $|\kappa_n| \le M_R R^{-n}$ for every $R > 1$, with M_R independent of the approximation. Therefore the compositions of the approximants converge to that of K_1 and K_2 locally uniformly in $\mathbb{C}$. The limit is not zero, since $\kappa_0 = \kappa_0^1\kappa_0^2 > 0$ (members of $\mathcal{K}$ are positive on $(-1, 1]$), so it lies in $\mathcal{K}$ by (K5). □

Lemma 4.7 (dimension raising)**.** *For every path P, $\frac{d}{dt}K_P^{(N)} = N\prod_{g\in P}(J_g/N)\,K_P^{(N+2)}$, where $K_P^{(N)}$ is the kernel of P in the O(N) model.*

Proof. Comparing coefficients in $\frac{d}{dt}e^{Jt} = Je^{Jt}$ gives $\frac{d}{dt}\big[d_\ell^{(N)}\widetilde{C}_\ell^{(N)}\big] = N\,d_{\ell-1}^{(N+2)}\widetilde{C}_{\ell-1}^{(N+2)}$ and $k_{\ell-1}^{(N+2)}(J) = (N/J)\,k_\ell^{(N)}(J)$. □

By Lemmas 4.6 and 4.7 and (K3), every O(4) path kernel lies in $\mathcal{K}$. More precisely it lies in $\mathcal{D} = \{A' : A \in \mathcal{K} \text{ nonconstant}\} \subset \mathcal{K}$, the class of derivatives.

4.4. **Theta graphs.**

Theorem 4.8. *Let G be a generalized theta graph: two vertices x, y joined by internally disjoint paths $P_1, \dots, P_k$, $k \geq 2$. For $N = 4$ and $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ for all edges $e \neq f$, and also for the pair (x, y) against any edge.*

Proof. By continuity in J we may assume that all couplings are positive. If e, f lie on the same path, $\{e, f\}$ is a 2-edge cut, and Lemmas 2.4 and 4.1 apply. If $e \in P_i$ and $f \in P_j$ with $i \neq j$, let R be the bundle of the other paths. Then $K_R = \prod_{l \neq i,j} K_{P_l} \in \mathcal{K}$ by (K2), so $K_R(\mathsf{X})$ is totally nonnegative by (K1). Lemma 4.2 applies with $\phi_1(a) = r_a(J_e)$ and $\phi_2(b) = r_b(J_f)$, which increase by Lemma 2.2. For the pair (x, y), add the edge xy with coupling 0^+ as one more path (Section 1.1). □

The same proof works for every even N, using Lemma 4.7 $N/2 - 1$ times and, for two edges on the same path, the nonnegativity of the linearization coefficients of Gegenbauer polynomials.

4.5. **Ladders and outerplanar strips.** For a polynomial $K = \sum_c \rho_c \, d_c \widetilde{C}_c = \sum_c (c+1)\rho_c U_c$, the Weyl numerator is

$$K(\cos\theta)\sin\theta = \sum_{m \geq 1} b_m \sin m\theta, \qquad b_m = m\,\rho_{m-1} = \kappa_{m-1} - \kappa_{m+1},$$

where κ_n are the Chebyshev coefficients of Section 4.3. In the variable $z = e^{i\theta}$ this reads $G(z) := (z - z^{-1})F(z) = \sum_m b_m z^m$ with $b_{-m} = -b_m$. Series composition at $N = 4$, written $K_1 \star_4 K_2$, multiplies the ρ_c, so it acts by

$$b_m = b^1_m \, b^2_m / m. \tag{5}$$

Lemma 4.9. (L0) *If $K \in \mathcal{K}$ is a polynomial, then $b_m \geq 0$ for all $m \geq 1$, and $b_1 > 0$ if K is not constant.*
(L1) $\mathcal{D} \star_4 \mathcal{D} \subset \mathcal{D}$.
(L2) $\mathcal{K} \star_4 \mathcal{D} \subset \mathcal{K}$.

Proof. (L0) By (K4), $z^m F$ is a real-rooted polynomial with nonpositive zeros, so (κ_n) is symmetric and log-concave without internal zeros (Newton's inequalities). Hence κ_n is nonincreasing for $n \geq 0$ and $b_m = \kappa_{m-1} - \kappa_{m+1} \geq 0$. Newton's inequalities are strict at the centre for nonconstant K, so $b_1 = \kappa_0 - \kappa_2 > 0$.

(L1) If $K = A'$ with $A = \kappa_0 + 2\sum \kappa_n T_n$, then $\rho_c(K) = 2\kappa_{c+1}$. Thus $A_1' \star_4 A_2' = 2\,(A_1 \star_2 A_2)'$, and $A_1 \star_2 A_2 \in \mathcal{K}$ by Lemma 4.6.

(L2) Let $K_1 \in \mathcal{K}$ and $K_2 = A'$ with $A \in \mathcal{K}$ nonconstant, and suppose first that K_1 and A are polynomials. By (5), $b_m = 2b^1_m \kappa^A_{|m|}$, that is $G = 2\,G_1 \odot F_A$. The polynomial $z^M G_1$ is real-rooted, with zeros at the zeros of F_1, at ± 1 and at 0. The polynomial $z^M F_A$ has only nonpositive zeros. By Lemma 4.5, G is real-rooted. Since G is odd under $z \mapsto 1/z$, $G = (z - z^{-1})F$ with F

symmetric and real-rooted. For $z > 0$, $G(z) = \sum_{m\geq 1} b_m(z^m - z^{-m})$ with all $b_m \geq 0$ and $b_1 > 0$, by (L0). Therefore G has no positive zeros except $z = 1$, and $F(1) = \sum_m m b_m > 0$; hence F has only negative zeros, and the composition lies in $\mathcal{K}$ by (K4). In general, approximate K_1 and A by polynomials in $\mathcal{K}$ as in (K5); the derivatives of the approximants of A converge to A'. As in the proof of Lemma 4.6, uniform Cauchy bounds on the coefficients make the compositions converge locally uniformly in $\mathbb{C}$, the limit has $\rho_0 > 0$, and (K5) applies. □

Single edges with $J > 0$ lie in $\mathcal{D}$, since $e^{Jt} = \frac{d}{dt}(e^{Jt}/J)$. Let $\mathcal{N}$ be the class of networks generated from single edges by parallel composition and by series composition with a path on either side. By (K2), (L1) and (L2), $K_X \in \mathcal{K}$ for every $X \in \mathcal{N}$, and so $K_X(\mathsf{X})$ is totally nonnegative by (K1).

Theorem 4.10. *Let G be a 2-connected outerplanar graph whose weak dual is a path. For $N = 4$ and $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ for all edges $e \neq f$.*

Proof. By continuity in J we may assume that all couplings are positive. Let $F_0, \ldots, F_k$ be the faces in the order of the weak dual, and let the chord $c_j = p_j q_j$ separate F_{j-1} from F_j. The boundary of F_j consists of c_j, c_{j+1} and two outer arcs, whose interior vertices have degree two. Let A_j be the two-terminal network between p_j and q_j on the side of F_0, and B_j the one on the side of F_k. Then $G = c_j \parallel A_j \parallel B_j$, and A_{j+1} is a path, followed by $c_j \parallel A_j$, followed by a path. Thus every A_j and every B_j lies in $\mathcal{N}$.

If $e = c_j$ is a chord, apply Lemma 4.2 at c_j, with f in A_j or B_j and the other part as R. If e and f lie in different faces, apply it at a chord between them, with $R = c_j$. In both cases $K_R(\mathsf{X})$ is totally nonnegative, and ϕ_1, ϕ_2 increase by Lemma 4.3, since every sibling met in the nesting is a chord or some A_l or B_l. If e and f lie in the same face, $\{e, f\}$ is a 2-edge cut, because the interior vertices of the arcs have degree two. Lemmas 2.4 and 4.1 apply. □

Ladders $P_2 \times P_n$, fans and triangulated strips are of this kind. The proof uses only the nesting of the A_j, so it applies unchanged when chords or arcs are subdivided.

Proof of Theorem 1.2. Part (a) follows from Lemma 2.3 and Theorems 4.8 and 4.10. In part (b), pairs sharing a vertex are covered by Theorem 3.2, and 2-edge cuts by Lemmas 2.4 and 4.1. □

4.6. **Series-parallel graphs.** In a general series-parallel network a series composition may join two networks that are not paths. Such compositions lie outside the scope of Theorem 4.10.

Conjecture 4.11. *If $K_1, K_2 \in \mathcal{K}$, then $K_1 \star_4 K_2 \in \mathcal{K}$.*

By (L1) and the identity $K_1 \star_4 K_2 = 2\,(A_1 \star_2 A_2)'$, which holds for any antiderivatives A_1, A_2, the conjecture says that O(2) series composition preserves the polynomials whose derivative lies in $\mathcal{K}$. Together with Duffin's

characterization of series-parallel graphs [10], it would extend the argument of Theorem 4.10 to every two-terminal series-parallel decomposition, and so give $\mathrm{Cov}(s_e, s_f) \geq 0$ on every series-parallel graph at $N = 4$.

Remark 4.12. (L2) proves the conjecture when one factor lies in $\mathcal{D}$. Every kernel of degree at most two lies in $\mathcal{D}$, so the first open case is two cubic kernels outside $\mathcal{D}$. We verified the conjecture in exact rational arithmetic on 2,033 pairs of polynomials in $\mathcal{K}$, chosen at random, with clustered zeros (including kernels outside $\mathcal{D}$), with zeros at -1, with widely spread zeros and mixed degrees, and in a search designed to find counterexamples, and found no violation. Real-rootedness was decided by Sturm sequences, and the composition was computed in two independent bases. The analogue for $N = 3$, with Legendre coefficients in place of Chebyshev ones, is the subject of a companion paper in preparation.

5. The O(3) model on graphs without a W_4 minor

In this section $N = 3$ and $\mathbb{N} = \{0, 1, 2, \dots\}$. We prove Theorem 1.3 for pair weights that need not be exponential, and in a form in which only the part of the graph between the two pairs has to avoid a W_4 minor. The section does not use Sections 2–4 or 6.

5.1. **Statements.** Let $G = (V, E)$ be a finite graph. In this section graphs may have parallel edges, but no loops. For spins $\sigma \in (S^2)^V$ and an edge e with ends x, y put $s_e = \sigma_x \cdot \sigma_y$ and $u_e = (1 + s_e)/2 \in [0, 1]$. Let $\mathcal{P}$ be the set of functions $W : [-1, 1] \to [0, \infty)$ of the form

$$W(t) = \sum_{m \geq 0} w_m \Big(\frac{1+t}{2}\Big)^m, \qquad w_m \geq 0, \qquad \sum_{m \geq 0} w_m < \infty,$$

with $w_m > 0$ for at least one m. Products of functions in $\mathcal{P}$ lie in $\mathcal{P}$, and $e^{Jt} = e^{-J} \sum_m \frac{(2J)^m}{m!} \big(\frac{1+t}{2}\big)^m$ lies in $\mathcal{P}$ for $J \geq 0$. Given weights $W_e \in \mathcal{P}$, $e \in E$, let $\mu_{G,W}$ be the probability measure on $(S^2)^V$ with density proportional to $\prod_{e \in E} W_e(s_e)$ with respect to $\prod_{x \in V} \omega(d\sigma_x)$. For $W_e(t) = e^{J_e t}$ it is the measure $\mu_{G,J}$ of Section 1.1.

For vertices $a \neq b$ and $c \neq d$, $G \cup \{ab, cd\}$ denotes the graph obtained from G by adding those of the edges ab and cd that are not edges of G. A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. The wheel W_4 consists of a 4-cycle and a hub joined to its four vertices.

Theorem 5.1. *Let G be a finite graph with weights $W_e \in \mathcal{P}$, and let $a \neq b$ and $c \neq d$ be vertices of G. If $G \cup \{ab, cd\}$ has no W_4 minor, then*

$$\mathrm{Cov}_{\mu_{G,W}}(\sigma_a \cdot \sigma_b,\ \sigma_c \cdot \sigma_d) \geq 0.$$

Theorem 1.3 is the case $W_e(t) = e^{J_e t}$. Let Γ_0 be a subgraph of a graph Γ. A *bridge* of Γ_0 (in Tutte's sense, not a cut edge as in Sections 1–3) in Γ is either an edge of Γ that is not in Γ_0 and has both ends in $V(\Gamma_0)$, or

a connected component K of $\Gamma - V(\Gamma_0)$ together with the edges joining K to $V(\Gamma_0)$ and their ends. The vertices of a bridge that lie in $V(\Gamma_0)$ are its *vertices of attachment*.

Theorem 5.2. *Let Γ be a finite graph with weights $W_e \in \mathcal{P}$, and let ab and cd be edges of Γ. Let $\Gamma_0 \subseteq \Gamma$ be a subgraph that contains ab and cd and whose bridges in Γ have at most two vertices of attachment each. Let Γ_0^+ be obtained from Γ_0 by adding, for each bridge with two vertices of attachment x, y, a new edge joining x and y. If Γ_0^+ has no W_4 minor, then $\mathrm{Cov}_{\mu_{\Gamma,W}}(s_{ab}, s_{cd}) \geq 0$.*

The bridges are arbitrary graphs; they may contain W_4 or any other minor. Theorem 5.1 is the case $\Gamma = G \cup \{ab, cd\}$ and $\Gamma_0 = \Gamma$, with weight 1 on the added edges, which does not change the measure.

Both theorems rest on a total positivity property. For $n \in \mathbb{N}^E$ define the *moment array*

$$U_G(n) = \int \prod_{e \in E} u_e^{n_e} \prod_{x \in V} \omega(d\sigma_x).$$

The integrand is continuous, nonnegative and equal to 1 when all spins coincide, so $U_G(n) > 0$. A function f on $\mathbb{N}^E$ is *MTP*$_2$ if $f(n \vee n')f(n \wedge n') \geq f(n)f(n')$ for all n, n', where $\vee$ and $\wedge$ are the coordinatewise maximum and minimum.

Theorem 5.3. *If G has no W_4 minor, then U_G is MTP$_2$ on $\mathbb{N}^E$.*

The second ingredient holds on every graph. For a graph H with distinct vertices x, y and weights $W_e \in \mathcal{P}$, the integral

$$K_H(\sigma_x, \sigma_y) = \int \prod_{e \in E(H)} W_e(s_e) \prod_{z \in V(H) \setminus \{x,y\}} \omega(d\sigma_z)$$

is invariant under simultaneous rotations of σ_x and σ_y, so it depends only on $\sigma_x \cdot \sigma_y$ (Lemma 5.13). We call it the *kernel* of H with *terminals* x, y.

Proposition 5.4. *For every finite graph H with distinct vertices x, y and weights $W_e \in \mathcal{P}$, the kernel K_H, as a function of $\sigma_x \cdot \sigma_y$, belongs to $\mathcal{P}$.*

Remark 5.5. The graphs without a W_4 minor include the graphs without a K_4 minor, K_4 and its subdivisions, and every graph obtained from cycles and copies of K_4 by repeatedly gluing two graphs along an edge. By Proposition 5.42, a graph has no W_4 minor if and only if each of its blocks with at least three vertices, after parallel edges are merged, is a spanning subgraph of a graph built from cycles and copies of K_4 glued along edges in a tree-like way. The grid $P_3 \times P_3$ is a subdivision of W_4 and is not covered; we treat it at uniform coupling in Section 6.

Remark 5.6. For exponential weights and two edges with a common vertex, Theorem 3.2 gives the inequality on every graph. Its proof conditions on the common spin and uses Dunlop's inequalities, which require exponential weights. For other weights in $\mathcal{P}$ the case of a common vertex is covered only under the hypothesis of Theorem 5.1.

The proof of Theorem 5.3 has four steps. (i) The kernel of a part of the graph attached at two vertices is a nonnegative combination of monomials u_{xy}^m (Proposition 5.18). (ii) Cycles and K_4 have totally positive transfer densities between any two edges, with monomial weights on the other edges (Sections 5.5–5.7). (iii) Gluing along an edge composes transfer densities, and Karlin's composition formula preserves total positivity (Lemmas 5.20 and 5.37). (iv) A graph without a W_4 minor decomposes into cycles and copies of K_4 glued along edges in a tree-like way (Proposition 5.42), and an induction over this tree combines (i) and (iii) (Theorem 5.39). Theorems 5.1 and 5.2 follow from Theorem 5.3 by the covariance criterion of Lemma 5.15. The transfer densities must be taken with monomial weights, because with other weights the transfer density of a triangle can fail to be totally positive (Remark 5.45).

5.2. **Total positivity.** A *chain* is a totally ordered set; we use subsets of $\mathbb{Z}$ and intervals of $\mathbb{R}$. On a product $X = X_1 \times \cdots \times X_d$ of chains, $x \vee y$ and $x \wedge y$ denote the coordinatewise maximum and minimum, and $\mathbf{1}_i$ is the i-th unit vector when $X_i \subseteq \mathbb{Z}$. A function $f : X \to [0,\infty)$ is *MTP*$_2$ if $f(x \vee y)f(x \wedge y) \geq f(x)f(y)$ for all $x, y \in X$. For $d = 2$ we also say *TP*$_2$; a kernel K on $X_1 \times X_2$ is TP_2 if and only if $K(x,y)K(x',y') \geq K(x,y')K(x',y)$ whenever $x < x'$ and $y < y'$.

The following *elementary facts* are immediate from the definition. Products of MTP_2 functions are MTP_2. A nonnegative function of one coordinate is MTP_2. The restriction of an MTP_2 function to a product of subchains is MTP_2. If f is MTP_2 on $X_1 \times \cdots \times X_d$ and $\varphi_i : Y_i \to X_i$ are nondecreasing, then $f(\varphi_1(y_1), \ldots, \varphi_d(y_d))$ is MTP_2, since $\varphi(y \vee y') = \varphi(y) \vee \varphi(y')$ and $\varphi(y \wedge y') = \varphi(y) \wedge \varphi(y')$.

Lemma 5.7 (four functions)**.** (a) *Let Y be a finite chain with counting measure, or $Y = \mathbb{R}$ with Lebesgue measure. Let $\varphi_1, \ldots, \varphi_4 : Y \to [0,\infty)$ be integrable and such that $\varphi_1(y)\varphi_2(y') \leq \varphi_3(y \vee y')\varphi_4(y \wedge y')$ for all $y, y' \in Y$. Then $\int \varphi_1 \int \varphi_2 \leq \int \varphi_3 \int \varphi_4$.*

(b) *Let X be a product of finitely many finite chains, and let $\alpha, \beta, \gamma, \delta : X \to [0,\infty)$ satisfy $\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$ for all $x, y \in X$. Then $\sum_X \alpha \sum_X \beta \leq \sum_X \gamma \sum_X \delta$.*

Proof. (a) For $y < y'$ put $a = \varphi_1(y)\varphi_2(y')$, $b = \varphi_1(y')\varphi_2(y)$, $c = \varphi_3(y')\varphi_4(y)$ and $d = \varphi_3(y)\varphi_4(y')$. The hypothesis for the pairs (y, y') and (y', y) gives $a \leq c$ and $b \leq c$. For the pairs (y, y) and (y', y') it gives $\varphi_1(y)\varphi_2(y) \leq \varphi_3(y)\varphi_4(y)$ and $\varphi_1(y')\varphi_2(y') \leq \varphi_3(y')\varphi_4(y')$, so $ab \leq cd$. If $c = 0$, then $a = b = 0 \leq d$. If $c > 0$, then $(c-a)(c-b) \geq 0$ gives $a + b \leq c + ab/c \leq c + d$. It follows that $a + b \leq c + d$ whenever $y < y'$. Now

$$\int \varphi_1 \int \varphi_2 = \iint_{y<y'} \big(\varphi_1(y)\varphi_2(y') + \varphi_1(y')\varphi_2(y)\big) + \int_\Delta \varphi_1\varphi_2,$$

where the last term is the integral over the diagonal. It vanishes for Lebesgue measure, and for counting measure it is at most $\int_\Delta \varphi_3\varphi_4$. The same decomposition of $\int\varphi_3\int\varphi_4$ and the inequality $a+b\le c+d$ give the claim.

(b) Induction on the number d of factors; $d=1$ is (a). Let $d\ge 2$, write $x=(x',t)$ with $x'\in X'=X_1\times\cdots\times X_{d-1}$ and $t\in X_d$, and put $\alpha'(x')=\sum_t\alpha(x',t)$, and similarly β',γ',δ'. For fixed $x',y'\in X'$ the functions $\varphi_1=\alpha(x',\cdot)$, $\varphi_2=\beta(y',\cdot)$, $\varphi_3=\gamma(x'\vee y',\cdot)$ and $\varphi_4=\delta(x'\wedge y',\cdot)$ satisfy the hypothesis of (a). Therefore $\alpha'(x')\beta'(y')\le\gamma'(x'\vee y')\delta'(x'\wedge y')$, and the induction hypothesis on X' gives the claim. □

Part (b) is a case of the four functions theorem of Ahlswede and Daykin [2].

Corollary 5.8 (FKG inequality)**.** *Let p be an MTP_2 probability measure on a product of finitely many finite chains, and let F,H be nondecreasing real functions. Then $\mathbb{E}_p[FH]\ge\mathbb{E}_p[F]\,\mathbb{E}_p[H]$.*

Proof. Adding constants does not change $\mathbb{E}_p[FH]-\mathbb{E}_p[F]\mathbb{E}_p[H]$, so we may assume $F,H\ge 0$. Put $\alpha=pF$, $\beta=pH$, $\gamma=pFH$ and $\delta=p$. Since $F(x)\le F(x\vee y)$ and $H(y)\le H(x\vee y)$, we have $\alpha(x)\beta(y)\le\gamma(x\vee y)\delta(x\wedge y)$, and Lemma 5.7(b) applies. □

This is the inequality of Fortuin, Kasteleyn and Ginibre [14].

Corollary 5.9 (marginals)**.** *Let X be a product of chains, Y a finite chain or $\mathbb{R}$, and $f:X\times Y\to[0,\infty)$ MTP_2 with $y\mapsto f(x,y)$ integrable for every x. Then $x\mapsto\int_Y f(x,y)\,dy$ is MTP_2 on X.*

Proof. For $x,x'\in X$ apply Lemma 5.7(a) to $f(x,\cdot)$, $f(x',\cdot)$, $f(x\vee x',\cdot)$ and $f(x\wedge x',\cdot)$. Its hypothesis is the MTP_2 inequality at the points (x,y) and (x',y'). □

This is due to Karlin and Rinott [22].

Lemma 5.10 (composition)**.** *Let X,Y,Z be chains, where Y is countable with counting measure ν or an interval of $\mathbb{R}$ with Lebesgue measure ν. Let $K:X\times Y\to[0,\infty)$ and $L:Y\times Z\to[0,\infty)$ be TP_2, and suppose that $M(x,z)=\int_Y K(x,y)L(y,z)\,\nu(dy)$ is finite for all x,z. Then M is TP_2.*

Proof. For $x<x'$ and $z<z'$, symmetrizing in (y,y'),

$$\begin{aligned}M(x,z)M(x',z')-M(x,z')M(x',z)&\\ =\tfrac12\iint\big(K(x,y)K(x',y')-K(x,y')K(x',y)\big)&\\ \times\big(L(y,z)L(y',z')-L(y,z')L(y',z)\big)\,\nu(dy)\,\nu(dy').&\end{aligned}$$

For $y<y'$ both factors are nonnegative, for $y>y'$ both are nonpositive, and for $y=y'$ both vanish. □

This is the case of order two of Karlin's basic composition formula [18].

Lemma 5.11 (local criterion)**.** (a) *If $f : \mathbb{N}^d \to (0,\infty)$ satisfies $f(x+\mathbf{1}_i+\mathbf{1}_j)f(x) \geq f(x+\mathbf{1}_i)f(x+\mathbf{1}_j)$ for all x and all $i \neq j$, then f is MTP_2.*

(b) *If $f : \mathbb{R}^d \to (0,\infty)$ is twice continuously differentiable with $\partial_i\partial_j \log f \geq 0$ for all $i \neq j$, then f is MTP_2.*

Proof. Let $g = \log f$. First, for $i \neq j$, a point x and $h,k \geq 0$ (integers in case (a)),

$$g(x+h\mathbf{1}_i+k\mathbf{1}_j) - g(x+h\mathbf{1}_i) - g(x+k\mathbf{1}_j) + g(x) \geq 0.$$

In case (a) the left side is a sum of the local second differences at the points $x + r\mathbf{1}_i + q\mathbf{1}_j$, $0 \leq r < h$, $0 \leq q < k$. In case (b) it equals $\int_0^h \int_0^k \partial_i\partial_j g(x + r\mathbf{1}_i + q\mathbf{1}_j)\, dq\, dr$. Now let $z = x \wedge y$, $\alpha = x - z$ and $\beta = y - z$. The vectors α and β have disjoint supports, and $x \vee y = z + \alpha + \beta$, so we must show $g(z+\alpha+\beta) - g(z+\beta) \geq g(z+\alpha) - g(z)$. Add the coordinates of α one at a time. At each step the increment of g in a direction i in the support of α is compared at two points that differ by β, a sum of multiples of unit vectors $\mathbf{1}_j$, $j \neq i$. Adding these one at a time, the displayed inequality shows that the increment does not decrease. □

See Topkis [33] and [22].

5.3. **Spinors, moment arrays and the covariance criterion.** For $\varphi, \psi \in \mathbb{C}^2$ write $\langle\varphi|\psi\rangle = \overline{\varphi^1}\psi^1 + \overline{\varphi^2}\psi^2$, and let $d\psi$ be the uniform probability measure on $S^3 = \{\psi \in \mathbb{C}^2 : |\psi| = 1\}$. The Hopf map is $s(\psi) = \big(2\,\mathrm{Re}(\overline{\psi^1}\psi^2),\, 2\,\mathrm{Im}(\overline{\psi^1}\psi^2),\, |\psi^1|^2 - |\psi^2|^2\big) \in S^2$.

Lemma 5.12. (a) *For $\varphi, \psi \in S^3$, $|\langle\varphi|\psi\rangle|^2 = \frac{1}{2}\big(1 + s(\varphi)\cdot s(\psi)\big)$.*

(b) *The image of $d\psi$ under s is ω.*

(c) *For $a, a', b, b' \in \mathbb{N}$, $\displaystyle\int (\psi^1)^a\, \overline{\psi^1}^{\,a'} (\psi^2)^b\, \overline{\psi^2}^{\,b'}\, d\psi = \delta_{aa'}\delta_{bb'}\frac{a!\,b!}{(a+b+1)!}$.*

Proof. (a) With the Pauli matrices $\vec{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3)$, a direct computation gives $|\psi\rangle\langle\psi| = \frac{1}{2}(I + s(\psi)\cdot\vec{\varsigma})$ for $|\psi| = 1$. Since $\mathrm{tr}\,\varsigma_i = 0$ and $\mathrm{tr}\,\varsigma_i\varsigma_j = 2\delta_{ij}$, we get $|\langle\varphi|\psi\rangle|^2 = \mathrm{tr}\,\big(|\varphi\rangle\langle\varphi|\,|\psi\rangle\langle\psi|\big) = \frac{1}{2}\big(1 + s(\varphi)\cdot s(\psi)\big)$. (b) For $g \in SU(2)$ there is $R_g \in SO(3)$ with $g(v\cdot\vec{\varsigma})g^* = (R_g v)\cdot\vec{\varsigma}$, and $g \mapsto R_g$ is onto $SO(3)$. Thus $s(g\psi) = R_g s(\psi)$, so the image of the $SU(2)$-invariant measure $d\psi$ is $SO(3)$-invariant and equals ω. (c) Under $\psi^1 \mapsto e^{i\theta}\psi^1$, which preserves $d\psi$, the integrand is multiplied by $e^{i\theta(a-a')}$, so the integral vanishes unless $a = a'$, and likewise unless $b = b'$. Let $Z = (Z_1, Z_2)$ have independent standard complex Gaussian coordinates. Then $Z/|Z|$ has law $d\psi$ and is independent of $|Z|^2$, which has the Gamma law with shape 2. Hence $a!\,b! = \mathbb{E}|Z_1|^{2a}|Z_2|^{2b} = (a+b+1)!\int |\psi^1|^{2a}|\psi^2|^{2b}\, d\psi$. □

By (a) and (b), $U_G(n) = \int \prod_{e=xy} |\langle\psi_x|\psi_y\rangle|^{2n_e} \prod_x d\psi_x$. By (c), $|\psi^1|^2$ is uniform on $[0,1]$ under $d\psi$, since its moments are $1/(m+1)$.

Lemma 5.13. *Let a be a bounded measurable function on $S^2 \times S^2$ with $a(R\sigma, R\sigma') = a(\sigma, \sigma')$ for all $R \in SO(3)$. Then $a(\sigma, \sigma') = \bar{a}\big((1 + \sigma \cdot \sigma')/2\big)$, where $\bar{a}(w) = a\big(\epsilon_3, (2\sqrt{w(1-w)}, 0, 2w-1)\big)$ and $\epsilon_3 = (0,0,1)$. For fixed σ, the variable $(1 + \sigma \cdot \sigma')/2$ is uniform on $[0,1]$ when σ' has law ω.*

Proof. $SO(3)$ acts transitively on the pairs of unit vectors with a given scalar product, and the scalar product of ϵ_3 with the second argument of a in the definition of $\bar{a}$ is $2w - 1$. For the second statement take $\sigma = s((1,0)) = \epsilon_3$, write $\sigma' = s(\psi)$ with ψ of law $d\psi$, and use Lemma 5.12: $(1 + \sigma \cdot \sigma')/2 = |\psi^1|^2$. □

Lemma 5.14. *Let G be a graph.*

(a) *For $F \subseteq E(G)$, U_{G-F} is the restriction of U_G to $\{n : n_f = 0 \text{ for } f \in F\}$. If U_G is MTP$_2$, so is U_{G-F}.*
(b) *$U_G(n + 2\mathbf{1}_e)\, U_G(n) \geq U_G(n + \mathbf{1}_e)^2$ for all n and e.*
(c) *Let e, e' be parallel edges. Then $U_G(n) = U_{G-e'}(\bar{n})$, where $\bar{n}$ agrees with n except that $\bar{n}_e = n_e + n_{e'}$. If $U_{G-e'}$ is MTP$_2$, so is U_G.*
(d) *If $G = G_1 \cup G_2$ with $E(G_1) \cap E(G_2) = \emptyset$ and $|V(G_1) \cap V(G_2)| \leq 1$, then $U_G(n) = U_{G_1}(n|_{E(G_1)})\, U_{G_2}(n|_{E(G_2)})$. Consequently U_G is MTP$_2$ if and only if the moment array of each block of G is MTP$_2$.*

Proof. (a) An edge with multiplicity 0 contributes the factor 1; restrictions preserve MTP$_2$. (b) This is the Cauchy–Schwarz inequality for the measure $\prod_g u_g^{n_g} \prod_x \omega(d\sigma_x)$ and the function u_e. (c) The identity holds because $u_e = u_{e'}$. We verify Lemma 5.11(a) for U_G. For a pair of coordinates other than $\{e, e'\}$, the local inequality for U_G at n is a local inequality for $U_{G-e'}$ at $\bar{n}$, in which a step in the coordinate e' becomes a step in the coordinate e. For the pair $\{e, e'\}$ it is the inequality (b) for $U_{G-e'}$ at $\bar{n}$. (d) If $V(G_1) \cap V(G_2) = \emptyset$, the integral factorizes. If $V(G_1) \cap V(G_2) = \{v\}$, integrate first over the spins of $V(G_2) \setminus \{v\}$. The result is a rotation-invariant function of σ_v, hence constant, and equal to $U_{G_2}(n|_{E(G_2)})$. The factorization over blocks follows by induction on the number of blocks, removing a block that meets the others in at most one vertex. The moment array of a block is a restriction of U_G, and a product of MTP$_2$ functions of disjoint sets of variables is MTP$_2$. □

Lemma 5.15 (covariance criterion)**.** *Let G be a graph with weights $W_e \in \mathcal{P}$ and $\mu = \mu_{G,W}$. If U_G is MTP$_2$ on $\mathbb{N}^{E(G)}$, then $\mathrm{Cov}_\mu(u_a, u_b) \geq 0$ for all $a, b \in E(G)$.*

Proof. Write $W_e(t) = \sum_m w_{e,m} u^m$ with $u = (1+t)/2$, and $w(n) = \prod_e w_{e,n_e}$. By Tonelli's theorem the normalization is $Z = \int \prod_e W_e(s_e)\, d\omega^{\otimes V} = \sum_n w(n)\, U_G(n)$, and $0 < Z \leq \prod_e W_e(1)$. Let $p(n) = w(n) U_G(n)/Z$, and let μ_n be the probability measure with density $U_G(n)^{-1} \prod_e u_e^{n_e}$. Then

$\mu = \sum_n p(n)\mu_n$, and the law of total covariance gives

$$\mathrm{Cov}_\mu(u_a, u_b) = \sum_n p(n)\,\mathrm{Cov}_{\mu_n}(u_a, u_b) + \mathrm{Cov}_p(R_a, R_b),$$

$$R_a(n) = \mathbb{E}_{\mu_n} u_a = \frac{U_G(n + \mathbf{1}_a)}{U_G(n)}.$$

For $a = b$ the claim is trivial; let $a \neq b$. Then $\mathrm{Cov}_{\mu_n}(u_a, u_b) = \big(U_G(n + \mathbf{1}_a + \mathbf{1}_b)U_G(n) - U_G(n + \mathbf{1}_a)U_G(n + \mathbf{1}_b)\big)/U_G(n)^2 \geq 0$. The function R_a is nondecreasing in each coordinate: in a coordinate $c \neq a$ by the MTP_2 property, and in the coordinate a by Lemma 5.14(b). The measure p is MTP_2, since w is a product of functions of single coordinates. For K large, $p(\{0, \dots, K\}^{E(G)}) > 0$, and p conditioned on this box is MTP_2, so Corollary 5.8 gives a nonnegative covariance of R_a and R_b under it. Since $0 \leq R_a, R_b \leq 1$, letting $K \to \infty$ gives $\mathrm{Cov}_p(R_a, R_b) \geq 0$. □

For the next lemma recall that a *circulation* on an oriented graph is an integer vector f on the edges such that at each vertex the sum of f over the outgoing edges equals the sum over the incoming edges.

Lemma 5.16 (latitudes)**.** *Let G be a graph, o a vertex and $n \in \mathbb{N}^{E(G)}$. Orient the edges of $G - o$ arbitrarily. Let $\mathcal{S}$ be the set of pairs (k, l) of integer vectors on the edges of $G - o$ with $0 \leq k_e, l_e \leq n_e$ such that $k - l$ is a circulation on $G - o$. For $(k, l) \in \mathcal{S}$ and $p \neq o$ put*

$$b_p = \tfrac{1}{2} \sum_{e \ni p,\ e \not\ni o} (k_e + l_e), \qquad d_p = \sum_{e \ni p} n_e, \qquad a_p = d_p - b_p.$$

Then b_p is an integer with $0 \leq b_p \leq d_p$, and for every bounded measurable F on $[0,1]^{V \setminus \{o\}}$, with $u_{op} = (1 + \sigma_o \cdot \sigma_p)/2$,

$$\int F\big((u_{op})_{p \neq o}\big) \prod_e u_e^{n_e} \prod_x \omega(d\sigma_x) = \int_{[0,1]^{V \setminus \{o\}}} F(w)\,\Psi(w)\,dw,$$

$$\Psi(w) = \sum_{(k,l) \in \mathcal{S}} \prod_e \binom{n_e}{k_e} \binom{n_e}{l_e} \prod_{p \neq o} w_p^{a_p} (1 - w_p)^{b_p}.$$

Proof. At $p \neq o$, the sum of $k_e + l_e$ over the edges of $G - o$ at p has the parity of the sum of $f_e = k_e - l_e$ over these edges. By the circulation condition the latter is twice the inflow of f at p, so $b_p \in \mathbb{Z}$, and $b_p \leq \sum_{e \ni p, e \not\ni o} n_e \leq d_p$.

By Lemma 5.12 the left side equals

$$\int F\big((|\langle \psi_o | \psi_p \rangle|^2)_{p \neq o}\big) \prod_{e = xy} |\langle \psi_x | \psi_y \rangle|^{2n_e} \prod_x d\psi_x.$$

The integrand is invariant under $\psi_x \mapsto g\psi_x$ for all x, $g \in SU(2)$, and $SU(2)$ acts transitively on S^3, so we may put $\psi_o = (1, 0)$. With Z as in the proof of Lemma 5.12, the arguments of Z_1, Z_2 are uniform and independent of each other and of $(|Z_1|, |Z_2|)$, and $|Z_1|^2/|Z|^2$ is uniform on $[0, 1]$. We may therefore write $\psi_p = (\sqrt{w_p}\, e^{i\alpha_p}, \sqrt{1 - w_p}\, e^{i\beta_p})$ with independent variables w_p

uniform on $[0,1]$ and $\theta_p = \alpha_p - \beta_p$ uniform on $\mathbb{R}/2\pi\mathbb{Z}$. Then $|\langle\psi_o|\psi_p\rangle|^2 = w_p$, and for $p, q \neq o$

$$|\langle\psi_p|\psi_q\rangle|^2 = \left|A_{pq} + B_{pq}e^{i(\theta_p-\theta_q)}\right|^2,$$
$$A_{pq} = \sqrt{w_p w_q}, \qquad B_{pq} = \sqrt{(1-w_p)(1-w_q)}.$$

For an edge e of $G-o$ oriented from p to q and $\phi = \theta_p - \theta_q$,

$$\left|A + Be^{i\phi}\right|^{2n_e} = \sum_{k,l=0}^{n_e} \binom{n_e}{k}\binom{n_e}{l} A^{2n_e-k-l}B^{k+l}e^{i(k-l)\phi}.$$

Multiplying over the edges of $G-o$ gives a sum over pairs (k,l), whose phases combine to $\exp(i\sum_p \theta_p(\partial f)_p)$, where $(\partial f)_p$ is the outflow minus the inflow of $f = k - l$ at p. The average of this phase over the θ_p is 1 if f is a circulation and 0 otherwise. For $(k,l) \in \mathcal{S}$, an edge $e = pq$ of $G-o$ contributes $(w_p w_q)^{n_e - m_e/2}\big((1-w_p)(1-w_q)\big)^{m_e/2}$ with $m_e = k_e + l_e$, and an edge joining o and p contributes $w_p^{n_e}$. Collecting the factors at each vertex gives $\prod_p w_p^{a_p}(1-w_p)^{b_p}$. Fubini's theorem completes the proof. □

5.4. **Two-terminal kernels.** A random variable ζ is standard complex normal if $\zeta = (X + iY)/\sqrt{2}$ with independent standard real normal X, Y.

Lemma 5.17. *Let $(\zeta_c)_{c\in I}$ be independent standard complex normal variables, and let $a_k, b_k \in I$ for $1 \le k \le n$. Then $\mathbb{E}\big[\prod_{k=1}^n \zeta_{a_k}\overline{\zeta_{b_k}}\big]$ is the number of permutations π of $\{1,\dots,n\}$ with $b_{\pi(k)} = a_k$ for all k.*

Proof. Let p_c and q_c be the numbers of k with $a_k = c$ and with $b_k = c$. By independence the left side is $\prod_c \mathbb{E}[\zeta_c^{p_c}\overline{\zeta_c}^{q_c}]$. The law of ζ_c is invariant under $\zeta \mapsto e^{i\theta}\zeta$, so a factor vanishes unless $p_c = q_c$, and $\mathbb{E}|\zeta_c|^{2p} = p!$ since $|\zeta_c|^2$ is exponential with mean 1. So the left side is $\prod_c \delta_{p_c q_c} p_c!$, which is also the number of the permutations in question. □

Let H be a graph with *terminals* $x \neq y$, let the other vertices be *internal*, let $n \in \mathbb{N}^{E(H)}$ and $d_z = \sum_{e\ni z} n_e$. For each edge e with ends p, q and each $j \le n_e$ introduce two *arcs*, one from p to q and one from q to p, and write $\mathsf{t}(\alpha)$ and $\mathsf{h}(\alpha)$ for the tail and the head of an arc α. An internal vertex z is the head of the d_z arcs in a set $A^-(z)$ and the tail of the d_z arcs in a set $A^+(z)$. A *transition system* is a family $\Theta = (\Theta_z)$ of bijections $\Theta_z : A^-(z) \to A^+(z)$, one for each internal vertex. Following $\alpha \mapsto \Theta_{\mathsf{h}(\alpha)}(\alpha)$, the arcs split into *closed trails* and *open trails*; an open trail starts with an arc whose tail is a terminal and ends with the first later arc whose head is a terminal. Let $L(\Theta)$ be the number of closed trails and $k(\Theta)$ the number of open trails from x to y. Put

$$K_H(n;u) = \int \prod_{e\in E(H)} u_e^{n_e} \prod_{z \text{ internal}} \omega(d\sigma_z),$$

a function of $u = u_{xy} = (1 + \sigma_x \cdot \sigma_y)/2$ by Lemma 5.13.

