

A LOWER BOUND FOR MOSER'S WORM PROBLEM WITH ROTATIONS AND TRANSLATIONS

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ABSTRACT. Moser's worm problem asks for the least area of a convex set in the plane that contains a copy of every arc of length 1. We consider the version in which the copies may be rotated and translated but not reflected, and give a computer-assisted proof that the least area M^+ satisfies $M^+ \geq 0.2405$. The previous lower bound, 0.239, was proved by Keller for the version with reflections and carries over to this one. Like the earlier lower bounds, ours uses that such a set contains the convex hull of copies of finitely many arcs. We take five polygonal arcs: a segment, an equilateral two-link arc, a three-link arc and two zigzag arcs that turn in opposite directions. Without reflections, a placed zigzag can serve only as a copy of the zigzags that turn in its own direction. On a box of placements we bound the hull area from below by signed areas of polygons spanned by vertices of the placed arcs. We bound these areas by second-order Taylor expansions with certified eigenvalue bounds and, where the hull changes its combinatorial type, by convex combinations of such bounds. Our branch and bound over a twelve-dimensional box of placements produced a certificate with 203,209,418 leaves, which a separate checker verifies in ball arithmetic.

1. INTRODUCTION

An *arc* is the image of a continuous map $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ of finite length. In 1966 Leo Moser asked for the smallest region of the plane that contains a copy of every arc of length 1 [8]. This is Moser's worm problem [1]. We consider convex regions. A convex region of least area exists by the Blaschke selection theorem [9], and the argument applies to both versions of the problem defined below [7]. The answer depends on which copies are admitted. Let $M^\pm$ be the infimum of the areas of convex sets that contain a congruent copy of every arc of length 1, and let M^+ be the infimum over convex sets that contain, for every such arc, a copy obtained by a rotation and a translation. This notation is Mazur's [7]. Since every copy of the second kind is congruent, $M^\pm \leq M^+$. The constant M^+ is constant 13a of the repository of optimization constants [12], which defines it as the infimal area of a convex domain that can contain "a rigid motion (translation + rotation; no reflections)" of every planar arc of length 1.

Lower bounds come from finite families of arcs. A convex set that contains copies of arcs $\gamma_1, \dots, \gamma_m$ contains the convex hull of these copies, so its area is at least the least area of such a hull. Wetzel [14] combined the unit segment with the width of Schaer's broadworm [11] and obtained 0.21946. Khandhawit and Sriswasdi [6] obtained 0.227498 from three arcs, and Khandhawit, Pagonakis and Sriswasdi [5] obtained 0.232239. In preprints with formalizations in Lean, Mazur [7] reports 0.23743658 for both $M^\pm$ and M^+ , and Keller [4] proved $M^\pm \geq 0.239$ with four polygonal arcs: a segment L , an equilateral two-link arc T , a three-link arc and a zigzag arc. The least hull area of his four arcs that he found numerically is 0.23943903. Since $M^\pm \leq M^+$, his bound also gives $M^+ \geq 0.239$.

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Upper bounds come from explicit covers. Panraksa and Wichiramala [10] proved Wetzel's conjecture that the circular sector of radius 1 and angle $\pi/6$ contains a congruent copy of every unit arc, so $M^\pm \leq \pi/12 < 0.2618$. Deng [2] reports, in a preprint, an isosceles triangle of area 0.257883595... with the same property. Both covers are symmetric about a line. A convex set Ω that is symmetric about a line and contains a congruent copy of an arc also contains a copy obtained by a rotation and a translation: if the copy is reflected, its mirror image in the axis of Ω is such a copy. These bounds therefore hold for M^+ as well. Keller [4] constructed a quadrilateral of area 0.24633... that contains a congruent copy of every unit arc, which shows $M^\pm \leq 0.24633...$. The quadrilateral, however, is not symmetric, and his proof uses reflections [4, Section 4], so it gives no bound for M^+ . The repository [12] lists $\pi/12$ as the best upper bound for M^+ , without Deng's triangle, and Keller's 0.239 as the best lower bound.

Our arcs are Keller's arcs L and T , an arc U close to his, a zigzag C and a zigzag S that turns the other way (Definition 2.1). When reflections are allowed, either zigzag can be reflected so that it turns like the other. When they are not, the two zigzags constrain the hull separately, and the smallest hull area of the five arcs that we found is 0.2420538 (Section 5).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be convex, and suppose that Ω contains, for each of the arcs L, T, U, C, S of Definition 2.1, a copy obtained by a rotation and a translation. Then $\text{area} \Omega \geq 0.2405$. Consequently $M^+ \geq 0.2405$.*

The method. We fix the segment L and normalize the placements of the other four arcs (Section 2). A placement is then a point $p \in \mathbb{R}^{12}$, given by an angle and a translation for each of T, U, C and S . Let $A(p)$ be the area of the convex hull of the placed arcs. A chord estimate of Keller confines the placements with $A(p) < 0.2405$ to a box X , so Theorem 1.1 follows once $A > 0.2405$ on X . We prove this by a branch and bound over subboxes of X . On a box B with centre c we follow finitely many vertices of the placed arcs and bound A from below by the signed area $F(p) = \frac{1}{2} \sum_j \det(x_j(p), x_{j+1}(p))$ of the polygon that they span. Three criteria ensure $F \leq A$ on B : the vertices stay in strict angular order around a point, winding once around it (Lemma 3.2); the list has at most four vertices (Lemma 3.1, due to Keller); or the list can be cut along a chord into pieces that have at most four vertices, form fans from one of their vertices, or can be cut again, as in Keller's chord decomposition (Lemmas 3.3 and 3.4). We bound F on B by its Taylor expansion at c , with a certified bound for the smallest eigenvalue of its Hessian and explicit bounds for the variation of the Hessian over B (Lemmas 3.7, 3.8 and 3.9). Where the hull changes its combinatorial type inside a box, we combine the bounds of several polygons convexly, so that their first-order losses can cancel (Lemma 3.10). Keller [4] bounds the polygon areas on a box by polynomials of degree at most two in the translations and in the cosines and sines of the angle offsets, with bounds for each coefficient [4, Lemma 3.6 and (3.5)], and he also combines several lists with weights [4, Section 3.4]. Instead of his polynomial bounds we use second-order Taylor bounds. Our Taylor bound extends that of our proof of a lower bound for Lebesgue's universal covering problem [13, Lemma 3.5] by two further estimates of the second-order term, and the combination bound is [13, Lemma 3.8]. The certificate has 405,370,260 nodes and 203,209,418 leaves, and 28,558,819 of its leaves use a convex combination.

Verification. A checker of about 610 lines of Python, which shares no code with the search program, verifies the certificate in ball arithmetic with the Arb library [3] through python-flint. It reads the data of the arcs from the certificate, checks the root box against the chord estimate, decodes the trees, checks that every leaf box equals $c \pm h$ for its double-precision centre c and half-widths h , and verifies at every leaf every inequality that the proof uses (Section 4). Two short programs check that the headers of the parts record the arcs of Definition 2.1 and a grid whose cells tile the root box, that all headers agree, and that the checked leaves cover every

part. Since a leaf is accepted only if inequalities verified in ball arithmetic prove $A > 0.2405$ on it, the certificate itself need not be trusted. The proof does, however, rest on these programs: it is not formalized in a proof assistant, and no independently written verifier has been run on the whole certificate. As a partial cross-check, a further implementation of the leaf bound, written without reference to the checker, accepted all 450 leaves on which it was run, among them 50 of those with the smallest margins (Section 4.7).

