

AN EXPLICIT BOUND BELOW $2/\pi$ FOR THE MAXIMUM TERM OF AN ENTIRE FUNCTION

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ABSTRACT. For a transcendental entire function f let $\mu(r)$ be its maximum term and $M(r)$ its maximum modulus on $|z| = r$. Erdős asked for the largest possible value B of $\liminf_{r \rightarrow \infty} \mu(r)/M(r)$. An argument of Clunie gives $B \leq 2/\pi$, and Clunie and Hayman proved that $B \leq 2/\pi - c$ for an absolute constant $c > 0$, which they did not compute. We prove $B \leq 0.63005$, while $2/\pi = 0.636619\dots$. The proof considers two consecutive radii at which the central index jumps and tests the function, at one or both of them, against trigonometric polynomials with five or six frequencies. When the two jumps have the same size, this leads to an optimization problem in six real parameters. When they differ, an averaging argument over roots of unity leads to a family of problems in seven parameters, indexed by the larger of the two jumps divided by their greatest common divisor. We bound both by branch and bound in ball arithmetic, and two independently written programs check the resulting certificates.

1. INTRODUCTION

Let $f(z) = \sum_{k \geq 0} a_k z^k$ be a transcendental entire function, and for $r > 0$ let

$$M(r) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \max_{k \geq 0} |a_k| r^k$$

be its maximum modulus and its maximum term. By Cauchy's inequality $\mu(r) \leq M(r)$. Erdős [3, p. 249] asked how large the ratio $\mu(r)/M(r)$ can stay, that is, for the value of

$$B := \sup_f \liminf_{r \rightarrow \infty} \frac{\mu(r)}{M(r)},$$

the supremum being over all transcendental entire functions (for polynomials the ratio can tend to 1). This is Problem #513 in Bloom's list [1] and Problem 2.14(c) in the Hayman–Lingham collection [5], and it appears as constant 51a in the repository of optimization constants [9].

Trivially $1/2 \leq B \leq 1$ [1], and an unpublished example of Kővári shows that $B > 1/2$ [2, p. 146]. Gray and Shah [4, Lemma 1] published an argument of Clunie that gives $B \leq 2/\pi$. Clunie and Hayman showed that the same argument gives $\mu(r)/M(r) \leq 2/\pi$ at every radius where the central index jumps, and that this is sharp for the upper limit over these radii [2, Theorem 1]. For the lower limit they proved $4/7 < B \leq 2/\pi - c$ with an absolute constant $c > 0$ [2, Theorem 2]. Their constant comes from a stability argument in which the dependence on the distance from equality is not made explicit: near equality in Clunie's bound at one jump forces the coefficients to resemble those of a square wave, so that the next jump must come soon, while near equality at the next jump forces it to come late [2, Lemmas 2.2–2.4]. The value of c is not computed there [2, p. 156], and as far as we know no explicit value has been recorded since [1, 5]. Recently He and Tang [6] generalized the Clunie–Hayman construction and proved $B > 1/1.70919 > 0.5850724$ [6, Theorem 2.8 and Proposition 3.5], and a certified computation reported by Sothanaphan [8] gives $B > 0.5850788$. Our result is an explicit upper bound.

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Theorem 1.1. *For every transcendental entire function f ,*

$$\liminf_{r \rightarrow \infty} \frac{\mu(r)}{M(r)} \leq 0.63005.$$

Consequently $B \leq 0.63005$.

Together with the lower bound of [8], $0.5850788 < B \leq 0.63005$. The bound improves $2/\pi = 0.636619\dots$ by more than 0.0065.

Where Clunie's bound loses. Clunie's argument is short. Write $c_k = a_k r^k$ and $g(\theta) = f(re^{i\theta}) = \sum_k c_k e^{ik\theta}$. At a radius where two indices $n < m$ both attain the maximum term, rotate θ so that c_n and c_m have the same argument. Then

$$2\mu(r) = |c_n + c_m| = \left| \frac{1}{2\pi} \int_0^{2\pi} g(\theta)(e^{-in\theta} + e^{-im\theta}) d\theta \right| \leq M(r) \cdot \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i(m-n)\theta}| d\theta = \frac{4}{\pi} M(r).$$

Since the central index tends to infinity, such radii are arbitrarily large, and $B \leq 2/\pi$. Near equality forces g to be close to a multiple of the square wave $\text{sgn} \cos((m-n)\theta/2)$, up to a phase and a frequency shift. The Fourier coefficients of that square wave decay only like $1/j$ along the progression $n + j(m-n)$. But the moduli $|a_k|$ are the same at every radius; only the factor r^k changes. At the next radius where the central index jumps, a slowly decaying coefficient profile would produce a term larger than the two that tie there. This is the mechanism of the Clunie–Hayman proof [2, §2]. We make it quantitative by testing g at two consecutive jump radii against trigonometric polynomials with five frequencies; when the two jumps have different sizes we use one of the radii and six frequencies.