Proposition 5.18 (loop expansion)**.** *For every graph H with terminals $x \neq y$ and every $n \in \mathbb{N}^{E(H)}$,*

$$K_H(n;u) = \prod_{z \text{ internal}} \frac{1}{(d_z+1)!} \sum_{\Theta} 2^{L(\Theta)}\, u^{k(\Theta)},$$

the sum running over all transition systems. In particular $K_H(n;\cdot)$ is a polynomial with nonnegative coefficients.

Proof. Fix unit spinors ψ_x, ψ_y with $s(\psi_x) = \sigma_x$ and $s(\psi_y) = \sigma_y$. Since $|\langle\psi_p|\psi_q\rangle|^2 = \langle\psi_p|\psi_q\rangle\langle\psi_q|\psi_p\rangle$, Lemma 5.12 gives $K_H(n;u_{xy}) = \int \prod_\alpha \langle\psi_{\mathsf{t}(\alpha)}|\psi_{\mathsf{h}(\alpha)}\rangle \prod_{z \text{ internal}} d\psi_z$.

Gaussian vectors. Let ξ_z, z internal, be independent random vectors in $\mathbb{C}^2$ with independent standard complex normal coordinates, and put $\xi_x = \psi_x$, $\xi_y = \psi_y$. Then $\xi_z = |\xi_z|\phi_z$ with ϕ_z of law $d\psi$, independent of $|\xi_z|^2$, which has the Gamma law with shape 2. Each internal z is the tail of d_z arcs and the head of d_z arcs, so $\prod_\alpha \langle\xi_{\mathsf{t}(\alpha)}|\xi_{\mathsf{h}(\alpha)}\rangle = \prod_z |\xi_z|^{2d_z} \prod_\alpha \langle\phi_{\mathsf{t}(\alpha)}|\phi_{\mathsf{h}(\alpha)}\rangle$ with $\phi_x = \psi_x$, $\phi_y = \psi_y$. Since $\mathbb{E}|\xi_z|^{2d} = (d+1)!$,

$$\mathbb{E}\prod_\alpha \langle\xi_{\mathsf{t}(\alpha)}|\xi_{\mathsf{h}(\alpha)}\rangle = \prod_z (d_z+1)!\; K_H(n;u_{xy}).$$

Wick expansion. Write $\langle\xi_p|\xi_q\rangle = \sum_{i=1}^2 \overline{\xi_p^i}\xi_q^i$ and expand the product over arcs as a sum over colourings c of the arcs by $\{1,2\}$. For fixed c, the factors containing an internal ξ_z are $\xi_z^{c(\alpha)}$, $\alpha \in A^-(z)$, and $\overline{\xi_z^{c(\beta)}}$, $\beta \in A^+(z)$. By Lemma 5.17 their expectation is the number of transition systems Θ with $c(\Theta_z(\alpha)) = c(\alpha)$ for all internal z and $\alpha \in A^-(z)$; call such c compatible with Θ. Therefore the left side of the last display equals

$$\sum_{\Theta} \sum_{c \text{ compatible with } \Theta} \prod_{\alpha:\, \mathsf{t}(\alpha)\in\{x,y\}} \overline{\psi_{\mathsf{t}(\alpha)}^{c(\alpha)}} \prod_{\alpha:\, \mathsf{h}(\alpha)\in\{x,y\}} \psi_{\mathsf{h}(\alpha)}^{c(\alpha)}.$$

Trails. A colouring is compatible with Θ if and only if it is constant on each trail of Θ. A closed trail has no terminal factor and contributes 2 colours. An open trail from a terminal p to a terminal q with colour i contributes $\overline{\psi_p^i}\psi_q^i$, and the sum over i gives $\langle\psi_p|\psi_q\rangle$. It follows that the inner sum is $2^{L(\Theta)}\prod\langle\psi_p|\psi_q\rangle$ over the open trails.

Counting. Let n_{pq} be the number of open trails from p to q, $p,q \in \{x,y\}$. Each arc with tail p starts one open trail and each arc with head p ends one. Since x is the tail of d_x arcs and the head of d_x arcs, $n_{xx}+n_{xy} = d_x = n_{xx}+n_{yx}$, so $n_{xy} = n_{yx}$. Since $\langle\psi_x|\psi_x\rangle = \langle\psi_y|\psi_y\rangle = 1$, the product over open trails is $\langle\psi_x|\psi_y\rangle^{n_{xy}}\langle\psi_y|\psi_x\rangle^{n_{yx}} = u_{xy}^{k(\Theta)}$. □

Proof of Proposition 5.4. Write $W_e(t) = \sum_m w_{e,m}u^m$ and $w(n) = \prod_e w_{e,n_e}$. By Tonelli's theorem $K_H = \sum_n w(n)K_H(n;u_{xy})$, and by Proposition 5.18 each term is a polynomial in u_{xy} with nonnegative coefficients. Thus $K_H = \sum_m \kappa_m u_{xy}^m$ with $\kappa_m \geq 0$, and the coefficients sum to the value at $u_{xy} = 1$, which is at most $\prod_e W_e(1) < \infty$. If $w(n) > 0$, then $K_H(n;1) > 0$, since

the integrand is continuous, nonnegative and equal to 1 when all spins equal $\sigma_x = \sigma_y$. Therefore some κ_m is positive, and $K_H \in \mathcal{P}$. □

5.5. **Transfer densities, gluing and cycles.** In this subsection a weight on an edge g is a bounded measurable function $W_g : [0,1] \to [0,\infty)$ of u_g.

Definition 5.19. Let H be a graph, $e \neq f$ edges of H, and W_g weights for $g \in E(H) \setminus \{e,f\}$. A *transfer density* of (e,f) is a bounded measurable function $D : [0,1]^2 \to [0,\infty)$ such that for all bounded measurable $\varphi, \chi \geq 0$ on $[0,1]$

$$\int \varphi(u_e)\,\chi(u_f) \prod_{g\neq e,f} W_g(u_g) \prod_{x\in V(H)} \omega(d\sigma_x) = \int_0^1\!\!\int_0^1 \varphi(v)\,\chi(w)\,D(v,w)\,dv\,dw.$$

It is unique up to null sets. For $n \in \mathbb{N}^{E(H)}$ with $n_e = n_f = 0$, $D^H_{e,f}[n]$ denotes the transfer density for the weights $W_g(u) = u^{n_g}$, when it exists.

If $D^H_{e,f}[n]$ exists, then $U_H(n + i\mathbf{1}_e + j\mathbf{1}_f) = \iint v^i w^j D^H_{e,f}[n](v,w)\,dv\,dw$ for $i,j \in \mathbb{N}$. Since $(i,v) \mapsto v^i$ is TP_2 on $\mathbb{N} \times (0,1)$, two applications of Lemma 5.10 show: *if $D^H_{e,f}[n]$ is TP_2 on $(0,1)^2$, then $(i,j) \mapsto U_H(n + i\mathbf{1}_e + j\mathbf{1}_f)$ is TP_2 on $\mathbb{N}^2$.*

Lemma 5.20 (gluing)**.** *Let $G = G_1 \cup G_2$, where $G_1 \cap G_2$ consists of an edge f and its ends x, y. Let $e \in E(G_1) \setminus \{f\}$, $e' \in E(G_2) \setminus \{f\}$, and let W_g be weights for $g \in E(G) \setminus \{e,e'\}$. Suppose that (e,f) has a transfer density A in G_1 and (f,e') has a transfer density B in G_2, with respect to the weights of the other edges of G_1 and of G_2. Then (e,e') has the transfer density $C(v,v') = \int_0^1 A(v,w)\,W_f(w)\,B(w,v')\,dw$ in G. If A and B are TP_2 on $(0,1)^2$, so is C. If A, B and W_f are polynomials, so is C.*

Proof. Let $\varphi_1, \varphi_2 \geq 0$ be bounded measurable on $[0,1]$. The integral $a(\sigma_x,\sigma_y)$ of $\varphi_1(u_e) \prod_{g\in E(G_1)\setminus\{e,f\}} W_g(u_g)$ over the spins of $V(G_1) \setminus \{x,y\}$ is invariant under simultaneous rotations, so $a = \bar{a}(u_{xy})$ by Lemma 5.13. Integrating also over σ_x, σ_y against $\chi(u_f)$, and using that u_{xy} is uniform on $[0,1]$, the definition of A gives $\int_0^1 \chi(w)\bar{a}(w)\,dw = \iint \varphi_1(v)\chi(w)A(v,w)\,dv\,dw$ for all χ, so $\bar{a}(w) = \int \varphi_1(v)A(v,w)\,dv$ for almost every w. In the same way the corresponding integral over $V(G_2) \setminus \{x,y\}$ is $\bar{b}(u_{xy})$ with $\bar{b}(w) = \int B(w,v')\varphi_2(v')\,dv'$. The two sets of internal vertices are disjoint, so by Fubini's theorem

$$\int \varphi_1(u_e)\varphi_2(u_{e'}) \prod_{g\neq e,e'} W_g(u_g) \prod_{z\in V(G)} \omega(d\sigma_z)$$
$$= \int_0^1 \bar{a}(w)W_f(w)\bar{b}(w)\,dw = \iint \varphi_1(v)\varphi_2(v')C(v,v')\,dv\,dv'.$$

C is bounded by $\sup A \cdot \sup W_f \cdot \sup B$. If A and B are TP_2, then so is $A(v,w)W_f(w)$, and Lemma 5.10 applies. □

For $M \in \mathbb{N}$ and $v, w \in [0,1]$ put

$$\tau_M(v,w) = \sum_{k=0}^{M} \binom{M}{k}^2 (vw)^{M-k} \big((1-v)(1-w)\big)^k,$$

the average over $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ of $\left|\sqrt{vw} + \sqrt{(1-v)(1-w)}\, e^{i\phi}\right|^{2M}$ (expand as in the proof of Lemma 5.16).

Lemma 5.21. *τ_M is TP_2 on $[0,1]^2$.*

Proof. On $(0,1)^2$ put $X = v/(1-v)$ and $Y = w/(1-w)$. Then $\tau_M(v,w) = \big((1-v)(1-w)\big)^M \Phi(XY)$ with $\Phi(z) = \sum_{j=0}^{M} \binom{M}{j}^2 z^j$. The function $g(r) = \log \Phi(e^r)$ is convex, since $g''(r)$ is the variance of j under the weights proportional to $\binom{M}{j}^2 e^{jr}$. For $r < r'$ and $q < q'$ the numbers $r+q'$ and $r'+q$ lie between $r+q$ and $r'+q'$ and have the same sum, so $g(r+q) + g(r'+q') \geq g(r+q') + g(r'+q)$. Hence $(X,Y) \mapsto \Phi(XY)$ is TP_2 on $(0,\infty)^2$. Since $v \mapsto X$ and $w \mapsto Y$ are increasing, the elementary facts show that τ_M is TP_2 on $(0,1)^2$, and continuity extends this to $[0,1]^2$. □

Lemma 5.22 (cycles)**.** *Let C be a cycle with vertices $z_0, \dots, z_{m-1}$ in cyclic order, $m \geq 3$, and edges $g_i = z_{i-1}z_i$ for $1 \leq i \leq m-1$ and $g_m = z_{m-1}z_0$.*

(a) *For multiplicities n_i on g_i, $2 \leq i \leq m-1$, the pair (g_1, g_m) has the transfer density*

$$D(v,v') = \int_{[0,1]^{m-3}} \prod_{i=2}^{m-1} \tau_{n_i}(w_{i-1}, w_i)\, dw_2 \cdots dw_{m-2}, \quad w_1 = v,\ w_{m-1} = v',$$

which is τ_{n_2} for $m = 3$.

(b) *For any two distinct edges of C and any multiplicities on the other edges, the transfer density exists, is a polynomial, and is TP_2 on $[0,1]^2$.*

(c) *U_C is MTP_2.*

Proof. (a) Apply Lemma 5.16 with $o = z_0$ and multiplicity 0 on g_1 and g_m. The graph $C - z_0$ is the path $z_1 \cdots z_{m-1}$, which carries no nonzero circulation, so $\mathcal{S}$ consists of the pairs with $k = l$. An edge g_i with $k_{g_i} = l_{g_i} = k$ contributes $\binom{n_i}{k}^2 (w_{i-1}w_i)^{n_i-k}\big((1-w_{i-1})(1-w_i)\big)^k$, so $\Psi(w) = \prod_{i=2}^{m-1} \tau_{n_i}(w_{i-1}, w_i)$ with $w_i = u_{z_0 z_i}$. Since $u_{g_1} = w_1$ and $u_{g_m} = w_{m-1}$, integrating out the other latitudes gives the claim.

(b) If the two edges share a vertex, label C so that they are g_1 and g_m. By (a) the density is a polynomial, and it is TP_2 by Lemma 5.21 and $m-3$ applications of Lemma 5.10. If they are disjoint, label them g_1 and g_j with $3 \leq j \leq m-1$, and let h be a new edge joining z_0 and z_{j-1}. The cycles $C' = z_0 z_1 \cdots z_{j-1}$ and $C'' = z_{j-1} z_j \cdots z_{m-1} z_0$, each closed by h, meet in h and its ends. In C' the edges g_1, h share z_0, and in C'' the edges h, g_j share z_{j-1}. Lemma 5.20 with $W_h = 1$ gives a polynomial TP_2 transfer density of

(g_1, g_j) in $C' \cup C''$, which is the transfer density in C, since h carries the weight 1.

(c) By (b) and the remark after Definition 5.19, U_C is TP_2 in each pair of coordinates when the others are fixed. Since $U_C > 0$, Lemma 5.11(a) applies. □

5.6. K_4: **opposite edges.** Let K_4 have vertices $0, 1, 2, 3$, write $n = (n_{01}, n_{02}, n_{03}, n_{12}, n_{13}, n_{23})$, and $U = U_{K_4}$.

Theorem 5.23. *Fix $n_{02}, n_{03}, n_{12}, n_{13} \in \mathbb{N}$. The pair $(01, 23)$, with these multiplicities on the other edges, has a transfer density that is a polynomial and is TP_2 on $[0,1]^2$. In particular $(n_{01}, n_{23}) \mapsto U(n)$ is TP_2 on $\mathbb{N}^2$.*

Binary forms. For $c \in \mathbb{N}$ let H_c be the space of homogeneous polynomials of degree c in $\xi = (\xi^1, \xi^2)$, with the inner product $\langle p, q\rangle_c = (c+1)\int \overline{p(\psi)}\, q(\psi)\, d\psi$. Put $e_a = (\xi^1)^a (\xi^2)^{c-a}$, $0 \le a \le c$, and $K_\varphi(\xi) = \langle \varphi | \xi \rangle^c$ for $\varphi \in \mathbb{C}^2$. For $g \in SU(2)$ let $\rho_c(g)p = p \circ g^{\mathsf{T}}$. On H_c define $S^+ = \xi^1 \partial_{\xi^2}$, $S^- = \xi^2 \partial_{\xi^1}$, $S^3 = \frac{1}{2}(\xi^1 \partial_{\xi^1} - \xi^2 \partial_{\xi^2})$, $S^1 = \frac{1}{2}(S^+ + S^-)$ and $S^2 = \frac{1}{2i}(S^+ - S^-)$.

Lemma 5.24. *Let $c \in \mathbb{N}$ and $j = c/2$.*

(a) *The e_a are orthogonal, and $\langle e_a, e_a\rangle_c = a!(c-a)!/c!$.*
(b) *$\langle K_\varphi, p\rangle_c = p(\varphi)$ for $p \in H_c$, and $\langle K_\varphi, K_\varphi\rangle_c = 1$ for unit vectors $\varphi \in \mathbb{C}^2$.*
(c) *ρ_c is a unitary representation of $SU(2)$ with $\rho_c(g)K_\varphi = K_{\bar{g}\varphi}$. Its derived representation maps the matrices E_{12}, E_{21} and $\frac{1}{2}\mathrm{diag}(1,-1)$ to S^+, S^- and S^3. We have $S^+ e_a = (c-a)e_{a+1}$, $S^- e_a = a\, e_{a-1}$, $S^3 e_a = (a-j)e_a$, $(S^\pm)^* = S^\mp$, $(S^1)^2 + (S^2)^2 + (S^3)^2 = j(j+1)$, and ρ_c is irreducible: it is the representation of spin j.*

Proof. (a) is Lemma 5.12(c). (b) Since $K_\varphi = \sum_a \binom{c}{a} \overline{\varphi^1}^{\,a}\, \overline{\varphi^2}^{\,c-a} e_a$, (a) gives $\langle K_\varphi, e_b\rangle_c = (\varphi^1)^b (\varphi^2)^{c-b} = e_b(\varphi)$, and linearity gives the first claim; then $\langle K_\varphi, K_\varphi\rangle_c = |\varphi|^{2c}$. (c) The map $\psi \mapsto g^{\mathsf{T}}\psi$ preserves $d\psi$, so $\rho_c(g)$ is unitary, and $\rho_c(gh) = \rho_c(g)\rho_c(h)$. Since $\langle \bar{g}\varphi | \xi\rangle = \langle \varphi | g^{\mathsf{T}}\xi\rangle$, we get $\rho_c(g)K_\varphi = K_{\bar{g}\varphi}$. Differentiating $p(\exp(tX)^{\mathsf{T}}\xi)$ at $t = 0$ gives $p \mapsto \sum_{i,k} X_{ki}\xi^k \partial_{\xi^i} p$, which sends the three matrices to S^+, S^-, S^3. The action on the e_a is direct, and with (a) it gives $\langle e_{a+1}, S^+ e_a\rangle_c = (a+1)!(c-a)!/c! = \langle S^- e_{a+1}, e_a\rangle_c$, so $(S^+)^* = S^-$. The value of the Casimir operator follows from $[S^+, S^-] = 2S^3$ and $S^- S^+ e_a = (a+1)(c-a)e_a$. Each weight $a - j$ has multiplicity one, and every invariant subspace contains a vector of weight 0 or $\frac{1}{2}$, so H_c is irreducible [4, Ch. II, §5]. □

The vectors $\hat{e}_a = \binom{c}{a}^{1/2} e_a$ form an orthonormal basis with $S^3 \hat{e}_a = (a-j)\hat{e}_a$. We identify $H_c \otimes H_{c'}$ with the polynomials in (ξ, η) that are homogeneous of degree c in ξ and c' in η, with the product inner product and the representation $\rho_c \otimes \rho_{c'}$, whose spin operators are $S_0 + S_1$ with $S_0 = S \otimes 1$ and $S_1 = 1 \otimes S$. Let $\Lambda(c, c') = \{\frac{|c-c'|}{2}, \frac{|c-c'|}{2} + 1, \ldots, \frac{c+c'}{2}\}$. By the Clebsch–Gordan decomposition [4, Ch. II, §5], $H_c \otimes H_{c'}$ is the orthogonal sum of

irreducible subspaces V_L of spin L, one for each $L \in \Lambda(c,c')$. Let P_L be the orthogonal projection onto V_L, the spectral projection of $(S_0+S_1)^2$ for the eigenvalue $L(L+1)$, and put $P_L = 0$ for $L \notin \Lambda(c,c')$.

The flip. For $\varphi, \psi \in \mathbb{C}^2$ put $[\varphi,\psi] = \varphi^1\psi^2 - \varphi^2\psi^1$ and $\psi^\sharp = (-\overline{\psi^2}, \overline{\psi^1})$. Then $[\psi^\sharp, \varphi] = -\langle\psi|\varphi\rangle$ and $\langle\psi^\sharp|\varphi^\sharp\rangle = \langle\varphi|\psi\rangle$, and $\psi \mapsto \psi^\sharp$ is a real-linear isometry of $\mathbb{C}^2$, so it preserves $d\psi$. Put

$$c_0 = n_{02}+n_{03}, \qquad c_1 = n_{12}+n_{13}, \qquad c_2 = n_{02}+n_{12}, \qquad c_3 = n_{03}+n_{13},$$

let $\mathcal{H} = H_{c_0} \otimes H_{c_1} \otimes H_{c_2} \otimes H_{c_3}$, realized as polynomials in $(\xi_0,\dots,\xi_3)$, with the representation $\rho = \rho_{c_0} \otimes \cdots \otimes \rho_{c_3}$, and $K_\varphi = K_{\varphi_0} \otimes \cdots \otimes K_{\varphi_3}$, so that $\langle K_\varphi, G\rangle_{\mathcal{H}} = G(\varphi)$ for $G \in \mathcal{H}$. The polynomial

$$F(\xi_0,\xi_1,\xi_2,\xi_3) = \prod_{x\in\{0,1\},\, y\in\{2,3\}} [\xi_x,\xi_y]^{n_{xy}}$$

lies in $\mathcal{H}$, and $\rho(g)F = F$ for $g \in SU(2)$, since $[g^{\mathsf{T}}\xi, g^{\mathsf{T}}\eta] = \det(g)[\xi,\eta]$. For $i,c,c' \in \mathbb{N}$ define the positive operator on $H_c \otimes H_{c'}$

$$T^{(i)}_{c,c'} = \iint |\langle\varphi|\varphi'\rangle|^{2i}\, |K_\varphi \otimes K_{\varphi'}\rangle\langle K_\varphi \otimes K_{\varphi'}|\, d\varphi\, d\varphi'.$$

Lemma 5.25. $U(n) = \big\langle F, \big(T^{(n_{01})}_{c_0,c_1} \otimes T^{(n_{23})}_{c_2,c_3}\big)F\big\rangle_{\mathcal{H}}$.

Proof. By the flip identities, $|\langle\psi_x|\psi_y\rangle| = |[\psi_x^\sharp, \psi_y]|$ for $x \in \{0,1\}$, $y \in \{2,3\}$, and $|\langle\psi_0|\psi_1\rangle| = |\langle\psi_0^\sharp|\psi_1^\sharp\rangle|$. Substituting $\varphi_x = \psi_x^\sharp$ for $x = 0,1$ and $\varphi_x = \psi_x$ for $x = 2,3$ in the spinor form of $U(n)$ gives $U(n) = \int |\langle\varphi_0|\varphi_1\rangle|^{2n_{01}} |\langle\varphi_2|\varphi_3\rangle|^{2n_{23}} |F(\varphi)|^2 \prod_x d\varphi_x$, and $|F(\varphi)|^2 = \langle F, K_\varphi\rangle_{\mathcal{H}}\langle K_\varphi, F\rangle_{\mathcal{H}}$. □

Lemma 5.26. $T^{(i)}_{c,c'} = \sum_{L\in\Lambda(c,c')} \tau^{(i)}_{c,c'}(L)\, P_L$ *with numbers* $\tau^{(i)}_{c,c'}(L) \geq 0$.

Proof. By Lemma 5.24(c) and the invariance of the measure and of the weight under $\varphi \mapsto \bar g^{-1}\varphi$, $\varphi' \mapsto \bar g^{-1}\varphi'$, the operator commutes with $\rho_c(g) \otimes \rho_{c'}(g)$. The V_L are irreducible and pairwise inequivalent, so by Schur's lemma it acts on each V_L as a scalar, which is nonnegative because the operator is positive. □

On $\mathcal{H}$ let $P^{01}_L = P_L \otimes 1$, with P_L acting on $H_{c_0} \otimes H_{c_1}$, and $P^{23}_L = 1 \otimes P_L$, with P_L acting on $H_{c_2} \otimes H_{c_3}$.

Lemma 5.27 (pair spins on invariants)**.** *If $G \in \mathcal{H}$ satisfies $\rho(g)G = G$ for all $g \in SU(2)$, then $P^{01}_L G = P^{23}_L G$ for every L.*

Proof. Let S_x be the spin operators of the x-th factor. Invariance under the group gives invariance under its Lie algebra, so $(S_0+S_1)^\alpha G = -(S_2+S_3)^\alpha G$ for $\alpha = 1,2,3$. Put $A = (S_0+S_1)^2$ and $B = (S_2+S_3)^2$. Operators on

different factors commute, so

$$\begin{aligned} AG &= \sum_{\alpha} (S_0+S_1)^{\alpha}(S_0+S_1)^{\alpha} G = -\sum_{\alpha} (S_2+S_3)^{\alpha}(S_0+S_1)^{\alpha} G \\ &= \sum_{\alpha} (S_2+S_3)^{\alpha}(S_2+S_3)^{\alpha} G = BG. \end{aligned}$$

Since A and B commute, $A^k G = B^k G$ for all k, so $q(A)G = q(B)G$ for every polynomial q. Take q equal to 1 at $L(L+1)$ and to 0 at $L'(L'+1)$ for every other $L' \in \Lambda(c_0,c_1) \cup \Lambda(c_2,c_3)$. □

Proposition 5.28. *Let $\Lambda' = \Lambda(c_0,c_1) \cap \Lambda(c_2,c_3)$ and $s_L = \|P_L^{01} F\|_{\mathcal{H}}^2$. Then $U(n) = \sum_{L \in \Lambda'} s_L \, \tau_{c_0,c_1}^{(n_{01})}(L)\, \tau_{c_2,c_3}^{(n_{23})}(L)$, and $s_L \geq 0$ depends only on $n_{02}, n_{03}, n_{12}, n_{13}$.*

Proof. By Lemmas 5.25 and 5.26,

$$U(n) = \sum_{L,L'} \tau_{c_0,c_1}^{(n_{01})}(L) \tau_{c_2,c_3}^{(n_{23})}(L') \langle F, P_L^{01} P_{L'}^{23} F \rangle_{\mathcal{H}}.$$

The vector $P_L^{01} F$ is invariant, so Lemma 5.27 gives $P_{L'}^{23} P_L^{01} F = P_{L'}^{01} P_L^{01} F = \delta_{LL'} P_L^{01} F$. Therefore only $L = L'$ contributes, with $\langle F, P_L^{01} F \rangle = s_L$. Since also $s_L = \|P_L^{23} F\|^2$, $s_L = 0$ unless $L \in \Lambda'$. □

For $c, c' \in \mathbb{N}$, $j = c/2$, $j' = c'/2$, $L \in \Lambda(c,c')$ and $0 \leq k \leq c'$ put $w_L(k) = \|P_L(\hat{e}_c \otimes \hat{e}_k)\|^2$, a squared Clebsch–Gordan coefficient, and

$$\pi_L^{(c,c')}(u) = \sum_{k=0}^{c'} \binom{c'}{k} u^k (1-u)^{c'-k} \, w_L(k).$$

Lemma 5.29. $\tau_{c,c'}^{(i)}(L) = \dfrac{1}{2L+1} \displaystyle\int_0^1 u^i \, \pi_L^{(c,c')}(u) \, du.$

Proof. By Lemma 5.26,

$$(2L+1)\tau_{c,c'}^{(i)}(L) = \operatorname{tr}(T_{c,c'}^{(i)} P_L) = \iint |\langle \varphi | \varphi' \rangle|^{2i} \Phi(\varphi,\varphi') \, d\varphi \, d\varphi'$$

with $\Phi(\varphi,\varphi') = \|P_L(K_\varphi \otimes K_{\varphi'})\|^2$. For unit vectors $\varphi, \varphi' \in \mathbb{C}^2$ choose $g \in SU(2)$ with $\bar{g}\varphi = (1,0)$. Since $\rho_c(g) \otimes \rho_{c'}(g)$ is unitary and commutes with P_L, $\Phi(\varphi,\varphi') = \Phi((1,0), \bar{g}\varphi')$. Write $\bar{g}\varphi' = (\alpha,\beta)$; then $|\alpha| = |\langle \varphi | \varphi' \rangle|$ and $|\beta|^2 = 1 - |\alpha|^2$. Now $K_{(1,0)} = \hat{e}_c$ and $K_{(\alpha,\beta)} = \sum_k \binom{c'}{k}^{1/2} \bar{\alpha}^k \bar{\beta}^{\,c'-k} \hat{e}_k$. The vectors $\hat{e}_c \otimes \hat{e}_k$ have the distinct total weights $j + k - j'$, and P_L commutes with $S_0^3 + S_1^3$, so the vectors $P_L(\hat{e}_c \otimes \hat{e}_k)$ are pairwise orthogonal and $\Phi((1,0),(\alpha,\beta)) = \pi_L^{(c,c')}(|\alpha|^2)$. Finally $|\langle \varphi | \varphi' \rangle|^2$ is uniform on $[0,1]$ under $d\varphi \, d\varphi'$. □

Lemma 5.30 (stretched Clebsch–Gordan coefficients)**.** *Let $a = L + j - j'$, $b = L - j + j'$ and $r = j + j' - L$, which are nonnegative integers. For*

$0 \le k \le c'$,

$$w_L(k) = \begin{cases} \dfrac{(2L+1)\,c!\,b!}{(j+j'+L+1)!\,a!\,r!} \cdot \dfrac{(c'-k)!\,(a+k)!}{k!\,(b-k)!}, & k \le b, \\ 0, & k > b. \end{cases}$$

Proof. If $k > b$, the weight $j + k - j'$ exceeds L, so $P_L(\hat{e}_c \otimes \hat{e}_k) = 0$. Let $k \le b$; note $c - r = a$ and $c' - r = b$. Let $J^{\pm} = S_0^{\pm} + S_1^{\pm}$.

Highest weight vector. Let $h = [\xi, \eta]^r (\xi^1)^a (\eta^1)^b$. Since J^+ is a derivation with $J^+[\xi, \eta] = 0$ and $J^+\xi^1 = J^+\eta^1 = 0$, $J^+ h = 0$, and every monomial of h has total weight $\frac{a}{2} + \frac{b}{2} = L$. Thus h spans the highest weight space of V_L. Expanding, $h = \sum_{i=0}^{r} \binom{r}{i} (-1)^i (\xi^1)^{c-i} (\xi^2)^i (\eta^1)^{b+i} (\eta^2)^{r-i}$, and Lemma 5.24(a) gives

$$\begin{aligned} \|h\|^2 &= \sum_{i=0}^{r} \binom{r}{i}^2 \frac{(c-i)!\,i!}{c!} \cdot \frac{(b+i)!\,(r-i)!}{c'!} \\ &= \frac{(r!)^2 a!\,b!}{c!\,c'!} \sum_{i=0}^{r} \binom{c-i}{r-i} \binom{b+i}{i} = \frac{(r!)^2 a!\,b!}{c!\,c'!} \binom{c+c'-r+1}{r}, \end{aligned}$$

the last sum being the coefficient of x^r in $(1-x)^{-(a+1)}(1-x)^{-(b+1)}$. The only monomial of h of degree c in ξ^1 is the term $i = 0$, so $\langle h, \hat{e}_c \otimes e_b \rangle = \|e_b\|^2 = \binom{c'}{r}^{-1}$, and with $\hat{e}_b = \binom{c'}{r}^{1/2} e_b$ and $v_L = h/\|h\|$,

$$|\langle v_L, \hat{e}_c \otimes \hat{e}_b \rangle|^2 = \frac{\|e_b\|^2}{\|h\|^2} = \frac{c!\,(2L+1)!}{a!\,(j+j'+L+1)!}.$$

Lowering. Let $M = j + k - j'$ and $m = L - M = b - k$. The vector $(J^-)^m v_L$ spans the weight-M space of V_L, and $\|(J^-)^m v_L\|^2 = m!\,(2L)!/(2L-m)!$, from $\|J^- v\|^2 = (L+M')(L-M'+1)$ for unit vectors $v \in V_L$ of weight M'. Since $\hat{e}_c \otimes \hat{e}_k$ has weight M, $w_L(k) = |\langle (J^-)^m v_L, \hat{e}_c \otimes \hat{e}_k \rangle|^2 / \|(J^-)^m v_L\|^2$. Since $(J^-)^* = J^+$ and $S_0^+ \hat{e}_c = 0$,

$$\langle (J^-)^m v_L, \hat{e}_c \otimes \hat{e}_k \rangle = \langle v_L, (J^+)^m (\hat{e}_c \otimes \hat{e}_k) \rangle = \langle v_L, \hat{e}_c \otimes (S^+)^m \hat{e}_k \rangle.$$

In $H_{c'}$, $S^+ \hat{e}_i = \big((c'-i)(i+1)\big)^{1/2} \hat{e}_{i+1}$, so $(S^+)^m \hat{e}_k = \big((c'-k)!\,b!/(r!\,k!)\big)^{1/2} \hat{e}_b$. Hence

$$w_L(k) = \frac{(c'-k)!\,b!}{r!\,k!} \cdot \frac{(2L-m)!}{m!\,(2L)!} \cdot \frac{c!\,(2L+1)!}{a!\,(j+j'+L+1)!},$$

and $2L - m = a + k$, $m = b - k$ give the stated formula. □

This is the case $m_1 = j_1$ of Racah's formula [28], in which a single term survives.

Lemma 5.31. *Let $c, c' \in \mathbb{N}$ and $\Lambda = \Lambda(c, c')$.*

(a) $(L, k) \mapsto w_L(k)$ *is TP_2 on $\Lambda \times \{0, \dots, c'\}$.*
(b) $(L, u) \mapsto \pi_L^{(c,c')}(u)$ *is TP_2 on $\Lambda \times (0, 1)$.*
(c) $(i, L) \mapsto \tau_{c,c'}^{(i)}(L)$ *is TP_2 on $\mathbb{N} \times \Lambda$.*

Proof. (a) Write a_L, b_L for the numbers a, b of Lemma 5.30; then $a_{L+1} = a_L + 1$, $b_{L+1} = b_L + 1$, and w_L is positive exactly on $\{0, \dots, \min(b_L, c')\}$. For $L, L+1 \in \Lambda$ and k in this set, $w_{L+1}(k)/w_L(k) = \kappa_L(a_L + 1 + k)/(b_L + 1 - k)$ with $\kappa_L > 0$ independent of k, which increases with k. Let $L < L'$ and $k < k'$. If $k' > b_L$, then $w_L(k') = 0$ and the inequality is trivial. Otherwise $w_{L'}(k)/w_L(k)$ is a product of such ratios, which are positive and increasing on the support of w_L, so $w_{L'}(k')/w_L(k') \geq w_{L'}(k)/w_L(k)$. (b) The kernel $\binom{c'}{k} u^k (1-u)^{c'-k} = \binom{c'}{k} (1-u)^{c'} \big(u/(1-u)\big)^k$ is TP_2 in (k, u). Apply Lemma 5.10 with counting measure and (a). (c) The kernel u^i is TP_2 in (i, u). Apply Lemma 5.10 with Lebesgue measure on $(0,1)$ and (b). □

Proof of Theorem 5.23. Let μ' be the image of the finite measure $u_{02}^{n_{02}} u_{03}^{n_{03}} u_{12}^{n_{12}} u_{13}^{n_{13}} \prod_x \omega(d\sigma_x)$ under (u_{01}, u_{23}), and

$$Q(v, w) = \sum_{L \in \Lambda'} \frac{s_L}{(2L+1)^2} \, \pi_L^{(c_0, c_1)}(v) \, \pi_L^{(c_2, c_3)}(w).$$

By Proposition 5.28 and Lemma 5.29, for all $i, i' \in \mathbb{N}$,

$$\int v^i w^{i'} \mu'(dv, dw) = U(n)|_{n_{01}=i,\, n_{23}=i'} = \iint v^i w^{i'} Q(v, w) \, dv \, dw.$$

Q is a polynomial, nonnegative on $[0,1]^2$, and polynomials are dense in $C([0,1]^2)$, so $\mu' = Q(v,w)\, dv\, dw$: Q is the transfer density. By Lemma 5.31(b), $(v, L) \mapsto \pi_L^{(c_0,c_1)}(v)$ and $(L, w) \mapsto s_L (2L+1)^{-2} \pi_L^{(c_2,c_3)}(w)$ are TP_2 on the subchain Λ', and Lemma 5.10 with counting measure on Λ' shows that Q is TP_2 on $(0,1)^2$, hence on $[0,1]^2$ by continuity. The last statement follows from the remark after Definition 5.19. □

5.7. K_4**: adjacent edges.** Fix multiplicities $n_{03}, n_{12}, n_{13}, n_{23}$ and orient the triangle 123 cyclically, with edges $12, 23, 31$; write $n_{31} = n_{13}$. A *configuration* C assigns to each of the n_e slots of each triangle edge e a pair of bits $(\kappa, \lambda) \in \{0,1\}^2$. Let k_e and l_e be the numbers of slots of e with $\kappa = 1$ and with $\lambda = 1$, and put $f_e = k_e - l_e$ and $m_e = k_e + l_e$. The configuration is *admissible* if $f_{12} = f_{23} = f_{31}$; let $\mathcal{C}$ be the set of admissible configurations, and

$$\nu_1 = m_{12} + m_{31}, \qquad \nu_2 = m_{12} + m_{23}, \qquad \nu_3 = m_{23} + m_{31},$$
$$\Omega(\eta) = \sum_{C \in \mathcal{C}} e^{-\frac{1}{2}\langle \eta, \nu(C) \rangle}, \quad \eta \in \mathbb{R}^3.$$

Since $m_e = 2l_e + f_e$, every ν_x of an admissible configuration is even. Reversing the orientation of the triangle exchanges κ and λ in every slot, which preserves admissibility and every m_e. Thus a permutation π of $\{1,2,3\}$ induces a bijection between the admissible configurations for the multiplicities (n_e) and those for the permuted multiplicities, which carries ν_x to $\nu_{\pi(x)}$. It follows that a statement about a pair $x \neq y$ that is proved for $(x, y) = (1, 2)$ and all multiplicities holds for every pair.

For $\eta \in \mathbb{R}^3$ let $\Pr_\eta$ be the probability measure on $\mathcal{C}$ proportional to $e^{-\frac{1}{2}\langle \eta, \nu(C)\rangle}$, with covariance Cov_η.

Theorem 5.32. *For $x \neq y$ in $\{1,2,3\}$ and $S \in \mathbb{N}^3$ let*

$$c_S^{xy} = \sum_{(C,C')\in\mathcal{C}^2:\ \nu(C)+\nu(C')=S} \big(\nu_x(C) - \nu_x(C')\big)\big(\nu_y(C) - \nu_y(C')\big).$$

Then $c_S^{xy} \geq 0$. Consequently $\mathrm{Cov}_\eta(\nu_x, \nu_y) \geq 0$ for all η, and Ω is MTP$_2$ on $\mathbb{R}^3$.

Proof. By the relabelling above it suffices to treat $(x,y) = (1,2)$. Then $\Delta_1 := \nu_1(C) - \nu_1(C') = a_{12} + a_{31}$ and $\Delta_2 := \nu_2(C) - \nu_2(C') = a_{12} + a_{23}$, where $a_e = m_e(C) - m_e(C')$.

Step 1: profiles. For an ordered pair (C, C') and a slot, the *slot sum* is $(\kappa+\kappa', \lambda+\lambda') \in \{0,1,2\}^2$, where (κ,λ) and (κ',λ') are the bits of C and C' there. The *profile* ϑ of (C,C') is the family of slot sums. A *position* is a slot with one of the labels κ, λ; it is *active* if the corresponding entry of the slot sum is 1. The profile determines $\nu(C) + \nu(C')$, since $m_e(C) + m_e(C')$ is the sum of the entries of the slot sums on e. Hence $c_S^{12} = \sum_\vartheta \sum_{(C,C')\in\mathcal{A}_\vartheta} \Delta_1\Delta_2$, where ϑ runs over the profiles with $\nu(C) + \nu(C') = S$ and $\mathcal{A}_\vartheta$ is the set of pairs of admissible configurations with profile ϑ. We fix ϑ with $\mathcal{A}_\vartheta \neq \emptyset$ and show that the inner sum is nonnegative.

Step 2: parametrization. Let A be the set of active positions. A pair with profile ϑ is determined by choosing, at each active position, which of C, C' carries the 1; we encode the choice by $\varepsilon \in \{\pm 1\}^A$, with $\varepsilon = +1$ at an active κ-position if C carries the 1, and $\varepsilon = +1$ at an active λ-position if C' carries the 1. Let A_e^κ, A_e^λ be the active κ- and λ-positions on e, $A_e = A_e^\kappa \cup A_e^\lambda$, $\alpha_e = |A_e|$, and $r_e = (|A_e^\kappa| - |A_e^\lambda|)/\alpha_e$, with $r_e = 0$ if $\alpha_e = 0$. Inactive positions contribute equally to C_ε and C'_ε, so

$$h_e := f_e(C_\varepsilon) - f_e(C'_\varepsilon) = \sum_{i\in A_e} \varepsilon_i, \qquad a_e = \sum_{i\in A_e^\kappa} \varepsilon_i - \sum_{i\in A_e^\lambda} \varepsilon_i,$$

while $g_e := f_e(C_\varepsilon) + f_e(C'_\varepsilon)$ depends only on ϑ. The pair is admissible if and only if $(g_e + h_e)/2$ and $(g_e - h_e)/2$ do not depend on e. Since $\mathcal{A}_\vartheta \neq \emptyset$, g_e does not depend on e, so $(C_\varepsilon, C'_\varepsilon) \in \mathcal{A}_\vartheta$ if and only if $h_{12} = h_{23} = h_{31}$.