Section 2 reduces Theorem 1.1 to a statement about a box of placements. Section 3 proves the polygon bounds and their versions on a box. Section 4 describes the certificate and the checker and proves Theorem 1.1. Section 5 records numerical observations that the proof does not use.

2. REDUCTION TO A BOX OF PLACEMENTS

For $a, b \in \mathbb{R}^2$ put $\det(a, b) = a_1b_2 - a_2b_1$. Let $u(\varphi) = (\cos \varphi, \sin \varphi)$, let R_φ be the rotation of $\mathbb{R}^2$ about the origin by the angle φ , and let $J = R_{\pi/2}$. Rotations preserve $\det$, and J commutes with every R_φ . We write area for the Lebesgue measure on $\mathbb{R}^2$, conv for the convex hull and $|\cdot|$ for the Euclidean norm. A *proper motion* is a map $x \mapsto R_\varphi x + t$. For points $v_0, \dots, v_n$ we write $[v_0, \dots, v_n]$ for the polygonal arc that runs through them in this order. For a point v , lengths $\ell_1, \dots, \ell_n > 0$ and angles $\alpha_1, \dots, \alpha_n$ in degrees, the polygonal arc *starting at v with these lengths and directions* is $[v_0, \dots, v_n]$ with $v_0 = v$ and $v_k = v_{k-1} + \ell_k u(\pi\alpha_k/180)$.

Definition 2.1. The arcs L, T, U, C, S are the polygonal arcs starting at the given point with the given lengths and directions:

Arc	Start	Lengths	Directions (degrees)
L	$(-1/2, 0)$	1	0
T	$(-1/4, 0)$	$1/2, 1/2$	$60, -60$
U	$(0, 0)$	$1/3, 1/3, 1/3$	$0, 84, 168$
C	$(0, 0)$	$3/20, 7/10, 3/20$	$0, -70, 0$
S	$(0, 0)$	$41/250, 84/125, 41/250$	$0, 75, 0$

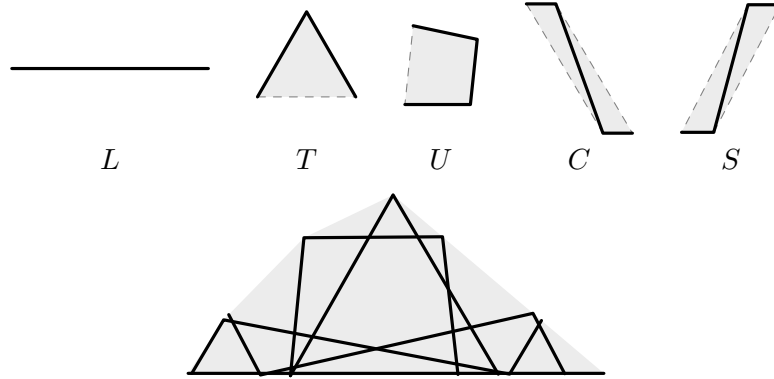


FIGURE 1. Top: the arcs L, T, U, C, S of Definition 2.1, at the same scale, with their convex hulls shaded. Bottom: the placement with the smallest hull area found, 0.2420538 (Section 5); the shaded region is the convex hull of the five placed arcs.

Each arc has length 1 (Figure 1). The convex hull of L is a segment, that of T is an equilateral triangle of side $1/2$, that of U is an isosceles trapezoid, and those of C and S are parallelograms,

because the first and the last link of these arcs are parallel and have equal length. Every vertex of T , U , C and S is a vertex of its convex hull. The zigzag C turns clockwise and then counterclockwise, and S turns counterclockwise and then clockwise. A proper motion cannot map one of these patterns to the other.

We number the arcs as bodies $0, \dots, 4$ in the order L, T, U, C, S . Let $v_{i,0}, \dots, v_{i,n_i}$ be the vertices of body i , and for $i \geq 1$ let c_i be the mean of all these vertices and $q_{i,j} = v_{i,j} - c_i$. For $i = 0$ put $q_{0,j} = v_{0,j}$. A *placement* is a vector

$$p = (\varphi_1, t_1, \varphi_2, t_2, \varphi_3, t_3, \varphi_4, t_4) \in \mathbb{R}^{12},$$

with $\varphi_i \in \mathbb{R}$ and $t_i \in \mathbb{R}^2$. We put $\varphi_0 = 0$ and $t_0 = 0$. In the placement p , vertex j of body i is at

$$x_{i,j}(p) = R_{\varphi_i} q_{i,j} + t_i,$$

and body i occupies the arc $\gamma_i(p)$ through these points, which is the image of the arc of Definition 2.1 under a proper motion. Let

$$K(p) = \text{conv}\{x_{i,j}(p) : 0 \leq i \leq 4, 0 \leq j \leq n_i\} \quad \text{and} \quad A(p) = \text{area } K(p).$$

The convex hull of a polygonal arc is the convex hull of its vertices, so $K(p)$ is the convex hull of the five placed arcs.

Lemma 2.2. *Let Ω be as in Theorem 1.1. Then $\text{area } \Omega \geq \inf_p A(p)$.*

Proof. Let $g_0, \dots, g_4$ be proper motions with $g_i(\gamma_i) \subseteq \Omega$, where γ_i is body i in the position of Definition 2.1. The set $g_0^{-1}(\Omega)$ is convex, has the same area as Ω , and contains L and the arcs $g_0^{-1}g_i(\gamma_i)$. The proper motion $g_0^{-1}g_i$ maps γ_i to $R_{\varphi}(\gamma_i - c_i) + t$ for some φ and t . So $g_0^{-1}(\Omega)$ contains the five arcs of some placement p , hence $K(p)$, and $\text{area } \Omega \geq A(p)$. $\square$

The next lemma is Keller's chord bound [4, Lemma 3.2]. The *extent* of a compact set $E \neq \emptyset$ in the direction of a unit vector u is $\max_{x \in E} u \cdot x - \min_{x \in E} u \cdot x$.

Lemma 2.3 (Chord bound). *Let $K \subseteq \mathbb{R}^2$ be convex and contain a segment of length $d > 0$ and a compact set $E \neq \emptyset$. If E has extent ω in the direction perpendicular to the segment, then $\text{area } K \geq d\omega/2$.*

Proof. Let E' be the union of E and the endpoints of the segment, and let $\ell_+, \ell_- \geq 0$ be the largest distances from the line of the segment of the points of E' in the two closed half-planes bounded by this line. Then $\ell_+ + \ell_- \geq \omega$. The segment and two points at these distances span triangles $\Delta_{\pm} \subseteq K$, possibly degenerate, with disjoint interiors and areas $d\ell_{\pm}/2$. $\square$

For $\tau > 0$ let $w_T = \sqrt{3}/4$, $X_{\tau} = 2\tau/w_T - 1/2$ and $Y_{\tau} = 2\tau$, and let $D_{\tau} \subset \mathbb{R}^{12}$ be the set of placements with

$$\begin{aligned} \varphi_1 &\in [0, \pi/3], & \varphi_2 &\in [0, 2\pi], & \varphi_3, \varphi_4 &\in [0, \pi], \\ t_i &\in [-X_{\tau}, X_{\tau}] \times [-Y_{\tau}, Y_{\tau}] & (1 \leq i \leq 4). \end{aligned}$$

Proposition 2.4. *Let $\tau > 0$. If $A(p) < \tau$ for some placement p , then $A(p') = A(p)$ for some $p' \in D_{\tau}$.*

Proof. The argument is Keller's [4, Section 3.2], without the reflections. The equilateral triangle $\text{conv } T$ is invariant under the rotation by $2\pi/3$ about its centre c_1 , and the parallelograms $\text{conv } C$ and $\text{conv } S$ are invariant under the half-turns about their centres c_3 and c_4 . The centre of a parallelogram is the mean of its vertices, and so is the centre of an equilateral triangle. Hence $K(p)$ does not change if φ_1 is changed by a multiple of $2\pi/3$, φ_3 or φ_4 by a multiple of π , or φ_2 by a multiple of 2π . The half-turn $x \mapsto -x$ is a proper motion that maps L onto itself, and it maps the placed arcs of p to those of the placement with angles $\varphi_i + \pi$ and translations $-t_i$,

with the same hull area. Since $\pi \equiv \pi/3 \pmod{2\pi/3}$, after this half-turn if necessary and the reductions above we may assume $\varphi_1 \in [0, \pi/3]$, $\varphi_2 \in [0, 2\pi]$ and $\varphi_3, \varphi_4 \in [0, \pi]$.