Method. At a radius r and for a finite set $S \subset \mathbb{Z}$ of frequencies, $M(r)$ is at least the smallest sup norm of a bounded function whose Fourier coefficients on S are the c_k (Section 2); dual trigonometric polynomials give lower bounds for this quantity. The radii where the central index jumps are the breakpoints of $t \mapsto \log \mu(e^t)$; they correspond to the edges of the Newton polygon of $\log |a_k|$ (Section 3). Two consecutive breakpoints involve three consecutive vertices $v_{j-1} < v_j < v_{j+1}$. If $v_j - v_{j-1} = v_{j+1} - v_j$, a dilation reduces everything to five consecutive coefficients, with six free real parameters. If the two gaps differ, we use a single radius, five coefficients on the lattice generated by the larger gap, and the coefficient of the remaining vertex, which lies off that lattice; averaging over the p -th roots of unity, where p is the larger gap divided by the greatest common divisor of the two gaps (Lemma 4.1), shows that the off-lattice coefficient can only help; a one-variable function r_p controls the cost of including it in the test polynomial. Both finite-dimensional problems are then bounded by a branch-and-bound computation with certified dual vectors (Sections 5 and 6).

The limit of the method. Numerically, the six-parameter problem has the value 0.630041, attained at a real configuration with the sign pattern of the Clunie–Hayman construction, and the largest value we found for the mixed problems is about 0.6281. Our bound is within 10^{-5} of the first of these values, so this formulation cannot improve the bound by more than about 10^{-5} . Wider windows may lower this limit: with seven coefficients centred at the middle of five consecutive vertices with equal gaps, tested at the four jump radii between them, the best configuration we found for the equal-gap problem has value about 0.6059 (Remark 7.2). The exact value of that problem is not known: it is at least this number and, by Lemma 2.4(d) applied at the two middle radii, at most the value 0.630041 of the six-parameter problem. Covering unequal gaps would also need a new idea.

2. THE WINDOW BOUND

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with the normalized measure $d\theta/2\pi$; $\|\cdot\|_1$ and $\|\cdot\|_\infty$ refer to it, and $\hat{h}(k) = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) e^{-ik\theta} d\theta$. For a finite set $S \subset \mathbb{Z}$ and $\tau \in \mathbb{C}^S$ put

$$T_\tau(\theta) = \sum_{k \in S} \tau_k e^{-ik\theta}, \quad \langle c, \tau \rangle = \sum_{k \in S} c_k \tau_k \quad (c \in \mathbb{C}^S).$$

Definition 2.1. For a finite $S \subset \mathbb{Z}$ and $c \in \mathbb{C}^S$ let $N_S(c) = \inf\{\|h\|_\infty : h \in L^\infty(\mathbb{T}), \hat{h}(k) = c_k \text{ for all } k \in S\}$. For $S = \{-2, \dots, 2\}$ we write N_5 and list $c = (c_{-2}, c_{-1}, c_0, c_1, c_2)$.

Lemma 2.2 (duality). *For every $\tau \in \mathbb{C}^S$ with $T_\tau \neq 0$, $N_S(c) \geq |\langle c, \tau \rangle| / \|T_\tau\|_1$.*

Proof. If $\hat{h}(k) = c_k$ on S , then $\langle c, \tau \rangle = \sum_k \tau_k \hat{h}(k) = \frac{1}{2\pi} \int_0^{2\pi} h T_\tau d\theta$, so $|\langle c, \tau \rangle| \leq \|h\|_\infty \|T_\tau\|_1$. $\square$

Remark 2.3. By the Hahn–Banach theorem the bound is sharp: $N_S(c) = \max |\langle c, \tau \rangle|$ over $\|T_\tau\|_1 \leq 1$. Only Lemma 2.2 is used in the proofs; sharpness explains why the numerical values in Sections 1 and 7 are the limits of the method.