Step 3: probabilistic form. Let ε be uniform on $\{\pm 1\}^A$ and $\mathcal{E} = \{h_{12} = h_{23} = h_{31}\}$. Then $\sum_{\mathcal{A}_\vartheta} \Delta_1\Delta_2 = 2^{|A|} \Pr(\mathcal{E})\, \mathbb{E}[\Delta_1\Delta_2 \,|\, \mathcal{E}]$. The restrictions $\varepsilon|_{A_e}$ are independent, so conditionally on $\mathcal{E}$ the common value h of the h_e has the law proportional to $\prod_e \beta_e(h)$ with $\beta_e(h) = \binom{\alpha_e}{(\alpha_e+h)/2}$ (zero unless $\alpha_e + h$ is even and $|h| \leq \alpha_e$), and given $\mathcal{E}$ and h the restrictions $\varepsilon|_{A_e}$ are independent and uniform on the sign vectors on A_e with sum h. Below all expectations are conditional on $\mathcal{E}$.

Step 4: one edge. Fix e and condition on h. If $\alpha_e = 0$, then $a_e = 0$ and $h = 0$. If $\alpha_e = 1$, then $a_e = r_e h$ with $r_e = \pm 1$. Let $\alpha_e \geq 2$ and let P be the number of $i \in A_e^\kappa$ with $\varepsilon_i = +1$. The set $\{i \in A_e : \varepsilon_i = +1\}$ is

a uniform random subset of size $(\alpha_e + h)/2$, so P is hypergeometric with $\mathbb{E}P = |A_e^\kappa|(\alpha_e + h)/(2\alpha_e)$ and $\operatorname{Var} P = \frac{\alpha_e+h}{2} \cdot \frac{\alpha_e-h}{2} \cdot \frac{|A_e^\kappa||A_e^\lambda|}{\alpha_e^2(\alpha_e-1)}$. Since $a_e = 4P - 2|A_e^\kappa| - h$ and $1 - r_e^2 = 4|A_e^\kappa||A_e^\lambda|/\alpha_e^2$,

$$\mathbb{E}[a_e \,|\, h] = r_e h, \qquad \operatorname{Var}(a_e \,|\, h) = (1 - r_e^2)\,\frac{\alpha_e^2 - h^2}{\alpha_e - 1} \quad (\alpha_e \geq 2),$$

and $\mathbb{E}[a_e|h] = r_e h$ holds for $\alpha_e \leq 1$ as well.

Step 5: the conditional covariance. Given h, the variables a_{12}, a_{23}, a_{31} are independent, so

$$\mathbb{E}[\Delta_1\Delta_2] = \mathbb{E}\big[\operatorname{Var}(a_{12} \,|\, h)\big] + \mathbb{E}[h^2]\,(r_{12} + r_{31})(r_{12} + r_{23}).$$

If $\alpha_{12} = 0$, then $h = 0$ and both terms vanish. If $\alpha_{12} = 1$, the first term vanishes, $r_{12} = \pm 1$, and since $|r_{31}|, |r_{23}| \leq 1$ both factors $r_{12} + r_{31}$ and $r_{12} + r_{23}$ have the sign of r_{12} or vanish. Let $\alpha_{12} \geq 2$. For $r \in [-1, 1]$ the function $(z, z') \mapsto (r + z)(r + z')$ on $[-1, 1]^2$ is affine in each variable, so its minimum is among the vertex values $(r + 1)^2$, $(r - 1)^2$ and $r^2 - 1$, and $(r + z)(r + z') \geq r^2 - 1$. Hence

$$\mathbb{E}[\Delta_1\Delta_2] \geq (1 - r_{12}^2)\Big(\frac{\alpha_{12}^2 - \mathbb{E}[h^2]}{\alpha_{12} - 1} - \mathbb{E}[h^2]\Big) = (1 - r_{12}^2)\,\frac{\alpha_{12}\,(\alpha_{12} - \mathbb{E}[h^2])}{\alpha_{12} - 1}.$$

Step 6: $\mathbb{E}[h^2] \leq \alpha_{12}$. Since $\mathcal{E} \neq \emptyset$, the numbers $\alpha_{12}, \alpha_{23}, \alpha_{31}$ have the same parity. On $\mathcal{H} = \{h \in \mathbb{Z} : |h| \leq \alpha_{12},\ h \equiv \alpha_{12} \bmod 2\}$ the law of h is proportional to $\beta_{12}\phi$ with $\phi = \beta_{23}\beta_{31}$. On $\mathcal{H}$ each β_e, and hence ϕ, is an even function of h that is nonincreasing in $|h|$. Under the law proportional to β_{12}, h is a sum of α_{12} independent uniform signs, so its second moment is α_{12}. Since h^2 is nondecreasing and ϕ nonincreasing in $|h|$, Corollary 5.8 on the chain of values of $|h|$ gives $\mathbb{E}_{\beta_{12}}[h^2\phi] \leq \mathbb{E}_{\beta_{12}}[h^2]\,\mathbb{E}_{\beta_{12}}[\phi]$, that is, $\mathbb{E}[h^2] \leq \alpha_{12}$. Steps 5 and 6 give $\mathbb{E}[\Delta_1\Delta_2 \,|\, \mathcal{E}] \geq 0$, so $c_S^{12} \geq 0$.

The consequences. Symmetrizing in C and C',

$$\operatorname{Cov}_\eta(\nu_x, \nu_y) = \frac{1}{2\Omega(\eta)^2} \sum_S c_S^{xy}\, e^{-\frac{1}{2}\langle \eta, S\rangle} \geq 0.$$

Since $\partial_x\partial_y \log \Omega = \frac{1}{4} \operatorname{Cov}_\eta(\nu_x, \nu_y)$, Lemma 5.11(b) shows that Ω is MTP_2. □

Lemma 5.33 (adjacent edges)**.** *The pair* $(01, 02)$ *of* K_4*, with multiplicities* $n_{03}, n_{12}, n_{13}, n_{23}$ *on the other edges, has a transfer density that is a polynomial and is* TP_2 *on* $[0, 1]^2$*.*

Proof. Apply Lemma 5.16 with $o = 0$ and multiplicity 0 on 01 and 02. The graph $K_4 - 0$ is the triangle 123, and with the cyclic orientation $(k, l) \in \mathcal{S}$ if and only if $k_{12} - l_{12} = k_{23} - l_{23} = k_{31} - l_{31}$. The number of configurations with given (k, l) is $\prod_e \binom{n_e}{k_e}\binom{n_e}{l_e}$, and $2b_x = \nu_x$. With $w_x = u_{0x}$, $d_1 = n_{12} + n_{13}$,

$d_2 = n_{12} + n_{23}$, $d_3 = n_{03} + n_{13} + n_{23}$ and $\ell(\eta) = (1 + e^{-\eta})^{-1}$,

$$\Psi(w) = \sum_{C \in \mathcal{C}} \prod_{x=1}^{3} w_x^{d_x - \nu_x/2} (1 - w_x)^{\nu_x/2} = \prod_{x=1}^{3} w_x^{d_x}\ \Omega(\eta) \qquad \text{for } w_x = \ell(\eta_x),$$

since $(1 - w_x)/w_x = e^{-\eta_x}$. Thus (u_{01}, u_{02}) has the polynomial transfer density $D(w_1, w_2) = \int_0^1 \Psi(w)\, dw_3$. By Theorem 5.32 and the elementary facts, $\eta \mapsto \prod_x \ell(\eta_x)^{d_x} \Omega(\eta)$ is MTP_2 on $\mathbb{R}^3$. Composing with the increasing maps $w_x \mapsto \ell^{-1}(w_x)$ shows that Ψ is MTP_2 on $(0,1)^3$, and so is $(w_1, w_2, \eta_3) \mapsto \Psi(w_1, w_2, \ell(\eta_3))\, \ell'(\eta_3)$ on $(0,1)^2 \times \mathbb{R}$. Its sections in η_3 are integrable with integrals $D(w_1, w_2)$, so Corollary 5.9 shows that D is TP_2 on $(0,1)^2$, hence on $[0,1]^2$ by continuity. □

Corollary 5.34. *Let H be a cycle or K_4, and let $e \neq f$ be edges of H. For all multiplicities on the other edges, the transfer density of (e, f) exists, is a polynomial and is TP_2 on $[0,1]^2$. Moreover U_H is MTP_2.*

Proof. For cycles this is Lemma 5.22. Two distinct edges of K_4 are opposite or adjacent, and a permutation of the vertices maps them to $(01, 23)$ or to $(01, 02)$, so Theorem 5.23 and Lemma 5.33 apply. The MTP_2 property follows as in Lemma 5.22(c). □

5.8. **Chains and trees.** Graphs in this subsection are simple unless stated otherwise. A *cycle* is a connected graph with at least three vertices in which every vertex has degree two. For subgraphs G_1, G_2 of a graph, $G_1 \cup G_2$ and $G_1 \cap G_2$ have the unions and intersections of the vertex sets and of the edge sets.

Definition 5.35. A *K_4-cycle decomposition* of a simple graph G is a finite tree $\mathcal{T}$ together with subgraphs $H_\mu \subseteq G$, $\mu \in V(\mathcal{T})$, each a cycle or a K_4, such that: (D1) $G = \bigcup_\mu H_\mu$; (D2) for every edge $\mu\nu$ of $\mathcal{T}$, $H_\mu \cap H_\nu$ consists of one edge and its two ends; (D3) if λ lies on the path of $\mathcal{T}$ between μ and ν, then $H_\mu \cap H_\nu \subseteq H_\lambda$. The H_μ are the *pieces*.

Lemma 5.36 (splitting)**.** *Let $(\mathcal{T}, (H_\mu))$ be a K_4-cycle decomposition of G, $\mu\nu$ an edge of $\mathcal{T}$, and $\mathcal{T}_\mu, \mathcal{T}_\nu$ the components of $\mathcal{T} - \mu\nu$ containing μ and ν. Put $G_\mu = \bigcup_{\lambda \in \mathcal{T}_\mu} H_\lambda$ and $G_\nu = \bigcup_{\lambda \in \mathcal{T}_\nu} H_\lambda$. Then $G = G_\mu \cup G_\nu$, $G_\mu \cap G_\nu = H_\mu \cap H_\nu$, and the restrictions of the decomposition to $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ are K_4-cycle decompositions of G_μ and G_ν.*

Proof. The first identity is (D1). For $\lambda \in \mathcal{T}_\mu$ and $\lambda' \in \mathcal{T}_\nu$ the path from λ to λ' passes through μ and ν, so (D3) gives $H_\lambda \cap H_{\lambda'} \subseteq H_\mu \cap H_\nu$; taking unions gives $G_\mu \cap G_\nu \subseteq H_\mu \cap H_\nu$, and the reverse inclusion is clear. Paths in a subtree are paths in $\mathcal{T}$, so (D2) and (D3) hold for the restrictions. □

Lemma 5.37 (chains)**.** *Let $H_0, \dots, H_k$, $k \geq 0$, be cycles or copies of K_4 with $G = H_0 \cup \cdots \cup H_k$, such that for $1 \leq i \leq k$ the graph $(H_0 \cup \cdots \cup H_{i-1}) \cap H_i$ consists of an edge $f_i \in E(H_{i-1})$ and its ends, and $f_i \neq f_{i+1}$ for $1 \leq i < k$. Let $c \in E(H_0)$ and $d \in E(H_k)$ be distinct, with $c \neq f_1$ and $d \neq f_k$ if*

$k \geq 1$. Then for every $n \in \mathbb{N}^{E(G)}$ with $n_c = n_d = 0$ the transfer density $D^G_{c,d}[n]$ exists, is a polynomial and is TP_2 on $[0,1]^2$. Consequently $(i,j) \mapsto U_G(n + i\mathbf{1}_c + j\mathbf{1}_d)$ is TP_2 on $\mathbb{N}^2$.

Proof. Induction on k; for $k = 0$ this is Corollary 5.34. Let $k \geq 1$ and $R = H_1 \cup \cdots \cup H_k$.

First, $H_0 \cap R$ consists of f_1 and its ends. Let p be a vertex or an edge of H_0 that lies in R, and let $j \geq 1$ be minimal with $p \in H_j$. If $j \geq 2$, then $p \in (H_0 \cup \cdots \cup H_{j-1}) \cap H_j$, so p is f_j or one of its ends, and $p \in H_{j-1}$ since $f_j \in E(H_{j-1})$, contradicting minimality. So $j = 1$ and $p \in H_0 \cap H_1$.

Next, the graphs $H_1, \ldots, H_k$, the edges $f_2, \ldots, f_k$ and the pair (f_1, d) satisfy the hypotheses. For $2 \leq i \leq k$, $(H_1 \cup \cdots \cup H_{i-1}) \cap H_i$ is contained in $(H_0 \cup \cdots \cup H_{i-1}) \cap H_i$ and contains f_i and its ends. The edge f_1 lies in H_1, and $f_1 \neq f_2$ if $k \geq 2$. Also $f_1 \neq d$: for $k = 1$ this is $d \neq f_k$; for $k \geq 2$, if $f_1 \in E(H_j)$ with $j \geq 2$ minimal, then $f_1 \in (H_0 \cup \cdots \cup H_{j-1}) \cap H_j$, so $f_1 = f_j \in E(H_{j-1})$, which contradicts minimality for $j \geq 3$ and $f_1 \neq f_2$ for $j = 2$; so $f_1 \notin E(H_k)$. Finally $c \notin E(R)$, since $E(H_0) \cap E(R) = \{f_1\}$ and $c \neq f_1$.

By induction (f_1, d) has a polynomial TP_2 transfer density B in R, and by Corollary 5.34 (c, f_1) has one, A, in H_0, both with the multiplicities n on the other edges. Lemma 5.20 with $G_1 = H_0$, $G_2 = R$ and $W_{f_1}(w) = w^{n_{f_1}}$ shows that $D^G_{c,d}[n](v,v') = \int_0^1 A(v,w)\, w^{n_{f_1}} B(w,v')\, dw$ is a polynomial and is TP_2. The last statement follows from the remark after Definition 5.19. □

Lemma 5.38 (hanging parts)**.** *Let $G = G' \cup X$ be a graph, not necessarily simple, with $E(G') \cap E(X) = \emptyset$ and $V(G') \cap V(X) = \{x, y\}$, $x \neq y$, and let G'' be G' with a new edge f joining x and y. For $n = (n', n^X) \in \mathbb{N}^{E(G')} \times \mathbb{N}^{E(X)}$ write $K_X(n^X; u) = \sum_{m=0}^{M} \kappa_m u^m$ for the kernel of X with terminals x, y. Then $U_G(n', n^X) = \sum_{m=0}^{M} \kappa_m U_{G''}(n', m)$. If $U_{G''}$ is MTP_2, then $n' \mapsto U_G(n', n^X)$ is MTP_2 for each n^X.*

Proof. Integrating first over the spins of $V(X) \setminus \{x, y\}$ gives $U_G(n', n^X) = \int \prod_{e \in E(G')} u_e^{n'_e} K_X(n^X; u_{xy}) \prod_{z \in V(G')} \omega(d\sigma_z)$, which is the identity. By Proposition 5.18, $\kappa_m \geq 0$. If $U_{G''}$ is MTP_2, then so is $(n', m) \mapsto \kappa_m U_{G''}(n', m)$ on $\mathbb{N}^{E(G')} \times \{0, \ldots, M\}$, and Corollary 5.9 with the finite chain $\{0, \ldots, M\}$ shows that the sum over m is MTP_2 in n'. □

Theorem 5.39. *If a simple graph G has a K_4-cycle decomposition, then U_G is MTP_2 on $\mathbb{N}^{E(G)}$.*

Proof. Induction on the number of nodes of $\mathcal{T}$. By Lemma 5.11(a) it suffices to show that for distinct edges c, d and every n with $n_c = n_d = 0$, the function $(i,j) \mapsto U_G(n + i\mathbf{1}_c + j\mathbf{1}_d)$ is TP_2. Choose nodes α, β with $c \in E(H_\alpha)$ and $d \in E(H_\beta)$ at minimal distance in $\mathcal{T}$, and let P be the path between them.

Case A: $\mathcal{T}$ has a node outside P. Then some edge $\mu\nu$ of $\mathcal{T}$ has $\mu \in P$ and $\nu \notin P$. With the notation of Lemma 5.36, let f be the edge of $H_\mu \cap H_\nu$,

with ends x, y. Then $P \subseteq \mathcal{T}_\mu$ and $c, d \in E(G_\mu)$. Put $X = G_\nu - f$; then $G = G_\mu \cup X$, $E(G_\mu) \cap E(X) = \emptyset$ and $V(G_\mu) \cap V(X) = \{x, y\}$. By induction U_{G_μ} is MTP_2; adding an edge parallel to f preserves this by Lemma 5.14(c); and Lemma 5.38 with $G' = G_\mu$ shows that U_G is MTP_2 in the coordinates of $E(G_\mu)$, in particular TP_2 in (n_c, n_d).

Case B: $\mathcal{T} = P$. Write $P = \mu_0 \cdots \mu_k$ with $\mu_0 = \alpha$, $\mu_k = \beta$. If $k = 0$, then $G = H_\alpha$ and Corollary 5.34 applies. Let $k \geq 1$ and let f_i be the edge of $H_{\mu_{i-1}} \cap H_{\mu_i}$.

Case B1: $f_i = f_{i+1}$ for some $1 \leq i < k$. Let x, y be the ends of f_i, $G' = \bigcup_{j \neq i} H_{\mu_j}$ and $X = H_{\mu_i} - f_i$. Lemma 5.36 for the edges $\mu_{i-1}\mu_i$ and $\mu_i\mu_{i+1}$ gives $G' \cap H_{\mu_i} = (H_{\mu_{i-1}} \cap H_{\mu_i}) \cup (H_{\mu_i} \cap H_{\mu_{i+1}})$, which consists of f_i and its ends. Thus $E(G') \cap E(X) = \emptyset$ and $V(G') \cap V(X) = \{x, y\}$. The path $\mu_0 \cdots \mu_{i-1}\mu_{i+1} \cdots \mu_k$ with the pieces H_{μ_j}, $j \neq i$, is a K_4-cycle decomposition of G': (D2) for the new edge $\mu_{i-1}\mu_{i+1}$ holds since $H_{\mu_{i-1}} \cap H_{\mu_{i+1}} \subseteq H_{\mu_{i-1}} \cap H_{\mu_i}$ by (D3) and both pieces contain $f_i = f_{i+1}$, and (D1), (D3) are inherited. Since $1 \leq i < k$, c and d lie in G'. By induction $U_{G'}$ is MTP_2, and Lemmas 5.14(c) and 5.38 show that U_G is TP_2 in (n_c, n_d).

Case B2: $f_i \neq f_{i+1}$ for $1 \leq i < k$. By the minimality of the distance, c lies in no H_{μ_i} with $i \geq 1$ and d in no H_{μ_i} with $i < k$; in particular $c \neq f_1$ and $d \neq f_k$. For $1 \leq i \leq k$, Lemma 5.36 applied to the decomposition of $H_{\mu_0} \cup \cdots \cup H_{\mu_i}$ by the path $\mu_0 \cdots \mu_i$ gives $(H_{\mu_0} \cup \cdots \cup H_{\mu_{i-1}}) \cap H_{\mu_i} = H_{\mu_{i-1}} \cap H_{\mu_i}$, which consists of $f_i \in E(H_{\mu_{i-1}})$ and its ends. Lemma 5.37 applies. □

5.9. **Graphs without a W_4 minor.** A graph is *simple* if it has no parallel edges; its *simplification* replaces each class of parallel edges by one edge and has the same simple minors. A graph is 2-*connected* if it is connected, has at least three vertices and has no cut vertex. A *block* is a maximal connected subgraph without a cut vertex. Each block is an isolated vertex, a set of parallel edges between two vertices, or a 2-connected subgraph; each edge lies in exactly one block, and two blocks share at most one vertex [9, Section 3.1]. For a simple graph H, H is a minor of G if and only if there are disjoint nonempty sets $B_h \subseteq V(G)$, $h \in V(H)$, each inducing a connected subgraph, such that for each edge hh' of H some edge of G joins B_h and $B_{h'}$ [9, Section 1.7]. We call them *branch sets*.

Lemma 5.40. *A simple 3-connected graph without a W_4 minor is K_4.*

Proof. Let G be simple and 3-connected. Then $|V(G)| \geq 4$ and G has minimum degree at least 3; if $|V(G)| = 4$, then $G = K_4$. Let $|V(G)| \geq 5$. Every 3-connected simple graph other than K_4 has an edge whose contraction, followed by the deletion of parallel edges, leaves a 3-connected graph [9, Lemma 3.2.1], [34]. Repeating this gives a 3-connected simple minor G_5 of G with five vertices. The minimum degree of G_5 is at least 3, so its complement is a matching with at most two edges. If the complement has

two edges 13 and 24, then G_5 is W_4 with hub 5 and rim 1234. Otherwise G_5 is K_5 or K_5 minus an edge, and both contain W_4 as a subgraph. □

Lemma 5.41. *Let H be a simple 3-connected graph.*

(a) *If $G = G_1 \cup G_2$, where $G_1 \cap G_2$ consists of an edge f and its ends, and H is a minor of G, then H is a minor of G_1 or of G_2.*
(b) *If $G = G_1 \cup G_2$ with $E(G_1) \cap E(G_2) = \emptyset$ and $|V(G_1) \cap V(G_2)| \leq 1$, and H is a minor of G, then H is a minor of G_1 or of G_2.*

Proof. (a) Let x, y be the ends of f, $A_i = V(G_i) \setminus \{x, y\}$, and (B_h) branch sets of H in G. No edge joins A_1 and A_2. Let S be the set of h with $B_h \cap \{x, y\} \neq \emptyset$, so $|S| \leq 2$. For $h \notin S$, B_h is connected in $G - \{x, y\}$, so it lies in A_1 or in A_2. The graph $H - S$ is connected and nonempty, since H is 3-connected with at least four vertices, and an edge hh' of $H - S$ requires an edge between B_h and $B_{h'}$. Therefore all B_h with $h \notin S$ lie in a common A_i, say A_1. Put $B'_h = B_h \cap V(G_1)$. For $h \in S$, B'_h contains a vertex of $\{x, y\}$, and it is connected in G_1: a path in B_h between two vertices of B'_h leaves G_1 only through $\{x, y\} \cap B_h$, and each excursion into A_2 can be removed if it returns to its starting vertex or replaced by f if it joins x and y. Let hh' be an edge of H. If $h \notin S$, an edge of G joins $B_h \subseteq A_1$ to $B_{h'}$; it lies in G_1, and its other end lies in $B'_{h'}$. If $h, h' \in S$, then B_h and $B_{h'}$ contain one of x, y each, and f joins B'_h and $B'_{h'}$. Therefore (B'_h) are branch sets of H in G_1.

(b) The same argument with $\{x, y\}$ replaced by $V(G_1) \cap V(G_2)$ applies. Now $|S| \leq 1$, and since $H - S$ is connected with at least three vertices, every edge of H meets $V(H) \setminus S$, so no edge between two branch sets that meet $V(G_1) \cap V(G_2)$ is needed. □

Proposition 5.42. *For a simple 2-connected graph G the following are equivalent: (a) G has no W_4 minor; (b) G is a spanning subgraph of a simple graph that has a K_4-cycle decomposition. Consequently a graph has no W_4 minor if and only if the simplification of each of its blocks with at least three vertices satisfies (b).*

Proof. (a)⇒(b). Induction on $|E(G)|$. If G is 3-connected, $G = K_4$ by Lemma 5.40. If $|V(G)| = 3$, G is a triangle. Otherwise G has a separating set $\{x, y\}$. Let C_1 be a component of $G - \{x, y\}$ and C_2 the union of the others. For $i = 1, 2$ let G_i consist of C_i, the vertices x, y and the edges with an end in C_i. Let e_0 be the edge xy if it exists in G, and a new edge joining x and y otherwise, and put $G'_i = G_i + e_0$, so that $G'_1 \cap G'_2$ consists of e_0 and its ends. Since G is 2-connected, each component of $G - \{x, y\}$ has neighbours at x and at y, so G'_i is 2-connected: removing x or y leaves C_i attached to the other one, and after removing a vertex z of C_i every other vertex of C_i is joined to x or y in $G_i - z$ by the initial segment of a path in $G - z$ to C_{3-i}, while x, y are joined by e_0. Moreover G_{3-i} contains a path from x to y with at least two edges and interior in C_{3-i}; contracting it to one edge and deleting the rest of G_{3-i} shows that G'_i is a minor of G, so it has no

W_4 minor, and $|E(G_i')| < |E(G)|$. By induction G_i' is a spanning subgraph of a simple graph G_i^+ with a K_4-cycle decomposition $(\mathcal{T}_i, (H_\mu^i))$, the added edges of G_1^+ and of G_2^+ being distinct. Then $G_1^+ \cap G_2^+$ consists of e_0 and its ends, since a common edge has both ends in $\{x, y\}$. Choose pieces $H_{\mu_1}^1$ and $H_{\mu_2}^2$ containing e_0 and join $\mathcal{T}_1$ and $\mathcal{T}_2$ by the edge $\mu_1\mu_2$. (D2) holds for it, because $H_{\mu_1}^1 \cap H_{\mu_2}^2 \subseteq G_1^+ \cap G_2^+$ and both pieces contain e_0. For (D3), let $\mu \in \mathcal{T}_1$, $\nu \in \mathcal{T}_2$ and λ on the path between them. Each element of $H_\mu^1 \cap H_\nu^2$ is one of x, y, e_0, and it lies in $H_\mu^1 \cap H_{\mu_1}^1$ and in $H_{\mu_2}^2 \cap H_\nu^2$; by (D3) in $\mathcal{T}_1$ and in $\mathcal{T}_2$ it lies in H_λ. Thus G is a spanning subgraph of $G_1^+ \cup G_2^+$, which has a K_4-cycle decomposition.

(b)⇒(a). A minor of a subgraph is a minor, so it suffices to show that a graph with a K_4-cycle decomposition has no W_4 minor. We use induction on the number of pieces. A cycle has no minor with a vertex of degree 3, and K_4 has four vertices, so neither has a W_4 minor. With at least two pieces, by Lemma 5.36 the graph is $G_\mu \cup G_\nu$ with $G_\mu \cap G_\nu$ an edge with its ends, and Lemma 5.41(a) applies, since W_4 is simple and 3-connected.

The last statement follows from Lemma 5.41(b), by which a graph has a W_4 minor if and only if one of its blocks has one, since blocks with at most two vertices have no W_4 minor and a graph and its simplification have the same simple minors. □

5.10. **Proofs of the theorems.**

Proof of Theorem 5.3. Let G have no W_4 minor. By Lemma 5.14(c), applied repeatedly, it suffices to treat the simplification of G, which has the same simple minors. By Lemma 5.14(d) it suffices to treat its blocks. A block with at most two vertices has at most one edge, and its moment array $n \mapsto 1/(n+1)$, or the constant 1, is MTP_2. A block B with at least three vertices is 2-connected and has no W_4 minor. By Proposition 5.42 it is a spanning subgraph of a simple graph B^+ with a K_4-cycle decomposition, U_{B^+} is MTP_2 by Theorem 5.39, and so is U_B by Lemma 5.14(a). □

Proof of Theorem 5.2. Let $\beta_1, \dots, \beta_r$ be the bridges of Γ_0 in Γ. They are edge-disjoint from each other and from Γ_0, two of them share only vertices of attachment, every vertex of Γ outside $V(\Gamma_0)$ lies in exactly one bridge, and every edge of Γ lies in Γ_0 or in exactly one bridge. Integrate the density $\prod_{e \in E(\Gamma)} W_e(s_e)$ over the spins of the vertices of β_j outside $V(\Gamma_0)$, one bridge at a time. If β_j has at most one vertex of attachment, the result is a positive constant by rotation invariance. If it has two vertices of attachment x_j, y_j, the result is the kernel of β_j with terminals x_j, y_j, which lies in $\mathcal{P}$ as a function of $\sigma_{x_j} \cdot \sigma_{y_j}$ by Proposition 5.4. Thus the law of $(\sigma_x)_{x \in V(\Gamma_0)}$ under $\mu_{\Gamma,W}$ is $\mu_{\Gamma_0^+, W^+}$, where W^+ agrees with W on $E(\Gamma_0)$ and is the kernel of β_j on the new edge of β_j. The observables s_{ab} and s_{cd} are functions of these spins. By Theorem 5.3, $U_{\Gamma_0^+}$ is MTP_2, and Lemma 5.15 gives $\mathrm{Cov}(u_{ab}, u_{cd}) \geq 0$. Since $s = 2u - 1$, the claim follows. □

Proof of Theorem 5.1 and of Theorem 1.3. Apply Theorem 5.2 to $\Gamma = G \cup \{ab, cd\}$, with weight $1 \in \mathcal{P}$ on the added edges, and $\Gamma_0 = \Gamma$. There are no bridges, so $\Gamma_0^+ = \Gamma$, and the added edges do not change $\mu_{G,W}$. Theorem 1.3 is the case $W_e(t) = e^{J_e t}$, $J_e \geq 0$; for $a = b$ or $c = d$ the covariance in (1) vanishes. □

Corollary 5.43. *Let G be a finite graph with couplings $J_e \geq 0$, and let $a \neq b$ be vertices such that $G \cup \{ab\}$ has no W_4 minor. Then $\langle \sigma_a \cdot \sigma_b \rangle_{G,J}$ is nondecreasing in each coupling J_e on $[0, \infty)^E$.*

Proof. For an edge e with ends c, d, $G \cup \{ab, cd\} = G \cup \{ab\}$, and $\partial_{J_e} \langle \sigma_a \cdot \sigma_b \rangle = \mathrm{Cov}(\sigma_a \cdot \sigma_b, s_e) \geq 0$ by Theorem 5.1. □

5.11. **Limits of the method.** The proof needs total positivity of transfer densities across every piece between the two pairs. Already for W_4 this fails.

Proposition 5.44 (†)**.** *Let W_4 have hub 0 and rim 1234. Give the rim edges $12, 23, 34, 41$ the multiplicities $2, 3, 2, 3$, the spoke 02 the multiplicity 0 and the spoke 04 the multiplicity 1, and let $\varrho(x, y)$ be the transfer density of the spokes $(01, 03)$. Then ϱ is a polynomial of degree 5 in x and 4 in y, with*

$$\varrho(0, y) = \tfrac{1}{252} - \tfrac{2}{315} y + \tfrac{1}{525} y^2 + \tfrac{1}{1050} y^3 - \tfrac{1}{2800} y^4,$$
$$\partial_x \varrho(0, y) = -\tfrac{1}{420} + \tfrac{29}{525} y - \tfrac{59}{700} y^2 + \tfrac{13}{350} y^3 - \tfrac{11}{2800} y^4.$$

In particular $\varrho(0, 1) = \frac{1}{8400}$ and $\partial_x \partial_y \log \varrho(0, 1) = -8$, so ϱ is not TP_2; for example $\varrho(0, \frac{99}{100})\, \varrho(\frac{1}{100}, 1) - \varrho(0, 1)\, \varrho(\frac{1}{100}, \frac{99}{100}) = -\frac{10524303486251}{3024000000000000000000000000}$. Likewise, with multiplicities 1 and 4 on the rim edges 23 and 34 and $4, 4, 0, 2$ on the spokes $01, 02, 03, 04$, the transfer density ϱ' of the rim edges $(12, 41)$ has $\varrho'(0, 1) = \frac{1}{69300}$ and $\partial_x \partial_y \log \varrho'(0, 1) = -\frac{32}{3}$.

Proof. By Lemma 5.16 with $o = 0$, ϱ is a polynomial, obtained by integrating out the latitudes of the rim vertices 2 and 4. It is computed in exact rational arithmetic in this way, and independently by inverting the moments $\iint x^i y^j \varrho = U_{W_4}(n + i\mathbf{1}_{01} + j\mathbf{1}_{03})$, $i, j \leq 5$, which determine a polynomial of degree at most 5 in each variable. From the displayed rows, $\partial_y \varrho(0, 1) = -\frac{1}{900}$, $\partial_x \varrho(0, 1) = \frac{1}{560}$ and $\partial_x \partial_y \varrho(0, 1) = -\frac{37}{2100}$, so $\varrho\, \partial_x \partial_y \varrho - \partial_x \varrho\, \partial_y \varrho = -8\varrho^2 < 0$ at $(0, 1)$. A positive TP_2 function of class C^2 on $(0, 1)^2$ has $\partial_x \partial_y \log \varrho \geq 0$, so by continuity ϱ is not TP_2. The same two computations, with the rim vertex 1 frozen, give ϱ'. □

Therefore the argument of Lemma 5.37 cannot pass through a W_4 piece, between two spokes or between two adjacent rim edges. Whether U_{W_4} itself is MTP_2 is open. A positive answer would give the inequality on W_4 for all weights in $\mathcal{P}$, by Lemma 5.15, but by Proposition 5.44 it would not by itself reach graphs in which a W_4 piece lies between the two pairs.

The transfer densities above are taken with monomial weights. For a triangle xzy whose edge zy carries a weight $g(u) = \sum_M g_M u^M$ in $\mathcal{P}$, the proof of Lemma 5.22(a) shows that the transfer density of (xz, xy) is $\tau_g = \sum_M g_M \tau_M$, and sums of TP_2 kernels need not be TP_2.

Remark 5.45 (†). Let $g(u) = \sum_{m=0}^{40}(4u)^m/m!$, the truncation of e^2e^{2t}. Then $\tau_g(\frac{97}{100},\frac{1}{100})\,\tau_g(\frac{99}{100},\frac{3}{100}) - \tau_g(\frac{97}{100},\frac{3}{100})\,\tau_g(\frac{99}{100},\frac{1}{100}) < 0$, with relative size -2.66×10^{-3}, while the corresponding minors of every τ_M are nonnegative by Lemma 5.21. The same holds for the exponential weight e^2e^{2t} itself: on $[0,1]^2$ we have $0 \le \tau_M \le 1$ and $\tau_g \ge g_0\tau_0 = 1$, and by hand estimates the omitted tail $\sum_{m>40} 4^m/m!$ is below 10^{-24}, so the minor changes by less than 10^{-22}, while its absolute value exceeds 2.6×10^{-3}.

This negative minor is why Lemma 5.38 expands every hanging part into monomials, and why Theorem 5.39 proves the MTP_2 property in all coordinates rather than total positivity in the two observed ones. The argument uses the Hopf map $S^3 \to S^2$ throughout, and it does not apply to other values of N.

6. Uniform coupling on $3 \times n$ grids

In this section all couplings equal J and $N \in \{3,4\}$. We label the vertices of $P_3 \times P_3$ by the hub C, the side midpoints $\mathrm{N},\mathrm{E},\mathrm{S},\mathrm{W}$ and the corners $\mathrm{NE},\mathrm{SE},\mathrm{SW},\mathrm{NW}$. The rim edges are $b_0 = \mathrm{S\,SE}$, $b_1 = \mathrm{SE\,E}$, $b_2 = \mathrm{E\,NE}$, $b_3 = \mathrm{NE\,N}$, $b_4 = \mathrm{N\,NW}$, $b_5 = \mathrm{NW\,W}$, $b_6 = \mathrm{W\,SW}$, $b_7 = \mathrm{SW\,S}$, and the spokes are $\mathsf{s}_X = CX$ for $X \in \{\mathrm{N},\mathrm{E},\mathrm{S},\mathrm{W}\}$.

6.1. **Mirror pairs.**

Lemma 6.1. *Let θ be a reflection of the box $P_m \times P_n$ in a horizontal or vertical line, or, if $m = n$, in a diagonal, and let the couplings be θ-invariant. Then $\mathrm{Cov}(s_e, s_{\theta e}) \ge 0$ for every edge e, for every $N \ge 1$.*

Proof. If the reflection line passes through vertices, no edge joins the two open half-boxes. Split every edge that lies on the line into two copies with half its coupling, one in each closed half; then the weight factorizes as $e^{H_+}\,\theta(e^{H_+})$ with H_+ depending only on the spins in one closed half. Thus $\langle F\,\theta F\rangle \ge 0$ for every function F of those spins, which is reflection positivity [13]. If the line passes between two rows, the crossing bonds have the form $J\sigma_x\cdot\sigma_{\theta x}$, and reflection positivity holds by the standard expansion of $e^{J\sigma_x\cdot\sigma_{\theta x}}$ into products of components [13]. Apply it to $F = s_e - \langle s_e\rangle$, using $\langle s_e\rangle = \langle s_{\theta e}\rangle$. □

6.2. **The corner identity.**

Lemma 6.2. *Let v be a vertex of degree two with neighbours x, y, and let $J_{vx} = J_{vy}$. Then $\mathrm{Cov}(s_{vx}, F) = \mathrm{Cov}(s_{vy}, F)$ for every bounded F that does not depend on σ_v. This holds for every $N \ge 2$ and every finite graph.*

Proof. Given all spins except σ_v, the law of σ_v has density $\propto \exp(J\sigma_v\cdot(\sigma_x+\sigma_y))$. The reflection of $\mathbb{R}^N$ that exchanges σ_x and σ_y fixes $\sigma_x+\sigma_y$, so it preserves this law and exchanges $\sigma_v\cdot\sigma_x$ and $\sigma_v\cdot\sigma_y$. It follows that $\mathbb{E}[s_{vx} \mid \sigma_{V\setminus v}] = \mathbb{E}[s_{vy} \mid \sigma_{V\setminus v}]$, and $\mathrm{Cov}(s_{vx},F) = \mathrm{Cov}(s_{vy},F)$. □

6.3. **The transfer operator.** Write a column of $P_3 \times P_n$ as $(\sigma_T, \sigma_M, \sigma_B)$ and put $X = \sigma_T \cdot \sigma_M$, $Y = \sigma_M \cdot \sigma_B$. Let $K_1 = \sum_\ell k_\ell(J)\Pi_\ell$ be the bond operator of (3), $K = K_1^{\otimes 3}$ the three horizontal bonds between consecutive columns, and V multiplication by $e^{J(X+Y)}$. The operator $T = K^{1/2}VK^{1/2}$ is positive and trace class on $L^2(\omega^{\otimes 3})$, and it preserves the subspace $\mathcal{H}$ of functions invariant under the diagonal action of $\mathrm{O}(N)$. This subspace has an orthonormal basis e_s indexed by the admissible label triples $s = (t, h, b)$ of the three horizontal bonds across a cut, with $e_s \in H_t \otimes H_h \otimes H_b$. In this basis $T(s, s') = (k_s k_{s'})^{1/2}\langle e_s | e^{J(X+Y)} | e_{s'}\rangle$ with $k_s = k_t k_h k_b$, and the matrix elements are rational combinations of recoupling coefficients. With $\mathbf{b} = K^{1/2}V\mathbf{1}$,

$$Z(P_3 \times P_n) = \langle \mathbf{b},\, T^{n-2}\mathbf{b}\rangle.$$

A horizontal bond of row $\tau \in \{T, M, B\}$ at a cut acts by $R_\tau = \sum_c r_c(J)\Pi_c^{(\tau)}$, the diagonal multiplication by $r_c(J)$ on its label. A vertical bond in a column replaces one factor T by $T_X = K^{1/2}XVK^{1/2}$ or $T_Y = K^{1/2}YVK^{1/2}$.

Let e lie to the left of f, with $m \geq 0$ free transfer steps between them. For a horizontal edge e of row τ at distance i from the left end put $L^1 = T^i\mathbf{b}$ and $L^E = R_\tau T^i\mathbf{b}$. For a vertical edge in column j put $L^1 = T^j\mathbf{b}$ and $L^E = T_X T^{j-1}\mathbf{b}$ (or T_Y). Define R^1, R^E for f in the mirror way. Then $Z\langle s_e s_f\rangle = \langle L^E, T^m R^E\rangle$, and the covariance is a 2×2 determinant:

$$\mathrm{Cov}(s_e, s_f)\, Z^2 = \langle L^E, T^m R^E\rangle\langle L^1, T^m R^1\rangle - \langle L^E, T^m R^1\rangle\langle L^1, T^m R^E\rangle.$$

Let $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$ be the eigenvalues of T on $\mathcal{H}$ and φ_l unit eigenvectors, with φ_1 positive. Put $u_l^X = \langle \varphi_l, L^X\rangle$ and

$$\varepsilon_L = \frac{L^1 - u_1^1\varphi_1}{u_1^1}, \qquad c_e = \frac{u_1^E}{u_1^1}, \qquad \tilde{u} = \frac{L^E - c_e L^1}{u_1^1}, \qquad \hat{a}_l = \langle \varphi_l, \tilde{u}\rangle,$$

so that $\tilde{u} \perp \varphi_1$. Define ε_R, $\tilde{v}$ and $\hat{b}_l$ on the right in the same way. The reflection of the column that exchanges top and bottom commutes with T, and each eigenvector is even or odd under it.