Let p' be the placement obtained in this way, so that $A(p') = A(p) < \tau$, let $t_1, \dots, t_4$ be its translations and let $K' = K(p')$. The triangle $\text{conv} T$ has extent at least $w_T = \sqrt{3}/4$ in every direction, and so does its placed copy in K' . Lemma 2.3, applied to a chord of K' of length $\text{diam } K'$ and to this copy, gives $\text{diam } K' \leq 2 \text{area } K' / w_T < 2\tau / w_T$. For $i \geq 1$ the vectors $q_{i,j}$ have mean 0, so t_i is the mean of the placed vertices of body i , and $t_i \in K'$. With $a_{\pm} = (\pm 1/2, 0)$, the endpoints of L , we get $|t_{i,1}| + 1/2 \leq \max_{\pm} |t_i - a_{\pm}| \leq \text{diam } K' < 2\tau / w_T$, and therefore $|t_{i,1}| < X_{\tau}$. The triangle $\text{conv}\{a_-, a_+, t_i\} \subseteq K'$ has area $|t_{i,2}|/2$, so $|t_{i,2}| < 2\tau = Y_{\tau}$. Hence $p' \in D_{\tau}$. $\square$

Corollary 2.5. *Let $\tau > 0$ and let X be a box with $D_{\tau} \subseteq X$. If $A > \tau$ on X , then every Ω as in Theorem 1.1 has $\text{area } \Omega \geq \tau$.*

Proof. If $A(p) < \tau$ for some p , then $A(p') < \tau$ for some $p' \in D_{\tau} \subseteq X$ by Proposition 2.4, a contradiction. Therefore $A \geq \tau$ everywhere, and Lemma 2.2 gives $\text{area } \Omega \geq \tau$. $\square$

3. THE BOX BOUND

In this section we bound A from below on a box of placements. A *box* is a set $B = \prod_{a=1}^{12} [c_a - h_a, c_a + h_a]$ with centre c and half-widths $h_a \geq 0$. A *vertex list* is a sequence $P = (g_1, \dots, g_m)$ of distinct vertices $g_j = (i_j, k_j)$ of the five bodies. At a placement p its points are $x_j(p) = x_{i_j, k_j}(p)$, indices are taken modulo m , and its *signed area* is

$$F_P(p) = \frac{1}{2} \sum_{j=1}^m \det(x_j(p), x_{j+1}(p)).$$

All points $x_j(p)$ lie in $K(p)$. The following four lemmas give conditions for $F_P \leq A$. The *angle* from a nonzero vector v to a nonzero vector v' with $\det(v, v') > 0$ is the number $\theta \in (0, \pi)$ with $v'/|v'| = R_{\theta}(v/|v|)$.

Lemma 3.1 (Small lists). *Let $x_1, \dots, x_m$ be points of a convex set K with $m \leq 4$. Then $\frac{1}{2} \sum_{j=1}^m \det(x_j, x_{j+1}) \leq \text{area } K$.*

Proof. For $m \leq 2$ the sum is 0. For $m = 3$ its absolute value is the area of the triangle $\text{conv}\{x_1, x_2, x_3\} \subseteq K$. For $m = 4$ the sum equals $\frac{1}{2} \det(x_3 - x_1, x_4 - x_2)$. If $x_1 = x_3$ this is 0. Otherwise its absolute value is $d\omega/2$, where $d = |x_3 - x_1|$ and ω is the extent of $\{x_2, x_4\}$ perpendicular to $[x_1, x_3]$, and Lemma 2.3 applies. For $m = 3, 4$ this is [4, Lemma 3.3]. $\square$

Lemma 3.2 (Fan lemma). *Let $K \subset \mathbb{R}^2$ be convex, let $x_1, \dots, x_m \in K$ with $m \geq 3$, and let $z \in \mathbb{R}^2$. Suppose that $\det(x_j - z, x_{j+1} - z) > 0$ for all j , and let θ_j be the angle from $x_j - z$ to $x_{j+1} - z$. If $\theta_1 + \dots + \theta_m = 2\pi$, then*

$$\text{area } K \geq \frac{1}{2} \sum_{j=1}^m \det(x_j - z, x_{j+1} - z) = \frac{1}{2} \sum_{j=1}^m \det(x_j, x_{j+1}).$$

Proof. This extends [13, Lemma 3.1], where $z \in K$ is assumed. Write $x_1 - z = r_1 u(\psi_1)$ with $r_1 > 0$, and put $\psi_{j+1} = \psi_j + \theta_j$. Then $x_j - z = r_j u(\psi_j)$ with $r_j > 0$, and $\psi_{m+1} = \psi_1 + 2\pi$. The intervals (ψ_j, ψ_{j+1}) are disjoint subintervals of $[\psi_1, \psi_1 + 2\pi)$ of length less than π . Every direction from z lies in one of the closed sectors $\{z + s u(\psi) : s \geq 0, \psi_j \leq \psi \leq \psi_{j+1}\}$, so the points $x_j - z$ do not lie in an open half-plane bounded by a line through 0, and $z \in \text{conv}\{x_1, \dots, x_m\} \subseteq K$. Hence the triangle $\Delta_j = \text{conv}\{z, x_j, x_{j+1}\}$ lies in K . Its interior lies in the open sector between the directions ψ_j and ψ_{j+1} , so the triangles have disjoint interiors,

and $\text{area } K \geq \sum_j \text{area } \Delta_j = \frac{1}{2} \sum_j \det(x_j - z, x_{j+1} - z)$. Finally, $\sum_j \det(x_j - z, x_{j+1} - z) = \sum_j \det(x_j, x_{j+1}) + \sum_j \det(z, x_j) - \sum_j \det(z, x_{j+1})$, and the last two sums are equal. $\square$

Lemma 3.3 (Fan from a vertex). *Let $x_1, \dots, x_m$ be points of a convex set K with $m \geq 3$, and put $\xi_j = x_j - x_1$. If $\det(\xi_2, \xi_j) > 0$ for $3 \leq j \leq m$ and $\det(\xi_j, \xi_{j+1}) > 0$ for $3 \leq j \leq m-1$, then $\frac{1}{2} \sum_{j=1}^m \det(x_j, x_{j+1}) = \sum_{j=2}^{m-1} \text{area conv}\{x_1, x_j, x_{j+1}\} \leq \text{area conv}\{x_1, \dots, x_m\}$.*

Proof. This is Keller's fan bound [4, Lemma 3.4]. The vectors ξ_j with $j \geq 3$ lie in the open half-plane to the left of ξ_2 , and their angles from ξ_2 increase strictly with j . Consequently, the triangles $\text{conv}\{x_1, x_j, x_{j+1}\}$ have disjoint interiors and positive areas $\frac{1}{2} \det(\xi_j, \xi_{j+1})$, and the shoelace sum of the list equals the sum of these areas. $\square$