Lemma 2.4. *Let $S \subset \mathbb{Z}$ be finite and $c, c' \in \mathbb{C}^S$.*

- (a) $N_S(\lambda c) = |\lambda| N_S(c)$ for $\lambda \in \mathbb{C}$.
- (b) (rotation) $N_S((c_k e^{ik\alpha})_k) = N_S(c)$ for $\alpha \in \mathbb{R}$.
- (c) (shift) for $j \in \mathbb{Z}$, the window $S + j$ with coefficients c_{k-j} has the same value $N_S(c)$.
- (d) (restriction) if $S' \subset S$ then $N_{S'}(c|_{S'}) \leq N_S(c)$.
- (e) (Lipschitz) $|N_S(c) - N_S(c')| \leq \sum_{k \in S} |c_k - c'_k|$.
- (f) (dilation) for an integer $d \geq 1$, the window $dS = \{dk : k \in S\}$ with coefficient c_k at dk has the same value $N_S(c)$.
- (g) (reversal and conjugation) the window $-S$ with coefficient c_{-k} at k has the value $N_S(c)$, and $N_S(\bar{c}) = N_S(c)$.

Proof. Each statement follows by transforming admissible functions: (a) $h \mapsto \lambda h$; (b) $h \mapsto h(\cdot + \alpha)$; (c) $h \mapsto e^{ij\theta} h$; (d) restriction of the constraints; (e) $h \mapsto h + \sum_{k \in S} (c'_k - c_k) e^{ik\theta}$; (g) $h \mapsto h(-\cdot)$ and $h \mapsto \overline{h(-\cdot)}$. For (f), if h is admissible for S then $h(d\cdot)$ is admissible for dS with the same norm. Conversely, if H is admissible for dS , the average $\tilde{H}(\theta) = \frac{1}{d} \sum_{j=0}^{d-1} H(\theta + 2\pi j/d)$ has $\|\tilde{H}\|_\infty \leq \|H\|_\infty$, Fourier support in $d\mathbb{Z}$ and the same coefficients on dS , so $\tilde{H} = h(d\cdot)$ with h admissible for S . $\square$

For an entire function $f = \sum_{k \geq 0} a_k z^k$ and $r > 0$ put $c_k(r) = a_k r^k$ for $k \geq 0$ and $c_k(r) = 0$ for $k < 0$.

Lemma 2.5 (window bound). *For every $r > 0$ and every finite $S \subset \mathbb{Z}$, $M(r) \geq N_S((c_k(r))_{k \in S})$.*

Proof. The function $g(\theta) = f(re^{i\theta})$ satisfies $\|g\|_\infty = M(r)$ and $\hat{g}(k) = c_k(r)$ for every $k \in \mathbb{Z}$. $\square$

3. VERTICES, EVENTS AND THE TWO REDUCTIONS

3.1. The Newton polygon. Let f be transcendental and $L(t) = \log \mu(e^t) = \max_k \ell_k(t)$, where $\ell_k(t) = \log |a_k| + kt$ for $a_k \neq 0$; the maximum is attained because $|a_k| e^{kt} \rightarrow 0$ as $k \rightarrow \infty$. The *central index* $\nu(r)$ is the largest k with $|a_k| r^k = \mu(r)$; it is nondecreasing in r . Call k a *vertex* if $\ell_k(t) = L(t)$ for some $t \in \mathbb{R}$, and let $v_0 < v_1 < v_2 < \dots$ be the vertices. For a vertex k the set $I_k = \{t : \ell_k(t) = L(t)\}$ is a closed interval, since $L - \ell_k$ is convex and nonnegative. If $k < k'$ are vertices, $t \in I_k$ and $t' \in I_{k'}$, then $t \leq t'$: otherwise $\ell_{k'}(t) - \ell_k(t) = \ell_{k'}(t') - \ell_k(t') + (k' - k)(t - t') > 0$, contradicting $\ell_k(t) = L(t)$. The indices attaining L on a bounded interval $[a, b]$ are at most $\nu(e^b)$, so only finitely many I_k meet $[a, b]$; and there are infinitely many vertices, since otherwise

$|a_k|r^k \leq \max_{j \leq N} |a_j|r^j$ for all r and $k > N$ would force $a_k = 0$ for $k > N$. As the I_{v_j} cover $\mathbb{R}$, are ordered and are locally finite, consecutive ones meet:

$$t_j := \max I_{v_{j-1}} = \min I_{v_j} \quad (j \geq 1),$$

so that $t_1 \leq t_2 \leq \dots$, $t_j \rightarrow \infty$, and at $r = e^{t_j}$ both v_{j-1} and v_j attain the maximum term.