Lemma 6.3. *With $\mathcal{N} = Z/(u_1^1 v_1^1 \lambda_1^m) > 0$,*

$$\begin{aligned} \mathrm{Cov}(s_e, s_f)\,\mathcal{N}^2 &= \sum_{l \geq 2} \Big(\frac{\lambda_l}{\lambda_1}\Big)^m \hat{a}_l \hat{b}_l + R_{CE}, \\ R_{CE} &= \sum_{2 \leq k < l} \Big(\frac{\lambda_k \lambda_l}{\lambda_1^2}\Big)^m (\hat{a}_k \varepsilon_{L,l} - \hat{a}_l \varepsilon_{L,k})(\hat{b}_k \varepsilon_{R,l} - \hat{b}_l \varepsilon_{R,k}), \end{aligned} \tag{6}$$

where $\varepsilon_{L,k} = \langle \varphi_k, \varepsilon_L\rangle$. Moreover $|R_{CE}| \leq (\lambda_2/\lambda_1)^{2m}\|\tilde{u}\|\,\|\varepsilon_L\|\,\|\tilde{v}\|\,\|\varepsilon_R\|$.

Proof. Expand the determinant by the Cauchy–Binet formula:

$$\mathrm{Cov}(s_e, s_f) Z^2 = \sum_{k<l} (\lambda_k\lambda_l)^m (u_k^E u_l^1 - u_l^E u_k^1)(v_k^E v_l^1 - v_l^E v_k^1).$$

Insert $u_1^E = c_e u_1^1$, $u_l^E = u_1^1 \hat{a}_l + c_e u_l^1$ and $u_l^1 = u_1^1 \varepsilon_{L,l}$ for $l \geq 2$, and the same on the right. The terms with $k = 1$ give the sum, and those with $k \geq 2$ give R_{CE}. The bound follows from the Cauchy–Schwarz inequality. $\square$

The vectors $\tilde{u}$ and ε_L depend only on the type of e (horizontal t, h, bb in the top, middle or bottom row; vertical a between top and middle, b between middle and bottom) and on its distance $i \geq 0$ to the left end. As the distance grows they converge geometrically, with $\|\varepsilon(i)\| \leq (\lambda_2/\lambda_1)^i \|\varepsilon(0)\|$. By the mirror symmetry of the grid, $\tilde{v}$, ε_R and $\hat{b}_l$ are the same functions of the type of f and its distance to the right end. Top types are t, a, bottom types are bb, b, and h is neither. A pair of edges is top–bottom if one of them has a top type and the other a bottom type.

Theorem 6.4. *Fix $J > 0$ and an integer $m_0 \geq 0$. Here a pair consists of a left edge e and a right edge f, each of any type and at any distance $i \geq 0$ from its end, with $m \geq 0$ free transfer steps between them. For a pair put $p_l = \hat{a}_l \hat{b}_l$ ($l = 2, 3, 4$), which does not depend on m, and for a top–bottom pair put $d = -p_3$ and $F(0) = p_2 + p_4 - d$. Let Q_5 be the spectral projection onto* $\operatorname{span}\{\varphi_l : l \geq 5\}$. *Suppose that:*

- (C1) *$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5$, and φ_3 is odd under the top–bottom reflection.*
- (C2) *$\lambda_2 + \lambda_4 - 2\lambda_3 > 0$.*
- (C3) *With suitable signs of $\varphi_2, \varphi_3, \varphi_4$, for every type and every distance: $\hat{a}_2 > 0$ and $\hat{a}_4 > 0$; $\hat{a}_3 > 0$ for top types, $\hat{a}_3 < 0$ for bottom types, and $\hat{a}_3 = 0$ for h.*
- (C4) *For every top–bottom pair, $p_2 \geq p_4$ and $F(0) > 0$.*
- (C5) *For every pair with $m \geq m_0$,*

$$B(m) > (\lambda_5/\lambda_1)^m \|Q_5\tilde{u}\| \, \|Q_5\tilde{v}\| + (\lambda_2/\lambda_1)^{2m} \|\tilde{u}\| \, \|\varepsilon_L\| \, \|\tilde{v}\| \, \|\varepsilon_R\|,$$

  where $B(m) = (\lambda_3/\lambda_1)^m F(0)$ for top–bottom pairs and $B(m) = (\lambda_4/\lambda_1)^m (p_2 + p_4)$ otherwise.

- (C6) *For every pair with $m < m_0$, the determinant above is positive:*

$$\langle L^E, T^m R^E \rangle \langle L^1, T^m R^1 \rangle - \langle L^E, T^m R^1 \rangle \langle L^1, T^m R^E \rangle > 0.$$

- (C7) *For every interior column,*

$$\langle L_0, T_{XY} R_0 \rangle \langle L_0, T R_0 \rangle - \langle L_0, T_X R_0 \rangle \langle L_0, T_Y R_0 \rangle > 0,$$

  where $T_{XY} = K^{1/2} XYVK^{1/2}$ and L_0, R_0 are the plain vectors on the two sides of the column.

Then $\operatorname{Cov}(s_e, s_f) > 0$ *for every pair of distinct edges of every grid $P_3 \times P_n$ with $n \geq 2$, at this J.*

Proof. Pairs of the two vertical edges of an interior column are covered by (C7), whose left side is $\operatorname{Cov}(s_a, s_b)$ times $\langle L_0, T R_0 \rangle^2$. A vertical corner edge is in series with a horizontal corner edge, and Lemma 6.2 replaces it by that edge; in an end column this is done for both vertical edges. For the two edges at the same corner, Lemma 2.4 gives the variance of $r_c(J)$

under the law of their common label, which is nonnegative and charges the labels 0 and 1 (Corollary 3.8, at $N = 3$ and $N = 4$), so their covariance is positive. The hypothesis of Corollary 3.8 holds: the two neighbours of a corner v are joined in $G - v$ by the path of length two through the middle vertex of the adjacent column, and its couplings equal $J > 0$. All other pairs are covered by Lemma 6.3, where same-cut pairs have $m = 0$. If $m < m_0$, the left side of (C6) is $\mathrm{Cov}(s_e, s_f)Z^2$, which is therefore positive. Let $m \geq m_0$, and write the sum in (6) as $S_{cl} + S_\infty$ with S_{cl} the terms $l = 2, 3, 4$. Since $0 \leq \lambda_l \leq \lambda_5$ for $l \geq 5$, the Cauchy–Schwarz inequality gives $|S_\infty| \leq (\lambda_5/\lambda_1)^m \|Q_5\tilde{u}\| \, \|Q_5\tilde{v}\|$. If the pair is not top–bottom, (C3) gives $p_2, p_4 > 0$ and $p_3 \geq 0$, so $S_{cl} \geq (\lambda_4/\lambda_1)^m(p_2 + p_4)$. If it is top–bottom, $S_{cl} = (\lambda_3/\lambda_1)^m F(m)$ with $F(m) = p_2(1+x)^m + p_4(1-y)^m - d$, $x = \lambda_2/\lambda_3 - 1$ and $y = 1 - \lambda_4/\lambda_3 \in [0, 1)$. Bernoulli's inequality gives $F(m) - F(0) \geq m(p_2 x - p_4 y)$, and $p_2 x - p_4 y = \big[(p_2 - p_4)(\lambda_2 - \lambda_3) + p_4(\lambda_2 + \lambda_4 - 2\lambda_3)\big]/\lambda_3 \geq 0$ by (C2) and (C4). In both cases the right side of (6) is positive by (C5). □

Theorem 6.5 (weak coupling, †)**.** *For $N \in \{3, 4\}$, $0 < J \leq 1/100$ and $n \geq 3$, $\mathrm{Cov}(s_e, s_f) > 0$ for every pair of distinct edges of $P_3 \times P_n$.*

The proof is in Appendix A. It verifies the conclusion of Theorem 6.4 with explicit constants. The step marked † is a certified enclosure of the eigendata of an explicit 4×4 matrix, uniformly in $J \in (0, 1/100]$.

Theorem 6.6 (†)**.** *For every $J \in [0.0099, 20.064]$ at $N = 4$ and every $J \in [0.0099, 20.070]$ at $N = 3$, the hypotheses of Theorem 6.4 hold with $m_0 = 0$ or $m_0 = 1$.*

Proof. The interval is covered by closed subintervals with overlapping ends. On each subinterval the matrix of T on the labels up to a cutoff is enclosed by a Taylor model in J with ball-arithmetic coefficients and a rigorous remainder. Eigenvalues and eigenvectors are enclosed by residual bounds together with the Weyl and Davis–Kahan inequalities. The labels above the cutoff enter through a Schur complement, whose norm is bounded through the tail $\sum_{a>L}(a+1)^2 k_a$ of the character expansion and, at larger J, through a weighted vector bound of Combes–Thomas type. The conditions are checked on classes: a class is a type with a distance up to a fixed depth, and one deep class per type encloses all larger distances, through the geometric convergence of the insertion vectors. On each subinterval an integer $m_0 \in \{0, 1\}$ is fixed. The inequalities (C1)–(C5) and (C7) are then checked with a positive margin on the whole subinterval, for every class and, in (C5), for every $m \geq m_0$. Condition (C6) is a direct evaluation: where $m_0 = 1$, the determinant at $m = 0$ is enclosed from the enclosures of L^E, L^1, R^E, R^1 and shown to be positive for every pair of classes. This is needed on the subintervals where (C5) cannot be certified at $m = 0$; elsewhere $m_0 = 0$. □

Proof of Theorem 1.4(ii). The ranges $(0, 1/100]$ and $[0.0099, 20]$ are covered by Theorems 6.5 and 6.6. □

6.4. **The 3×3 grid at strong coupling.** The dihedral group of the square acts on the 66 pairs of edges of $P_3 \times P_3$, and the covariance is constant on each orbit.

Proposition 6.7. *For every $J \geq 0$ and $N \in \{3,4\}$, $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of edges outside the orbits of $(b_0, \mathsf{s}_\mathrm{N})$ and $(b_0, \mathsf{s}_\mathrm{W})$, and these two orbits have the common covariance $\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}})$.*

Proof. Pairs sharing a vertex are covered by Theorem 3.2: the orbits of (b_0, b_7), (b_0, b_1), $(b_0, \mathsf{s}_\mathrm{S})$, $(\mathsf{s}_\mathrm{N}, \mathsf{s}_\mathrm{S})$ and $(\mathsf{s}_\mathrm{N}, \mathsf{s}_\mathrm{E})$. The orbits of (b_0, b_3) and (b_0, b_5) consist of mirror pairs for a side reflection and for a diagonal reflection, and Lemma 6.1 applies to them. The corners have degree two, and by Lemma 6.2 the remaining orbits are identified with covered ones. At NE it gives $\mathrm{Cov}(s_{b_0}, s_{b_2}) = \mathrm{Cov}(s_{b_0}, s_{b_3})$. At NW it gives $\mathrm{Cov}(s_{b_0}, s_{b_4}) = \mathrm{Cov}(s_{b_0}, s_{b_5})$. At SE, with $F = s_{\mathsf{s}_\mathrm{E}}$, it gives $\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{E}}) = \mathrm{Cov}(s_{b_1}, s_{\mathsf{s}_\mathrm{E}})$, a pair sharing a vertex. Finally, at SE with $F = s_{\mathsf{s}_\mathrm{N}}$, $\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}}) = \mathrm{Cov}(s_{b_1}, s_{\mathsf{s}_\mathrm{N}})$, and the reflection in the diagonal through NW, C and SE maps $(b_1, \mathsf{s}_\mathrm{N})$ to $(b_0, \mathsf{s}_\mathrm{W})$. □

We now describe the representation in which the remaining covariance is certified. Fix the hub spin $\sigma_C = e$ and decompose $L^2(S^{N-1})$ under the stabilizer of e: the degree-l harmonics split into sectors $m = 0, \dots, l$, each sector carrying an irreducible representation of dimension D_m of the stabilizer, once. Multiplication by a function of $\sigma \cdot e$ acts in sector m by a function of the Jacobi matrix $\mathsf{X}^{(m)}$ of the weight $(1-u^2)^{m+\lambda-1/2}$, and a zonal operator $\sum_l w_l \Pi_l$ acts by $\mathrm{diag}(w_l)_{l\geq m}$.

Lemma 6.8 (hub trace)**.** *For couplings $\beta_e \geq 0$ on $P_3 \times P_3$ and $N \geq 3$,*

$$Z = \sum_{m\geq 0} D_m \ \mathrm{tr}_{\ell^2(l\geq m)} \big[M_\mathrm{N} W_\mathrm{NE} M_\mathrm{E} W_\mathrm{SE} M_\mathrm{S} W_\mathrm{SW} M_\mathrm{W} W_\mathrm{NW}\big],$$

with $M_X = \exp(\beta_{cX} \mathsf{X}^{(m)})$ for each side X and, for each corner c with sides X, Y,

$$W_c = \mathrm{diag}\big(k_l(\beta_{cX}) k_l(\beta_{cY})\big)_{l\geq m}.$$

The series converges absolutely, and every entry of every factor is nonnegative.

Proof. Integrating a corner spin with (3) and (2) gives the zonal kernel $\sum_l k_l(\beta_{cX}) k_l(\beta_{cY}) \Pi_l$ between its two sides. The remaining integral over the four side spins is the trace of a product of multiplication operators and zonal operators, and each factor commutes with the stabilizer of e. By Schur's lemma the trace splits by sectors. □

Differentiating term by term inserts $\mathrm{diag}(r_l)$ at a corner for a rim edge and $\mathsf{X}^{(m)}$ next to M_X for a spoke. At uniform coupling, with $V = \exp(\beta \mathsf{X}^{(m)}) \,\mathrm{diag}(k_l^2)$, the pair $(b_0, \mathsf{s}_\mathrm{N})$ corresponds to the cyclic trace $\mathrm{tr}(V_c V V_s V)$ with one corner insertion and one spoke insertion three slots apart. Positivity of $\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}})$ is equivalent to positivity of

$N_\tau = Z^0_\tau Z - Z^0_e Z^0_f$, where the superscript 0 denotes insertions centred by arbitrary constants. No division is needed.

Theorem 6.9 (†)**.** *For $N \in \{3,4\}$ and every $J = \beta \in [20, 100.68]$, $\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}}) > 0$ on $P_3 \times P_3$.*

Proof. The interval is covered by closed subintervals. On each, the sector traces of Lemma 6.8 are enclosed by Taylor models in β. The sectors and labels beyond explicit cutoffs are bounded by tail lemmas for the powers of $\mathsf{X}^{(m)}$ and for the diagonals k_l, and the sign of N_τ is certified with a positive margin. □

Lemma 6.10 (strong coupling, †)**.** *On $P_3 \times P_3$, $J^2 \,\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}}) \geq 2.24 \cdot 10^{-4}$ for every $J \geq 80$ at $N = 3$, and $J^2 \,\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}}) \geq 4.63 \cdot 10^{-4}$ for every $J \geq 100$ at $N = 4$.*

Proof. Put $\xi_e = J(1 - s_e)$. Then $J^2 \,\mathrm{Cov}(s_e, s_f) = \mathrm{Cov}(\xi_e, \xi_f)$, which is a rational function of the four expectations $\mathbb{E}[1], \mathbb{E}[\xi_e], \mathbb{E}[\xi_f]$ and $\mathbb{E}[\xi_e \xi_f]$ relative to a common normalisation. Each is expanded in $t = 1/J$ to fourth order in a chart around the aligned configurations. The coefficients are exact rationals, obtained from Gaussian moments. The remainders are bounded by explicit functions of J: a Taylor remainder on balls of fixed radii, Chernoff bounds for the tails, and bounds for the annulus and far regions. With the radii fixed, each bound is nonincreasing in J, so its value at J_1 bounds it for all $J \geq J_1$. The polynomial main part is enclosed on $t \in [0, 1/J_1]$ by subdivision, which gives the lower bound uniformly in $J \geq J_1$. As $J \to \infty$, $J^2 \,\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}})$ tends to $1/576$ at $N = 3$ and to $1/384$ at $N = 4$. □

Proof of part (a) of Theorem 6.34, which is stated in Section 6.6. At $J = 0$, $\mathrm{Cov}(s_e, s_f) = 0$ for $e \neq f$. Pairs with a common vertex have $\mathrm{Cov}(s_e, s_f) \geq 0$ at every J by Theorem 3.2. For the other pairs and $0 < J \leq 20$, Theorem 1.4(ii) with $n = 3$ applies. For $J \geq 20$, Proposition 6.7 reduces the claim to $\mathrm{Cov}(s_{b_0}, s_{\mathsf{s}_\mathrm{N}}) \geq 0$, which Theorem 6.9 gives on $[20, 100.68]$ and Lemma 6.10 for $J \geq 80$ at $N = 3$ and $J \geq 100$ at $N = 4$. □

6.5. **Strips for $20 \leq J \leq 300$.** In this subsection $N \in \{3,4\}$ and all couplings of $P_3 \times P_n$ equal J. We write $\sigma_1, \sigma_2, \sigma_3$ for the top, middle and bottom spins of a column. The horizontal edges of the three rows have the types th, mh and bh, and the vertical edges the types tv (rows $1, 2$) and bv (rows $2, 3$). We prove $\mathrm{Cov}(s_e, s_f) > 0$ for $20 \leq J \leq 300$ ($20 \leq J \leq 350$ when $N = 4$), every $n \geq 4$ and every pair of distinct edges other than the two vertical edges of the same column (Theorem 6.19). As in Section 6.3 the covariance is a 2×2 determinant of matrix elements of a transfer operator. Here the operator is written in the hub frame of each column, whose axis is the middle spin, and the determinant is read as a matrix element of the second exterior power of the operator. Pairs less than two transfer steps apart are evaluated directly. For all other pairs the leading modes of the

exterior power decide the sign, uniformly in the distance between the edges and in their distances to the ends.

The column transfer. Let $\mathcal{H}$ be the space of $\mathrm{O}(N)$-invariant functions in $L^2(\omega^{\otimes 3})$, put $u_1 = \sigma_1 \cdot \sigma_2$ and $u_3 = \sigma_2 \cdot \sigma_3$, let V be multiplication by $e^{J(u_1+u_3)}$, and let K be the integral operator with kernel $\prod_{r=1}^{3} e^{J\sigma_r \cdot \sigma_r'}$. Then $T_0 = V^{1/2} K V^{1/2}$ is compact, self-adjoint and positive on $\mathcal{H}$, and $Z(P_3 \times P_n) = \langle v, T_0^{n-1} v \rangle$ with $v = V^{1/2}\mathbf{1}$. It has the same nonzero eigenvalues as the operator T of Section 6.3. A horizontal edge of row r between the columns c and $c+1$ replaces the c-th factor K by its derivative K_r in the coupling of row r. A vertical edge of column c inserts multiplication by s_e between the two factors $V^{1/2}$ of column c, and two edges in the same slot insert the product of their insertions.

The hub frame. Fix $\sigma_2 = e$. As in Section 6.4, the harmonics of degree l in σ_1 split under the stabilizer $\mathrm{O}(N-1)$ of e into sectors $m = 0, \dots, l$, and those of sector m are $f_{l,m}(u_1) Y_{m,j}(\omega_1)$, $1 \le j \le D_m$, where ω_1 is the direction of the part of σ_1 orthogonal to e and the $Y_{m,j}$ are an orthonormal basis of the sector. For $N - 1 \in \{2, 3\}$ every sector is absolutely irreducible, so by Schur's lemma an invariant function of (σ_1, σ_3) pairs equal sectors of σ_1 and σ_3 through the canonical pairing. Therefore $\mathcal{H}$ has the orthonormal basis

$$\Phi_{l_1 l_3 m} = D_m^{-1/2} \sum_{j=1}^{D_m} f_{l_1,m}(u_1) Y_{m,j}(\omega_1)\, f_{l_3,m}(u_3) Y_{m,j}(\omega_3),$$

with $0 \le m \le \min(l_1, l_3)$, and multiplication by u_1 (by u_3) acts in sector m as $\mathsf{X}^{(m)}$ on the label l_1 (on l_3).

Lemma 6.11. *Let $l_1, l_3 \ge 0$, $s = \min(l_1, l_3) + 1$ and $\mathcal{H}_{l_1 l_3} = \operatorname{span}\{\Phi_{l_1 l_3 m} : 0 \le m < s\}$.*

(i) *The Laplacian Δ_2 in σ_2 preserves $\mathcal{H}_{l_1 l_3}$, and $-\Delta_2$ acts there by the tridiagonal matrix $C = C^{(l_1, l_3)}$ with, for $N = 3$,*

$$C_{mm} = l_1(l_1+1) + l_3(l_3+1) - 2m^2,$$
$$C_{m,m+1} = -\sqrt{(l_1 - m)(l_1 + m + 1)(l_3 - m)(l_3 + m + 1)},$$

the off-diagonal entry multiplied by $\sqrt{2}$ when $m = 0$, and, for $N = 4$,

$$C_{mm} = l_1(l_1+2) + l_3(l_3+2) - 2m(m+1),$$
$$C_{m,m+1} = -2(m+1)\sqrt{\frac{\left((l_1+1)^2 - (m+1)^2\right)\left((l_3+1)^2 - (m+1)^2\right)}{(2m+1)(2m+3)}}.$$

(ii) *The off-diagonal entries are nonzero, and the eigenvalues of C are $l_2(l_2 + N - 2)$ for $l_2 = |l_1 - l_3| + 2j$, $0 \le j < s$, each simple.*

(iii) *Let $Q = Q^{(l_1, l_3)}$ be the orthogonal matrix of unit eigenvectors of C, with columns labelled by l_2 and $Q^{(l_3, l_1)} := Q^{(l_1, l_3)}$. The product singlets $\Psi_{l_1 l_2 l_3} = \sum_m Q_{m, l_2} \Phi_{l_1 l_3 m}$ form an orthonormal basis of $\mathcal{H}$,*

and

$$K\Psi_{l_1l_2l_3} = k_{l_1}k_{l_2}k_{l_3}\Psi_{l_1l_2l_3}, \qquad K_r\Psi_{l_1l_2l_3} = r_{l_r}k_{l_1}k_{l_2}k_{l_3}\Psi_{l_1l_2l_3},$$

with $k_l = k_l(J)$ *as in* (3), $r_l = \partial_J \log k_l(J)$ *and* l_r *the label of row* r.

Proof. Take $e = e_N$ and write $L^{(i)}_{ab}$ for the generators of rotations of σ_i and $B^{(i)}_a = L^{(i)}_{aN}$. On $\mathcal{H}$, $L^{(1)} + L^{(2)} + L^{(3)} = 0$, so $(L^{(2)}_{ab})^2 = (L^{(1)}_{ab} + L^{(3)}_{ab})^2$ there, and

$$-\Delta_2 = -\Delta_1 - \Delta_3 - 2\sum_{a<b<N} L^{(1)}_{ab}L^{(3)}_{ab} - 2\sum_{a<N} B^{(1)}_a B^{(3)}_a.$$

In sector m, $\sum_{a<b<N} L^{(1)}_{ab}L^{(3)}_{ab} = m(m+N-3)$, which gives the diagonal. The last sum maps sector m to the sectors $m \pm 1$. Write $f_{l,m}(u) = c_{l,m}(1-u^2)^{m/2}p^{(m)}_{l-m}(u)$ with $c_{l,m} > 0$ and $p^{(m)}_n$ orthonormal for the weight $(1-u^2)^{m+\lambda-1/2}$. The component in sector $m+1$ comes from the three-term rule $tP_m = \alpha'_m P_{m+1} + \gamma'_m P_{m-1}$ of the zonal polynomials of S^{N-2} and from the ladder $(p^{(m)}_n)' = \kappa\, p^{(m+1)}_{n-1}$, $\kappa > 0$, whose size is fixed by the Sturm–Liouville identity of the Gegenbauer equation. This gives $C_{m,m+1} = -2\alpha'_m\sqrt{D_m/D_{m+1}}\,\rho_{l_1,m}\rho_{l_3,m}$ with $\rho^2_{l,m} = (l-m)(l+m+N-2)$, which are the stated entries. For $m < s-1$ they are nonzero, so C is an unreduced Jacobi matrix with s simple eigenvalues. Each eigenvalue is $l_2(l_2+N-2)$ for a degree l_2 in σ_2 that carries an invariant of $H_{l_1}\otimes H_{l_2}\otimes H_{l_3}$. Such an l_2 satisfies $l_2 \le l_1 + l_3$, because $\Phi_{l_1l_3m}$ is a polynomial of degree l_1+l_3 in σ_2, and by symmetry in the three variables also $l_1 \le l_2+l_3$ and $l_3 \le l_1+l_2$; and $l_1+l_2+l_3$ is even, because $-1 \in \mathrm{O}(N)$ acts on H_l by $(-1)^l$. There are exactly s such l_2, so each occurs once. For (iii), (3) and (2) give $K = k(\Delta_1)k(\Delta_2)k(\Delta_3)$ with $k(-l(l+N-2)) = k_l$, and Δ_2 commutes with Δ_1, Δ_3 and $\mathrm{O}(N)$; K_r replaces the factor k_{l_r} of row r by $\partial_J k_{l_r}$. □

The sign of the off-diagonal entries does not depend on choices of sign inside a sector. Computations may use the basis $(-1)^m\Phi_{l_1l_3m}$, in which these entries are positive; this changes Q by a diagonal of signs and none of the quantities below. For $s \ge 2$ a column $Q_{\cdot,l_2}$, with its sign fixed by $\langle Q_{\cdot,l_2}, U\rangle > 0$ and the same sign used for (l_3, l_1), is enclosed from a computed vector U by $\|Q_{\cdot,l_2} - U\| \le \sqrt{2}\,\|CU - \mu U\|/(g\|U\|) + |\|U\| - 1|$, with $\mu = l_2(l_2+N-2)$ and $g \ge 4|l_1 - l_3| + 2N$ its distance to the other eigenvalues; for $s = 1$, $Q = (1)$.

Frames and truncation. Put $\hat{k}_l = c\,k_l$ with a single constant $c > 0$ for all l, $\tilde{E}_m = e^{J(\mathsf{X}^{(m)}-1)}$, $\hat{E} = \bigoplus_m \tilde{E}^{1/2}_m \otimes \tilde{E}^{1/2}_m$ (on the labels l_1 and l_3), $\mathsf{D} = \mathrm{diag}(\hat{k}_{l_1}\hat{k}_{l_2}\hat{k}_{l_3})$ on the product singlets and $H = Q\mathsf{D}Q^\top$. Since $\hat{E} = e^{-J}V^{1/2}$ and H is a positive multiple of K, the operator $T = \hat{E}H\hat{E}$ is a positive multiple of T_0. With $S = H^{1/2}\hat{E}$ we have $T = S^*S$, and we work mostly with $\tilde{T} = SS^*$, which has the same nonzero eigenvalues. For a cutoff L let H_L be the part of H on the blocks with $\max(l_1, l_3) \le L$, $S_L = H_L^{1/2}\hat{E}$,

$T_L = S_L^* S_L$, and let

$$\hat{T} = \mathsf{D}^{1/2} Q^\top G Q\, \mathsf{D}^{1/2}, \qquad G = \bigoplus_m (\tilde{E}_m)_{[m,L]} \otimes (\tilde{E}_m)_{[m,L]},$$

on the product singlets with $\max(l_1, l_3) \le L$, where $(\cdot)_{[m,L]}$ is the compression to the labels $m \le l \le L$. A horizontal edge e of row r acts by $R_e = \mathrm{diag}(r_{l_r})$ on the product singlets. A vertical edge acts by U_e, multiplication by u_1 (tv) or by u_3 (bv); we put $\tilde{T}_e = S U_e S^*$ and $\hat{T}_e = \mathsf{D}^{1/2} Q^\top G_e Q\, \mathsf{D}^{1/2}$, where $G_e = \bigoplus_m (\mathsf{X}^{(m)} \tilde{E}_m)_{[m,L]} \otimes (\tilde{E}_m)_{[m,L]}$ for tv and the two factors are exchanged for bv. The reflection ρ that exchanges σ_1 and σ_3 acts by $\rho \Psi_{l_1 l_2 l_3} = \Psi_{l_3 l_2 l_1}$; we write $\pm$ for its two eigenspaces, the parity classes.

Lemma 6.12. (i) $\|T - T_L\| \le \varepsilon_T := \hat{k}_{L+1} \hat{k}_0^2$.

(ii) *T_L and $\hat{T}$ have the same nonzero eigenvalues, with multiplicities. If χ (written in the product singlets) is a unit eigenvector of $\hat{T}$ with eigenvalue $\nu > 0$, then $S_L^* \chi / \sqrt{\nu}$ is a unit eigenvector of T_L. In the same coordinates $S_L U_e S_L^*$ is $\hat{T}_e$.*

(iii) *T, T_L and $\hat{T}$ are positive and commute with ρ, and in each parity class the k-th eigenvalues satisfy $|\lambda_k^\pm(T) - \lambda_k^\pm(\hat{T})| \le \varepsilon_T$, where $\lambda_k^\pm(\hat{T}) = 0$ beyond the rank.*

(iv) *Let g be the distance from an eigenvalue of T_L to the other eigenvalues of T_L in its parity class, zero included. If $g > 2\varepsilon_T$, then T has exactly one eigenvalue of that parity class within ε_T of it, and the unit eigenvectors, with signs aligned, differ by at most $\sqrt{2}\, \varepsilon_T / (g - \varepsilon_T)$.*

Proof. The coefficients $\hat{k}_l$ decrease in l, since $k_{l+1}/k_l = I_{l+\lambda+1}(J)/I_{l+\lambda}(J) < 1$ [3, (16)]. The part of H on the blocks with $\max(l_1, l_3) \ge L+1$ is positive, with norm at most $\hat{k}_{L+1}\hat{k}_0^2$, and $\|\hat{E}\| \le 1$ because $\|\mathsf{X}^{(m)}\| \le 1$; this gives (i). For (ii), $S_L S_L^* = H_L^{1/2} P_L \tilde{E} P_L H_L^{1/2}$, where $\tilde{E} = \hat{E}^2$ and P_L projects onto the labels $l_1, l_3 \le L$; so $P_L \tilde{E} P_L$ is the direct sum of the compressions, and conjugation by Q gives $\hat{T}$. The same computation with $U_e \tilde{E}$ in place of $\tilde{E}$ gives $\hat{T}_e$: it is the compression of the product $\mathsf{X}^{(m)} \tilde{E}_m$, not the product of compressions. For (iii), $\hat{E}$, H and H_L commute with ρ, because D is symmetric in l_1 and l_3 and $Q^{(l_3,l_1)} = Q^{(l_1,l_3)}$, and Weyl's inequality applies in each parity class. For (iv), the eigenvalue ν of T is isolated by Weyl's inequality. If ψ and ψ^L are the eigenvectors and $P_\perp$ projects off ψ^L inside the parity class, then $(T_L - \nu) P_\perp \psi = -P_\perp (T - T_L)\psi$, and every other eigenvalue of T_L in the parity class is at distance at least $g - \varepsilon_T$ from ν, so $\sin\angle(\psi, \psi^L) \le \varepsilon_T / (g - \varepsilon_T)$ and $\|\psi - \psi^L\| \le \sqrt{2} \sin\angle(\psi, \psi^L)$. □

The truncation thus enters once, through ε_T, in the frame of T, where it is linear, while $\hat{T}$ and $\hat{T}_e$ need only the compressions of $\tilde{E}_m$ and $\mathsf{X}^{(m)} \tilde{E}_m$, which are enclosed with proven tails as in the proof of Theorem 6.9.

Objects. Put $w = Sv$ and $w_p = \tilde{T}^p w$ for $p \geq 0$. For a horizontal type e put $w_{e,p} = R_e w_p$; for a vertical type e put $w_{e,0} = SU_e v$ and $w_{e,p} = \tilde{T}_e w_{p-1}$ for $p \geq 1$. An edge has index p from the left end if it lies between the columns $p+1$ and $p+2$ (horizontal) or in the column $p+1$ (vertical); its index from the right end is defined in the mirror way, and the mirror symmetry of the strip gives the same vectors at the right end. For $a, b, c, d \in \mathcal{H}$ put $\langle a \wedge b, c \wedge d\rangle = \langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle$, and let $\Lambda^2\tilde{T}$ act by $a \wedge b \mapsto \tilde{T}a \wedge \tilde{T}b$.

Lemma 6.13. *Let e have index p from the left end and f index q from the right end, with $k \geq 0$ complete factors $\tilde{T}$ between their insertions, so that $n = p+k+q+2$. Then $Z = \langle w_p, \tilde{T}^k w_q\rangle$ is a positive multiple of $Z(P_3 \times P_n)$, and*

$$\mathrm{Cov}(s_e, s_f)\, Z^2 = \big\langle w_p \wedge w_{e,p}, (\Lambda^2\tilde{T})^k (w_q \wedge w_{f,q})\big\rangle.$$

The right side does not change if $w_{e,p}$ is replaced by $w'_{e,p} = w_{e,p} - \alpha_e w_p$, or $w_{f,q}$ by $w'_{f,q} = w_{f,q} - \alpha_f w_q$, for any real α_e, α_f. Two horizontal edges of the same junction are the case $k = 0$. Every pair of distinct edges of $P_3 \times P_n$, $n \geq 4$, other than the two vertical edges of the same column, arises in this way.

Proof. For $m \geq 1$, $T^m = S^*\tilde{T}^{m-1}S$ and $ST^m = \tilde{T}^m S$, so the partition function is a positive multiple of $\langle w, \tilde{T}^{n-2} w\rangle$. A horizontal insertion turns one factor T into $S^* R_e S$, because R_e commutes with $H^{1/2}$; a vertical insertion in a column $c \geq 2$ turns one factor $\tilde{T}$ into $\tilde{T}_e$, and in the first column it gives $w_{e,0}$. Collecting the factors on both sides of the insertions gives Z, $Z_e = \langle w_{e,p}, \tilde{T}^k w_q\rangle$, $Z_f = \langle w_p, \tilde{T}^k w_{f,q}\rangle$ and $Z_{ef} = \langle w_{e,p}, \tilde{T}^k w_{f,q}\rangle$ for the derivatives of Z in the couplings of e and f, and $\mathrm{Cov}(s_e, s_f)Z^2 = ZZ_{ef} - Z_e Z_f$ is the stated matrix element of $(\Lambda^2\tilde{T})^k$. The centring holds because $a \wedge (b - \alpha a) = a \wedge b$. For two horizontal edges of the same junction the factor of that junction carries $R_e R_f$, and $\langle w_p, R_e R_f w_q\rangle = \langle w_{e,p}, w_{f,q}\rangle$. Two edges in different slots have their insertions in different factors; within a slot, the only two distinct edges other than a pair at a junction are the two vertical edges of the same column. □

Spectral enclosures.

Lemma 6.14. (i) *For $x \neq 0$ and real μ, $\hat{T}$ has an eigenvalue within $\|\hat{T}x - \mu x\|/\|x\|$ of μ.*

(ii) *If disjoint intervals $I_1, \ldots, I_K$ each contain an eigenvalue of $\hat{T}$ in the parity class $\pm$, then every other eigenvalue of that parity class is at most $\mathrm{tr}(\hat{T}\Pi_\pm) - \sum_i \max(\min I_i, 0)$, where $\Pi_\pm = (1 \pm \rho)/2$, $\mathrm{tr}(\hat{T}\Pi_\pm) = (\mathrm{tr}\,\hat{T} \pm \mathrm{tr}\,\hat{T}\rho)/2$ and*

$$\mathrm{tr}\,\hat{T} = \sum \mathsf{D}_{l_1 l_2 l_3} \sum_m Q^2_{m,l_2}\, \tilde{E}_m(l_1, l_1)\tilde{E}_m(l_3, l_3),$$

$$\mathrm{tr}\,\hat{T}\rho = \sum \mathsf{D}_{l_1 l_2 l_3} \sum_m Q^2_{m,l_2}\, \tilde{E}_m(l_1, l_3)^2,$$

the outer sums running over the product singlets with $\max(l_1, l_3) \le L$.

(iii) *Let* x *lie in a single parity class, let* ν *be a simple eigenvalue of* $\hat{T}$ *in that parity class with unit eigenvector* χ_ν*, and let every other eigenvalue of the parity class lie at distance at least* $\gamma > 0$ *from* μ*. Then* $\sin\angle(x, \chi_\nu) \le \|\hat{T}x - \mu x\|/(\gamma\|x\|)$.

(iv) *The largest eigenvalue of* T *is simple, its eigenvector* ψ_0 *is positive and* ρ*-even, and* $\langle v, \psi_0\rangle > 0$.

Proof. For (i) and (iii), $\|(\hat{T} - \mu)x\|$ is at least the distance from μ to the spectrum times $\|x\|$, and at least γ times the norm of the part of x orthogonal to χ_ν. (ii) holds because $\hat{T} \ge 0$. The trace formulas hold because G is diagonal in m and ρ maps the entry of G at (l_1, l_3, m) to the entry at (l_3, l_1, m), while $\tilde{E}_m$ is symmetric. For (iv), the kernel of T_0 on $(S^{N-1})^3$ is continuous and strictly positive, so by Jentzsch's theorem its largest eigenvalue is simple with a positive eigenvector; by uniqueness this eigenvector is invariant under $\mathrm{O}(N)$ and under ρ, so it lies in $\mathcal{H}$; and $\langle v, \psi_0\rangle > 0$ because $v > 0$. □

Certified modes. On a subinterval of J we obtain disjoint intervals that contain eigenvalues $\nu_0 > \nu_1 > \cdots > \nu_5$ of $\tilde{T}$, each simple in its parity class. We also obtain unit eigenvectors $\chi_0, \dots, \chi_5$ enclosed in angle and a number u_{rem} that bounds every other eigenvalue of $\tilde{T}$ in both parity classes. The intervals, the eigenvectors and the number u_{rem} come from Lemma 6.14 for $\hat{T}$ (with mode vectors that lie exactly in a single parity class) and are transferred to $\tilde{T}$ by Lemma 6.12: every interval and the bounds for both parity classes are widened by ε_T, and the angles carry the Davis–Kahan term of Lemma 6.12(iv) after transport by $\chi_a = S\psi_a/\sqrt{\nu_a}$. We check that the interval of ν_0 lies above all other bounds, that χ_1 is ρ-even, and that $u_{\mathrm{rem}} < \nu_1$.

Lemma 6.15. *Put* $\theta_a = \langle \chi_a, w\rangle$ *and* $w_\perp = w - \sum_{a\le 5} \theta_a \chi_a$*, assume* $\theta_0 \ne 0$*, and for* $P \ge 1$ *let*

$$\eta(P) = \sum_{a=1}^{5} \Big(\frac{\nu_a}{\nu_0}\Big)^P \Big|\frac{\theta_a}{\theta_0}\Big| + \Big(\frac{u_{\mathrm{rem}}}{\nu_0}\Big)^P \frac{|w_\perp|}{|\theta_0|}.$$

(i) *For every* $p \ge P$, $w_p = \nu_0^p \theta_0 (\chi_0 + w_p^\circ)$ *with* $|w_p^\circ| \le \eta(P)$.