Lemma 3.4 (Chord decomposition). *Let $x_1, \dots, x_m$ be points of a convex set K , and let $1 \leq s, s' \leq m$ with $2 \leq s' - s \pmod{m} \leq m-2$. Let P' be the list $x_s, x_{s+1}, \dots, x_{s'}$ and P'' the list $x_{s'}, x_{s'+1}, \dots, x_s$, with indices modulo m . Suppose that $x_s \neq x_{s'}$, that the points of P' strictly between x_s and $x_{s'}$ satisfy $\det(x_{s'} - x_s, x - x_s) \leq 0$ and those of P'' strictly between $x_{s'}$ and x_s satisfy $\det(x_{s'} - x_s, x - x_s) \geq 0$, and that the signed areas of P' and P'' are at most the areas of the convex hulls of their points. Then $\frac{1}{2} \sum_{j=1}^m \det(x_j, x_{j+1}) \leq \text{area conv}\{x_1, \dots, x_m\}$.*

Proof. This is Keller's chord decomposition [4, Lemma 3.5]. The signed area of the list is the sum of the signed areas of P' and P'' , because the chord appears in the two shoelace sums with opposite orientations. The convex hulls of P' and P'' lie in the two closed half-planes bounded by the line through x_s and $x_{s'}$, so they meet in a set of area 0, and both lie in $\text{conv}\{x_1, \dots, x_m\}$. $\square$

By Lemmas 3.1 and 3.3, the hypothesis of Lemma 3.4 on P' and P'' holds if each of them has at most four points, satisfies the conditions of Lemma 3.3 after a cyclic shift, or can itself be cut by Lemma 3.4. We call a vertex list together with such a recipe, or with a point z for Lemma 3.2, a *structure*, and we say that a structure is *valid* at p if the hypotheses of the corresponding lemma hold for the points $x_j(p)$. If a structure with vertex list P is valid at p , then $F_P(p) \leq A(p)$.

On a box the validity conditions are strict inequalities between determinants of the points $x_j(p)$. We bound how much they can change. For a body $i \geq 1$ let r_i be the half-width of B in φ_i , and let $\delta_i = (h_{i,1}^2 + h_{i,2}^2)^{1/2}$, where $h_{i,1}, h_{i,2}$ are the half-widths of B in the coordinates of t_i . Put $r_0 = \delta_0 = 0$. For a vertex $g = (i, k)$ let $\mu(g) = |q_{i,k}| r_i + \delta_i$. For a function ϕ on B and $p \in B$ write $\Delta\phi = \phi(p) - \phi(c)$.

Lemma 3.5 (Variation of determinants). *Let B be a box with centre c and $p \in B$.*

- (a) $\|R_{\varphi_i(p)} - R_{\varphi_i(c)}\| \leq r_i$, $|\Delta t_i| \leq \delta_i$ and $|\Delta x_g| \leq \mu(g)$ for every vertex g .
- (b) For vertices g, g' let $\mu(g, g') = |q_{i,k'} - q_{i,k}| r_i$ if $g = (i, k)$ and $g' = (i, k')$ lie on the same body, and $\mu(g, g') = \mu(g) + \mu(g')$ otherwise. Then $|\Delta(x_{g'} - x_g)| \leq \mu(g, g')$.
- (c) For vertices a, b, g put $u = x_b - x_a$ and $v = x_g - x_a$. Then $|\Delta \det(u, v)| \leq \mu(a, b)|v(c)| + \mu(a, g)|u(c)| + \mu(a, b)\mu(a, g)$, and $\Delta \det(u, v) = 0$ if a, b, g lie on one body.
- (d) For a point z and consecutive vertices $g = (i, k)$, $g' = (i', k')$ of a list, with positions x and y , let $\eta = \det(x - z, y - z)$. If $i = i' = 0$, then $\Delta\eta = 0$. If $i = i' \geq 1$, then $|\Delta\eta| \leq |q_{i,k'} - q_{i,k}|(\delta_i(1 + r_i) + |t_i(c) - z| r_i)$. If $i \neq i'$, then $|\Delta\eta| \leq \mu(g)(|y(c) - z| + \mu(g')) + |x(c) - z| \mu(g')$.

Proof. Parts (a) and (d) are [13, Lemma 3.6(a),(b)], with $\mu(g)$ in place of $e(i, q)$. (a) $\|R_\varphi - R_{\varphi'}\| = 2|\sin((\varphi - \varphi')/2)| \leq |\varphi - \varphi'|$, so $|\Delta x_g| \leq \|\Delta R_{\varphi_i}\| |q_{i,k}| + |\Delta t_i| \leq \mu(g)$. (b) On one body $x_{g'} - x_g = R_{\varphi_i}(q_{i,k'} - q_{i,k})$; otherwise use (a) twice. (c) $\Delta \det(u, v) = \det(\Delta u, v(c)) + \det(u(c), \Delta v) + \det(\Delta u, \Delta v)$ and (b). If $a = (i, a')$, $b = (i, b')$ and $g = (i, g')$ lie on one body, then $\det(u, v) = \det(q_{i,b'} - q_{i,a'}, q_{i,g'} - q_{i,a'})$ is constant. (d) If $i = i' \geq 1$, then $\eta = \det(q_{i,k}, q_{i,k'}) + \det(t_i - z, R_{\varphi_i} w)$ with $w = q_{i,k'} - q_{i,k}$, because $\det(Ra + t, Rb + t) = \det(a, b) + \det(t, R(b - a))$ for a rotation R .

Writing $R = R_{\varphi_i}$ and $R_c = R_{\varphi_i(c)}$, $\Delta\eta = \det(\Delta t_i, R_c w) + \det(t_i(c) - z, (R - R_c)w) + \det(\Delta t_i, (R - R_c)w)$, and (a) bounds the three terms. If $i \neq i'$, then $\Delta\eta = \det(\Delta x, y - z) + \det(x(c) - z, \Delta y)$ and $|y - z| \leq |y(c) - z| + \mu(g')$. $\square$

Lemma 3.6 (Validity on a box). *Let B be a box with centre c , and let a structure satisfy the following. A fan has at least three points, a fan from a vertex at least three and a small list or piece at most four, and every cut satisfies the condition on s and s' of Lemma 3.4. At c , every determinant in its validity conditions that must be positive exceeds the bound of Lemma 3.5(c) or (d) for its change, every determinant that must be ≤ 0 or ≥ 0 satisfies the corresponding inequality with this bound added or subtracted, every pair of chord endpoints satisfies $|x_{s'}(c) - x_s(c)| > \mu(g_s, g_{s'})$, and, for a fan, the angles at c add up to 2π . Then the structure is valid at every $p \in B$, and $F_P \leq A$ on B .*

Proof. The conditions on the numbers of points do not depend on p . By Lemma 3.5 every other condition holds on B , except possibly the angle sum of a fan. For the angle sum we use the argument of [13, Lemma 3.3]. For a fan let $\Theta(p)$ be the sum of the angles at p . It is continuous on B , because the determinants are positive there, and it is an integer multiple of 2π , since rotating the direction of $x_1(p) - z$ successively by the angles returns it to itself. As B is connected, $\Theta = \Theta(c) = 2\pi$ on B . $\square$

We now bound F_P on a box. We split the cyclic vertex list into maximal *runs* of consecutive vertices on the same body. Suppose that the vertices lie on at least two bodies; then consecutive runs lie on different bodies, also cyclically. Let run ϱ lie on body ι_ϱ and consist of the vertices with positions $x_{\alpha_\varrho}, \dots, x_{\beta_\varrho}$, with q -values $q_{\alpha_\varrho}, \dots, q_{\beta_\varrho}$. The identity $\det(Ra + t, Rb + t) = \det(a, b) + \det(t, R(b - a))$, summed over a run, gives

$$(3.1) \quad F_P(p) = \frac{1}{2} \sum_{\varrho} \left(C_{\varrho} + \det(t_{\iota_\varrho}, R_{\varphi_{\iota_\varrho}} w_{\varrho}) + \det(x_{\beta_\varrho}(p), x_{\alpha_{\varrho+1}}(p)) \right),$$

where $C_{\varrho} = \sum_{j=\alpha_\varrho}^{\beta_\varrho-1} \det(q_j, q_{j+1})$ and $w_{\varrho} = q_{\beta_\varrho} - q_{\alpha_\varrho}$. We call $\frac{1}{2} \det(t_{\iota_\varrho}, R_{\varphi_{\iota_\varrho}} w_{\varrho})$ a *run term*; it depends on the parameters of one body and vanishes for the segment. We call $\frac{1}{2} \det(x_{\beta_\varrho}(p), x_{\alpha_{\varrho+1}}(p))$ a *transition term*; it depends on the parameters of two different bodies. This is [13, Lemma 3.4 and equation (3.1)], with all four bodies rotating.

Lemma 3.7 (Taylor bound). *Let B be a box with centre c and half-widths h , and let F be twice continuously differentiable near B . Put $H = \nabla^2 F(c)$ and $M = \text{diag}(h) H \text{diag}(h)$, let W be a symmetric matrix with $|\partial_a \partial_b F(\xi) - H_{ab}| \leq W_{ab}$ for all $\xi \in B$, and let n_h be the number of a with $h_a > 0$. Let Λ be one of the numbers*

$$\Lambda_1 = \min(0, \lambda_{\min}(H)) \sum_a h_a^2, \quad \Lambda_2 = \sum_a \min(0, M_{aa}) - \sum_{a \neq b} |M_{ab}|, \quad \Lambda_3 = \min(0, \lambda_{\min}(M)) n_h,$$

or a lower bound for one of them, and put $\varepsilon = -\frac{\Lambda}{2} + \frac{1}{2} \sum_{a,b} W_{ab} h_a h_b$. Then $F(p) \geq F(c) + \nabla F(c) \cdot (p - c) - \varepsilon \geq F(c) - \sum_a |\partial_a F(c)| h_a - \varepsilon$ for all $p \in B$.

Proof. Let $p \in B$ and $v = p - c = \text{diag}(h) y$ with $|y_a| \leq 1$, where we take $y_a = 0$ if $h_a = 0$. Taylor's theorem gives $\xi \in B$ with $F(p) = F(c) + \nabla F(c) \cdot v + \frac{1}{2} v^T \nabla^2 F(\xi) v$. Here $v^T (\nabla^2 F(\xi) - H) v \geq -\sum_{a,b} W_{ab} h_a h_b$, and $v^T H v = y^T M y$ is at least each of Λ_1, Λ_2 and Λ_3 : $v^T H v \geq \lambda_{\min}(H) |v|^2 \geq \Lambda_1$; $y^T M y = \sum_a M_{aa} y_a^2 + \sum_{a \neq b} M_{ab} y_a y_b \geq \Lambda_2$; and $y^T M y \geq \lambda_{\min}(M) |y|^2 \geq \Lambda_3$, since $|y|^2 \leq n_h$. Finally, $\nabla F(c) \cdot v \geq -\sum_a |\partial_a F(c)| h_a$. $\square$

For the bound W we need the second derivatives of the terms of (3.1) and their variation on B .

Lemma 3.8 (Derivatives and variation bounds). *Let B be a box with centre c , and let $p \in B$.*

- (a) *Let $G = \frac{1}{2} \det(t_i, R_{\varphi_i} w)$ be a run term with $i \geq 1$. Its second derivatives that do not vanish identically are $\partial^2 G / \partial \varphi_i^2 = -\frac{1}{2} \det(t_i, R_{\varphi_i} w)$ and $\partial^2 G / \partial \varphi_i \partial t_{i,a} = \frac{1}{2} \det(e_a, JR_{\varphi_i} w)$, where e_1, e_2 is the standard basis of $\mathbb{R}^2$, and on B*

$$\left| \Delta \frac{\partial^2 G}{\partial \varphi_i^2} \right| \leq \frac{|w|}{2} (\delta_i(1 + r_i) + |t_i(c)| r_i), \quad \left| \Delta \frac{\partial^2 G}{\partial \varphi_i \partial t_{i,a}} \right| \leq \frac{|w|}{2} r_i \quad (a = 1, 2).$$

- (b) *Let $G = \frac{1}{2} \det(x, y)$ be a transition term, where x is the position of a vertex $g = (i, k)$ and y that of a vertex $g' = (i', k')$ with $i \neq i'$, and put $q = q_{i,k}$, $q' = q_{i',k'}$, $\mu = \mu(g)$ and $\mu' = \mu(g')$. The second derivatives of G with respect to the coordinates of t_i and $t_{i'}$ are constant. The other second derivatives ϕ of G that do not vanish identically, and bounds for $|\Delta \phi|$ on B , are as follows, where $a, b \in \{1, 2\}$:*

$$\begin{aligned} \phi &= \frac{\partial^2 G}{\partial \varphi_i \partial t_{i',b}} = \frac{1}{2} \det(JR_{\varphi_i} q, e_b), & |\Delta \phi| &\leq \frac{1}{2} |q| r_i, \\ \phi &= \frac{\partial^2 G}{\partial t_{i,a} \partial \varphi_{i'}} = \frac{1}{2} \det(e_a, JR_{\varphi_{i'}} q'), & |\Delta \phi| &\leq \frac{1}{2} |q'| r_{i'}, \\ \phi &= \frac{\partial^2 G}{\partial \varphi_i \partial \varphi_{i'}} = \frac{1}{2} \det(JR_{\varphi_i} q, JR_{\varphi_{i'}} q'), & |\Delta \phi| &\leq \frac{1}{2} |q| |q'| (r_i + r_{i'}), \\ \phi &= \frac{\partial^2 G}{\partial \varphi_i^2} = -\frac{1}{2} \det(R_{\varphi_i} q, y), & |\Delta \phi| &\leq \frac{1}{2} |q| (r_i(|y(c)| + \mu') + \mu'), \\ \phi &= \frac{\partial^2 G}{\partial \varphi_{i'}^2} = -\frac{1}{2} \det(x, R_{\varphi_{i'}} q'), & |\Delta \phi| &\leq \frac{1}{2} |q'| (r_{i'}(|x(c)| + \mu) + \mu). \end{aligned}$$

Derivatives with respect to φ_i or t_i occur only for $i \geq 1$.

Proof. This is [13, Lemma 3.6(c),(d)], where body 0 is a disk and body 1 does not rotate; the proof is the same. We have $\partial_\varphi R_\varphi = R_\varphi J$ and $J^2 = -I$, which gives the derivatives. Their variations are bounded with Lemma 3.5(a): for instance $|\Delta \det(R_{\varphi_i} q, y)| \leq r_i |q| |y| + |q| |\Delta y| \leq |q| (r_i(|y(c)| + \mu') + \mu')$, and $\det(Ja, Jb) = \det(a, b)$ gives the third line. $\square$

For a vertex list on at least two bodies, let W be the symmetric matrix obtained by adding, over all run terms and transition terms of (3.1), the bounds of Lemma 3.8 at the corresponding entries, symmetrically. Since $\nabla^2 F_P$ is the sum of the Hessians of these terms, W satisfies the hypothesis of Lemma 3.7. All entries of W are nonnegative.