(ii) *Let* e *be horizontal,* $R' = R_e - \alpha_e$ *and* $z_e = R'\chi_0 - \langle \chi_0, R'\chi_0\rangle \chi_0$*. The norms* $\|R'\chi_a\|$ *and* $\|R'\tilde{T}\|$ *are finite, and for every* $p \ge P$

$$w_p \wedge w'_{e,p} = (\nu_0^p \theta_0)^2 \big(\chi_0 \wedge z_e + \Delta_e(p)\big),$$
$$|\Delta_e(p)| \le m_e + \eta(P)\|R'\chi_0\| + \eta(P) m_e,$$
$$m_e = \sum_{a=1}^{5} \Big(\frac{\nu_a}{\nu_0}\Big)^P \Big|\frac{\theta_a}{\theta_0}\Big| \, \|R'\chi_a\| + \frac{\|R'\tilde{T}\| \, u_{\mathrm{rem}}^{P-1} |w_\perp|}{\nu_0^P |\theta_0|}.$$

(iii) *Let e be vertical and $T' = \tilde{T}_e - \alpha_e \tilde{T}$, so that $w'_{e,p} = T' w_{p-1}$. For every $p \geq P \geq 2$, $w_p \wedge w'_{e,p} = \nu_0(\nu_0^{p-1}\theta_0)^2(\chi_0 \wedge z_e + \Delta_e(p))$, where z_e is the part of $T'\chi_0$ orthogonal to χ_0 and $\Delta_e(p)$ obeys the bound of (ii) with T' in place of R' and $P-1$ in place of P in m_e.*

Proof. (i) $w_p = \sum_{a\leq 5} \nu_a^p \theta_a \chi_a + \tilde{T}^p w_\perp$, and $\tilde{T}$ maps the orthogonal complement of $\chi_0, \dots, \chi_5$ into itself with norm at most $u_{\rm rem}$; every term of w_p° decreases in p. (ii) The operator R' is unbounded, since r_l grows like l/J, but $\chi_a = \tilde{T}\chi_a/\nu_a$ lies in the range of $H^{1/2}$, on which R' is bounded because $\hat{k}_l$ decays faster than any geometric sequence, and $\|R'\tilde{T}\| \leq \|R'S\|\,\|S^*\|$. Since $R'w_p^\circ = \sum_{a\geq 1}(\nu_a/\nu_0)^p(\theta_a/\theta_0)R'\chi_a + (R'\tilde{T})\tilde{T}^{p-1}w_\perp/(\nu_0^p\theta_0)$, we have $\|R'w_p^\circ\| \leq m_e$ for $p \geq P$. Expanding $w_p \wedge R'w_p$ into four terms and using $|a \wedge b| \leq |a||b|$ gives the bound; the projection in z_e does not change the wedge with χ_0. (iii) is the same computation with $w_p = \nu_0 \cdot \nu_0^{p-1}\theta_0(\chi_0 + w_p^\circ)$ and $T'w_{p-1} = \nu_0^{p-1}\theta_0(T'\chi_0 + T'w_{p-1}^\circ)$, where T' is bounded. □

Let D_e be a bound for $|\Delta_e(p)|$, $p \geq P_e$, as in Lemma 6.15, where $P_e = 9$ for mh and $P_e = 6$ for the other types. Since $\|\Lambda^2\tilde{T}\| = \nu_0\nu_1$, for every $\Upsilon = x \wedge y$ and every $p \geq P_e$ the value $\langle w_p \wedge w'_{e,p}, (\Lambda^2\tilde{T})^k\Upsilon\rangle$ is a positive multiple of $\langle \chi_0 \wedge z_e, (\Lambda^2\tilde{T})^k\Upsilon\rangle + E$ with $|E| \leq D_e(\nu_0\nu_1)^k|x||y|$. One check of the representative $\chi_0 \wedge z_e$ with this error therefore covers every index $p \geq P_e$. We call (e,∞) the infinity object of type e. When both objects of a pair are infinity objects, the product of their two errors is added. In the computation χ_0 is replaced by its computed unit direction, and D_e includes the resulting error.

Near and far pairs. For object pairs $\Xi = x \wedge y$ and $\Upsilon = x' \wedge y'$, with x, x' plain and y, y' centred inserted objects, put $S(k) = \langle \Xi, (\Lambda^2\tilde{T})^k\Upsilon\rangle$. For $k < 2$ it is computed directly as a 2×2 determinant of inner products of the objects and of their images under $\tilde{T}$. For $k \geq 2$ the leading modes of $\Lambda^2\tilde{T}$ decide its sign.

Lemma 6.16. *Put $X_{ab} = \langle \chi_a \wedge \chi_b, \Xi\rangle$ and $Y_{ab} = \langle \chi_a \wedge \chi_b, \Upsilon\rangle$ for $0 \leq a < b \leq 5$, and let $\Xi^\perp$, $\Upsilon^\perp$ be the parts of Ξ, Υ orthogonal to all $\chi_a \wedge \chi_b$ with $a < b \leq 5$. Then for $k \geq 0$*

$$\frac{S(k)}{(\nu_0\nu_1)^k} = X_{01}Y_{01} + \sum_{(a,b)\neq(0,1)} r_{ab}^k\, X_{ab}Y_{ab} + R_k, \qquad |R_k| \leq q^k|\Xi^\perp||\Upsilon^\perp|,$$

where $r_{ab} = \nu_a\nu_b/(\nu_0\nu_1) < 1$ for $(a,b) \neq (0,1)$ and $q = u_{\rm rem}/\nu_1 < 1$. If ℓ is a lower bound for $X_{01}Y_{01}$, then $g(k) = \ell - \sum_{(a,b)\neq(0,1)} r_{ab}^k|X_{ab}Y_{ab}| - q^k|\Xi^\perp||\Upsilon^\perp|$ is nondecreasing in k and $g(k) \leq S(k)/(\nu_0\nu_1)^k$; so $g(k_*) > 0$ gives $S(k) > 0$ for every $k \geq k_*$. Moreover $|\Xi^\perp| \leq |x_C||y_\perp| + |x_\perp||y|$, where x_C and $x_\perp$ are the parts of x in and orthogonal to $\operatorname{span}\{\chi_0, \dots, \chi_5\}$*, and likewise for y.*

Proof. $\Lambda^2\tilde{T}$ is compact, self-adjoint and positive, with eigenvectors $\chi_a \wedge \chi_b$, $a < b$, and eigenvalues $\nu_a\nu_b$, where the χ_a run over an eigenbasis of

$\tilde{T}$ that extends $\chi_0, \dots, \chi_5$. A pair with an index beyond 5 has $\nu_a \nu_b \le \nu_0 u_{\mathrm{rem}}$, so the Cauchy–Schwarz inequality bounds the sum over these pairs by $(\nu_0 u_{\mathrm{rem}})^k |\Xi^{\perp}||\Upsilon^{\perp}|$. Each r_{ab} and q is less than one, so every subtracted term in g decreases in k. Finally $x \wedge y - x_C \wedge y_C = x_C \wedge y_{\perp} + x_{\perp} \wedge y$. $\square$

For an infinity object the errors of the preceding paragraph, divided by $(\nu_0 \nu_1)^k$, are bounded uniformly in k and are subtracted in g as well. If χ_1 were ρ-odd, then X_{01} would vanish for every ρ-even object pair, and the check could only fail.

Lemma 6.17. (NORM) *If every object is replaced by a positive multiple, where the factor may depend on J and on the number of factors of the object, then every value $S(k)$ and every amplitude X_{ab} is multiplied by a positive number. The same holds if $\tilde{T}$ is replaced by a positive multiple, which does not change the ratios ν_a/ν_b.*

(PAR) *ρ commutes with $\tilde{T}$, $\hat{T}$, R_{mh} and with scalars, exchanges R_{th} with R_{bh} and $\tilde{T}_{\mathrm{tv}}$ with $\tilde{T}_{\mathrm{bv}}$, and fixes w. Hence the plain objects and all objects of type mh are ρ-even, and $\langle \chi_a, x \rangle = 0$ whenever χ_a is ρ-odd and x is ρ-even.*

Proof. (NORM) Each product of two inner products in $S(k)$ contains one plain and one inserted object from each side, and X_{ab} is bilinear in the plain and the inserted object. (PAR) D and G are symmetric under $l_1 \leftrightarrow l_3$, the multiplier r_{l_2} of R_{mh} depends only on the middle label, and $w = e^{J} H^{1/2} \tilde{E}\mathbf{1}$, where $\tilde{E}\mathbf{1}$ lies in sector $m = 0$ and is symmetric in l_1 and l_3; $H^{1/2} = Q \mathsf{D}^{1/2} Q^{\top}$ commutes with ρ by the convention $Q^{(l_3,l_1)} = Q^{(l_1,l_3)}$. For ρ-odd χ_a and ρ-even x, $\langle \chi_a, x \rangle = \langle \rho\chi_a, \rho x \rangle = -\langle \chi_a, x \rangle$. $\square$

The computed frame. We work on closed subintervals $b_- \le J \le b_+$, $b_{\pm} = b_0 \pm h$, with $t = J - b_0$ and the normalization $\hat{k}_l = e^{-t} k_l(J)/k_0(b_0)$. Here $\hat{k}_0$ decreases in t, and $I_{l+\lambda}(J)/I_{\lambda}(J)$, a product of ratios $I_{\nu+1}/I_{\nu}$, increases in J (proof of Lemma 2.2). Thus, with $A = \sup \hat{k}_0$ and $\hat{k}_{0,\mathrm{lo}} = \inf \hat{k}_0$ over the subinterval,

$$\varepsilon_T \le A^3 \frac{I_{L+1+\lambda}(b_+)}{I_{\lambda}(b_+)},$$

$$\varepsilon_e := A^3 \frac{I_{L+\lambda}(b_+)}{I_{\lambda}(b_+)},$$

$$\varepsilon_W := \max\Big(\frac{2A\varepsilon_T}{\hat{k}_{0,\mathrm{lo}}}, \Big(1 + \frac{L+1}{b_-}\Big)\varepsilon_e \Big),$$

and we put $\varepsilon_* = \max(\varepsilon_T, \varepsilon_e, \varepsilon_W)$. Every object is formed in the truncated frame, with $\hat{T}$, $\hat{T}_e$ and R_e on the labels $\max(l_1, l_3) \le L$ in place of $\tilde{T}$, $\tilde{T}_e$ and R_e; for an object x of the exact frame we write x_L for the computed one and p_x for its number of factors T or $S^* R' S$. Let $[\nu_0^-, \nu_0^+]$ be the interval of ν_0, widened by ε_T, and $|\theta_0|^-$ a lower bound for $|\langle \chi_0^L, w_L \rangle|$, with χ_0^L the top eigenvector of $\hat{T}$. The insertion factor f_x of an object is 1 for plain

objects, $\max(\max_l |r_l - \alpha_e|, 1)$, the maximum over the computed labels, for horizontal ones, and $1 + |\alpha_e|$ for vertical ones.

Lemma 6.18. *Assume $0 \le \alpha_e \le 1$ for the horizontal types and $p_{\max}^2 \varepsilon_* / \nu_0^- \le 10^{-6}$, where $p_{\max} = \max_x p_x$. Put $c = \max_x \big((\nu_0^+/\nu_0^-)(1 + \varepsilon_T/\nu_0^-)\big)^{p_x}$, $\kappa = |v|\sqrt{\nu_0^+}/|\theta_0|^-$ and $\varrho = \max_x f_x |w_{p_x,L}|/|x_L|^-$, where $|x_L|^-$ is a lower bound for $|x_L|$; a mode vector χ_a enters with $p_x = 0$, $x_L := P_L \chi_a$ and $(\nu_0^+/\nu_a^-)^{1/2}$ in place of $f_x |w_{p_x,L}|$. Then for all objects x, y, the mode vectors and the representatives of the infinity objects included,*

$$|\langle x, y\rangle - \langle x_L, y_L\rangle| \le \varepsilon_{\mathrm{sl}} |x_L||y_L|, \qquad \varepsilon_{\mathrm{sl}} = \frac{\varepsilon_*}{\nu_0^-}\big(2 p_{\max} c\kappa\varrho + c^2\kappa^2\varrho^2\big)(1 + 10^{-6}).$$

Proof. Each difference between an exact and a truncated operator, $T - T_L$, $S^* R' S - S_L^* R' S_L$ and $S^* R' R'' S - S_L^* R' R'' S_L$, is $\hat{E}$ times the tail of an operator that is diagonal on the product singlets times $\hat{E}$. Its norm is at most the largest tail entry, a product of three factors $\hat{k}_{l_r}$ in which an inserted row carries $r_{l_r} - \alpha_e$ once or twice. We use $|r_l - \alpha_e| \le \max(r_l, 1)$, $r_l \hat{k}_l \le \hat{k}_{l-1}$ and $r_l < 1 + l/J$, and that both $\hat{k}_l$ and $t_l = (1 + l/J)\hat{k}_{l-1}$ decrease in l: for t_l, $l \ge 1$, the Bessel ratio bound $I_{\nu+1}(J)/I_\nu(J) < J/(\nu + \frac{1}{2} + J)$ from the proof of Lemma 2.2 and $\lambda \ge \frac{1}{2}$ give $t_{l+1}/t_l < (J+l+1)J/(J+l)^2 < 1$. It follows that these entries are at most ε_T, ε_e and ε_W, and the tails satisfy $\|P_\perp S\|^2 \le \varepsilon_T$ and $\|R' P_\perp S\|^2 \le \varepsilon_W$, where $P_\perp$ projects onto the labels with $\max(l_1, l_3) > L$. Telescoping the word of an object, with the computed factors to the left of each difference, gives $|P_L x - x_L| \le p_x(\varepsilon_*/\nu_0^-)\kappa c_x f_x |w_{p_x,L}|$ with $c_x = \big((\nu_0^+/\nu_0^-)(1 + \varepsilon_T/\nu_0^-)\big)^{p_x}$, and the tail gives $|P_\perp x| \le (\varepsilon_*/\nu_0^-)^{1/2}\kappa c_x f_x |w_{p_x,L}|$; the factor κ converts $(\nu_0^+)^{p_x}|v|$ into the norm of the computed plain object, since $|w_{p,L}| \ge (\nu_0^-)^p |\theta_0|^-$. Then $\langle x, y\rangle = \langle P_L x, P_L y\rangle + \langle P_\perp x, P_\perp y\rangle$ gives the claim. For a mode vector $x_L = P_L \chi_a$, so only its tail enters: $\chi_a = S\psi_a/\sqrt{\nu_a}$ and $\|P_\perp S\|^2 \le \varepsilon_T$ give $|P_\perp \chi_a| \le (\varepsilon_T/\nu_a^-)^{1/2} \le (\varepsilon_*/\nu_0^-)^{1/2}\kappa\,(\nu_0^+/\nu_a^-)^{1/2}$, since $\kappa \ge 1$: indeed $|\theta_0|^- \le |w_L| \le \|S_L\|\,|v| \le \sqrt{\nu_0^+}\,|v|$. □

The coefficient of an object x along a certified mode is enclosed from a computed mode vector u_a by

$$\langle \chi_a, x\rangle \in \frac{\langle u_a, x_L\rangle}{|u_a|} \pm \sqrt{2}\, s_a |x_L| \pm \varepsilon_{\mathrm{sl}} |x_L|,$$

where $\sqrt{2}\, s_a$ bounds $|P_L \chi_a - u_a/|u_a||$. The number s_a is the Krylov–Weinstein and Davis–Kahan bound for the angle between u_a and the eigenvector of $\hat{T}$, plus the transport of Lemma 6.12(iv) to $\tilde{T}$, namely $(\nu_0^+/\nu_a^-)^{1/2}\big(\sqrt{2}\,\varepsilon_T/(g_a - \varepsilon_T) + \varepsilon_T/(2\nu_a^-)\big)/\sqrt{2}$ with g_a the gap of Lemma 6.12(iv) for ν_a.

Vectors depending on J are carried as Taylor models in t of degree four, with exact floating-point coefficients and a rigorous bound for the remainder in the 2-norm; each operator step, multiplication by a polynomial in t

and linear combination adds its truncation and rounding errors to the remainder. The inner products of the coefficient vectors come from a single Gram matrix product in floating point. The product is formed by a classical algorithm (not a Strassen-type one), and the error of each entry $\langle x, y\rangle$ is at most $\gamma_{n+2}\|x\|\|y\| + (n+2)\eta$ for n states and the underflow unit η. The resulting polynomials in t are evaluated on subballs of the subinterval in ball arithmetic. With $\varepsilon_{\mathrm{sl}}|x_L||y_L|$ added to every inner product, every computed value of Lemmas 6.15 and 6.16, and every direct value $S(k)$ for $k < 2$, has a lower end that bounds the true value from below on the subball.

Theorem 6.19 (†)**.** *For $N = 3$ and every $J \in [20, 300]$, for $N = 4$ and every $J \in [20, 350]$, every $n \geq 4$ and every pair of distinct edges e, f of $P_3 \times P_n$ other than the two vertical edges of the same column,* $\mathrm{Cov}(s_e, s_f) > 0$.

Proof. By Lemma 6.13 and (NORM) it suffices to prove $S(k) > 0$ for every pair of objects, where an object is (e, p) with $p < P_e$ or the infinity object (e, ∞), which stands for every $p \geq P_e$. The interval $[20, 300]$ ($[20, 350]$ for $N = 4$) is covered by closed subintervals. On each of them, separately for $N = 3$ and $N = 4$, the following is certified.

(a) Inputs: Taylor models of degree four in J with rigorous remainders for $\hat{k}_l$, $\hat{k}_l^{1/2}$, r_l, $\tilde{E}_m$ and $\mathsf{X}^{(m)}\tilde{E}_m$ (built with the enclosures used for Theorem 6.9), and the blocks Q with the radius bound above.

(b) Spectral data: the modes $\chi_0, \ldots, \chi_5$ and u_{rem}, with the checks listed above and the gap condition of Lemma 6.12(iv); the coefficients that vanish by (PAR) are set to zero.

(c) Objects: w_p and $w'_{e,p}$ for $p < P_e$, the representatives of the infinity objects with their errors D_e, and the images of all of these under $\tilde{T}$. Every propagation step is multiplied by a polynomial $\varpi(t)$ close to $1/\nu_0$, with a rigorous lower bound $\varpi > 0$ on the subinterval, as (NORM) allows; upper bounds of ϖ enter only the error terms.

(d) Checks: for every unordered pair of objects and every $k \geq 0$ with $p + q + k \geq 2$, the direct value $S(k)$ for $k < 2$. For $k \geq 2$, the lower bound for $S(k)/(\nu_0\nu_1)^k$ from the expansion of Lemma 6.16 must be positive at $k = 2, \ldots, k_* - 1$, and the monotone bound must satisfy $g(k_*) > 0$ for some $k_* \leq 60$, which covers every $k \geq k_*$. An infinity object paired with itself stands for all indices $p, q \geq P_e$, unequal ones included, and is checked like every other pair. The pairs at $k = 0$ of the same horizontal type, which are not pairs of distinct edges, are checked as well.

The checks are divided into part A, the checks for $k \geq 2$, and part B, the checks for $k < 2$. Part B may also be carried out as two parts, Bh and Bv, which together contain its checks. A range of J is covered when each of its points meets two conditions. First, the point lies in a closed subinterval on which every check of part A holds. Second, it lies in a closed subinterval on which every check of part B holds, or else both in one on which every

check of Bh holds and in one on which every check of Bv holds. Every centring constant α_e of a horizontal type used on such a subinterval lies in $[0, 1 - 10^{-9}]$, so that Lemma 6.18 applies. For $N = 3$ and $N = 4$ the range $[20, 300]$ is covered, and for $N = 4$ the range $[300, 350]$ is covered as well. The controls of the runs and the program that formed the unions of these subintervals are described in the supplementary document. □

Corollary 6.20. *For $N = 3$ and every $J \in [20, 300]$, for $N = 4$ and every $J \in [20, 350]$, and every $n \geq 1$, $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of distinct edges of $P_3 \times P_n$. For $n \geq 4$ the inequality is strict for every pair other than the two vertical edges of the same column.*

Proof. For $n \geq 4$ this is Theorem 6.19, together with Theorem 3.2 for the two vertical edges of the same column, which share the middle vertex. For $n = 3$ it is Theorem 6.34(a). For $n = 2$ the graph $P_3 \times P_2$ is the ladder with three rungs, a 2-connected outerplanar graph whose weak dual is a path: Theorem 1.2(a) applies at $N = 4$, and Theorem 1.3 at $N = 3$, since an outerplanar graph has no K_4 minor, hence no W_4 minor, and $G \cup \{e, f\} = G$ for edges e, f of G. For $n = 1$ the graph is a path, whose two edges lie in different blocks, so $\mathrm{Cov}(s_e, s_f) = 0$ by Lemma 2.3. □

Part (b) of Theorem 6.34 follows from Corollary 6.20 together with the case $J = 0$, Theorem 1.4(ii) for $0 < J \leq 20$ and, for $n \leq 2$, Theorems 1.2, 1.3 and 3.2. The assembly of the theorem is at the end of Section 6.6.

6.6. **Strips at strong coupling.** In this subsection $N \in \{3, 4\}$, all couplings of $P_3 \times P_n$ equal $J \geq 300$, and $t = 1/J$. We prove Theorem 6.21, state Theorem 6.22 for the pairs with a horizontal edge of the middle row, and assemble Theorem 1.4(i) as well as parts (e) and (f) of Theorem 6.34. The steps S1–S8 below are computations in rigorous arithmetic, and so are the steps S4(c) and S9 of Appendix B, as well as the certificates of the middle row in that appendix.

Theorem 6.21 (†)**.** *For $N \in \{3, 4\}$, every $n \geq 2$ and every $J \geq 300$, $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of distinct vertical edges of $P_3 \times P_n$.*

Theorem 6.22 (middle row, †)**.** *Let $N \in \{3, 4\}$ and $n \geq 6$, let all couplings equal J, and let $J \geq 300$ at $N = 3$ and $J \geq 350$ at $N = 4$. Then $\mathrm{Cov}(s_e, s_f) \geq 0$ for every pair of distinct edges e, f of $P_3 \times P_n$ at least one of which is a horizontal edge of the middle row. $\mathrm{Cov}(s_e, s_f) > 0$ for every such pair that has no common vertex, is not a pair of mirror images in the vertical midline, and is not one of the end pairs (h_1, t_1), (h_1, bb_1), (h_{n-1}, t_{n-1}) and (h_{n-1}, bb_{n-1}), where t_i, h_i and bb_i are the horizontal edges of the top, middle and bottom row between columns i and $i + 1$.*

The proof is in Appendix B.

Remark 6.23. With parts (b), (e) and (f) of Theorem 6.34, Theorem 6.22 gives Theorem 1.4(i): at $N = 3, 4$ the inequality holds for every pair of edges

of every strip $P_3 \times P_n$ and for every $J \geq 0$. For $J > 20$, strict inequality is proved for the pairs named in Remark 6.35; for the other pairs it is not proved here. The case $N \geq 5$, wider grids $P_m \times P_n$ with $m \geq 4$, and nonuniform couplings are not treated here.

For $n \geq 4$, the proof of Theorem 6.21 uses S0 (pairs with a common vertex), S7 (vertical pairs away from the ends) and S8 (vertical pairs near the ends), with the inputs S1–S3 (Laplace bounds and the vacuum), S4(a) and S4(b) (twisted traces), S5 and S6 (flip pairs in the bulk). It does not use S4(c) or any part of S9. At $N = 4$ and $300 \leq J \leq 350$, S8 is replaced by Corollary 6.20. For $n \leq 3$ the theorem follows from Theorems 6.34(a), 1.2(a) and 1.3.

The middle-row pairs. The pairs of Theorem 6.22 contain a horizontal edge of the middle row. For such a pair the leading term of the covariance comes from the second of the two flip-even cluster modes of the transfer operator, since the middle-row edge meets the first of them only at order t (the first mode carries the pairs with a top-row or bottom-row edge). Appendix B bounds each class of pairs in the strict clause of Theorem 6.22 by one inequality in these two modes and in the rest of the flip-even spectrum, and it covers every n by monotonicity in the end exponents and in the separation.

Setting. We use the column transfer operator $T = K^{1/2} V K^{1/2}$ of Section 6.3, with K and V divided by e^{3J} and e^{2J}, on the $\mathrm{O}(N)$-invariant column space $\mathcal{H}$, its top eigenvalue λ_0 with unit eigenvector Ω_0, and $S = T/\lambda_0$. The flip F exchanging the top and bottom rows commutes with T and splits $\mathcal{H}$ into an even and an odd part. In the bond variables $\xi_{uv} = J(1 - \sigma_u \cdot \sigma_v)$ the kernel of K is $e^{-(\xi_T + \xi_M + \xi_B)}$, one horizontal bond in each row, and $V = e^{-(\xi_a + \xi_b)}$, with ξ_a, ξ_b the vertical bonds of the column, so that $0 < V \leq 1$. An edge e has the centred insertion vector $v_e = Q_0 T_e \Omega_0 / \lambda_0^{w_e}$, where Q_0 projects off Ω_0, T_e is T with the insertion of ξ_e, and $w_e = 2$ for a horizontal and $w_e = 1$ for a vertical edge. In the bulk of the infinite strip, $J^2 \operatorname{Cov}(s_{e_0}, s_{f_m}) = \langle v_e, S^k v_f \rangle$ with $k = m - w_e$ for the edges e_0 and f_m at offset m. Near the ends the vacuum on one side is replaced by the end vector of Section 6.3.

We name the edges by type: the horizontal edges t_i, h_i and bb_i of the top, middle and bottom row join columns i and $i+1$, and the vertical edges a_j (top–middle) and b_j (middle–bottom) lie in column j. For a pair of edges whose columns lie in $\{j, \dots, j'\}$ with $2 \leq j \leq j' \leq n-1$, the end exponents are $d_L = j - 2$ and $d_R = n - 1 - j'$. An edge with a vertex in column 1 or n lies inside an end vector, and the pairs containing such an edge are treated with the ends.

S0. Pairs with a common vertex, and mirror pairs. By Theorem 3.2, $\operatorname{Cov}(s_e, s_f) \geq 0$ at every J for every pair of edges with a common vertex. This includes the two vertical edges a_j, b_j of the same column, which share the middle vertex, and two consecutive horizontal edges of the same row. The reflection in the middle row maps t_i to bb_i, and Lemma 6.1 gives $\operatorname{Cov}(s_{t_i}, s_{bb_i}) \geq 0$ at every J, every n and every N; these mirror pairs have

no common vertex. Lemma 6.1 with the reflection θ in the vertical midline gives $\mathrm{Cov}(s_e, s_{\theta(e)}) \geq 0$ for the left-right mirror pairs $e, \theta(e)$ at every J, every n and every N.

S1–S2. Laplace bounds with insertions. Every finite-volume quantity below is an integral over the spins of a graph of two to four columns, normalised by its Gaussian main term. For such an integral I with insertion F and fixed radii $\rho_v \in (0,1)$, put $\Omega = \{tX_v \leq \rho_v^2 \text{ for all } v\}$, where $X_v = |\varphi_v|^2$ and $\varphi_v = \sqrt{J}\,\pi_v$ is the scaled chart coordinate of σ_v in the centre-of-mass gauge ($\sum_v \varphi_v = 0$), so that $tX_v = |\pi_v|^2$; here π_v is the component of σ_v orthogonal to the unit vector along $\sum_w \sigma_w$, on the hemisphere around that vector.

Proposition 6.24 (S1: Laplace bounds, †)**.** *For every integral I of this form that S5–S8 and S9(i) use and every $K \geq 0$, $I/\mathrm{main} = \sum_{k \leq K} a_k t^k + R$, where the a_k are exact Gaussian moments and $|R| \leq B_{\mathrm{rem}} + B_{\mathrm{tail}} + B_{\mathrm{ann}} + B_{\mathrm{far}}$: a remainder of order t^{K+1} built from derivative majorants and a tilted Gaussian, the Gaussian mass of the complement of Ω, an annulus term at an outer radius $\rho' > \max_v \rho_v$, and a far term for the configurations outside the outer radius or with a spin in the lower hemisphere.*

Proposition 6.25 (S2: third-order exactness, †)**.** *For the same integrals the same holds with $K = 3$ and a refined quartic remainder: $I/\mathrm{main} = \sum_{k \leq 3} a_k t^k + R$, with $a_k = \sum_{i+j=k} E_\gamma[F_i E_j]$ exact and $|R| \leq B_{\mathrm{rem}} + B_{\mathrm{tail}} + B_{\mathrm{ann}} + B_{\mathrm{far}}$, where B_{rem} now uses the expansion of e^H to third order with the bound $|\rho_4(x)| \leq x^4 e^{x_+}/24$ on its remainder.*

S3. The vacuum. Let $\hat{\psi} = K^{1/2}\varphi$ be the explicit trial vector of the vacuum, with φ a Gaussian weight times a polynomial of degree three in t, let λ_h be the Gaussian main term of the column, and put $\hat{\mu} = \lambda_h \hat{m}$ with $\hat{m} = 1 + m_1 t + m_2 t^2 + m_3 t^3$, whose coefficients are fixed together with those of $\hat{\psi}$. The residual $g = VK\varphi - \hat{\mu}\varphi$ satisfies $(T - \hat{\mu})\hat{\psi} = K^{1/2} g$. More generally, $K^{1/2} w V K^{1/2} \hat{\psi} = \hat{\mu} K^{1/2} w \varphi + K^{1/2} w g$ for every multiplication w; we call this the residual identity. It replaces a transfer step that acts on the trial vacuum by $\hat{\mu}$ and a term in g.

Proposition 6.26 (S3: vacuum, †)**.** *For $N \in \{3, 4\}$ and every $J \geq 300$: (a) $\mathrm{tr}_{\mathcal{H}}\, T \leq R_{\mathrm{tr}}\, \hat{\mu}$ with a single constant R_{tr}; (b) $\|K^{1/2}|g|\| \leq \varepsilon_r \hat{\mu} \|\hat{\psi}\|$, which implies $\|(T - \hat{\mu})\hat{\psi}\| \leq \varepsilon_r \hat{\mu} \|\hat{\psi}\|$ because K has a positive kernel; (c) $\lambda_0 \geq (1 - \varepsilon_r)\hat{\mu}$ and $\lambda_1 \leq \gamma \hat{\mu}$ with $\gamma < 1 - \varepsilon_r$, so λ_0 is simple; (d) $\|\hat{\psi} - \Omega_0'\| \leq s\|\hat{\psi}\|$, where Ω_0' is the projection of $\hat{\psi}$ onto the λ_0-eigenline.*

Part (b) bounds $|g|$ rather than g, because S5–S7 apply it to the products wg with their weights w, through $|wg| \leq |w|\,|g|$. The certificate of Proposition B.10 uses Proposition 6.26(b) in the same way.

Lemma 6.27 (S3(c+))**.** *Let $c = |\langle \Omega_0, \hat{\psi} \rangle| / \|\hat{\psi}\|$ and suppose that $c > 0$. If $\lambda_1 \leq (1 - \varepsilon_r/c)\hat{\mu}$, then $|\lambda_0 - \hat{\mu}| \leq \varepsilon_r \hat{\mu}$; in particular $\lambda_0 \leq (1 + \varepsilon_r)\hat{\mu}$.*

Proof. Write $\hat{\psi} = a\Omega_0 + e$ with $e \perp \Omega_0$, so that $|a| = c\|\hat{\psi}\| > 0$. Since T is self-adjoint and $T\Omega_0 = \lambda_0\Omega_0$, T maps the orthogonal complement of Ω_0 into itself, and $(T-\hat{\mu})\hat{\psi} = a(\lambda_0-\hat{\mu})\Omega_0 + (T-\hat{\mu})e$ is an orthogonal sum. By S3(b),

$$\varepsilon_r^2\hat{\mu}^2\|\hat{\psi}\|^2 \geq |a|^2(\lambda_0-\hat{\mu})^2 + \|(T-\hat{\mu})e\|^2.$$

If $\varepsilon_r = 0$, this gives $a(\lambda_0-\hat{\mu}) = 0$, hence $\lambda_0 = \hat{\mu}$. Let $\varepsilon_r > 0$. Dropping the last term gives $|\lambda_0-\hat{\mu}| \leq \varepsilon_r\hat{\mu}/c$. On the complement of Ω_0 the spectrum of T lies in $[0,\lambda_1]$, and $\lambda_1 < \hat{\mu}$ by the hypothesis, so $\|(T-\hat{\mu})e\| \geq (\hat{\mu}-\lambda_1)\|e\| \geq (\varepsilon_r\hat{\mu}/c)\|e\| \geq |\lambda_0-\hat{\mu}|\,\|e\|$. Inserting this into the display gives $\varepsilon_r^2\hat{\mu}^2\|\hat{\psi}\|^2 \geq (\lambda_0-\hat{\mu})^2(|a|^2+\|e\|^2) = (\lambda_0-\hat{\mu})^2\|\hat{\psi}\|^2$. □

For $J \geq 300$ the hypothesis holds. By the proof of S3(c), $\lambda_1 \leq (R_{\mathrm{tr}} - 1 + \varepsilon_r)\hat{\mu}$, and S3(d) holds with $s = \varepsilon_r/(2-R_{\mathrm{tr}}-\varepsilon_r)$, which gives $c \geq (1-s^2)^{1/2}$. The hypothesis therefore follows from

$$2 - R_{\mathrm{tr}} - \varepsilon_r - \varepsilon_r(1-s^2)^{-1/2} > 0. \tag{H}$$

The left side decreases as ε_r increases, and ε_r is nonincreasing in J; so (H) at $J = 300$ gives (H) for every $J \geq 300$, and it is certified (†). S6 and S7 use the upper bound $\lambda_0 \leq (1+\varepsilon_r)\hat{\mu}$.

S4. The twisted traces. Let P_4 be the span of $\hat{\psi}$ and the three cluster trial states e_1, e_2, e_3, $Q_4 = 1 - P_4$ and $\tau_\infty = \|Q_4TQ_4\|$. Of these states e_2 is flip-odd; $\hat{\psi}$, e_1 and e_3 span the trial space P of S9 in the flip-even parity class. Let Π project onto P and $Q = 1-\Pi$ within the flip-even parity class, let $\tilde{u}_k$ be the Ritz vectors of S on P and s_x the top cluster eigenvalue of S in this parity class, and put $\bar{c} = (\sum_k \|QS\tilde{u}_k\|^2)^{1/2}$, which bounds $\|QS\Pi\|$.

Proposition 6.28 (S4: twisted traces, †)**.** *For $N \in \{3,4\}$ and every $J \geq 300$: (a) $\mathrm{tr}_{\mathcal{H}}\,T^2 \leq U_N\lambda_h^2$ with a single constant U_N, and $\mathrm{tr}_{\mathcal{H}}\,T^2 \leq \big(U_N^{(3)}(t) + R_N\big)\lambda_h^2$, where $U_N^{(3)}$ is a polynomial of degree three in t with certified coefficient enclosures and R_N bounds a sum of remainders, each nonincreasing in J, by their values at $J = 300$; both bounds come from the two-column twisted trace $\mathrm{tr}_{\mathcal{H}}\,T^2$; (b) the one-column twisted trace with the insertion ξ_a of a vertical edge, that is, the trace on $\mathcal{H}$ of the operator T_{ξ_a} defined before Lemma B.3, is at most $\mathrm{TRB}_N\lambda_h$, with TRB_N independent of $J \geq 300$; (d) for a horizontal edge e of the top row, the two-column twisted trace with the insertion $p_L(\xi_e)$, that is, the trace on $\mathcal{H}$ of the operator obtained from $T^2 = (VK^{1/2})^*K(VK^{1/2})$ by multiplying the kernel K by $p_L(\xi_e)$, is at most $\mathrm{TRB}_N^H\lambda_h^2$, with TRB_N^H independent of $J \geq 300$; here $p_L(\xi) = \frac{N-1}{2} - \xi + \frac{\xi^2-(N-1)\xi}{2J}$.*

Step S4(c), a bound on $\|QSQ\|$ in this notation, is Proposition B.8 in Appendix B, where it is used. Part (d) is stated for the top row only; the bottom-row case follows by the flip.

S5–S7. Vertical pairs. For the vertical edges a and b, v_a and $v_b = Fv_a$ split into flip-even and flip-odd parts $v^{\pm}$. Two vertical edges of the same type at offset m have $J^2 \operatorname{Cov} = \langle v, S^{m-1} v\rangle \geq 0$ in the bulk, exactly, since $S \geq 0$. The following statements concern the flip pairs, an edge a_j and an edge $b_{j'}$. S5 and S6 are statements about the infinite strip; S7 carries them to $P_3 \times P_n$ away from the ends, and S8 treats the remaining vertical pairs.

Proposition 6.29 (S5: the binding pair, †)**.** *For $N \in \{3, 4\}$ and every $J \geq 300$, in the bulk, the edges a_j and b_{j+1} satisfy $J^2 \operatorname{Cov}(s_{a_j}, s_{b_{j+1}}) \geq c_5(N)$ for a constant $c_5(N) > 0$.*

We call the flip pair (a_j, b_{j+1}) at offset one the binding pair: by S6, a lower bound on its covariance gives a lower bound on the covariance of the flip pairs at every offset $m \geq 2$. Here c_5 is the minimum, over the certified subintervals of J, of the Bessel lower bound on $\|v^+\|^2/t^2$ minus the bound from the residual identity on $\|v^-\|^2/t^2$ and the vacuum correction.

Proposition 6.30 (S6: flip pairs in the bulk, †)**.** *For $N \in \{3, 4\}$, every $J \geq 300$ and every offset $m \geq 2$, with $R^+ = \langle v^+, Sv^+\rangle/\|v^+\|^2$ and $\rho \geq \lambda_1^{\mathrm{odd}}/\lambda_0$: the certified lower bound for R^+ is at least ρ, and therefore $J^2 \operatorname{Cov}_m \geq J^2 \operatorname{Cov}_1 \rho^{m-1} > 0$.*

Proposition 6.31 (S7: vertical pairs away from the ends, †)**.** *For $N \in \{3, 4\}$, every $J \geq 300$ and every n, every vertical pair of $P_3 \times P_n$ without a common vertex whose two end exponents are at least $d_N = 3$ (flip pairs) or at least $d'_N = 1$ (pairs of the same type) has $\operatorname{Cov} > 0$.*

The proof uses Jensen's inequality at the offsets $1 \leq m \leq m_J$ and the top even mode at $m \geq m_M$, the two ranges overlapping.

The data of S6 and S7. On a piece with left end J_{lo}, the Ritz data come from the certified Gram and transfer matrices G and T of the flip-even trial functions $(\hat{\psi}, e_1, e_3)$, taken in a fixed basis. The floating-point eigenvector matrix U of $Tu = \mu Gu$ at $t_c = 1/J_{\mathrm{lo}}$ is computed from the midpoints of the certified coefficients and used as an exact matrix, and the Ritz matrix is $A' = L^{-1}(U^{\mathsf{T}}TU)L^{-\mathsf{T}}$ with L the Cholesky factor of $U^{\mathsf{T}}GU$. All Ritz quantities are in units of λ_h, so that $\hat{\mu}$ becomes $\hat{m}$. With μ_0 the largest lower end of the diagonal of A' on the piece, the lower bound for λ_0 is $\lambda_{0,\mathrm{lo}} = \max(\mu_0, (1 - \varepsilon_r/c)\hat{m})$. The even bound on the complement of Ω_0 is

$$L_1^{\mathrm{even}} = \big(U_N^{(3)}(t) + R_N - A'^2_{i_0 i_0} - (A'_{i_y i_y} - R_y)^2 - A'^2_{zz}\big)^{1/2},$$

where $U_N^{(3)}(t) + R_N$ is the trace bound of S4(a), i_0 and i_y are the rows with the largest and the smallest diagonal entry at the midpoint of the piece (the vacuum and the lowest even cluster state), row i_y is certified to have the smallest Gershgorin lower end on the whole piece, R_y is a Gershgorin radius of row i_y valid on the whole piece and for every admissible input, and A'_{zz} is the Ritz value of the flip-odd trial function. The odd bound is the

Hilbert–Schmidt bound

$$L_1^{\mathrm{odd}} = \big(U_N^{(3)}(t) + R_N - A'^2_{i_0 i_0} - A'^2_{i_x i_x} - A'^2_{i_y i_y}\big)^{1/2},$$

with i_x the remaining even row. Each of the two arguments is the smaller of its interval value on the piece and a mean-value form about the midpoint (Lemma 6.33); where the condition on row i_y fails, the even bound is replaced by the minimum over rows, each with its own radius. With $L_1 = \max(L_1^{\mathrm{even}}, L_1^{\mathrm{odd}})$, $\rho_1 = L_1/\lambda_{0,\mathrm{lo}}$ bounds S on the complement of Ω_0 and $\rho = L_1^{\mathrm{odd}}/\lambda_{0,\mathrm{lo}}$ bounds it on the odd parity class. The amplitude a_x and the Rayleigh bound q'_x of the top even mode are computed instead in the raw Cholesky basis of G, and q'_x uses the upper bound $(1+\varepsilon_r)\hat{m}$ of Lemma 6.27.