Lemma 3.9 (Eigenvalue certificate). *Let H be a real symmetric $d \times d$ matrix, let $\lambda \in \mathbb{R}$ and $N = H - \lambda I$. For $a = 1, \dots, d$ define recursively*

$$L_{ab} = \frac{1}{D_b} \left(N_{ab} - \sum_{\gamma < b} L_{a\gamma} L_{b\gamma} D_\gamma \right) \quad (b < a), \quad D_a = N_{aa} - \sum_{\gamma < a} L_{a\gamma}^2 D_\gamma.$$

If the recursion yields $D_1, \dots, D_d > 0$, then $\lambda_{\min}(H) > \lambda$.

Proof. This is [13, Lemma 3.7]. The recursion gives $N = LDL^T$ with L unit lower triangular and $D = \text{diag}(D_1, \dots, D_d)$. For $v \neq 0$ we have $L^T v \neq 0$, so $v^T N v = \sum_a D_a ((L^T v)_a)^2 > 0$. $\square$

If the entries of H are known only up to enclosing intervals and the recursion is evaluated in interval arithmetic, positive lower endpoints of the computed D_a certify $\lambda_{\min}(H) > \lambda$, as in [13].

Lemma 3.10 (Combination bound). *Let B be a box with centre c and half-widths h . For $s = 1, \dots, n$ let a structure with vertex list P_s be valid on B , let $F_s = F_{P_s}$, and let $\varepsilon_s \in \mathbb{R}$ satisfy $F_s(p) \geq F_s(c) + \nabla F_s(c) \cdot (p - c) - \varepsilon_s$ for all $p \in B$. Let $\kappa_1, \dots, \kappa_n \geq 0$ with $\kappa_1 + \dots + \kappa_n = 1$. Then for all $p \in B$*

$$A(p) \geq \sum_{s=1}^n \kappa_s (F_s(c) - \varepsilon_s) - \sum_a \left| \sum_{s=1}^n \kappa_s \partial_a F_s(c) \right| h_a.$$

Proof. $A(p) \geq F_s(p)$ for every s , so $A(p) \geq \sum_s \kappa_s F_s(p) \geq \sum_s \kappa_s (F_s(c) - \varepsilon_s) + u \cdot (p - c)$ with $u = \sum_s \kappa_s \nabla F_s(c)$, and $u \cdot (p - c) \geq -\sum_a |u_a| h_a$. $\square$

By Lemma 3.7, the number ε defined there can serve as ε_s . For $n = 1$ the lemma gives the bound $F(c) - \sum_a |\partial_a F(c)| h_a - \varepsilon$ of Lemma 3.7.

4. THE CERTIFICATE AND ITS VERIFICATION

4.1. The certificate. The certificate for $\tau = 0.2405$ consists of 16,384 parts. Each part has a header, an array of codes and a file of leaf records. The header records the exact data of the five arcs as in Definition 2.1 (start points and lengths as rational numbers, directions in degrees as integers), the numbering of their vertices, the angle periods $2/3$, 2 , 1 and 1 of bodies 1 to 4 in units of π , τ , a root box X given by its lower and upper corners $X^-, X^+ \in \mathbb{R}^{12}$, a grid $(s_1, \dots, s_{12})$, the number of parts and the index of the part, and the largest length of a small list, here 4 . The grid is $(4, 2, 2, 16, 2, 2, 8, 2, 2, 8, 2, 2)$. In coordinate a the endpoints of the cells are the numbers $X_a^- + (X_a^+ - X_a^-)k/s_a$ for $k = 0, \dots, s_a$, computed in double precision. For this grid and root box they are exact, so X is cut into $s_1 \cdots s_{12} = 1,048,576$ equal cells. The cells are listed in lexicographic order of their grid positions, and part j holds the cells whose position in this list is congruent to j modulo 16,384. The array of a part describes, for each of its cells in turn, a binary tree in depth-first preorder. A code $a \in \{0, \dots, 11\}$ splits the current box at the midpoint of coordinate $a + 1$, and the trees of the lower and the upper half follow. The codes 254 and 253 mark leaves. For each leaf, in order, the leaf file has a record with a list of structures and a list of nonnegative weights. A structure is a vertex list together with its type: a point z for Lemma 3.2, the marker of a small list, or a tree of cuts by Lemma 3.4 whose pieces are small lists or fans from a vertex (Lemma 3.3) with a given first vertex. A leaf with code 254 has one structure, and a leaf with code 253 has several, to be combined by Lemma 3.10.

4.2. The search. The search program, which is not part of the proof, is a depth-first branch and bound in double precision. At a box it forms candidate structures from the vertices of the convex hull at the centre and of sub-hulls of selected bodies, from small lists and from chord decompositions of the hull list, and evaluates the bound of Lemma 3.7. If no single structure closes the box, it chooses the weights of Lemma 3.10 for several valid structures by a linear program, clips them at 0 and normalizes them. A box is closed when its bound is at least $\tau + 10^{-7}$. Otherwise it is split in the widest coordinate if no candidate structure is valid, and else in the coordinate with the largest contribution to the loss in the bound. If the largest contribution is below 10^{-5} or the contributions explain less than half of the gap to the hull area at the centre, the box is split instead in the coordinate in which the vertices move most, and a coordinate whose half-width is below 10^{-12} is never split.

4.3. The checker. The checker¹ works in ball arithmetic at 100 bits of precision. The box coordinates are double-precision numbers, which are exact in Arb. Every comparison that decides whether a part is accepted is made in ball arithmetic or exactly. Two steps leave ball arithmetic, a fallback for square roots and the rounding of the number Λ below, and we bound their effect at the end of this subsection. The checker reads τ as the shortest decimal number that represents the header value, here 0.2405. For each part it performs the following steps.

- (1) *Arcs.* From the header it computes the vertices $v_{i,j}$ and the vectors $q_{i,j}$ in ball arithmetic. It verifies that each arc has length at most 1, that body 0 is the unit segment L , that body 1 is the arc T , that each body whose angle range is taken modulo π has links (a, m, a) with directions (α, β, α) , so that its convex hull is a parallelogram whose centre is the mean of its vertices, that the numbering lists every vertex once, and that the largest length of a small list is at most 4.
- (2) *Root box.* It verifies that $X \supseteq D_\tau$: the range of φ_1 contains $[0, \pi/3]$, for $i = 2, 3, 4$ the range of φ_i contains $[0, \pi p_i]$, where p_i is the period of body i recorded in the header, that is, $[0, 2\pi]$, $[0, \pi]$ and $[0, \pi]$, and the ranges of the coordinates of every t_i contain $[-X_\tau, X_\tau]$ and $[-Y_\tau, Y_\tau]$, with X_τ and Y_τ evaluated in ball arithmetic.
- (3) *Tree.* It computes the cells of the part from the grid, the number of parts and the index of the part as in Section 4.1, decodes the codes and computes the boxes, splitting each box at the double-precision midpoint of the split coordinate. It checks that every code is one of $0, \dots, 11, 253$ and 254 , that the array consists of complete trees with no data after the last one, that the leaf file has exactly one record for each leaf, and that the record of each leaf with code 254 has exactly one structure.
- (4) *Leaves.* At each leaf it computes the centre c and the half-widths h in double precision and verifies, in rational arithmetic, that $c \pm h$ are exactly the endpoints of the leaf box. It requires the record to contain at least one structure and one weight for each structure, every weight to be a nonnegative number, and at least one weight to be positive. Structures of weight 0 are ignored. For every structure of positive weight it verifies that the vertices of its list are distinct, and it verifies the hypotheses of Lemma 3.6 in ball arithmetic: for a fan, the determinants $\det(x_j(c) - z, x_{j+1}(c) - z)$ exceed the bounds of Lemma 3.5(d) and the computed sum of the angles at c differs from 2π by less than 1; for a small list, it has at most four vertices; for a chord decomposition, every cut and every piece satisfies the conditions of Lemmas 3.4 and 3.3 with the bounds of Lemma 3.5(b) and (c). For each such structure it evaluates $F_P(c)$, $\nabla F_P(c)$ and $\nabla^2 F_P(c)$ through (3.1), computes W , and computes ε by Lemma 3.7 with Λ the largest of Λ_2 and the available numbers among Λ_1 and Λ_3 , where $\lambda_{\min}(\nabla^2 F_P(c))$ and $\lambda_{\min}(M)$ are bounded from below by Lemma 3.9 at a floating-point estimate lowered by $10^{-9}(1 + |\lambda|) + 10^{-12}$. It divides the positive weights by their sum in ball arithmetic and verifies that the right-hand side of Lemma 3.10 for the structures of positive weight, with these weights, exceeds τ .

If the factorization of Lemma 3.9 fails for H or for M , the corresponding number Λ_1 or Λ_3 is not used. The checker also reports, for each part, the smallest lower endpoint of the differences between the accepted bounds and τ . For the certificate the smallest of them is 1.0×10^{-7} .

Square roots. The checker computes lengths such as $|q_{i,k}|$, $|t_i(c)|$ and those in Lemma 3.5 as square roots of balls for sums of squares. If such a ball is positive, it takes the square root in ball arithmetic. Otherwise the ball contains 0. Let U be its upper endpoint. If U rounds to 0

¹In the archive, the programs are `wcheck.py` (the checker), `wcheck_v2.py` (its variant for ranges of leaves), `wcheck_presence_v2.py` (the presence program) and `check_headers.py` (the header program). The checker itself checks that the files of all parts exist only outside the distributed runs, which leave this to the presence program. All parts of the certificate were made by the same version of the search program.

in double precision, the checker returns 0; else it returns the ball $[0, s]$, where s is computed in double precision from the rounded value of U , with a relative margin of 10^{-12} and an absolute margin of 10^{-300} . For $U \geq 10^{-300}$ the three roundings involved, of U , of its square root and of the product with $1 + 10^{-12}$, change s by a relative amount of at most $3 \cdot 2^{-53}$, so $s > \sqrt{U}$, and $[0, s]$ contains the square root of every nonnegative number in the ball. A ball of this kind is either exactly 0, and then 0 is the correct value, or it carries the error of a cosine or a sine of a nonzero angle computed at 100 bits, so that U is far above 10^{-300} .²

Rounding of Λ . For a list on one body the matrices H and W computed by the checker are not used (see the proof of Theorem 1.1). For a list on at least two bodies, the number Λ that the checker uses is the largest of up to three double-precision numbers obtained by rounding to nearest: the lower endpoints of the balls computed for Λ_2 and Λ_1 , and the product of the certified bound for $\lambda_{\min}(M)$ with n_h . In the ball for Λ_2 , the lower endpoint of a diagonal entry M_{aa} whose ball contains 0 is rounded in the same way. A rounding to nearest raises a number by at most 2^{-53} times its absolute value. The absolute values of the rounded diagonal entries add up to at most $\sum_a |M_{aa}|$, and, since $\Lambda \geq \Lambda_2$ and both are at most 0, the rounded number that is used has absolute value at most $|\Lambda_2| \leq \sum_{a,b} |M_{ab}|$, in both cases up to the radii of the balls, which are far smaller than the slack in the bound $\sum_{a,b} |M_{ab}| < 18.8$ shown below. Hence $\Lambda - 2^{-52} \cdot 19$ is a lower bound for one of Λ_1 , Λ_2 and Λ_3 , and Lemma 3.7 holds with $\varepsilon + 10^{-14}$ in place of the number ε that the checker computes from Λ . As the weights add up to 1, the bound of Lemma 3.10 that the checker accepts exceeds a valid lower bound for A on the leaf by less than 10^{-14} , which is far below the smallest margin 1.0×10^{-7} . For the bound on M , note that F_P is the sum of at most 17 terms $\frac{1}{2} \det(x_j, x_{j+1})$. Up to a constant, each of them is a run term with $|w| \leq 1$ or a transition term as in Lemma 3.8. In particular, it depends on the coordinates of at most two bodies, and on X each of its second derivatives has absolute value at most $1/2$, since $|q_{i,k}| \leq 1/2$ for the arcs of Definition 2.1 and $|t_i| < 0.78$ on X . The half-widths of a leaf are at most those of a cell, that is, at most 0.19635 for the angles, 0.3055 for $t_{i,1}$ and 0.2406 for $t_{i,2}$. So $\sum_{a,b} |M_{ab}| \leq 17 \cdot \frac{1}{2} \cdot (2(0.19635 + 0.3055 + 0.2406))^2 < 18.8$.

4.4. Headers and coverage. The distributed check runs used a variant of the checker that checks the leaves in a given range of a part. It differs from the checker only in its command line and in the routine that checks a part, which decodes the whole tree of the part in every run and, for a leaf outside the range, reads the line of its record and nothing more, so that the test of the code 254 and step (4) are made in the run whose range contains the leaf. Two short programs complete the verification. For each of the 16,384 parts, the presence program verifies (i) that its three files exist, (ii) that its header records the index of the part and agrees with that of part 0 in the arcs, the numbering of the vertices, τ , X , the grid, the number of parts and the largest length of a small list, and (iii) that, among the passing lines of the check logs, those whose SHA-256 hashes of the three files equal the hashes of the archived files report a single leaf count and have leaf ranges that together cover all leaves of the part. The header program verifies (i) that the arcs in every header are exactly those of Definition 2.1, with the angle periods $2\pi/3$, 2π , π and π of bodies 1 to 4 that step (2) uses, and that the numbering lists every vertex once, (ii) that the grid consists of twelve positive integers and that there is one header for each part, and (iii) that in every coordinate a the cell endpoints computed as in the checker start at X_a^- , end at X_a^+ , increase with k and are equal to the rational numbers $X_a^- + (X_a^+ - X_a^-)k/s_a$. Adjacent cells share the endpoint computed by the same expression, so the cells then tile X .

²On parts 0 to 9 of the certificate the smallest such U is 1.8×10^{-59} .

4.5. Proof of Theorem 1.1.

Proposition 4.1. *For the certificate described above, the checker accepts all parts, the presence program and the header program accept the certificate and the check logs, and the smallest margin in the passing lines of the logs, which the presence program computes, is 1.0001483×10^{-7} .*

We verified Proposition 4.1 by running the checker, the presence program and the header program on the archived certificate (Table 1).