In $P_3 \times P_n$ the covariance of two vertical edges X and Y at offset m is an exact identity in the centred insertions $B_X = (T_{\xi_X} - c_b T)/\lambda_0$, c_b the bulk mean, between the end states $\Omega_0 + \varepsilon_L$ and $\Omega_0 + \varepsilon_R$, with $\|\varepsilon_L\| \le e = R_w \rho_1^{d_L}$, R_w the relative size of the end vector off Ω_0, and likewise on the right. For two edges of the same type the Jensen step bounds the main term below by $P(P-2\eta) - \eta^2$, where P is the Jensen lower bound for $\|B_X \Omega_0\|_m$, $\|z\|_m^2 = \langle z, S^{m-1} z\rangle$, and

$$\eta = \rho_1^{(m-1)/2}\, n_B\, e, \qquad n_B \ge \|B_X\|,$$

a bound on the whole insertion, both parity parts: since $-c_b T \le T_{\xi_X} - c_b T \le T_{\xi_X}$, $\|B_X\| \le \max(\|T_{\xi_X}\|/\lambda_0, c_b)$. The step requires $P \ge \eta$ and $P(P-2\eta) - \eta^2 - t_2 > 0$, where t_2 collects the second-order terms of the identity.

S8. Vertical pairs near the ends.

Proposition 6.32 (S8: vertical pairs near the ends, †)**.** *For $N = 3$ and every $J \ge 300$, for $N = 4$ and every $J \ge 350$, and every $n \ge 4$, a pair of distinct vertical edges of $P_3 \times P_n$ without a common vertex has* Cov > 0 *if one of its edges lies in column* 1 *or* n*, or if* $\min(d_L, d_R)$ *is below the threshold of S7, d_N for flip pairs and d'_N for pairs of the same type. This includes every such pair of the short strips, in which no pair reaches those thresholds.*

In S8 we treat the flip pairs with $\min(d_L, d_R) \le 2$, for $N = 3$ and for $N = 4$, and from $\min(d_L, d_R) = 3$ on we use S7. The pairs of the same type are treated in S8 at $\min(d_L, d_R) = 0$, and in S7 from $\min(d_L, d_R) = 1$ on. On the first piece of its range (near $J = 350$ for $N = 4$ and near $J = 300$ for $N = 3$), the certificate of S8 uses the trace bound of S4(a) as certified and would fail with a slightly larger bound.

The pairs with a top-row or bottom-row edge. For $n \ge 5$, Appendix B treats the pairs outside S0 with a top-row or bottom-row edge and no middle-row horizontal edge: away from the ends by S9(h), which rests on the steps in the bulk; near the ends by S9(i) and Proposition B.13; and by whole-graph certificates for the overlapping pairs at $n = 5$ and in the rows that S9(i) excludes.

Uniformity in J. Every certified quantity above has the following form; the steps refer back to this paragraph. On each piece of a partition of $t \in (0, 1/300]$ the quantity is a polynomial in t with certified coefficient enclosures, plus remainders; each remainder is nonincreasing in J for the reason stated with it (frozen tilts, Chernoff factors, or $J^p e^{-cJ}$ with $J \geq J_0$) and is bounded on the piece by its value at the smallest J of the piece. The last piece contains $t = 0$. Radii, tilts, quadrature tolerances and Chernoff parameters are frozen at $J = 300$. Composite quantities, such as ε_r, s, τ and the Ritz data, are evaluated on each piece in interval arithmetic from these enclosures. For the centred enclosures of S5–S8 we use the following mean-value lemma.

Lemma 6.33 (mean-value form)**.** *Let $Z \subset \mathbb{R}^d$ be a box with centre z_m and half-widths $r_1, \dots, r_d$, let Θ be any set, and let $G : Z \times \Theta \to \mathbb{R}$ be such that $z \mapsto G(z, \theta)$ is C^1 on a neighbourhood of Z for every $\theta \in \Theta$. Let F be an interval that contains $G(z_m, \theta)$ for every $\theta \in \Theta$, and $D_1, \dots, D_d$ intervals with $\partial G/\partial z_j(z, \theta) \in D_j$ for all $z \in Z$ and $\theta \in \Theta$. Then $G(z, \theta) \in F + \sum_j D_j[-r_j, r_j]$ for all $z \in Z$ and $\theta \in \Theta$.*

Proof. Fix z and θ. The segment from z_m to z lies in Z, and $g(\sigma) = G(z_m + \sigma(z - z_m), \theta)$ is C^1 near $[0, 1]$. By the mean value theorem $g(1) = g(0) + g'(\sigma_0)$ for some $\sigma_0 \in (0, 1)$, with $g(0) \in F$ and $g'(\sigma_0) = \sum_j \partial_j G(\zeta, \theta)(z_j - z_{m,j}) \in \sum_j D_j[-r_j, r_j]$ for a point $\zeta \in Z$. □

Since θ is frozen at the target value of t, a true deviation needs only to lie in a single box valid on the whole piece: if a target Φ satisfies $\Phi(t) \leq G(z^*(t), \theta^*(t))$ with $z^*(t) \in Z$ and $\theta^*(t) \in \Theta$ for every t in a piece, then $\Phi(t) \leq \sup\big(F + \sum_j D_j[-r_j, r_j]\big)$ there. Such deviations include the offset of an input inside the radius of its certified ball or inside an allowance (a certified margin on both sides of a computed value that covers an error with a proven bound). The lemma is applied under the following conditions, tagged (M1)–(M6). Each true input is a polynomial in t with coefficients in certified balls plus an offset in a single box valid on the piece; the form G bounds the target at every admissible input; every constant inside G that stands for a bound holds on all of $Z \times \Theta$; and the fixed choices of G (orderings, Gershgorin rows, bases) are part of its definition (M1). F and the D_j are inclusion-isotone interval evaluations of G and of its derivatives, with the same choices and constants (M2–M4). Every square-root argument and Cholesky pivot is certified positive and every divisor nonzero, with no absolute value, maximum or minimum at a kink; otherwise the naive bound is used on that piece (M5). For a one-sided target the smaller of the naive and centred bounds is used, and for a two-sided one their intersection; an empty intersection stops the computation (M6).

Theorem 6.34 (special cases of Theorem 1.4)**.** *Let $N \in \{3, 4\}$ and let all couplings of $P_3 \times P_n$ equal J, with the notation of Theorem 1.4.*

(a) *For* $n = 3$, $\mathrm{Cov}(s_e, s_f) \geq 0$ *for every pair of edges* $e \neq f$ *and every* $J \geq 0$.
(b) *For every* $n \geq 1$, $\mathrm{Cov}(s_e, s_f) \geq 0$ *for every pair of edges* $e \neq f$ *and every* $0 \leq J \leq 300$ *(*$0 \leq J \leq 350$ *at* $N = 4$*).*
(c) *For every* $n \geq 1$ *and every* $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ *for every pair of vertical edges* $e \neq f$*, and for every pair of edges* $e \neq f$ *if* $n \leq 2$.
(d) *For* $n = 4$ *and every* $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ *for every pair of edges* $e \neq f$.
(e) *For* $n \leq 5$ *and every* $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ *for every pair of edges* $e \neq f$.
(f) *For every* $n \geq 1$ *and every* $J \geq 0$, $\mathrm{Cov}(s_e, s_f) \geq 0$ *for every pair of edges* $e \neq f$ *neither of which is a horizontal edge of the middle row.*

Theorem 1.4(i) contains parts (a)–(f). Parts (e) and (f) contain parts (a), (c) and (d), which enter the proofs of parts (e) and (f).
The strips at every coupling. Theorem 1.4 collects the results on the strips at every coupling, and Theorem 6.34 collects the special cases that enter the proof of Theorem 1.4. Part (b) combines the case $J = 0$, Theorem 1.4(ii) and Corollary 6.20. Part (c) adds Theorem 6.21. For $n \leq 2$ both parts use Theorems 1.2, 1.3 and 3.2. Pairs with a common vertex have $\mathrm{Cov}(s_e, s_f) \geq 0$ at every J by Theorem 3.2. Part (d) follows for $J \leq 300$ ($J \leq 350$ at $N = 4$) from part (b). For larger J, Theorem 6.21 covers the pairs of $P_3 \times P_4$ with two vertical edges, S0 covers the top-bottom and left-right mirror pairs as well as the pairs with a common vertex, and Proposition 6.36 covers every other pair. Parts (e) and (f) follow from part (b) for $J \leq 300$ ($J \leq 350$ at $N = 4$). For larger J and $n \leq 4$, both follow from parts (a), (c) and (d). For $n = 5$, part (e) follows from S0, Theorem 6.21 and Proposition B.14. For $n \geq 5$, part (f) follows from S0, Theorem 6.21 and the steps for the pairs with a top-row or bottom-row edge. Theorem 1.4(i) follows from part (b) for $J \leq 300$ ($J \leq 350$ at $N = 4$) and from part (e) for $n \leq 5$; for $n \geq 6$ and larger J it follows from part (f) and Theorem 6.22. The following table lists both the steps for the pairs with a top-row or bottom-row edge and the other proved steps. At $N = 4$ the steps S8 and S9(i) start at $J = 350$, and so do Propositions B.13, B.14, B.15, B.22 and B.24; Corollary 6.20 covers $300 \leq J \leq 350$.

pairs	steps	statement
a common vertex; the mirror pairs	S0	Thm. 6.21, Thms. 1.4 and 6.34
two vertical edges without a common vertex, both end exponents at least the thresholds	S7 (S5, S6)	Thm. 6.21
two vertical edges near an end	S8	Thm. 6.21
a top-row or bottom-row edge, no middle-row horizontal edge, $n \geq 6$, away from the ends	S9(h) (S4(c), S9 in the bulk)	Thm. 6.34(f)
a top-row or bottom-row edge with an edge in overlapping columns, no middle-row horizontal edge, $n \geq 6$, near an end, except four rows at $N = 4$ (rows in the sense of Appendix B) and their mirror images	S9(i)	Thm. 6.34(f)
the four rows at $N = 4$ and their mirror images, $n = 6, 7$	Prop. B.15	Thm. 6.34(f)
a top-row or bottom-row edge, no middle-row horizontal edge, separated, $n \geq 5$, near an end	Prop. B.13	Thm. 6.34(f)
a horizontal edge of the middle row, no common vertex, not a mirror pair, not an end pair, $n \geq 6$	Props. B.22, B.24 (Props. B.17, B.18)	Thm. 6.22, Thm. 1.4(i)
the end pairs of Theorem 6.22, $n \geq 6$	Lemma 6.2, S0	Thm. 6.22, Thm. 1.4(i)
$n = 5$: a horizontal edge, no common vertex, not a mirror pair	Prop. B.14	Thm. 6.34(e), (f)
$n = 4$: a horizontal edge, no common vertex, not a mirror pair	Prop. 6.36	Thm. 6.34(d)

Remark 6.35. For $J > 20$ strict inequality in Theorem 1.4 is certified for the following pairs. For $n \geq 4$ and $20 < J \leq 300$, or $20 < J \leq 350$ at $N = 4$, the inequality is strict for every pair other than the two vertical edges of the same column (Corollary 6.20). For $n \geq 4$ and $J \geq 300$, it is strict for every pair of vertical edges other than the two vertical edges of the same column (Propositions 6.31 and 6.32, with Corollary 6.20 in place of Proposition 6.32 at $N = 4$ and $300 \leq J \leq 350$). For $J \geq 300$ ($J \geq 350$ at $N = 4$), Propositions 6.36, B.1, B.10, B.13, B.14 and B.15 give strict inequality on the pairs they cover, as does the strict clause of Theorem 6.22. The whole-graph certificates bound $J^2 \operatorname{Cov}(s_e, s_f)$ below by a positive constant.

Proposition 6.36 ($P_3 \times P_4$ at strong coupling, †)**.** *For $N = 3$ and every $J \geq 300$, and for $N = 4$ and every $J \geq 350$, $\operatorname{Cov}(s_e, s_f) > 0$ for every pair of distinct edges of $P_3 \times P_4$ that contains a horizontal edge, has no common vertex and is not a mirror pair for the reflection in the middle row or in the vertical midline.*

Proof. The proof is that of Lemma 6.10, with the expansion carried to third order on the whole graph and $J_1 = 300$ ($J_1 = 350$ at $N = 4$). Both reflections preserve the model with all couplings equal, so $\mathrm{Cov}(s_e, s_f)$ is constant on each orbit of pairs under them, and the 72 pairs of the statement fall into 19 orbits. For one pair of each orbit, at $N = 3$ and at $N = 4$, the certified lower bound for $J^2 \,\mathrm{Cov}(s_e, s_f)$ is positive on every piece of $[J_1, \infty)$; the smallest of these bounds is $8.62 \cdot 10^{-6}$ at $N = 3$ and $1.06 \cdot 10^{-7}$ at $N = 4$. □

7. The computations

Every step marked † is a computer calculation in rigorous arithmetic. Recoupling coefficients, series coefficients and Gaussian moments are exact rationals. Bessel functions, exponentials and square roots are evaluated in midpoint-radius ball arithmetic [17] or in interval arithmetic [26], and every final scalar is rounded in the direction that keeps the inequality valid. Dependence on the coupling is handled by Taylor models with rigorous remainders. The certificates of the middle row combine the certified enclosures of their inputs in interval arithmetic, with decimal endpoints rounded outward. For each step the range and the arithmetic are listed in the table. The controls are deliberately falsified inputs, which the checks must reject and all of which were detected, or independent recomputations; they are listed step by step in the supplementary document, together with the record and checksum of every certified run. An interval computation certifies the enclosures of Lemmas A.2 and A.3, with evaluations at 50 and 60 digits as controls; the range of ν_k and the lower end of μ_1 follow by hand in the proof of Lemma A.2. A computation in exact rationals certifies Proposition 5.44 and the minor for the truncated weight in Remark 5.45, and it is also listed in the supplementary document. The passage to the exponential weight in that remark is by hand.

step	range	arithmetic
Lemmas A.2, A.3, $N = 3, 4$	$J \in (0, 1/100]$	intervals, monotone power series
Theorem 6.6, $N = 4$	$J \in [0.0099, 20.064]$	balls, Taylor models in J
Theorem 6.6, $N = 3$	$J \in [0.0099, 20.070]$	balls, Taylor models in J
Theorem 6.9, $N = 3, 4$	$\beta \in [20, 100.68]$	balls, Taylor models in β
Lemma 6.10, $N = 3$	$J \geq 80$	exact moments, balls, subdivision in $1/J$
Lemma 6.10, $N = 4$	$J \geq 100$	exact moments, balls, subdivision in $1/J$
Theorem 6.19, $N = 3, 4$	$J \in [20, 300]$ ($[20, 350]$ for $N = 4$)	balls, Taylor models in J
S1, S2 (Props. 6.24, 6.25), $N = 3, 4$	$J \geq 300$	exact moments, balls
S3 (Prop. 6.26), $N = 3, 4$	$J \geq 300$	exact moments, balls
S4 (Prop. 6.28), $N = 3, 4$	$J \geq 300$	exact moments, balls
S5–S8 (Props. 6.29–6.32; Theorem 6.21), $N = 3, 4$	$J \geq 300$ ($J \geq 350$ for S8 at $N = 4$)	balls, subdivision in $1/J$
S9(i) (Prop. B.1), $N = 3, 4$	$J \geq 300$ ($J \geq 350$ at $N = 4$)	balls, subdivision in $1/J$
S4(c), S9(b), (d′), (e), (f), (g), (j) (Props. B.8, B.9), $N = 3, 4$	$J \geq 300$	balls, subdivision in $1/J$
S9(h) (Prop. B.10), $N = 3, 4$	$J \geq 300$	balls, subdivision in $1/J$
separated pairs near the ends (Prop. B.13), $N = 3, 4$	$J \geq 300$ ($J \geq 350$ at $N = 4$)	end states, balls, subdivision in $1/J$
$P_3 \times P_5$; the rows of $P_3 \times P_6$, $P_3 \times P_7$ at $N = 4$ (Props. B.14, B.15)	$J \geq 300$ ($J \geq 350$ at $N = 4$)	exact moments, balls, subdivision in $1/J$ (orders three to five)
the middle row (Props. B.22, B.24), $N = 3, 4$	$J \geq 300$ ($J \geq 350$ at $N = 4$)	balls, outward-rounded intervals, subdivision in $1/J$
Lemma B.4 (last claim), $N = 3, 4$	$J \geq 300$	products of certified upper ends, rounded up
Proposition 6.36 ($P_3 \times P_4$), $N = 3, 4$	$J \geq 300$ ($J \geq 350$ at $N = 4$)	exact moments, balls, subdivision in $1/J$ (third order)
Proposition 5.44	fixed multiplicities	exact rationals
Remark 5.45	four points	exact rationals

Data availability

The deposit at Zenodo, `https://doi.org/10.5281/zenodo.24410346`, contains the programs with their inputs and outputs, the Lean sources of the formal verification, and checksums for every step marked †, together

with the evidence for Conjecture 4.11. The supplementary document *Run protocol, records and controls of the certified computations*, `https://doi.org/10.5281/zenodo.24410344`, gives the rules by which the runs of Theorem 6.19 were accepted and by which the unions of their subintervals were formed; the record, checksum, outcome and controls of every certified run, including the runs of Appendix B; and the scope and assumptions of the formal verification.

Appendix A. Proof of Theorem 6.5

Throughout, $0 < J \le J_0 := 1/100$ and $n \ge 3$. We give the proof for $N = 4$ and list the changes for $N = 3$ in Section A.8. We use the notation of Section 6.3, with $\mathbf{b}$ the boundary vector.

A.1. **Levels and the four-state block.** The level of e_s is $(t + h + b)/2$. Level 0 is spanned by $e_0 = \mathbf{1}$, and level 1 by $e_{110} = \sqrt{N}X$, $e_{101} = \sqrt{N}\,\sigma_T{\cdot}\sigma_B$ and $e_{011} = \sqrt{N}Y$. Let $\mathcal{H}_4$ be their span, P its projection, $Q = 1 - P$, and P_0, P_1 the projections onto levels 0 and 1. Put $T_4 = PTP$, $T_\infty = QTQ$ and $C = QTP$, and let $\mu_1 \ge \cdots \ge \mu_4$ be the eigenvalues of T_4 with unit eigenvectors ψ_k.

Let X be a coordinate of a uniform point of S^{N-1}, and put $k_0 = \mathbb{E}e^{JX}$, $k_1 = \mathbb{E}Xe^{JX}$, $m_2 = \mathbb{E}X^2e^{JX}$, $m_3 = \mathbb{E}X^3e^{JX}$, $A = (k_0 - m_2)/(N-1)$, $B = (Nm_2 - k_0)/(N-1)$, $\kappa_0 = k_0^3$ and $\kappa_1 = k_1^2k_0$. Given σ_M, the variables X and Y are independent, $\mathbb{E}[\sigma_T e^{JX} \mid \sigma_M] = k_1\sigma_M$ and $\mathbb{E}[\sigma_T\sigma_T^{\mathsf{T}} e^{JX} \mid \sigma_M] = A\,I + B\,\sigma_M\sigma_M^{\mathsf{T}}$. Hence, in the basis $(e_0, e_{110}, e_{101}, e_{011})$, $T_4 = D^{1/2}V_4D^{1/2}$ with $D = \operatorname{diag}(\kappa_0, \kappa_1, \kappa_1, \kappa_1)$ and

$$V_4 = \begin{pmatrix} k_0^2 & \sqrt{N}k_0k_1 & \sqrt{N}k_1^2 & \sqrt{N}k_0k_1 \\ \sqrt{N}k_0k_1 & Nk_0m_2 & Nk_1m_2 & Nk_1^2 \\ \sqrt{N}k_1^2 & Nk_1m_2 & N(NA^2 + 2AB + B^2) & Nk_1m_2 \\ \sqrt{N}k_0k_1 & Nk_1^2 & Nk_1m_2 & Nk_0m_2 \end{pmatrix}.$$

The blocks of T_X and T_{XY} follow in the same way, with $A' = (k_1 - m_3)/(N-1)$ and $B' = (Nm_3 - k_1)/(N-1)$:

$$(XV)_4 = \begin{pmatrix} k_0k_1 & \sqrt{N}k_0m_2 & \sqrt{N}k_1m_2 & \sqrt{N}k_1^2 \\ \sqrt{N}k_0m_2 & Nk_0m_3 & Nk_1m_3 & Nk_1m_2 \\ \sqrt{N}k_1m_2 & Nk_1m_3 & N(NAA' + A'B + AB' + BB') & Nm_2^2 \\ \sqrt{N}k_1^2 & Nk_1m_2 & Nm_2^2 & Nk_1m_2 \end{pmatrix},$$

$$(XYV)_4 = \begin{pmatrix} k_1^2 & \sqrt{N}m_2k_1 & \sqrt{N}m_2^2 & \sqrt{N}m_2k_1 \\ \sqrt{N}m_2k_1 & Nm_3k_1 & Nm_3m_2 & Nm_2^2 \\ \sqrt{N}m_2^2 & Nm_3m_2 & N(NA'^2 + 2A'B' + B'^2) & Nm_2m_3 \\ \sqrt{N}m_2k_1 & Nm_2^2 & Nm_2m_3 & Nm_3k_1 \end{pmatrix}.$$

In the basis e_0, $e_+ = (e_{110} + e_{011})/\sqrt{2}$, e_{101}, $e_- = (e_{110} - e_{011})/\sqrt{2}$, the first three vectors are even and e_- is odd under the top–bottom reflection, and

$T_4 = E_3 \oplus (\mu_3)$ with $\mu_3 = \kappa_1 h_-$,

$$E_3 = \begin{pmatrix} k_0^5 & a & b \\ a & \kappa_1 h_+ & \kappa_1 \omega \\ b & \kappa_1 \omega & \kappa_1 h_1 \end{pmatrix},$$

where

$$a = \sqrt{2N}\, k_0^3 k_1^2, \qquad b = \sqrt{N}\, k_0^2 k_1^3, \qquad h_\pm = N(k_0 m_2 \pm k_1^2),$$
$$\omega = \sqrt{2}\, N\, k_1 m_2, \qquad h_1 = N(NA^2 + 2AB + B^2).$$

A.2. **Scalar and level bounds.** The coordinate X has density proportional to $(1-x^2)^{(N-3)/2}$, so $\mathbb{E}X^{2p} = C_p/4^p$ at $N=4$, with C_p the Catalan numbers, and $1/(2p+1)$ at $N=3$. Therefore $k_0 - 1$, k_1/J, m_2, m_3/J and k_2/J^2 are power series in J^2 with nonnegative coefficients, and the ratios of consecutive terms beyond the first are at most $J^2/19$. For $N=4$ this gives

$$1 + \tfrac{J^2}{8} \le k_0 \le 1 + \tfrac{J^2}{8} + \tfrac{J^4}{190}, \qquad \tfrac{J}{4} \le k_1 \le \tfrac{J}{4} + \tfrac{J^3}{47}, \qquad \tfrac{1}{4} \le m_2 \le \tfrac{1}{4} + \tfrac{J^2}{15.9},$$
$$\tfrac{J}{8} \le m_3 \le \tfrac{J}{8} + \tfrac{J^3}{76}, \qquad k_2 \le \tfrac{J^2}{24}\big(1 + \tfrac{J^2}{15.9}\big).$$

Since $k_\ell = (J/2)^\ell \sum_j a_j J^{2j}$ with $a_j \ge 0$, we have $\ell/J \le r_\ell \le \ell/J + J(1 + J^2)/(2(\ell+2))$, and $\rho := r_1 - r_0 \in [1/J - J/12 - J^3,\ 1/J - J/12 + J^3]$. Also $e^{2J} \le 1.0203$ and $\rho_2 := e^{2J} - 1 - 2J \le 2.014 J^2$.

Let $\kappa_2 = \sup\{k_s :\ s \text{ of level} \ge 2\}$. The triples $(2,1,1)$ and their permutations give $k_1^2 k_2$. For $(2,2,0)$ and its permutations, $k_2^2 k_0 < (J/2)^4/24 \le k_1^2 k_2$, and every triple of level at least 3 has $k_s \le (J/2)^{2\cdot\text{level}} e^{3J^2/8}/216 < (J/2)^4/24$. Hence

$$\kappa_0 \le 1 + 0.3751 J^2, \quad \kappa_1 \in \big[\tfrac{J^2}{16}, \tfrac{J^2}{16}(1 + 0.3J^2)\big], \quad \kappa_2 = k_1^2 k_2 \le \tfrac{J^4}{384}(1 + 0.24 J^2).$$

Since $0 < V \le e^{2J}$, we have $0 \le T \le e^{2J} K$ and $\|P_A T P_B\| \le e^{2J}(\kappa_A \kappa_B)^{1/2}$ for sets A, B of basis states with $\kappa_A = \max_A k_s$. Write $V = 1 + JW + R_2$ with $W = X + Y$ and $|R_2| \le \rho_2$. Multiplication by W changes the level by at most one. Thus

$$\tau_\infty := \|T_\infty\| \le \tfrac{J^4}{376}, \qquad c_0 := \|CP_0\| \le \rho_2 \sqrt{\kappa_0 \kappa_2} \le 0.1028 J^4,$$
$$c_1 := \|CP_1\| \le (2J + \rho_2)\sqrt{\kappa_1 \kappa_2} \le 0.0258 J^4,$$

and $c := \|C\| \le 0.1286 J^4$.

A.3. **Spectral location.**

Lemma A.1. *For $\lambda > \tau_\infty$ put $F(\lambda) = T_4 + C^*(\lambda - T_\infty)^{-1} C$.*

(i) *The number of eigenvalues of T above λ equals the number of eigenvalues of $F(\lambda)$ above λ. In particular $\lambda_5 \le \tau_\infty$.*

(ii) *$\lambda_k \ge \mu_k$ for $k \le 4$, and if $\mu_k > \tau_\infty$ and $\mu_{k-1} > \mu_k$ (or $k = 1$), then $\lambda_k \le \mu_k + c^2/(\mu_k - \tau_\infty)$.*

(iii) *If $\lambda_k > \tau_\infty$ is simple with eigenvector $\varphi_k = \hat{\psi}_k \oplus \chi_k$, $\hat{\psi}_k \in \mathcal{H}_4$, then $\chi_k = (\lambda_k - T_\infty)^{-1} C\hat{\psi}_k$ and $F(\lambda_k)\hat{\psi}_k = \lambda_k \hat{\psi}_k$ with $\|F(\lambda_k) - T_4\| \le g_k := c^2/(\lambda_k - \tau_\infty)$. If $\delta_k := \min_{j\neq k} |\mu_j - \mu_k| > |\lambda_k - \mu_k|$, the component w_k of $\hat{\psi}_k$ orthogonal to ψ_k satisfies $\|w_k\| \le g_k \|\hat{\psi}_k\|/(\delta_k - |\lambda_k - \mu_k|)$.*

Proof. The lower block $T_\infty - \lambda$ of $T - \lambda$ is negative definite. Congruence by a block-triangular map turns $T - \lambda$ into $(F(\lambda) - \lambda) \oplus (T_\infty - \lambda)$, and Haynsworth's inertia additivity gives (i). The first part of (ii) is the min–max principle for the compression T_4. For the second, $F(\lambda) \le T_4 + c^2/(\lambda - \tau_\infty)$, so at $\lambda = \mu_k + c^2/(\mu_k - \tau_\infty)$ at most $k - 1$ eigenvalues of $F(\lambda)$ exceed λ. Part (iii) is the Schur-complement form of the eigenvalue equation, projected onto $\psi_k^\perp$, where $|T_4 - \lambda_k| \ge \delta_k - |\lambda_k - \mu_k|$. □

A.4. **The four-state eigen-data.**

Lemma A.2 (†). *For $N = 4$ and all $J \in (0, J_0]$, with $\hat{\mu} = \mu/\kappa_1$:*

(a) $1 + 0.625J^2 \le \mu_1 \le 1 + 0.6253J^2$ *and* $\hat{\mu}_3 - 1 \in J^2[0.1249, 0.1270]$.
(b) $\hat{\mu}_2 - \hat{\mu}_3 \ge 0.3522J$, $\hat{\mu}_3 - \hat{\mu}_4 \ge 0.3516J$, $\hat{\mu}_2, \hat{\mu}_3, \hat{\mu}_4 \in [1 - 0.36J, 1 + 0.36J]$, *and* $\mu_2 + \mu_4 - 2\mu_3 \ge 0.12\,\kappa_1 J^2$.
(c) $\psi_1 = N_1(1, \beta, \gamma, \beta)$ *with* $N_1 = (1 + 2\beta^2 + \gamma^2)^{-1/2}$, $\beta \in \frac{J^2}{8}[0.9999, 1.0001]$ *and* $\gamma \in \frac{J^3}{32}[0.99986, 1.0003]$; $\psi_3 = e_-$; *and the even cluster vectors are* $\psi_2 = (\psi_2(0), \nu_2 \cos\theta_2/\sqrt{2}, \nu_2 \sin\theta_2, \nu_2 \cos\theta_2/\sqrt{2})$ *and* $\psi_4 = (\psi_4(0), \nu_4 \sin\theta_4/\sqrt{2}, -\nu_4 \cos\theta_4, \nu_4 \sin\theta_4/\sqrt{2})$, *with* $\theta_k = \pi/4 + \epsilon_k$ *and* $\epsilon_k \in J[0.0866, 0.0886]$ *for* $k = 2, 4$, $|\psi_k(0)| \le 0.1768J^2$ *and* $\nu_k = (1 - \psi_k(0)^2)^{1/2} \in [1 - J^4/63, 1]$.

Proof. Each of $(k_0 - 1)/J^2$, k_1/J, $(k_1/J - 1/N)/J^2$, $(m_2 - 1/N)/J^2$, m_3/J and k_2/J^2 is a power series in J^2 with nonnegative coefficients. It is therefore monotone on $(0, J_0]$ and lies between its values at 0 and at J_0. The value at J_0 is summed in interval arithmetic [26], with the tail bounded by twice the last term. The quantities in the lemma are rational expressions in these remainders, in $J \in [0, J_0]$ and in $J^2 \in [0, J_0^2]$, treated as independent intervals. The leading constants are cancelled by hand before evaluation, so no interval that contains zero is a divisor. The even cluster eigenvalues are the eigenvalues of the Schur complement of E_3 onto (e_+, e_{101}), evaluated at z with $|z/\kappa_1 - 1| \le 0.6J$, and they lie in that range. For instance, the enclosures give $(\hat{\mu}_3 - 1)/J^2 \in [0.124995, 0.125006]$, $\epsilon_k/J \in [0.088354, 0.088425]$ and $(\mu_2 + \mu_4 - 2\mu_3)/(\kappa_1 J^2) \in [0.12494, 0.12506]$. Here ν_k is fixed by normalization, and $1 - \nu_k \le \psi_k(0)^2/(2 - \psi_k(0)^2)$. The lower end of μ_1 is exact: $\mu_1 \ge \langle e_0, E_3 e_0\rangle = k_0^5 \ge (1 + J^2/8)^5 \ge 1 + 5J^2/8$. □

With $\sigma = N_1^2(\beta^2+\gamma^2)$ and $\sigma_h = 2N_1^2\beta^2$, the centred four-state vectors of horizontal edges in the deep interior are

$$\tilde u^{(t)} = \rho N_1(-\sigma, \beta(1-\sigma), \gamma(1-\sigma), -\sigma\beta),$$
$$\tilde u^{(bb)} = \rho N_1(-\sigma, -\sigma\beta, \gamma(1-\sigma), \beta(1-\sigma)),$$
$$\tilde u^{(h)} = \rho N_1(-\sigma_h, \beta(1-\sigma_h), -\sigma_h\gamma, \beta(1-\sigma_h)),$$

and the vertical ones are $\tilde u^{(a)} = (T_{X,4} - c_a T_4)\psi_1/\mu_1$ with $c_a = \langle \psi_1, T_{X,4}\psi_1\rangle/\mu_1$, and its reflection $\tilde u^{(b)}$. Near an end, the four-state part of $\mathbf{b}$ is exactly $k_0^{7/2}(1, \beta_b, \gamma_b, \beta_b)$ with $\beta_b = \sqrt{N}\, k_1^2/k_0^2$ and $\gamma_b = \sqrt{N}\, k_1^3/k_0^3$.

Lemma A.3 (†)**.** *In the four-state model, $\hat a_k = \langle \psi_k, \tilde u\rangle = \frac{J}{16}(\zeta_k + \delta_k)$ for $k = 2,3,4$, where $\zeta = (1, \sqrt{2}, 1)$ for t, a, $\zeta = (1, -\sqrt{2}, 1)$ for bb, b, and $\zeta = (2, 0, 2)$ for h, and for all $J \in (0, J_0]$:*

(a) *for t, bb: $\delta_2/J \in [0.26377, 0.26579]$, $\delta_4/J \in [-0.26656, -0.26453]$; for a, b: $\delta_2/J \in [0.26313, 0.26579]$, $\delta_4/J \in [-0.26719, -0.26453]$; for h: $\delta_2/J \in [-0.17902, -0.17670]$, $\delta_4/J \in [0.17454, 0.17685]$, and $\hat a_3 = 0$;*

(b) *for every top–bottom pair, deep or at one or both ends,*

$$f(0) := \sum_{k=2}^{4} \hat a_k \hat b_k = \langle \tilde u, \tilde v\rangle \geq 0.9994\, J^4/1024.$$

The pairing in (b) is, up to $O(J^8)\rho^2$, the product of the e_{101} components of the two centred vectors, each close to $\rho\gamma \approx J^2/32$.

A.5. **Uniform error bounds.**

Lemma A.4. *For $N = 4$, every edge type and every position:*

(E0) $|\lambda_k - \mu_k| \leq 0.266J^6$ $(k = 2,3,4)$, $|\lambda_1 - \mu_1| \leq 0.017J^8$, $\lambda_5 \leq J^4/376$, $\|w_k\| \leq 12.3J^3$ *and* $\|\chi_k\| \leq 0.415J^2$ $(k = 2,3,4)$, $\|\chi_1\| \leq 0.103J^4$, *and φ_3 is odd while $\varphi_1, \varphi_2, \varphi_4$ are even.*

(E1) $\|\varepsilon_L\| \leq 0.102J^4$, *and* $\|\varepsilon_L\| \leq (\lambda_2/\lambda_1)^i\, 0.102J^4$ *at distance $i \geq 1$ from the end. Also $\langle \varphi_1, \mathbf{b}\rangle \geq 0.9999$.*

(E2) *The level-≥ 2 part of $\tilde u$ has norm at most $0.45J^3$.*

(E3) $\|\tilde u\| \leq 0.181J$.

(E4) $\|Q_5\tilde u\| \leq 0.78J^3$.

(E5) $\hat a_l = \frac{J}{16}(\zeta_l + \delta_l)$ *with $|\delta_l| \leq 0.29J$ for top and bottom types and $|\delta_l| \leq 0.20J$ for h, $\delta_2 - \delta_4 \geq 0.48J$ for top and bottom types, and $\hat a_3 = 0$ for h.*

Proof. (E0) follows from Lemmas A.1 and A.2 with the constants of Section A.2. The parities are exact because T commutes with the reflection. (E1) Write $\varphi_1 = c\, K^{1/2}V(e_0 + \eta)$ with $\|\eta\| \leq (\kappa_1^{1/2}\|P_1\varphi_1\| + \kappa_2^{1/2}\|\chi_1\|)/(\kappa_0^{1/2}\varphi_1(0)) \leq 0.0443J^3$. Then $\mathbf{b} = c^{-1}\varphi_1 - K^{1/2}V\eta$, and the splitting $V = 1 + JW + R_2$ gives $\|\mathbf{b} - \langle\varphi_1, \mathbf{b}\rangle\varphi_1\| \leq 0.101J^4$. Each transfer

step multiplies the component orthogonal to φ_1 by at most λ_2/λ_1. (E2) For a horizontal edge at an end, the level-≥ 2 part is $(R_\tau - c_e)Q\mathbf{b}/\langle\varphi_1, \mathbf{b}\rangle$, and $\|R_\tau K^{1/2} Q\| = \sup_{\text{level}\geq 2} r(\tau_s) k_s^{1/2} \leq J/9.79$ with $\|QV\mathbf{1}\| \leq \rho_2$. The deep and vertical cases are similar. (E3) and (E4) follow from Lemma A.3, (E0) and (E2). (E5) combines Lemma A.3(a) with the boundary shift $\rho\|P\varepsilon_L\| \leq 0.0163J$ (in units of $J/16$), the difference between the model and T, at most $0.0036J$, and a possible rotation within the cluster, at most $12.3J^3$. □

A.6. **Pairs at distinct positions.** Let e lie to the left of f with $m \geq 0$ free steps, as in Lemma 6.3, and write S_{cl} for the terms $l = 2, 3, 4$ of (6). By (E3) and (E4), $|S_\infty| \leq (\lambda_5/\lambda_1)^m (0.78J^3)^2$ and $|R_{CE}| \leq (\lambda_2/\lambda_1)^{2m} (0.181J)^2 (0.102J^4)^2$.

Pairs that are not top–bottom. By (E5), $\hat{a}_l \hat{b}_l \geq (J/16)^2(1 - 0.29J)^2 \geq 0.994(J/16)^2$ for $l = 2, 4$, and $\hat{a}_3\hat{b}_3 \geq 0$ (it vanishes if h is involved, and both factors have the same sign otherwise). Hence $S_{cl} \geq 1.988(J/16)^2(\lambda_4/\lambda_1)^m$. Since $\lambda_5 \leq \lambda_4$ and $\lambda_2^2 \leq \lambda_1\lambda_4$, the remainder is at most $(\lambda_4/\lambda_1)^m(0.61J^6 + 3.5 \cdot 10^{-4} J^{10})$, which is less than $80J^4\, S_{cl}$.

Top–bottom pairs. Here $S_{cl} = (\lambda_3/\lambda_1)^m F(m)$ as in the proof of Theorem 6.4. By (E0) and Lemma A.2(b), $\lambda_2 + \lambda_4 - 2\lambda_3 \geq 0.12\kappa_1 J^2 - 1.07J^6 > 0$. By (E5), $p_2 - p_4 = (J/16)^2\big[(\delta_2 - \delta_4)^{(e)} + (\delta_2 - \delta_4)^{(f)} + \delta_2^{(e)}\delta_2^{(f)} - \delta_4^{(e)}\delta_4^{(f)}\big] \geq (J/16)^2(0.96J - 0.17J^2) > 0$. Thus $F(m) \geq F(0)$ for every m, by Bernoulli's inequality. Next, $F(0) = \langle\tilde{u}, P_{cl}\tilde{v}\rangle$, where P_{cl} projects onto $\varphi_2, \varphi_3, \varphi_4$. By (E0), $\|PP_{cl}P - (1 - \psi_1\psi_1^{\mathsf{T}})\| \leq 0.54J^4$. The cross terms with the level-≥ 2 parts are at most $2 \cdot 0.047J^6$, the product of those parts is at most $0.21J^6$, and the term $\langle\tilde{u}, \psi_1\rangle\langle\psi_1, \tilde{v}\rangle$ is $O(J^{14})$ because $\tilde{u} \perp \varphi_1$. The four-state value $\langle\tilde{u}_4, \tilde{v}_4\rangle$ is $f(0)$ of Lemma A.3(b), up to $0.0023J^6 + 0.4J^8$. It follows that $F(0) \geq 0.9994J^4/1024 - 0.33J^6 \geq 9.3 \cdot 10^{-4} J^4$, and the remainder is at most $656J^2 \leq 0.066$ times S_{cl}.

A.7. **Pairs at the same position.** *Same cut.* Both energies are diagonal at the cut, so $\mathrm{Cov}(s_e, s_f)$ is the covariance of $r(\tau_s)$ and $r(\tau_s')$ under the law of the cut triple s. For (t, h) the state 110 contributes at least $(J^2/8 - 0.105J^4)^2 N_1^2 (1/J - J/3)^2 \geq 0.999J^2/64$, the states $0, 011, 101$ at least $-J^6/2000$, and the levels ≥ 2 at most $0.054J^6$ in absolute value, by (E2). Therefore $\mathrm{Cov}(s_t, s_h) \geq 0.99J^2/64$, and likewise for (h, bb). For (t, bb) the state 101 gives $\mathrm{Cov}(s_t, s_{bb}) \geq 0.9J^4/1024$.

Same column. With $q(M) = \langle\varphi_1 + \varepsilon_L, M(\varphi_1 + \varepsilon_R)\rangle$, $\mathrm{Cov}(s_a, s_b) = \big[q(T_{XY})q(T) - q(T_X)q(T_Y)\big]/q(T)^2$. In the four-state model the single-column terms cancel exactly and the value is $J^4/512 + O(J^6)$. Since $\|QT_{XY}P\| \leq 0.0134J^2$, the difference from T is at most $0.006J^6 + 0.03J^8$, so $\mathrm{Cov}(s_a, s_b) \geq 0.9J^4/512$.

Corners. A vertical corner edge is in series with the adjacent horizontal corner edge (Lemma 6.2), and the covariance of the two is the variance of

$r_c(J)$ under the law of their common label, which is nonnegative and charges the labels 0 and 1 (Corollary 3.8); so it is positive.
This proves Theorem 6.5 for $N = 4$.