Proof of Theorem 1.1. Let $\tau = 0.2405$. By the presence program all parts carry the same header, and by the header program its arcs are those of Definition 2.1, its angle periods are those used in the definition of D_τ , and the cells of its grid tile its root box X . By step (2), $D_\tau \subseteq X$. By the presence program, the passing check runs of each part, for the archived files, cover all leaves of the part. By step (3), the leaves of the tree of a cell tile the cell, since each split replaces a box by its two halves. Let B be a leaf; by step (4) it has centre c and half-widths h . Consider a structure of positive weight in its record, with vertex list P . By step (4) and Lemma 3.6, the structure is valid on B and $F_P \leq A$ on B . If the vertices of P lie on at least two bodies, then (3.1) holds, the matrix W computed by the checker bounds the variation of $\nabla^2 F_P$ on B by Lemma 3.8, the number $\Lambda - 2^{-52} \cdot 19$ is a lower bound for one of $\Lambda_1, \Lambda_2, \Lambda_3$ by Lemma 3.9 and Section 4.3, and Lemma 3.7 gives $F_P(p) \geq F_P(c) + \nabla F_P(c) \cdot (p - c) - \varepsilon - 10^{-14}$ on B , where $\varepsilon = -\Lambda/2 + \frac{1}{2} \sum_{a,b} W_{ab} h_a h_b$ with the double-precision number Λ that the checker uses and the matrix W whose entries it encloses in balls. If they lie on one body, then F_P is constant, because the sum of the identity of (3.1) over the whole cyclic list equals $\sum_j \det(q_j, q_{j+1})$. The checker computes the gradient of F_P as the sum of the derivatives of its run term and of the term that closes the run, so its value encloses $\nabla F_P(c) = 0$, and the number ε , defined as above with the nonnegative matrix that the checker computes, is at least 0 because $\Lambda \leq 0$. Thus the same inequality holds. We apply Lemma 3.10 to the structures of positive weight, with their weights divided by their sum, which are nonnegative and add up to 1, and with $\varepsilon_s + 10^{-14}$, where ε_s is the number ε of structure s . The ball computed by the checker contains the right-hand side of Lemma 3.10 with the numbers ε_s , and by Proposition 4.1 its lower endpoint exceeds τ by at least 1.0×10^{-7} . Lemma 3.10 therefore gives $A \geq \tau + 1.0 \times 10^{-7} - 10^{-14} > \tau$ on B . Hence $A > \tau$ on X , and Corollary 2.5 proves the theorem. $\square$

4.6. Data of the certificate. Table 1 lists the data of the certificate and of the checker runs.

4.7. Further checks. To test that the checker rejects invalid input, we ran it on deliberately altered records. Its self-test runs it on two parts of a test certificate for the same arcs and the target 0.24, whose root box is cut into 65,536 parts, and on modified copies of their leaf records that it must reject; one of the two parts contains a chord decomposition. The checker accepts the original records and rejects all eight modifications.³ None of these modifications reaches the test of the angle sum of a fan. A second test reorders every fan of odd length $m \geq 5$ in star order, putting the vertex in position $2j \bmod m$ in position j , so that the list winds twice around z . On the four parts of the test certificate in the archive and on parts 0 to 9 of the certificate, 61 of these reorderings have positive determinants at the centre of the leaf box, so that only the angle sum can reject them. The checker rejects all of them and accepts the original fans. At the leaf that contains the placement of Section 5 with the smallest hull area found, the leaf routine of the checker, called with τ replaced, accepts the target 0.2405 with a margin of 7.0×10^{-5} and rejects the targets 0.2406 and 0.2421, the last above the numerical minimum of the family. Every

³A fan that repeats a vertex, a fan in clockwise order, a fan whose centre lies outside the polygon, a small list of five vertices, a negative weight, a target $\tau = 10$, a leaf whose box centre is moved so that two consecutive vertices of a fan coincide, and a chord decomposition with the two sides exchanged.

TABLE 1. The certificate of Theorem 1.1 and its verification.

Parameters; target τ	12; 0.2405
Root box: translation ranges	$ t_{i,1} \leq 0.6108219 \dots, t_{i,2} \leq 0.481$ (rounded outwards)
Grid; cells; parts	(4, 2, 2, 16, 2, 2, 8, 2, 2, 8, 2, 2); 1,048,576; 16,384
Nodes; leaves	405,370,260; 203,209,418
leaves closed by Lemma 3.10	28,558,819
Smallest difference between an accepted bound and τ	1.0×10^{-7}
Search: wall-clock time; servers	about 1.9 h; up to 6,432 vCPUs
Checker: parts accepted	all 16,384
Checker: wall-clock time of the distributed runs	about 36 min
Checker: CPU time of a complete repetition	92.5 core-hours
Header program and presence program (Section 4.4)	passed
Smallest hull area found by local search	0.2420538

header with $\tau > 0.240500004$ is rejected already at step (2), because D_τ is then not contained in the root box.

Further checks concern the checker itself. First, we rederived the lemmas on which it rests, independently of the development of the certificate and the checker, and reviewed the checker, the presence program and the header program against them. Second, we repeated all 16,533 check runs on other machines and obtained the same results, and we decoded the trees of all parts with a separate program. Third, we ran the further implementation of the leaf bound of Section 1, which bounds the second-order term differently, by certified eigenvalue bounds for an interval Hessian over the box. It accepted 450 leaves: the five leaves with the smallest margins in each of the ten check runs with the smallest margins, and 400 leaves chosen at random. Copies of the certificate and of the logs into which we had inserted errors were rejected.

5. REMARKS

The least hull area of the family. A lower bound obtained from the five arcs cannot exceed $\inf_p A(p)$. Local minimization of the exact hull area from 300 random starting points found no placement with hull area below 0.2420538. Rounded to seven decimals, a placement with this hull area has (φ_i, t_i) equal to (0.0071380, -0.0049090 , 0.1405348) for T , (1.6826541, -0.0535512 , 0.1616042) for U , (1.0341517, -0.0706697 , 0.0634469) for C and (2.0551541, 0.0017476, 0.0702157) for S , and the full values are in the archive. Numerically, therefore, the five arcs cannot give more than about 0.24205, which is 1.55×10^{-3} above the bound of Theorem 1.1.

Reflections. If reflections are allowed, a copy of S may be reflected, and its mirror image is a zigzag that turns like C , with other lengths and another angle. We have not computed the least hull area of the five arcs in this setting. In numerical experiments, other families of five or six arcs, with reflections allowed, had least hull areas between 0.2400 and 0.2403, so a certified bound $M^\pm > 0.239$ by this method seems possible, but with a much smaller margin.

Upper bounds. The best upper bound for M^+ that we know is Deng's 0.257883595... [2], from a symmetric cover described in the introduction, and the best refereed one is $\pi/12$ [10].

Keller’s quadrilateral shows $M^\pm \leq 0.24633\dots$. Since $0.2405 < 0.24633\dots$, Theorem 1.1 does not decide whether $M^\pm < M^+$.

DATA AVAILABILITY

The certificate, the checker, the presence program, `check_headers.py`, the tests, the search programs and the lines of the check logs for the archived files are archived at doi:10.5281/zenodo.23543309, together with the SHA-256 hashes of the programs that produced the certificate and the logs and a list of the changes in the released copies. These concern docstrings, comments and the lines that locate imported modules. The programs are also available at <https://github.com/tejasvi/moser-worm-rotations-bound>. The lines of the check logs were produced by `wcheck.v2.py`, with SHA-256 hash beginning with `9374fec5`, and the presence program has hash beginning with `9e7cbbd3`; `wcheck.py`, with hash beginning with `520e291e`, is the version from which `wcheck.v2.py` was made. The released copies of the two checkers differ from these files only in docstrings and in one comment, and the presence program is identical. The program `check_headers.py` was written after the check runs. The check runs used Python 3.12 with python-flint 0.9.0. The repetition of all check runs in the audit used Python 3.13 with python-flint 0.9.0 and took 92.5 core-hours.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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