A.8. **The case $N = 3$.** The structure is unchanged: positivity of T, the level selection rules, the exact four-state blocks of Section A.1 with $N = 3$, the top–bottom reflection, and the Bernoulli step. On three spins the SO(3)-invariant functions also include $\det[\sigma_T, \sigma_M, \sigma_B]$, but $\mathbf{b}$ and T are invariant under all of O(3), so the dynamics stays in the reflection-even part. At the corners the law of the common label is nonnegative and charges the labels 0 and 1 by Corollary 3.8, as at $N = 4$. Overlaps are measured in units of $\sqrt{3}J/18$ instead of $J/16$. The constants become:

quantity	$N = 4$	$N = 3$
κ_2	$\le \frac{J^4}{384}(1 + 0.24J^2)$	$\le \frac{J^4}{135}(1 + 0.3J^2)$
τ_∞; c	$\le J^4/376$; $\le 0.1286J^4$	$\le J^4/132$; $\le 0.2313J^4$
$\lvert\lambda_k - \mu_k\rvert$; $\lVert w_k\rVert$; $\lVert\chi_k\rVert$ ($k \ge 2$)	$0.266J^6$; $12.3J^3$; $0.415J^2$	$0.484J^6$; $9.3J^3$; $0.53J^2$
$\lVert\varepsilon_L\rVert$; $\lVert\tilde{u}\rVert$; $\lVert Q_5\tilde{u}\rVert$	$0.102J^4$; $0.181J$; $0.78J^3$	$0.215J^4$; $0.28J$; $0.59J^3$
$\lvert\delta_l\rvert$ (top, bottom); $\delta_2 - \delta_4$	$0.29J$; $\ge 0.48J$	$0.4J$; $\ge 0.67J$
$(\mu_2 + \mu_4 - 2\mu_3)/(\kappa_1 J^2)$ †	$[0.12494, 0.12506]$	$[0.19990, 0.20010]$
$f(0)$ †; $F(0)$	$\ge 0.9994\frac{J^4}{1024}$; $\ge 9.3 \cdot 10^{-4}J^4$	$\ge 0.9994\frac{J^4}{243}$; $\ge 0.99\frac{J^4}{243}$
remainder/main: not top–bottom; top–bottom	$80J^4$; $656J^2$	$19J^4$; $85.4J^2$
same column; $\lVert QT_{XY}P\rVert$	$\ge 0.9\frac{J^4}{512}$; $0.0134J^2$	$\ge 0.9999\frac{2J^4}{243} - 0.02J^6 - 0.03J^8$; $0.0253J^2$

At $N = 3$ the same-cut leading term (t, h) is $J^2/27$, against corrections of at most $0.2J^6$. The four-state enclosures at $N = 3$ are certified in the same way as Lemmas A.2 and A.3, for instance $(\hat{\mu}_3 - 1)/J^2 \in [0.133326, 0.133342]$, $\epsilon_k/J \in [0.106020, 0.106115]$, and, for the vertical types, $\delta_2/J \in [0.36210, 0.36633]$ and $\delta_4/J \in [-0.36856, -0.36433]$.
This proves Theorem 6.5 for $N = 3$.

Appendix B. Certified steps for Theorem 1.4(i) and parts (e) and (f) of Theorem 6.34

At strong coupling, for Theorem 1.4(i) and for parts (e) and (f) of Theorem 6.34, this appendix proves the certified steps not contained in Section 6.6. Proposition B.1 is S9(i), for the overlapping pairs near the ends. Propositions B.8 and B.9 are the steps in the bulk: S4(c), and S9(b), (d′), (e), (f), (g), (j). Proposition B.10 is S9(h), for the pairs away from the ends, and Proposition B.13 treats the separated pairs near the ends. Propositions B.14 and B.15 are the whole-graph certificates for $P_3 \times P_5$ and for

the rows that S9(i) excludes. The last part proves Theorem 6.22: Propositions B.17 and B.18 bound each pair of a channel class or of an interior same-bond orbit by one inequality, and Propositions B.22 and B.24 certify these inequalities for $6 \le n \le 11$ and for every $n \ge 6$, respectively. The notation is that of Section 6.6, and every certified quantity below has the form described in that section under the heading "Uniformity in J".

Overlapping pairs near the ends. Consider a pair of distinct edges of $P_3 \times P_n$, $n \ge 6$, outside S0, with a top-row or bottom-row edge and no middle-row horizontal edge. With $k = m - w_e$ as in Section 6.6, the pair is separated if $k \ge 0$ and overlapping if $k < 0$. The overlapping pairs form three families, each together with its image under the flip: D3, the pairs $(t_i, bb_{i\pm1})$; D4, the pairs (t_i, b_i); and D4M, the pairs (t_i, b_{i+1}). The two insertions of such a pair form a single joint centred insertion on a window of two or three columns. We say that these pairs are of the joint kind. Proposition B.1 and Proposition B.10 each state their threshold on $\min(d_L, d_R)$ for the joint kind. A row is a family, a value of N and a pair (d_L, d_R) of end exponents, as in the paragraph "The rows" below.

Proposition B.1 (S9(i): overlapping pairs near the ends, †)**.** *For $N = 3$ and every $J \ge 300$, for $N = 4$ and every $J \ge 350$, and every $n \ge 6$, an overlapping pair as above has $\mathrm{Cov}(s_e, s_f) > 0$ if one of its edges has a vertex in column 1 or n, or if $\min(d_L, d_R)$ is below the threshold of the joint kind,* 3 *at $N = 3$ and* 4 *at $N = 4$; at $N = 4$ the pairs of the rows $(0, 1)$ and $(1, 1)$ of D3 and of the rows $(1, 1)$ of D4 and D4M are excluded, together with their mirror images.*

As in S8, the proof near the ends uses exact end states. Away from the ends, S9(h) covers the pairs of the joint kind with $\min(d_L, d_R) \ge 3$ at both values of N: at $N = 3$ these are exactly the joint pairs that Proposition B.1 leaves, and at $N = 4$ the pairs with $\min(d_L, d_R) = 3$ lie in both steps. The separated pairs ($k \ge 0$) near the ends are treated in Proposition B.13 by the end-state method of S8 with horizontal insertions. Step S9(i) itself concerns the overlapping pairs, for which the certificate is organised by rows.

The rows. The mirror maps D3 to itself and D4 to D4M, and it exchanges d_L and d_R, so it suffices to take $d_L \le d_R$. A row consists of a family, a value of N and a pair (d_L, d_R) allowed by $n \ge 6$, where either $d_R \le 6$ or the row stands for every $d_R \ge 7$. For each row and each piece of the partition of S8 a certified ball lo bounds the covariance of the row from below at every J of the piece. The row is covered on the piece when the lower end of lo is positive; no margin beyond positivity is required.

The trial vacuum in the windows. Near the ends the windows are evaluated with the normalised trial vacuum $\hat{u} = \hat{\psi}/\|\hat{\psi}\|$ of S3 in place of Ω_0, and $\mu_R = \langle \hat{u}, T\hat{u} \rangle$. With the sign of Ω_0 chosen so that $\langle \Omega_0, \hat{u} \rangle > 0$, S3(d) gives $\hat{u} - \Omega_0 = -\delta\,\Omega_0 + d_\perp$ with $d_\perp \perp \Omega_0$, $\|d_\perp\| \le s'$ and $\delta = \|\hat{u} - \Omega_0\|^2/2 \le s'^2/2$, where $s' = s(1 + s)$. These bounds give the cost of replacing $\hat{u}$ by the vacuum in each window. Call an end of a window modelled if the

true end F_i is replaced there by its model $\tilde{F}_i$ (see below). Each modelled end contributes an error bound in signed form: by the Cauchy–Schwarz form when no horizontal bond sits at the transfer step of that end, and by Lemma B.2 below when one does. These error bounds use the certified inequalities $\|K^{1/2}|fg|\| \le \varepsilon(f)\hat{\mu}\|\hat{\psi}\|$ for the end weights f, with the residual g of S3 and $\varepsilon(1) = \varepsilon_r$.

A bond at a modelled end. For a horizontal edge e on row u between two columns, write $K \circ \xi_e$ for the operator with kernel $K(\sigma,\tau)\, J(1-\sigma_u\cdot\tau_u)$. Put $A = (N-1)/2$, and for real c let $m_c = A + |A-c|$ and $P_c = K \circ (m_c - \xi_e)$.

Lemma B.2 (a bond at a modelled end)**.** *For every real c: (R1) $P_c \ge 0$ and $|K \circ (\xi_e - c)| \le P_c$; (R2) $|\langle a, K \circ (\xi_e - c)\, b\rangle| \le \langle a, P_c a\rangle^{1/2}\langle b, P_c b\rangle^{1/2}$ for real a and b; (E-R) $\langle v, P_c v\rangle \le (2A + |A-c|)\,\langle |v|, K|v|\rangle$ for real v.*

Proof. On row u both $K \circ \xi_e$ and K are zonal. On the spherical harmonics of degree l of that row they act by $a_l c_l$ and c_l, respectively, times the kernel of the other two rows, with $c_l > 0$ and $a_l = J(1 - R_{l+\lambda}(J)) - l$, where $R_\mu = I_{\mu+1}/I_\mu$ is the ratio of Lemma 2.2. The bound $R_\mu(J) > 1 - (\mu + \frac{1}{2})/J$ for $\mu \ge \frac{1}{2}$ gives $a_l < A$ for every l; at $N = 3$ and $l = 0$ it reads $\coth J - 1/J > 1 - 1/J$. So $K \circ \xi_e \le AK$ as forms, and hence $|a_l - c| \le m_c - a_l$ for every l: for $c \le A$ by the two signs of $a_l - c$, and for $c \ge A$ with equality. This is (R1). The operators $K \circ (\xi_e - c)$ and P_c commute, which gives (R2). For (E-R), both kernels are nonnegative, so $\langle v, P_c v\rangle = m_c\langle v, Kv\rangle - \langle v, (K \circ \xi_e)v\rangle \le m_c\langle |v|, K|v|\rangle + \langle |v|, (K \circ \xi_e)|v|\rangle$, and $K \circ \xi_e \le AK$ bounds the last term by $A\langle |v|, K|v|\rangle$. □

For an end with weight f, (E-R) gives $\langle fg, P_c fg\rangle \le (2A + |A - c|)\,(\varepsilon(f)\hat{\mu}\|\hat{\psi}\|)^2$, so no integral beyond those of S3(b) is needed for Lemma B.2. An uncentred bond is written $\xi_e = (\xi_e - c) + c$, and the constant part c is bounded like a step without a bond.

Operator bounds at a modelled end. Let $Z = K^{1/2}V^{1/2}$, so that $T = ZZ^*$, and for a bounded multiplication f that commutes with the rotations put $T_f = ZfZ^*$. Let ξ_b be the vertical bond between the middle and bottom spins of a column, and let $\mathrm{A2}_N$ be the bound for the one-column twisted trace with the insertion ξ_b^2, normalised as TRB_N in S4(b). The bound is uniform in $J \ge 300$; it is certified at $N = 3$ and $N = 4$ by runs listed in the supplementary document.

Lemma B.3 (operator factors)**.** *(a) If $f \ge 0$, then $\|T_f\|_{\mathcal{H}} \le \mathrm{tr}_{\mathcal{H}}\, T_f$; in particular $\|T_{\xi_b}\|_{\mathcal{H}} \le \mathrm{TRB}_N\lambda_h$ and $\|T_{\xi_b^2}\|_{\mathcal{H}} \le \mathrm{A2}_N\lambda_h$. (b) For a horizontal bond between two columns that carry V, $G(c) = V^{1/2}(K \circ \xi - cK)V^{1/2}$ satisfies $\|G(c)\| \le (A + |c|)\lambda_0$. (c) $\lambda_0 \le \mu_R/(1 - s^2)$.*

Proof. (a) $T_f = (Zf^{1/2})(Zf^{1/2})^* \ge 0$ maps $\mathcal{H}$ into itself, so its norm there is at most its trace there, $\int dg \int d\sigma\, f(\sigma)V(\sigma)K(\sigma, g\sigma)$. For ξ_b and ξ_b^2 these are the twisted traces of S4(b) and of $\mathrm{A2}_N$, since the reflection in the middle row preserves V, K and $\mathcal{H}$ and exchanges ξ_a and ξ_b. (b) The kernel

of $V^{1/2}(K \circ \xi)V^{1/2}$ is symmetric and nonnegative, so its norm is its top eigenvalue, and $K \circ \xi \le AK$ (proof of Lemma B.2) bounds that eigenvalue by $A\|V^{1/2}KV^{1/2}\| = A\lambda_0$; the term $cV^{1/2}KV^{1/2}$ has norm $|c|\lambda_0$. (c) Write $\hat{u} = c_0\Omega_0 + e$ with $e \perp \Omega_0$. Then $\mu_R \ge c_0^2\lambda_0$, and $c_0^2 \ge 1 - s^2$ by S3(d). $\square$

At a modelled end at column 1 of a chain of three columns, the rest is $\Theta' = f_2VK_{23}(f_3F)$ with $F = VK\varphi$, where K_{23} is the step between the second and third columns, with or without a bond. Then $K^{1/2}\Theta' = Zf_2G_{23}f_3Z^*\hat{\psi}$ with $G_{23} = V^{1/2}K_{23}V^{1/2}$, and hence

$$\|K^{1/2}\Theta'\| \le \|T_{f_2^2}\|^{1/2}\,\|G_{23}\|\,\langle\hat{\psi}, T_{f_3^2}\hat{\psi}\rangle^{1/2},$$

with every factor uniform in J by Lemma B.3. When both ends of a window are modelled, write $F_i = \tilde{F}_i + \Delta_i$ for the true end F_i, its model $\tilde{F}_i$ and the residual Δ_i, and R for the middle of the window. The identity

$$W - \tilde{W} = \langle\Delta_1, RF_3\rangle + \langle F_1, R\Delta_3\rangle - \langle\Delta_1, R\Delta_3\rangle$$

is exact. Each first-order term has a rest that ends in a column carrying V, so Lemma B.3(b) applies to every bond inside it. The last term is of second order. A bond inside it is bounded pointwise by $0 \le \xi \le 2J$, and we check, for each residual bound, the rate condition under which the factor $2J$ stays bounded on every piece.

The window F_A. For D4 and D4M the in-span components of the left chain enter through the window F_A that starts at the trial end states c_j of the chain of S8; F_A has no bond on its first link. Its rest at c_j is $\Theta'_j = f_1VK_{01}c_j$, where K_{01} is the plain transfer step. With $g_{jj} = \langle c_j, K_{01}c_j\rangle$,

$$\|K^{1/2}\Theta'_j\| \le \|T_{f_1}\|\,g_{jj}^{1/2} \quad\text{and}\quad \langle\Theta'_j, P_c\Theta'_j\rangle \le (2A + |A - c|)\,\lambda_0\,\|T_{f_1^2}\|\,g_{jj}$$

at an end without and with a bond at its transfer step, respectively; for an uncentred bond the centre adds $|c|\,\|T_{f_1}\|\,g_{jj}^{1/2}$ to the error bound. The values g_{jj} for $j = 1, 2, 3$ are two-column integrals in the form of the entries of S8, with certified upper ends on every piece.

The rows with $d_L \ge 1$. Here $\mathrm{lo} = C_{\mathrm{lo}} + T_{\mathrm{end}} - B_{\mathrm{end}} - P$, where C_{lo} is the certified lower end of the bulk covariance of the pair at the piece, T_{end} is the sum of the certified end terms, B_{end} is the sum of the error bounds of both ends, and P is the product of the bounds on $|m_e|$ and $|m_f|$ that the display below gives. The sum $C_{\mathrm{lo}} + T_{\mathrm{end}} - B_{\mathrm{end}}$ bounds from below the window expectation $E[X_c]$ of the centred insertion $X_c = (\xi_e - c_e)(\xi_f - c_f)$, for every centre in the certified ball. For D3 the end terms use a certified bound q on the norm of the part of the window orthogonal to the vacuum. We call this use of q route 0, and q the route-0 bound. For D4 and D4M the end terms use the corresponding bounds for the windows of both sides and the window F_A. The two-sided in-span term is at most $n_X e_L e_R$, where e_L and e_R bound the in-span components of the chains of S8 on the two sides and $n_X = \min(n_X^{\mathrm{dir}}, n_X^{\mathrm{HS}})$, with n_X^{dir} a direct certified bound on $\|X_c\|$ and n_X^{HS} the certified bound on $\|X_c\|$ from the Hilbert–Schmidt bound of the class

operator. Since $E[X_c] = \mathrm{Cov}(\xi_e, \xi_f) + E[\xi_e - c_e]\, E[\xi_f - c_f]$, the covariance of the row is at least $C_{\mathrm{lo}} + T_{\mathrm{end}} - B_{\mathrm{end}} - |m_e||m_f| \geq \mathrm{lo}$, where $m_e = E[\xi_e] - c_e^*$ is the shift of the mean of the bond from its bulk value c_e^*, and likewise m_f. Write the chains of S8 as $a = \Omega_0 + \varepsilon_L + r_L$ and $b = \Omega_0 + \varepsilon_R + r_R$, with in-span parts of norms at most e_L, e_R and residuals of norms at most ρ_L, ρ_R. Then

$$|m_e| \leq \frac{O_e + W_e e_L e_R + (\rho_L + \rho_R) U_e + W_e(\rho_L e_R + e_L \rho_R + \rho_L \rho_R)}{1 - (e_L + \rho_L)(e_R + \rho_R)},$$

where O_e is the one-sided in-span part and, for the window operator X_e of the bond centred at c_e^* in the normalisation of the chain, $U_e \geq \max(\|Q_0 X_e \Omega_0\|, \|Q_0 X_e^* \Omega_0\|)$ and $W_e \geq \|X_e\|$. For a horizontal bond $W_e = A + |c_e^*|$ by Lemma B.3(b), and $U_e = p_t$, the certified bulk value for the top bond, which serves the bottom bond by the reflection; for a vertical bond $W_e = a_N + |c_e^*|$ and $U_e = a_N/2$ by Lemma B.4 below. The bound $O_e \leq (e_L + e_R) U_e$ is used for every class; for D4 and D4M the exact values from the single-bond windows of the family of F_A are available but not used.

The base at $d_L \leq 0$. For $d_L \leq 0$ the windows are evaluated against the exact left end state a and the right chain b of S8. For an insertion Y in the window put $E[Y] = \langle a, Yb\rangle / \langle a, X_1 b\rangle$, where X_1 is the window with the insertion 1. Then $E[X_c] = \mathrm{Cov}(\xi_e, \xi_f) + E[\xi_e - c_e]\, E[\xi_f - c_f]$, so the base is the truncated covariance

$$\mathrm{Cov}_{\mathrm{main}} = E_{\mathrm{main}}[X_c] - m_e m_f,$$

with $m_e = E_{\mathrm{main}}[\xi_e - c_e]$ and $m_f = E_{\mathrm{main}}[\xi_f - c_f]$, evaluated in this centred form. Here $E_{\mathrm{main}}[Y] = N_Y / D$ is computed exactly on the span of the right chain of S8,

$$N_Y = W_Y[\mathrm{cp}_0] + \sum_{k \geq 1} \alpha_k W_Y[\mathrm{cp}_k], \qquad D = W_1[\mathrm{cp}_0] + \sum_{k \geq 1} \alpha_k W_1[\mathrm{cp}_k],$$

where $W_Y[\mathrm{cp}_k]$ is the window with the insertion Y that ends in the trial state cp_k of the chain of S8, all windows of a row share a common normalisation, and the α_k are the components of the chain. No transfer factor enters, since the normalisations cancel in the ratios. If R_X, R_e and R_f bound $|E[X_c] - E_{\mathrm{main}}[X_c]|$, $|E[\xi_e - c_e] - m_e|$ and $|E[\xi_f - c_f] - m_f|$, then

$$|\,\mathrm{Cov}(\xi_e, \xi_f) - \mathrm{Cov}_{\mathrm{main}}\,| \leq R_X + |m_e| R_f + |m_f| R_e + R_e R_f,$$

and lo is the lower end of $\mathrm{Cov}_{\mathrm{main}}$ less this bound. For $\bullet \in \{X, e, f\}$, $R_\bullet = R_\bullet^{\mathrm{ch}} + R_\bullet^{\mathrm{id}} + R_\bullet^{\mathrm{vac}}$, where the three terms bound the residual of the chain of S8, the use of the residual identity of S3 at the end of the window and the step from $\hat{u}$ to the vacuum, each relative to the vacuum denominator $\langle a, X_1 \Omega_0\rangle$. The denominator of each $R_\bullet$ allows for the difference between D and the vacuum denominator, and its term of second order in s' involves the window with the vacuum on its right, which the main term does not bound; that term uses a bound on $\|X_c^* \hat{a}\|$ instead, and on the single-bond windows

in R_e and R_f, with $\hat{a}$ the multiple of a normalised by $\langle \hat{a}, X_1\Omega_0\rangle = 1$. At $d_L = -1$ each window starts at the free end column and ends in the trial state. At $d_L = 0$ the window of D4 or D4M is preceded by the free column, and that of D3 consists of the free column and two interior columns, and the residual identity of S3 replaces the transfer step at the right end of the window.

Single-bond inputs. Write X_e for the window with the insertion $\xi_e - c_e$ alone. The bound R_e needs $q_e = \|Q_0 X_e^* \hat{a}\|$, and R_f the same for f. At $d_L = -1$ both come from single-bond forms of the lemmas that bound the joint window, with the insertion of the other bond removed. At $d_L = 0$, $q_e \le p_e + W_e Q_a$, where $p_e \ge \max(\|Q_0 X_e \Omega_0\|, \|Q_0 X_e^* \Omega_0\|)$ bounds the bulk parts of the bond, $W_e \ge \|X_e\|$, and Q_a is the bound of S8 on $\|Q_0\hat{a}\|$. For a horizontal bond $W_e = A + |c_e|$ by Lemma B.3(b), and $p_e = p_t$, the certified bulk value for the top bond, which serves the bottom bond by the reflection; for the vertical bond the following lemma gives both.

Lemma B.4 (vertical bond, †)**.** *Let ξ_b be the vertical bond between the middle and bottom spins of a column, $A_b = T_{\xi_b}/\lambda_0$ and $a_N = \mathrm{TRB}_N \lambda_h/\lambda_0$. Then $0 \le A_b \le a_N$ on $\mathcal{H}$, and $\|Q_0 A_b \Omega_0\| \le a_N/2$. For $N = 3, 4$ and every $J \ge 300$, $a_N < 2$.*

Proof. Since $\xi_b \ge 0$, $T_{\xi_b} \ge 0$, and Lemma B.3(a) gives $\|A_b\|_{\mathcal{H}} \le a_N$. For $0 \le B \le \beta$ one has $B^2 \le \beta B$, so with $c = \langle \Omega_0, B\Omega_0\rangle$, $\|Q_0 B\Omega_0\|^2 = \langle \Omega_0, B^2\Omega_0\rangle - c^2 \le c(\beta - c) \le \beta^2/4$. The last claim follows from the certified value of TRB_N in S4(b) and a certified bound on λ_h/λ_0, uniform in $J \ge 300$, from a Rayleigh quotient of $V^{1/2} K V^{1/2}$. □

By Lemma B.4, $W_e = a_N + |c_e|$ and $p_e \le a_N/2$ for the vertical bond, at $d_L \le 0$ and at $d_L \ge 1$ alike. These bounds cover the adjoint windows too. Since $Z^* Z Z^* \Omega_0 = \lambda_0 Z^* \Omega_0$, each of $X_e\Omega_0$ and $X_e^*\Omega_0$ is $B\Omega_0$ or $SB\Omega_0$, with B the bond's operator in the normalisation of the chain; $S = T/\lambda_0$ commutes with Q_0 and has norm at most 1, so both norms are at most $\|Q_0 B\Omega_0\|$, which is bounded by p_t and $a_N/2$. The allowance $4s'$ of S8 on its centres rests on Lemma B.4 as well: since $a_N < 2$ and $s' < 1/3$, the shift of a centre is at most $a_N s'(1 + 2s')/(1 - s'^2) < 4s'$. Here $s' < 1/3$ because the certified s of S3(d) is below 10^{-5} at every $J \ge 300$.

D3 at $d_L = 0$. In this window the trial state cp_k sits on the last column of the observable, one column further from the right end than in the other windows. We therefore pair the window X that ends at the right half-gap of the observable with the right chain vector of S8 $b = b(d_R - 1)$ at that half-gap, keep only the vacuum part of the main term, and count all of $Q_0 b$ in the error. Write $a = \Omega_0 + \varepsilon_0$ with $\|\varepsilon_0\| \le R_w$, let e_1 and e_3 be the bounds of S8 on $\|S\varepsilon_0\|$ and $\|S^3\varepsilon_0\|$, and put $r_X = \|Q_0 b(d_R - 1)\|$ and $r'_X = \|Q_0 b(d_R)\|$.

Since $S \geq 0$ and $SQ_0 b(d) = Q_0 b(d+1)$ along the chain of S8,

$$\|Z^* \varepsilon_0\|^2 = \lambda_0 \langle \varepsilon_0, S\varepsilon_0 \rangle \leq \lambda_0 R_w e_1,$$
$$\|Z^* Q_0 b\|^2 = \lambda_0 \langle Q_0 b, SQ_0 b \rangle \leq \lambda_0 r_X r'_X.$$

The window $X = ZG_T G_B Z^*/\lambda_0^3$ has Z^* at both ends. With $F = (A + |c_t|)(A + |c_{bb}|)$, $q_w = (R_w e_1)^{1/2}$, U the route-0 bound and n_X as above, the $Q_0 b$ part of the error of the numerator is at most

$$\min\big(U r_X,\, F(r_X r'_X)^{1/2}\big) + \min\big(F q_w (r_X r'_X)^{1/2},\, n_X R_w r_X\big),$$

and the $Q_0 b$ part of the error of the denominator is at most $e_3 r_X$. The same damping gives $q \leq U + \min(F q_w,\, n_X R_w)$, and R_e and R_f take the same form with each single bond. The window of D4M also begins with Z^*, so its q is damped on the left as well, with a Hilbert–Schmidt bound on ZM, where M multiplies by the centred vertical bond, and $A + |c_t|$ in place of F. The residual term at the end of the window. For D3 at $d_L = 0$ the window meets the trial vacuum through $VK\varphi = \hat{\mu}\varphi + g$ (S3), and the part that comes from g is $G = \langle h, K_{(f_{bb})} g \rangle$ with $h = V_1 K_{(f_t)} V_0 K^{1/2} a$. Here f_t and f_{bb} are the factors of the insertion on the two gaps, $K_{(f)}$ is the transfer step with the insertion f, and V_0, V_1 are the potentials of the two columns. By Cauchy–Schwarz through $V_1^{1/2}$, relative to the vacuum denominator,

$$|G| \leq \vartheta\, Y\, T, \qquad \vartheta = \frac{1}{1 - \varepsilon_r},$$
$$Y = \frac{\|V_1^{1/2} K_{(f_t)} V_0 K^{1/2} a\|}{\lambda_0^{3/2}}, \qquad T = \frac{\|V_1^{1/2} K_{(f_{bb})} |g|\|}{\lambda_0^{1/2} \hat{\mu} \|\hat{\psi}\|}.$$

For $f_t = 1$, $Y \leq (1 + Q_a^2)^{1/2}$. For the centred top bond, Lemma B.3(b) gives

$$Y \leq Y_c = \big(E2_t - 2b_{\mathrm{lo}}^2 + b_{\mathrm{hi}}^2\big)^{1/2} + (A + b_{\mathrm{hi}}) Q_a,$$

where $E2_t$ bounds $E_{\Omega_0}[\xi_t^2]$ and the interval $[b_{\mathrm{lo}}, b_{\mathrm{hi}}]$, with $b_{\mathrm{lo}} \geq 0$, contains c_t and the bulk mean of ξ_t. No tail of the top bond is needed. In the damped form q_w replaces Q_a in Y_c, and $Y \leq (1 + R_w e_3)^{1/2}$ for $f_t = 1$. On the side of g, $T \leq \varepsilon_r$ for $f_{bb} = 1$; for the centred bottom bond, $T \leq \varepsilon_r L$ on $\{\xi_{bb} \leq L\}$ with L at least the upper end of c_{bb}, and the part on $\{\xi_{bb} > L\}$ is bounded as follows.

Lemma B.5 (one-sided tail)**.** *Let $c_{bb} \in [c_{\mathrm{lo}}, c_{\mathrm{hi}}]$ with $c_{\mathrm{lo}} \geq 0$, and $L \geq c_{\mathrm{hi}}$. Let c_K bound $\|K\|$ through the Schur norms of the three rows, $\kappa = c_K/((1 - \varepsilon_r)\lambda_h)$ at $J = 300$, $\Phi_\varepsilon \geq c_K \|\varphi\|^2/\|\hat{\psi}\|^2$, and let $\tilde{\tau}_0(L)$ bound the Schur norm of the kernel $\xi e^{-\xi}$ of the bottom row on $\{\xi > L\}$ relative to that of $e^{-\xi}$, uniformly in $J \geq 300$. Then the part of G from $\{\xi_{bb} > L\}$ is at most $Y \tau_1(L)$, where $\tau_1(L) = \vartheta\, \kappa^{1/2}(1 + \kappa)\, \Phi_\varepsilon^{1/2}\, \tilde{\tau}_0(L)$.*

Proof. On $\{\xi_{bb} > L\}$, $0 < \xi_{bb} - c_{bb} \leq \xi_{bb}$, and the kernels are nonnegative, so the cut step is dominated by the step with the insertion $\xi_{bb} \mathbf{1}\{\xi_{bb} > L\}$, whose norm is at most $k_1^3 \tilde{\tau}_0(L)$ by Schur's test, with k_1 the Schur norm of one

row and $k_1^3 \le c_K$. Since $0 \le V_1 \le 1$ and $\|g\| \le (c_K + \hat{\mu})\|\varphi\|$, the tail part of T is at most $\tilde{\tau}_0(L)\, k_1^3 (c_K + \hat{\mu})(\Phi_\varepsilon / c_K)^{1/2} / (\lambda_0^{1/2} \hat{\mu})$. Finally $c_K/\lambda_0 \le \kappa$ and $c_K/\hat{\mu} \le \kappa$ for every $J \ge 300$, because c_K/λ_h does not depend on J, $\lambda_0 \ge (1-\varepsilon_r)\hat{\mu}$, $\hat{m} \ge 1$ and ε_r is nonincreasing in J. This bounds the tail part of T by $\kappa^{1/2}(1+\kappa)\Phi_\varepsilon^{1/2}\tilde{\tau}_0(L)$, and $|G| \le \vartheta Y T$ completes the proof. □

The rows with $d_R \ge 7$. In the row that stands for every $d_R \ge 7$, the right side enters only through the chain bound $R_w \rho_1^7$ of S8 on the residual and on the in-span component, and the main term of this row does not depend on d_R. Each of its certified lower ends therefore holds for every $d_R \ge 7$, provided that this chain bound holds at every chain level from 6 on. The covariance at $t = 0$ is evaluated on 64 columns. It serves only for comparison and enters no certified bound; the following lemma bounds its truncation error.

Lemma B.6 (tail)**.** *Let $\mu > 0$, $\cosh\kappa = 1 + \mu/2$, and let G_n be the Green's function of $L+\mu$, with L the Laplacian of the path on the columns $0, \dots, n-1$ with free ends, and G_∞ that of the half-line. For $0 \le x, y \le X < n$,*

$$0 \le G_n(x,y) - G_\infty(x,y) \le \frac{2e^{-\kappa(2n-2X-1)}}{\sinh\kappa\,(1-e^{-2\kappa n})}.$$

Proof. Reflection about $-\frac{1}{2}$ and $n-\frac{1}{2}$ gives $G_n(x,y) = \sum_{j\in\mathbb{Z}} \big(g(x-y+2nj) + g(x+y+1+2nj)\big)$ with $g(z) = e^{-\kappa|z|}/(2\sinh\kappa)$, and G_∞ is the term $j = 0$. Each term with $j \ne 0$ is positive and at most $e^{-\kappa(2n|j|-2X-1)}/(2\sinh\kappa)$, and there are four such terms for each $|j|$. Summing over $|j| \ge 1$ gives the bound. □

The rows $(1,1)$ of D4 and D4M. At $d_L = d_R = 1$ the two-sided in-span term is $\alpha_L^{\mathrm{T}} W \alpha_R$, with W the compression of X_c onto the Gram-orthonormal span of the chain of S8, and α_L, α_R the components of the two sides. Since $\|W\| \le \|X_c\| \le n_X$, the term is at least $-n_X e_L e_R$, the form used in every other row. At $N = 3$ this crude form covers both rows. At $N = 4$ it does not: an evaluation outside the certificate, at the certified inputs, shows that on the first piece even a two-sided term of zero leaves the lower end negative, so no bound on that term, the matrix W included, makes the lower ends of these two rows positive there.

Coverage near the ends. The overlapping pairs are covered by rows as follows. The rows covered by Proposition B.1 are exactly those on which the certified lower end of lo is positive on every piece, and these rows are marked as `pass` in the certified record. They are all the rows with the listed family, N and d_L, and with every allowed $d_R \ge d_L$, except four rows at $N = 4$: the rows $(0,1)$ and $(1,1)$ of D3 and the rows $(1,1)$ of D4 and D4M. An evaluation of the same kind shows that on the first piece the three rows $(1,1)$ keep a negative lower end even with an in-span term of zero, and that the row $(0,1)$ of D3 would need a route-0 bound below the certified one.

family	N	d_L	form
D3	$3,4$	-1	end form, window from the free column
D3	$3,4$	0	end form, residual identity at the end; vacuum main term
D3	3	$1,2$	bulk form, route 0
D3	4	$1,2,3$	bulk form, route 0
D4, D4M	$3,4$	-1	end form, window from the free column
D4, D4M	$3,4$	0	end form, free column before the window
D4, D4M	3	$1,2$	bulk form with F_A; crude form at $(1,1)$
D4, D4M	4	$1,2,3$	bulk form with F_A

The steps in the bulk. The remaining steps concern a pair of a top-row or bottom-row edge e with an edge f that is not a middle-row horizontal edge. In the bulk, pairs of two edges of the same row are settled exactly by $\langle v, S^k v\rangle \geq 0$, and the flip pairs $(t_i, bb_{i'})$ take the variational route of S5 and S6. For a vertical edge f the far part is controlled through the flip-even eigenvector u_x of the top cluster eigenvalue s_x of S. Write $v^{\pm}$ for the flip-even and flip-odd parts of a vector v, let $e_x = \langle u_x, v_e\rangle$ and $f_x = \langle u_x, v_f\rangle$, and write $P_\perp$ for the projection of the flip-even sector off Ω_0 and u_x. Since S commutes with the flip, the bulk covariance at k steps is the sum of the x channel, the flip-even remainder and the flip-odd part, without cross terms, so that

$$J^2 \operatorname{Cov}(s_{e_0}, s_{f_m}) \;\geq\; X_{\mathrm{lo}}\, s_{x,\mathrm{lo}}^k - \|P_\perp v_e^+\|\,\|P_\perp v_f^+\|\,(s_y^{\mathrm{U}})^k - \big|\langle v_e^-, S^k v_f^-\rangle\big|, \tag{7}$$

where $X_{\mathrm{lo}} \leq e_x f_x$, $s_{x,\mathrm{lo}} \leq s_x$, and s_y^{U} bounds the flip-even spectrum of S below s_x and satisfies $s_y^{\mathrm{U}} < s_{x,\mathrm{lo}}$. The amplitudes are computed for the Ritz vectors $\tilde{u}_k$ of S4, with Ritz values $\tilde{s}_k$, and are transferred to the eigenvectors by the following lemma, in which Π, Q and s_x are as in S4.

Lemma B.7. *Let $s_x > \tau \geq \|QSQ\|$, $c \geq \|QS\Pi\|$, $r_k = \|QS\tilde{u}_k\|$, $q = c/(s_x - \tau)$ and $\varepsilon_k = r_k q/|\tilde{s}_k - s_x|$ for the Ritz values $\tilde{s}_k$ other than $\tilde{s}_x$. If $\langle \tilde{u}_x, u_x\rangle > 0$, then for every v in the flip-even sector*

$$|\langle u_x, v\rangle - \langle \tilde{u}_x, v\rangle| \leq q\|Qv\| + \sum_{k\neq x} \varepsilon_k |\langle \tilde{u}_k, v\rangle| + \Big(q^2 + \sum_{k\neq x} \varepsilon_k^2\Big)|\langle \tilde{u}_x, v\rangle|.$$

Proof. Applying Q to $Su_x = s_x u_x$ gives $(s_x - QSQ)Qu_x = QS\Pi u_x$, so $\|Qu_x\| \leq q$. Applying Π gives $(\tilde{s}_k - s_x)\langle \tilde{u}_k, u_x\rangle = -\langle QS\tilde{u}_k, Qu_x\rangle$ for $k \neq x$, so $|\langle \tilde{u}_k, u_x\rangle| \leq \varepsilon_k$. The bound follows on expanding u_x in $\tilde{u}_x$, the other Ritz vectors and Qu_x, and using $1 \geq \langle \tilde{u}_x, u_x\rangle \geq 1 - q^2 - \sum_{k\neq x} \varepsilon_k^2$. □

Proposition B.8 (S4(c), †)**.** *For $N \in \{3,4\}$ and every $J \geq 300$, $\|QSQ\| \leq \tau$ for a number $\tau < s_x - \bar{c}$, with $\bar{c}$ the bound of S4 on $\|QS\Pi\|$, certified in Proposition B.9(b).*

Proof. With $Q_{\mathcal{H}} = 1 - \Pi$ on $\mathcal{H}$, $\operatorname{tr}_{\mathcal{H}} T^2 = \operatorname{tr}(\Pi T \Pi)^2 + 2\operatorname{tr}(\Pi T Q_{\mathcal{H}} T \Pi) + \operatorname{tr}(Q_{\mathcal{H}} T Q_{\mathcal{H}})^2$, so the norm of $Q_{\mathcal{H}} T Q_{\mathcal{H}}$, and hence also the norm of QTQ, is at most $\big(\operatorname{tr}_{\mathcal{H}} T^2 - \sum_k \theta_k^2\big)^{1/2}$, the sum running over the Ritz values θ_k of T on P. Since $\sum_k \theta_k^2 \geq \lambda_h^2 \sum_i A_{ii}^2$, the trace bound of S4(a) bounds this

square root by $\lambda_h L_1^{\mathrm{odd}}$, with L_1^{odd} of S7, so that $\|QSQ\| \le \lambda_h L_1^{\mathrm{odd}}/\lambda_0 \le L_1^{\mathrm{odd}}/\lambda_{0,\mathrm{lo}} = \rho$. With $\tau = \rho$, the inequality $\tau < s_{x,\mathrm{lo}} - \bar{c}$ is checked on every piece of $[300, \infty)$. □

Proposition B.9 (S9 in the bulk, †)**.** *For $N \in \{3, 4\}$ and every $J \ge 300$, in the bulk:*

- (b) *the residuals $r_k = \|QS\tilde{u}_k\|$, the bound $\bar{c}$ on $\|QS\Pi\|$ and the norms $\|Qv\|$ of the insertion vectors of the top-row and vertical edges have certified bounds, with which Lemma B.7 bounds the errors of their x amplitudes;*
- (j) *for a top-row edge t and a vertical edge a, $X_{\mathrm{lo}} = \tilde{t}_x \tilde{a}_x - R_c^{ta} > 0$, where R_c^{ta} is the error bound for $t_x a_x$ that (b) gives;*
- (d′) *for a top-row or bottom-row edge e and a vertical edge f, $X_{\mathrm{lo}} > \|P_\perp v_e^+\| \, \|P_\perp v_f^+\|$;*
- (e) *for such e and f and every $k \ge 1$, $|\langle v_e^-, S^k v_f^- \rangle|$ is smaller than the lower bound $X_{\mathrm{lo}} s_{x,\mathrm{lo}}^k - \|P_\perp v_e^+\| \, \|P_\perp v_f^+\| \, (s_y^{\mathrm{U}})^k$ of the flip-even part in (7);*
- (g) *for a top-row edge t, a lower bound A_t for $|\langle u_x, v_t \rangle|$ and an upper bound O_t for $\|v_t^-\|^2$ satisfy $A_t^2 \ge 1.05\, O_t$, and $\rho \le s_{x,\mathrm{lo}}$;*
- (f) *the pairs $(t_i, bb_{i\pm1})$, (t_i, b_i) and (t_i, b_{i+1}), together with their images under the flip, have $\mathrm{Cov}(s_e, s_f) > 0$, and so has every pair of a top-row or bottom-row edge with a vertical edge at $k = 0$.*

The check of (e) is made at $k = 1$. It covers every $k \ge 1$ because the ratios of s_y^{U} and of ρ to $s_{x,\mathrm{lo}}$ are below 1, so that in units of $s_{x,\mathrm{lo}}^k$ the subtracted terms do not increase in k. In the bulk these steps settle the pairs of two edges of the same row exactly, with $\mathrm{Cov}(s_e, s_f) \ge 0$, and give $\mathrm{Cov}(s_e, s_f) > 0$ for every other pair of a top-row or bottom-row edge with an edge that is not a middle-row horizontal edge, outside S0: for the flip pairs $(t_i, bb_{i'})$ with $|i - i'| \ge 2$ by (g), since $\langle v_t^+, S^k v_t^+ \rangle \ge A_t^2 s_x^k \ge 1.05\, O_t \rho^k$ while $|\langle v_t^-, S^k v_t^- \rangle| \le O_t \rho^k$; for a vertical partner at $k = 0$ by (f) and at $k \ge 1$ by (7), (d′) and (e); and for the pairs in overlapping columns by (f).

Proposition B.10 (S9(h): away from the ends, †)**.** *For $N \in \{3, 4\}$, every $J \ge 300$ and every $n \ge 6$, $\mathrm{Cov}(s_e, s_f) > 0$ for every pair of distinct edges of $P_3 \times P_n$ outside S0 with a top-row or bottom-row edge, no middle-row horizontal edge, no vertex in column 1 or n, and both end exponents at least the threshold of its kind. For the separated pairs ($k \ge 0$) the threshold is 2 for a top-row and a bottom-row edge, 1 for two edges of the same row, and, for a horizontal and a vertical edge, 3 at $N = 4$ and 2 at $N = 3$; for the overlapping pairs ($k < 0$) it is 3.*

Proof. For the separated pairs, S9(h) uses the finite-strip identity of S7 with horizontal insertions. The overlapping pairs are those of the families D3, D4 and D4M; their two insertions form one joint centred insertion on a window of two or three columns, and the step is made for that insertion, with the

bulk covariance of Proposition B.9(f) and the norms of the joint insertion as inputs. Each separated kind gives one inequality, which is checked at $k = 0$ and whose subtracted terms do not increase in k. The joint kind gives one inequality for each family; in every kind the slack does not increase in the end factor $e = R_w \rho_1^d$ of S7, so that the threshold of a kind covers every larger end exponent. □

Two operator bounds for a horizontal edge. For a horizontal edge e let $H_e = ZV^{1/2}(K \circ \xi_e)V^{1/2}Z^*$, which is $T^2 = ZV^{1/2}KV^{1/2}Z^*$ with the bond ξ_e inserted in its middle kernel, and let $P_L = ZV^{1/2}(K \circ p_L(\xi_e))V^{1/2}Z^*$, the operator of S4(d); Z is as in Lemma B.3 and $A = (N-1)/2$. As in the proof of Lemma B.2, $K \circ \xi_e$ and K act on the spherical harmonics of degree l of the row of e by $a_l c_l$ and c_l, respectively, times the kernel of the other two rows, with $a_l = J(1 - R_{l+\lambda}(J)) - l$.

Lemma B.11 (zonal sandwich)**.** *Let $N \geq 3$, $J > 0$ and $c_N = 2 + (N-1)(N-3)/8$. Then $a_l > A - c_N/J - l(l+N-2)/(2J)$ for every $l \geq 0$, and*

$$(A - c_N/J)\, T^2 - P_L \;\leq\; H_e \;\leq\; A\, T^2, \qquad P_L \geq 0.$$

Proof. Put $\mu = l + \lambda$, $a = \mu + \frac{1}{2} = l + A$ and $r = R_\mu$. The recurrences of the modified Bessel functions give $r' = F(x, r)$ with $F(x, y) = 1 - 2ay/x - y^2$, and $r(x) = x/(2a+1) + O(x^3)$ as $x \to 0$. If $\varphi' \geq F(x, \varphi)$ on $(0, \infty)$ and $\varphi > r$ near 0, then $\varphi > r$ on $(0, \infty)$, because $d = \varphi - r$ satisfies $d' \geq -(2a/x + \varphi + r)\, d$. Take $\varphi(x) = x/(a + S)$ with $S = (x^2 + a^2 + a)^{1/2}$. A direct computation gives $(\varphi' - F(x, \varphi))(a + S)^2 = (a^2 + a)/S > 0$, and $\varphi(x) = x/(a + (a^2+a)^{1/2}) + O(x^3)$ with $a + (a^2+a)^{1/2} < 2a + 1$; hence $r(J) < J/(a + S)$, now with $S = (J^2 + a^2 + a)^{1/2}$. Since $l(l+N-2) = \mu^2 - \lambda^2$, the quantity $E_l = A - a_l - l(l+N-2)/(2J)$ equals $a - J + Jr(J) - (\mu^2 - \lambda^2)/(2J)$, so that $E_l < F_J + (\lambda^2 - \frac{1}{4})/(2J)$ with $F_J = a - J + J^2/(a+S) - (a^2 - a)/(2J)$. With $u = S - J > 0$, so that $a^2 + a = u(2J + u)$, an exact rearrangement gives

$$JF_J = 2u - \tfrac{1}{2}u^2 - \frac{u^2 + au + a}{J + u + a} < 2u - \tfrac{1}{2}u^2 \leq 2.$$

Hence $E_l < (2 + (\lambda^2 - \frac{1}{4})/2)/J = c_N/J$, the first claim; the upper bound $a_l < A$ is that of Lemma B.2. Multiplying by $c_l > 0$ gives, on the row of e, $(A - c_N/J)K_r - L_r^2 K_r/(2J) \leq K_r \circ \xi_e \leq AK_r$, where K_r is the factor of K on that row and $L_r^2 = -\Delta$ acts on its harmonics of degree l by $l(l+N-2)$. These operators commute, $L_r^2 K_r \geq 0$, and the kernel of $L_r^2 K_r/(2J)$ is $p_L(\xi_e)K_r$. Tensoring with the positive factors of the other two rows and conjugating by $V^{1/2}Z^*$ keep the inequalities, which then give the sandwich and $P_L \geq 0$. □

Lemma B.12 (centred bound)**.** *For a horizontal edge e and real c put $B_c = (H_e - cT^2)/\lambda_0^2$, $\kappa = A - c$ and $R = \kappa S^2 - B_c = (AT^2 - H_e)/\lambda_0^2$, which does not depend on c, and let $\tau' = (c_N/J)\rho_1^2 + \big(\operatorname{tr}_{\mathcal{H}} P_L - \langle \Omega_0, P_L \Omega_0 \rangle\big)/\lambda_0^2$. Then $X = Q_0 R Q_0$ satisfies $0 \leq X \leq \tau'$ on the complement of Ω_0, and for*

every $\varepsilon \perp \Omega_0$

$$Q_0 B_c \varepsilon = \kappa S^2 \varepsilon - \tfrac{\tau'}{2}\,\varepsilon - Y\varepsilon, \qquad Y = X - \tfrac{\tau'}{2}, \qquad \|Y\| \le \tfrac{\tau'}{2}.$$

Proof. Lemma B.11 gives $0 \le R \le (c_N/J)S^2 + P_L/\lambda_0^2$. Compressing by Q_0 gives $0 \le X \le \tau'$, since ρ_1 bounds S on the complement of Ω_0 (S7) and since the positive operator $Q_0 P_L Q_0$ is bounded by its trace $\mathrm{tr}_{\mathcal{H}}\, P_L - \langle \Omega_0, P_L \Omega_0 \rangle$. The spectrum of Y therefore lies in $[-\tau'/2, \tau'/2]$. For $\varepsilon \perp \Omega_0$ the identity follows from $Q_0 R\varepsilon = X\varepsilon$ and from the fact that $S^2\varepsilon$ is orthogonal to Ω_0. □

We bound τ' above by S4(d) ($\mathrm{tr}_{\mathcal{H}}\, P_L \le \mathrm{TRB}_N^H \lambda_h^2$) and by Jensen's inequality for the vacuum term: $\langle \Omega_0, P_L \Omega_0 \rangle / \lambda_0^2$ is the bulk mean of $p_L(\xi_e)$ and is at least $A - \bar{\xi}(1 + (N-1)/(2J)) + \bar{\xi}^2/(2J)$ for the bulk mean $\bar{\xi}$ of ξ_e; this lower bound decreases in $\bar{\xi}$ for $\bar{\xi} < J + A$ and is taken at a certified upper bound of $\bar{\xi}$. The centre c is a fixed rational, so that κ is exact, and Y is one operator for each edge, the same in every use of the lemma.

Proposition B.13 (separated pairs near the ends, †)**.** *For every $n \ge 5$, at $N = 3$ for every $J \ge 300$ and at $N = 4$ for every $J \ge 350$, $\mathrm{Cov}(s_e, s_f) > 0$ for every separated pair ($k \ge 0$) of distinct edges of $P_3 \times P_n$ outside S0 with a top-row or bottom-row edge and no middle-row horizontal edge, if one of its edges has a vertex in column 1 or n, or if one of its end exponents is below the threshold of its kind in Proposition B.10.*

Proof. The proof follows S8 with horizontal insertions, near one end or both. The chain $l(d) = S^{d+1} l(-1)$ of S8 starts at the end vector, whose part $\varepsilon(-1)$ off Ω_0 has norm R_w, and it has the same component along Ω_0 at every level. Slot i, the insertion of a horizontal edge at end exponent i, is $B_c\, l(i-2)$; the chain is read in the frame of S8, in which B_c is replaced by a unitary image that leaves Lemma B.12 unchanged. Slots 0 and 1 come from exact entries in the form of those of S8. Slots $2, \dots, 7$ and the limit slot come from Lemma B.12: slot i differs from slot 1 by $B_c(\varepsilon(i-2) - \varepsilon(-1))$, and the lemma gives the part of this vector off Ω_0 explicitly up to one Euclidean ball of radius $(\tau'/2)\|\varepsilon(-1) - \varepsilon(i-2)\| \le (\tau'/2) R_w$. Each class with separation below $m_0 \le 8$ is bounded separately, and classes with larger separations are bounded by the top-mode form of S8. A class whose other end is far is closed for every larger end exponent by the better of two bounds: a crude perturbation of the end vector at one far exponent D (route A), and the two-sided chain at the exponents D, $D+1$ and $D+2$ with the crude perturbation from $D+3$ on (route B). The classes at infinity use the limit slot. The certified lower bounds are positive on every piece. □

Proposition B.14 ($P_3 \times P_5$ at strong coupling, †)**.** *For $N = 3$ and every $J \ge 300$, and for $N = 4$ and every $J \ge 350$, $\mathrm{Cov}(s_e, s_f) > 0$ for every pair of distinct edges of $P_3 \times P_5$ that contains a horizontal edge, has no common vertex and is not a mirror pair for the reflection in the middle row or in the vertical midline.*

Proof. As in the proof of Proposition 6.36, with the expansion carried to order three, four or five on the whole graph $P_3 \times P_5$. The 138 pairs of the statement fall into 36 orbits under the two reflections. For one pair of each orbit, at $N = 3$ and at $N = 4$, the certified lower bound for $J^2 \operatorname{Cov}(s_e, s_f)$ is positive on every piece of $[J_1, \infty)$; the smallest of these bounds is $2.56 \cdot 10^{-6}$ at $N = 3$ and $3.77 \cdot 10^{-6}$ at $N = 4$. □

Proposition B.15 (the rows that S9(i) excludes, †)**.** *For $N = 4$ and every $J \geq 350$, $\operatorname{Cov}(s_e, s_f) > 0$ for the pairs of the four rows that Proposition B.1 excludes and of their mirror images. These pairs occur only in $P_3 \times P_6$ and $P_3 \times P_7$.*

Proof. A row fixes the family, the window and both end exponents, and so fixes n: the row $(0, 1)$ of D3 and the rows $(1, 1)$ of D4 and D4M lie in $P_3 \times P_6$, and the row $(1, 1)$ of D3 lies in $P_3 \times P_7$. The proof follows that of Proposition B.14, with the expansion carried to third order on these whole graphs. The 10 pairs fall into 3 orbits, and the smallest certified lower bound for $J^2 \operatorname{Cov}(s_e, s_f)$ is $3.74 \cdot 10^{-3}$. □

Pairs with a horizontal edge of the middle row. The rest of this appendix proves Theorem 6.22. Let $n \geq 6$, and consider in $P_3 \times P_n$ a pair of distinct edges that contains a horizontal edge of the middle row, has no common vertex and is not a pair of mirror images in the vertical midline. Up to the reflection in the vertical midline and the flip, the pair is (h_i, f) with $f = t_j$ and $j \geq i$, with $f = a_j$ and $j \geq i + 2$, or with $f = h_j$ and $j \geq i + 2$. The top vertex of column n has degree two, so Lemma 6.2 gives $\operatorname{Cov}(s_{h_i}, s_{a_n}) = \operatorname{Cov}(s_{h_i}, s_{t_{n-1}})$, and we take $j \leq n - 1$ for a_j. The end exponents of Section 6.6 extend to edges with a vertex in column 1 or n: $d = i - 2$ for h_i, $d' = n - j - 2$ for both t_j and h_j, and $d' = n - j - 1$ for a_j, so that $d, d' \geq -1$. Put $m = j - i$ for both t_j and h_j, and $m = j - i - 1$ for a_j. Then $n = d + m + d' + 4$, and m is the number of free transfer steps between the two edges, as in Section 6.3. The pairs with $m \geq 1$ form the *channel classes* (T, d, m, d'), where $T \in \{t, a, h\}$ is the type of f. The pairs with $m = 0$ are the same-bond pairs (h_i, t_i). For $2 \leq i \leq n - 2$ they form the *interior same-bond orbits*. The *end pairs* are (h_1, t_1) and (h_{n-1}, t_{n-1}) together with their images under the flip, (h_1, bb_1) and (h_{n-1}, bb_{n-1}).

The two even modes. We use the notation of Section 6.6: $S = T/\lambda_0$ on $\mathcal{H}$, its top eigenvector Ω_0, the projection Q_0 off Ω_0, and the flip, which commutes with S. The insertion of ξ_e is written X_e. For a horizontal edge it is the multiplication M_e by ξ_e at the cut and is diagonal in the labels; for a vertical edge it is the column step T_{ξ_e}/λ_0, which replaces one factor S. Put $c_e = \langle \Omega_0, X_e \Omega_0 \rangle$, $B_e = M_e - c_e$ for a horizontal edge and $B_e = X_e - c_e S$ for a vertical edge, and let $\bar{v}_e = Q_0 B_e \Omega_0$ be the bulk vector of e; for a horizontal edge, $S\bar{v}_e$ is the vector v_e of Section 6.6. On the flip-even part of the complement of Ω_0, let $s_x > s_y$ be the two largest eigenvalues of S, with unit eigenvectors u_x and u_y, and let P_{r} project onto the rest of that part.

With the trial space P, the projection Q and the Ritz vectors $\tilde{u}_k$ of S4, the signs of u_x and u_y are fixed by $\langle \tilde{u}_x, u_x\rangle > 0$ and $\langle \tilde{u}_y, u_y\rangle > 0$, where $\tilde{u}_x$ and $\tilde{u}_y$ have positive coefficients on the trial states e_1 and e_3, respectively. Since S is positive, $\|SP_{\mathrm{r}}\|$ is the fourth largest flip-even eigenvalue of S, after 1, s_x and s_y. Since P has dimension three, the min-max principle bounds this fourth largest eigenvalue by $\|QSQ\|$, and $\|QSQ\|$ is bounded by a certified number $\tau < s_y$. A middle-row edge is fixed by the flip, and at $t = 0$ its bulk vector lies along u_y (Lemma B.21 below). Its component along u_x is therefore of order t, while the bulk vectors of the top-row and vertical edges have components of order one along u_x. For a pair with a middle-row edge, the positive term of the bound below comes from the channel of u_y; the channel of u_x enters through the negative part of its contribution, and that negative part vanishes where the signs of the two components along u_x are certified to agree.

Lemma B.16 (centred identity)**.** *Let e lie to the left of f with $m \geq 1$ free transfer steps between them, and let $l_L = \Omega_0 + p$ and $l_R = \Omega_0 + q$, with $p, q \perp \Omega_0$, be the end states at e and at f, normalised as in Section 6.3. For a vertical edge, the end state at the edge is S applied to the end state l' at the adjacent cut on the side of the end, and $X_e l_L$ stands for $X_e l'$. Let $a = Q_0 X_e l_L - \langle \Omega_0, X_e l_L\rangle p$ and $b = Q_0 X_f l_R - \langle \Omega_0, X_f l_R\rangle q$ be the centred vectors of e and f. Then*

$$J^2 \operatorname{Cov}(s_e, s_f) = \frac{F_1}{Z} - \frac{F_2 F_3}{Z^2},$$

where $F_1 = \langle a, S^m b\rangle$, $F_2 = \langle a, S^m q\rangle$, $F_3 = \langle S^m p, b\rangle$ and $Z = 1 + \langle S^m p, q\rangle$.

Proof. Since $J^2 \operatorname{Cov}(s_e, s_f) = \operatorname{Cov}(\xi_e, \xi_f)$, the determinant formula of Section 6.3, divided by the normalisations, gives $J^2 \operatorname{Cov}(s_e, s_f) = N_1/Z - N_2 N_3/Z^2$ with $N_1 = \langle X_e l_L, S^m X_f l_R\rangle$, $N_2 = \langle X_e l_L, S^m l_R\rangle$, $N_3 = \langle l_L, S^m X_f l_R\rangle$ and $Z = \langle l_L, S^m l_R\rangle$. Write $X_e l_L = x\Omega_0 + a + xp$ and $X_f l_R = y\Omega_0 + b + yq$. Then $N_1 = F_1 + yF_2 + xF_3 + xyZ$, $N_2 = F_2 + xZ$ and $N_3 = F_3 + yZ$, so that $ZN_1 - N_2 N_3 = ZF_1 - F_2 F_3$. □

Proposition B.17 (channel bound)**.** *In Lemma B.16 let $e = h_i$. Let a_x, a_y and b_x, b_y be the components of a and b along u_x and u_y, and let b^+ be the flip-even part of b. Suppose that $a_y \geq \alpha_L > 0$, $b_y \geq \alpha_R > 0$, $s_x^m a_x b_x \geq -X$, $|\langle P_{\mathrm{r}} a, S^m P_{\mathrm{r}} b\rangle| \leq R$, $\|Sa\| \leq N_a$, $\|Sb^+\| \leq N_b$, $\|p\| \leq e_L$, $\|q\| \leq e_R$ and $\delta = s_x^m e_L e_R < 1$. If the margin*

$$s_y^m \alpha_L \alpha_R - X - R - \frac{s_x^{2m-2} N_a N_b e_L e_R}{1 - \delta}$$

is positive, then $\operatorname{Cov}(s_{h_i}, s_f) > 0$.

Proof. The vectors a, p and q are flip-even and orthogonal to Ω_0, since M_{h_i} commutes with the flip and since the end states are flip-even. As S commutes with the flip, only b^+ meets them. On the flip-even part of the complement of Ω_0, $S = s_x |u_x\rangle\langle u_x| + s_y |u_y\rangle\langle u_y| + SP_{\mathrm{r}}$, so that $F_1 =$

$s_x^m a_x b_x + s_y^m a_y b_y + \langle P_{\mathrm{r}} a, S^m P_{\mathrm{r}} b\rangle \geq s_y^m \alpha_L \alpha_R - X - R$. On the same space $\|S\| = s_x$, hence $|F_2| = |\langle Sa, S^{m-1} q\rangle| \leq s_x^{m-1} N_a e_R$, $|F_3| \leq s_x^{m-1} e_L N_b$ and $Z \geq 1 - \delta > 0$. By Lemma B.16, $Z^2 J^2 \operatorname{Cov}(s_{h_i}, s_f) = Z(F_1 - F_2 F_3/Z)$, and $F_1 - F_2 F_3/Z$ is at least the margin. □

Since P_{r} commutes with S, we take $R = \tau^{m-2} \|SP_{\mathrm{r}} a\| \, \|SP_{\mathrm{r}} b\|$ for $m \geq 2$, and for $m = 1$ the smaller of $\|P_{\mathrm{r}} a\| \, \|SP_{\mathrm{r}} b\|$ and $\|SP_{\mathrm{r}} a\| \, \|P_{\mathrm{r}} b\|$, taken among the products whose two factors are both bounded. The bound N_a comes from $\|Sa\|^2 = s_x^2 a_x^2 + s_y^2 a_y^2 + \|SP_{\mathrm{r}} a\|^2$. The x term X is s_x^m times the negative part of the lower end of the certified enclosure of $a_x b_x$, and $X = 0$ where that enclosure is nonnegative. For two middle-row edges, $a_x = v + E$ and $b_x = v + E'$ with the same bulk component $v = \langle u_x, \bar{v}_h\rangle$, and the inequality $(v + E)(v + E') \geq -(|E| + |E'|)^2/4$ gives X without any condition on signs.

Proposition B.18 (same bond)**.** *Let $2 \leq i \leq n-2$, and let $\Omega_0 + p$ and $\Omega_0 + q$ be the normalised end states from the left and from the right at the bond of h_i and t_i. With $c_h = c_{h_i}$, $c_t = c_{t_i}$, $\bar{v}_h = \bar{v}_{h_i}$ and $\bar{v}_t = \bar{v}_{t_i}$, put $\Xi = M_{t_i} M_{h_i}$, $A_0 = \langle \Omega_0, \Xi \Omega_0\rangle$, $w = Q_0 \Xi \Omega_0 - c_t \bar{v}_h - c_h \bar{v}_t$, $\Xi_c = \Xi - c_t M_{h_i} - c_h M_{t_i}$ and $\delta = \langle p, q\rangle$. Then, with $Z = 1 + \delta$,*

$$Z^2 J^2 \operatorname{Cov}(s_{h_i}, s_{t_i}) = A_0 - c_h c_t + \langle p + q, w\rangle + \big[\langle p, \Xi_c q\rangle + A_0 \delta - B_1 C_1\big] + \big[\delta A_1 - B_1 C_2 - B_2 C_1\big] + \big[\delta A_2 - B_2 C_2\big],$$

where $A_1 = \langle \Xi \Omega_0, p + q\rangle$, $A_2 = \langle p, \Xi q\rangle$, $B_1 = \langle \bar{v}_h, p + q\rangle$, $B_2 = \langle p, M_{h_i} q\rangle$, $C_1 = \langle \bar{v}_t, p + q\rangle$ and $C_2 = \langle p, M_{t_i} q\rangle$.

Proof. $J^2 \operatorname{Cov}(s_{h_i}, s_{t_i}) = A/Z - BC/Z^2$ with $A = \langle \Omega_0 + p, \Xi(\Omega_0 + q)\rangle = A_0 + A_1 + A_2$, $B = c_h + B_1 + B_2$ and $C = c_t + C_1 + C_2$. Expanding $ZA - BC$ and collecting the terms of order zero, one, two, three and four in p and q gives the identity. □

Each term after $A_0 - c_h c_t$ is bounded through the components of p, q and w along u_x and u_y, through the norms of the parts of p, q and w in the range of P_{r}, and through the norms of $SB_{h_i} S$, $SB_{t_i} S$ and $S\Xi_c S$ on the complement of Ω_0. The certified condition is that $A_0 - c_h c_t$ exceeds the sum of the bounds on these terms.

The inputs. The ends of h_i at $d \leq 0$, of t_j at $d' = -1$ and of a_j at $d' = 0$ are *exact*. For these we compute the components of the centred vectors on the trial states and bound the residual off them; Lemma B.7 and Lemma B.20(b) below then give the components along u_x and u_y, and the norms of the parts in the range of P_{r}. At larger exponents the ends are *crude*, and the following expansion about the bulk vector applies.

Lemma B.19 (end expansion)**.** *Let e have the end state $\Omega_0 + \varepsilon$ before its insertion, and put $\varepsilon' = \varepsilon$ for a horizontal edge and $\varepsilon' = S\varepsilon$ for a vertical edge. Then the centred vector of e is $a = \bar{v}_e + Q_0 B_e \varepsilon - \langle \bar{v}_e, \varepsilon\rangle \varepsilon'$.*

Proof. With $l = \Omega_0 + \varepsilon$, $\langle \Omega_0, X_e l\rangle = c_e + \langle \bar v_e, \varepsilon\rangle$ and $Q_0 X_e l = \bar v_e + Q_0 B_e \varepsilon + c_e \varepsilon'$. The centred vector subtracts from $Q_0 X_e l$ the multiple $\langle \Omega_0, X_e l\rangle$ of ε', which is the part off Ω_0 of the end state at e. □

At end exponent d, $\varepsilon = \varepsilon(d) = S^{d+1}\varepsilon(-1)$ with $\|\varepsilon(-1)\| \le R_w$ and $\|\varepsilon(d)\| \le R_w \rho_1^{d+1}$, as in S7 and S8; a vertical edge uses $\varepsilon(d-1)$. The components $c_x(d)$ and $c_y(d)$ of $\varepsilon(d)$ along u_x and u_y are $s_x^{d+1} c_x(-1)$ and $s_y^{d+1} c_y(-1)$, and $\|P_{\rm r}\varepsilon(d)\| \le \tau^{d+1}\|P_{\rm r}\varepsilon(-1)\|$ with $\|P_{\rm r}\varepsilon(-1)\|^2 \le R_w^2 - c_x(-1)^2 - c_y(-1)^2$. Lemma B.19 bounds the components of a crude end along u_x and u_y, and the norms of its parts in the range of $P_{\rm r}$, through these numbers for $\varepsilon(d)$, through the matrix elements of B_e between u_x and u_y, and through norms of B_e on u_x, u_y and the range of $P_{\rm r}$. The matrix elements come from the trial states with the errors of Lemma B.7, which holds with u_y and s_y in place of u_x and s_x by the same proof, since $\tau < s_y$; the norms come from the following lemma.

Lemma B.20 (two operator bounds)**.** *(a) Let M be a multiplication operator at a cut, with K and V normalised as in Section 6.6, and $C = \|MK^{1/2}\|\lambda_0^{-1/2}$. Then $\|QSM\| = \|MSQ\| \le C\|QSQ\|^{1/2}$ for every orthogonal projection Q on $\mathcal{H}$; in particular $\|Mu_\alpha\| \le C s_\alpha^{-1/2}$ for $\alpha \in \{x, y\}$. (b) Let the Ritz values $\tilde s_k$ of S on P exceed τ, and let $r_k = QS\tilde u_k$. Then $\|P_{\rm r} w\| \le \|Qw\| + \sum_k |\langle \tilde u_k, w\rangle|\, \|r_k\|/(\tilde s_k - \tau)$ for every flip-even w.*

Proof. (a) Since $0 < V \le 1$, $V^2 \le V$, and

$$\|MSy\|^2 = \lambda_0^{-2}\|MK^{1/2}VK^{1/2}y\|^2 \le C^2\lambda_0^{-1}\langle K^{1/2}y, VK^{1/2}y\rangle = C^2\langle y, Sy\rangle.$$

With $y = Qz$ this gives $\|MSQz\|^2 \le C^2\|QSQ\|\,\|z\|^2$, and QSM is the adjoint of MSQ. For $Q = |u_\alpha\rangle\langle u_\alpha|$, MSQ has norm $s_\alpha\|Mu_\alpha\|$. (b) Write $w = Qw + \sum_k \langle \tilde u_k, w\rangle \tilde u_k$. As $P_{\rm r}$ commutes with S and as $(S - \tilde s_k)\tilde u_k = r_k$, we have $(S - \tilde s_k)P_{\rm r}\tilde u_k = P_{\rm r} r_k$, and $0 \le S \le \tau < \tilde s_k$ on the range of $P_{\rm r}$; hence $\|P_{\rm r}\tilde u_k\| \le \|r_k\|/(\tilde s_k - \tau)$. □

The x component of a crude end of h needs more care: it is of order t, while Lemma B.19 bounds the terms of this component only by quantities of order one. Put $E(d) = \langle u_x, a\rangle - \langle u_x, \bar v_h\rangle$ for the end of h at exponent d.

Lemma B.21 (order zero)**.** *At $t = 0$ the bulk vector of h is a multiple of u_y, and $E(d)$ vanishes for every $d \ge -1$.*

Outline. At $t = 0$, in the Gaussian limit of the column variables, the transverse fluctuations of a column split into the three normal modes of the path P_3: $\pi_1 = (\pi_T - \pi_B)/\sqrt{2}$, $\pi_2 = (\pi_T - 2\pi_M + \pi_B)/\sqrt{6}$ and the uniform mode, on which no rotation-invariant state depends. The limit of S is a tensor product $S_1 \otimes S_2$, with $\Omega_0 = \omega_1 \otimes \omega_2$, $u_x = x_1 \otimes \omega_2$ and $u_y = \omega_1 \otimes y_2$, where ω_k is the ground state of the mode π_k, and x_1 and y_2 are the invariant two-quantum states of π_1 and π_2, respectively; the end vector factorises as well. The centred multiplication by ξ_h acts only on π_2, since the middle spin has no part in π_1. This multiplication maps ω_2 to a

multiple $\eta\, y_2$; so $\bar{v}_h = \eta u_y$ and $B_h u_x = \eta\, x_1 \otimes y_2$. Lemma B.19 then gives $E(d) = \eta\,(\langle x_1 \otimes y_2, \varepsilon(d)\rangle - c_x(d)c_y(d))$. Both terms are $(s_x s_y)^{d+1}$ times their values at $d = -1$, and the two values at $d = -1$ agree, because the end vector is a product whose component along Ω_0 is 1. □

The identities for $B_h u_x$ and for the end vector at $t = 0$ are confirmed in exact arithmetic over $\mathbb{Q}(\sqrt{5}, \sqrt{21})$. In Proposition B.22 the bound of $E(d)$ is that of Lemma B.19, with the overlap of $P_{\mathrm{r}} B_h u_x$ and $P_{\mathrm{r}}\varepsilon(d)$ bounded through the windowed operator SB_hS, and with sharper component bounds for $d = 1, 2, 3$. In Proposition B.24 the bound is $|E(d)| \le t\,\hat{e}(d)$, with $\hat{e}(d)$ bounded on every piece, the last one included. Below a coupling J_s, fixed for each N, $\hat{e}(d)$ is the window bound of Proposition B.22 divided by t. From J_s on, $\hat{e}(d)$ comes from an expansion of the end vector on the trial space enlarged by one state, whose part in the range of P_{r} is $x_1 \otimes y_2$ at $t = 0$. Every term of this expansion carries a factor t, and, with Q the projection off the enlarged space, Lemma B.20(a) bounds the factors free of t. In both forms $\hat{e}(d)$ is bounded row by row for $d \le 9$ and by a geometric envelope beyond. Every x component of an end of h is then t times a coefficient with a certified enclosure; for a crude end the enclosure of the coefficient is that of $\langle u_x, \bar{v}_h\rangle/t$ widened by $\hat{e}(d)$.

Proposition B.22 (middle row, $6 \le n \le 11$, †)**.** *Let $N \in \{3, 4\}$ and $6 \le n \le 11$, let all couplings equal J, and let $J \ge 300$ at $N = 3$ and $J \ge 350$ at $N = 4$. Then every pair of a channel class or of an interior same-bond orbit has* $\mathrm{Cov}(s_e, s_f) > 0$.

Proof. The other pairs of a class or an orbit are images of one of its pairs under the reflections in the middle row and in the vertical midline, which preserve the Gibbs measure. For one pair of each channel class the hypotheses of Proposition B.17 are checked with a positive margin, and for one pair of each interior same-bond orbit the inequality that follows Proposition B.18 is checked, in both cases with the inputs above at the outward ends of their enclosures. The checks are made on every piece of the partition at $N = 3$, and on every piece that meets $[350, \infty)$ at $N = 4$. □

Every n. For larger n the classes are infinitely many, and they are covered by monotonicity in the end exponents and in m. Every upper bound given by Lemma B.19 for a crude end at exponent d is a nondecreasing function of s_x^{d+1}, s_y^{d+1}, τ^{d+1} and ρ_1^{d+1} that does not otherwise depend on d, and every lower bound is a nonincreasing one; the four bases s_x, s_y, τ and ρ_1 are below 1 on every piece. So each upper bound is nonincreasing and each lower bound nondecreasing in d, and their values at an exponent $D_0 \ge 1$ bound the end at every $d \ge D_0$ from the unfavourable side. We call these values the *envelope* at D_0. For the x coefficient of an end of h the envelope uses $\sup_{d \ge D_0} \hat{e}(d)$. An *envelope class* is a channel class in which the end of h, the end of f or both are replaced by their envelopes at D_0; it stands for the classes in which the replaced ends have exponents at least D_0. The

sign condition on a piece is the following: every x coefficient of an end of h, whether exact, crude below D_0 or envelope, has one strict sign; every x component of an end of a top-row or vertical edge has one strict sign; and the two signs agree. Under the sign condition we have $s_x^m a_x b_x \geq 0$, and $X = 0$ for every channel class.

Lemma B.23 (envelopes and the tail in m)**.** *Assume the sign condition on a piece.*

(a) *If an envelope class has a positive margin in Proposition B.17, every class that it stands for has a positive margin.*

(b) *Let $M_0 \geq 2$, and suppose that $q_R = \tau/s_y < 1$ and $q_T = s_x^2/s_y < 1$. For each type of end let A^* be the least lower bound of its y component over all exponents, whether exact, crude or envelope, and let N'^* and N^* be the largest bounds of $\|SP_{\mathrm{r}}a\|$ and of $\|Sa\|$; let $e^* = R_w$ and $\delta^* = s_x^{M_0} e^{*2} < 1$. If for every type T*

$$A_h^* A_T^* - q_R^{M_0} \tau^{-2} N_h'^* N_T'^* - q_T^{M_0} s_x^{-2} N_h^* N_T^* \frac{e^{*2}}{1-\delta^*} > 0,$$

then every channel class with $m \geq M_0$ has a positive margin.

Proof. (a) The first term of the margin increases with the lower bounds of the y components, while R and the last term increase with the norm bounds and with e_L and e_R; each factor is bounded by the envelope from the unfavourable side at every $d \geq D_0$, and $X = 0$. (b) Divide the margin by s_y^m. Then $R/s_y^m \leq q_R^m \tau^{-2} N_L' N_R'$, the last term divided by s_y^m is at most $q_T^m s_x^{-2} N_a N_b e_L e_R/(1-\delta)$, and $\delta \leq \delta^*$. The first two bounds decrease in m, so for $m \geq M_0$ the margin divided by s_y^m is at least the left side of the inequality assumed in (b). □

A channel class with $d < D_0$, $d' < D_0$ and $m < M_0$ has $n \leq n_0 = 2D_0 + M_0 + 1$. A class with $m \geq M_0$ falls under Lemma B.23(b), and a class with $m < M_0$ and an end exponent at least D_0 falls under Lemma B.23(a), with the other end below D_0 or in its envelope. The lower bound that follows Proposition B.18 decreases in each bound of the end states, so for the interior same-bond orbits with an end exponent at least D_0 it suffices to check the orbits with that end in its envelope; the other orbits have $n \leq 2D_0 + 2 \leq n_0$.

Proposition B.24 (middle row, every n, †)**.** *Let $N \in \{3, 4\}$ and $n \geq 6$, let all couplings equal J, and let $J \geq 300$ at $N = 3$ and $J \geq 350$ at $N = 4$. Then every pair of a channel class or of an interior same-bond orbit has* $\mathrm{Cov}(s_e, s_f) > 0$.

Proof. Take $(D_0, M_0) = (2, 5)$, so that $n_0 = 10$. On every piece, with the x component of each crude end of h factored as above into t times a coefficient, the following are checked:

(1) the hypotheses of Proposition B.17 with a positive margin for every channel class with $6 \leq n \leq 10$, and the inequality that follows Proposition B.18 for every interior same-bond orbit with $6 \leq n \leq 10$;

(2) the sign condition;
(3) a positive margin for every envelope class with $m < M_0$;
(4) the inequality of Lemma B.23(b);
(5) the inequality that follows Proposition B.18 for the interior same-bond orbits with an end in its envelope.

By Lemma B.23 and the covering above, every channel class and every interior same-bond orbit has a positive margin at every $n \geq 6$, and the pairs of a class or an orbit are related by the two reflections, as in the proof of Proposition B.22. □

Proof of Theorem 6.22. Let $n \geq 6$, and let $J \geq 300$ at $N = 3$ and $J \geq 350$ at $N = 4$. A pair with a common vertex has $\mathrm{Cov}(s_e, s_f) \geq 0$ by Theorem 3.2, and a pair of mirror images in the vertical midline by Lemma 6.1. In the end pair (h_1, t_1) the edge t_1 ends at the top vertex of column 1, which has degree two; Lemma 6.2 replaces t_1 by the vertical edge a_1, which shares a vertex with h_1, so that $\mathrm{Cov}(s_{h_1}, s_{t_1}) \geq 0$ by Theorem 3.2. The other end pairs are images of (h_1, t_1) under the two reflections. Every remaining pair with a horizontal edge of the middle row is, up to the two reflections, a pair of a channel class or of an interior same-bond orbit, or the pair (h_i, a_n), which has the covariance of (h_i, t_{n-1}). Such pairs have $\mathrm{Cov}(s_e, s_f) > 0$ by Proposition B.22 for $n \leq 11$ and by Proposition B.24 for every $n \geq 6$.

Appendix C. The identity in Campbell's Lemma 1

We quote Lemma 1 from Campbell's preprint [6] of [5]. In its notation the Hamiltonian is $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_l + \mathcal{H}_m + \mathcal{H}_{lm}$, with weight $e^{-\beta\mathcal{H}}$, and for sites k, l, m, n the lemma asserts

$$I \equiv \langle (\boldsymbol{s}_k \cdot \boldsymbol{s}_l)(\boldsymbol{s}_m \cdot \boldsymbol{s}_n) \rangle = \frac{1}{\mathcal{Z}} \int_{[0,\pi]^{|\mathbb{S}|-2}} \prod_{i \neq l,m} d\lambda_i \, e^{-\beta \mathcal{H}_0^1} \sin\phi_k^N \sin\phi_n^N \times \int d^{|\mathbb{S}|}t \; e^{-\beta(\mathcal{H}_0^{N-1} + \mathcal{H}_l + \mathcal{H}_m + \mathcal{H}_{lm})} (\boldsymbol{t}_k \cdot \boldsymbol{T}_l)(\boldsymbol{T}_m \cdot \boldsymbol{t}_n), \tag{14}$$

where $\boldsymbol{s}_i = \cos\phi_i^N \hat{\boldsymbol{x}}^N + \sin\phi_i^N \boldsymbol{t}_i$ for $i \neq l, m$, the vectors $\boldsymbol{t}_i, \boldsymbol{T}_l, \boldsymbol{T}_m$ are unit vectors in $\mathbb{R}^{N-1}$ with the normalized uniform measure, $d\lambda_i = d\phi_i^N \sin^{N-3}(\phi_i^N)$, $\mathcal{H}_{lm} = -J_{lm} \boldsymbol{T}_m \cdot \boldsymbol{T}_l$, and $\mathcal{H}_0^1, \mathcal{H}_0^{N-1}, \mathcal{H}_l, \mathcal{H}_m$ collect the remaining couplings. The two spins $\boldsymbol{s}_l, \boldsymbol{s}_m$ are thus placed in the plane orthogonal to $\hat{\boldsymbol{x}}^N$.

The identity fails already for $N = 3$ on three sites k, l, m, with $n = k$ and $J_{lm} = J > 0$ the only nonzero coupling. The left side is $\frac{1}{3}\big(\coth\beta J - \frac{1}{\beta J}\big)$, because $\boldsymbol{s}_k$ is uniform and independent of $(\boldsymbol{s}_l, \boldsymbol{s}_m)$. On the right side $\mathcal{H}_0^1 = \mathcal{H}_0^{N-1} = \mathcal{H}_l = \mathcal{H}_m = 0$, the outer integral over ϕ_k^3 is a positive constant, and the inner integral over $\boldsymbol{t}_k, \boldsymbol{T}_l, \boldsymbol{T}_m \in S^1$ equals $\frac{1}{2} I_1(\beta J)$. Since $\mathcal{Z}$ is a constant multiple of $\sinh(\beta J)/(\beta J)$, the right side is a constant multiple of $\beta J \, I_1(\beta J)/\sinh(\beta J)$, which grows like $(\beta J)^{1/2}$. The left side stays below $\frac{1}{3}$.

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