Definition 3.1. The *event* at $j \geq 1$ consists of the vertices $v_{j-1} < v_j < v_{j+1}$, the gaps $d = v_j - v_{j-1}$ and $d' = v_{j+1} - v_j$, the radii $r_0 = e^{t_j} \leq r_1 = e^{t_{j+1}}$, and $s = t_{j+1} - t_j \geq 0$. It is an *equal-gap* event if $d = d'$ and a *mixed* event otherwise.

At an event we use only the following facts:

- (E1) $|c_k(r)| \leq \mu(r)$ for all $k \in \mathbb{Z}$ and $r > 0$;
- (E2) $|c_{v_{j-1}}(r_0)| = |c_{v_j}(r_0)| = \mu(r_0)$ and $|c_{v_j}(r_1)| = |c_{v_{j+1}}(r_1)| = \mu(r_1)$;
- (E3) $c_k(r_1) = c_k(r_0)e^{ks}$ for all k .

3.2. Equal gaps. Let $K = [0, \infty) \times [0, 1]^2 \times \mathbb{R}^3$, with points $\xi = (\sigma, x, y, \varphi_{-2}, \varphi_{-1}, \varphi_2)$, and

$$C_0(\xi) = (xe^{i\varphi_{-2}}, e^{i\varphi_{-1}}, 1, e^{-\sigma}, ye^{-2\sigma}e^{i\varphi_2}),$$

$$C_1(\xi) = (xe^{-2\sigma}e^{i\varphi_{-2}}, e^{-\sigma}e^{i\varphi_{-1}}, 1, 1, ye^{i\varphi_2}), \quad \Lambda(\xi) = \min_{i=0,1} \frac{1}{N_5(C_i(\xi))}.$$

($N_5(C_i(\xi)) \geq 1$ by Lemma 2.2 with $T_\tau = 1$.)

Proposition 3.2. *At an equal-gap event there is $\xi \in K$ with $\min\{\mu(r_0)/M(r_0), \mu(r_1)/M(r_1)\} \leq \Lambda(\xi)$.*

Proof. Let $n = v_j$, $\lambda = c_n(r_0)$ and $b_l = c_{n+dl}(r_0)/\lambda$ for $l = -2, \dots, 2$; so $|\lambda| = \mu(r_0)$, $|b_{-1}| = |b_0| = 1$ and $x := |b_{-2}| \leq 1$. By (E2) and (E3), $|b_1|e^{ds} = 1$, so $|b_1| = e^{-\sigma}$ with $\sigma = ds$, and (E1) at r_1 gives $y := |b_2|e^{2\sigma} \leq 1$. Choose α with $d\alpha = -\arg b_1$ and replace b_l by $b_l e^{idl\alpha}$; then $b_0 = 1$, $b_1 = e^{-\sigma}$, and with $\varphi_l = \arg b_l$ ($l = -2, -1, 2$; arbitrary if $b_l = 0$) we have $b = C_0(\xi)$. Lemma 2.5 with $S = \{n + dl : |l| \leq 2\}$ and Lemma 2.4(a)–(c), (f) give $M(r_0) \geq \mu(r_0)N_5(C_0(\xi))$. At r_1 we have $\mu(r_1) = |c_n(r_1)|$ and $c_{n+dl}(r_1)/c_n(r_1) = b_l e^{l\sigma}$, which after the same rotation is $C_1(\xi)$; so $M(r_1) \geq \mu(r_1)N_5(C_1(\xi))$. $\square$

3.3. Mixed gaps. At a mixed event let $g = \gcd(d, d')$, $a = d/g$, $b = d'/g$ (so $\gcd(a, b) = 1$, $a \neq b$), and

$$p := \max(a, b) \geq 2, \quad \sigma := gs.$$

For $p \geq 2$, $W \geq 0$ and $\varepsilon \geq 0$ let

$$R_p(W, \varepsilon) = \max_{\varphi \in \mathbb{R}} \frac{1}{p} \sum_{j=0}^{p-1} |W + \varepsilon e^{i(\varphi + 2\pi j/p)}| - W,$$

and for $\tau \in \mathbb{C}^5$ (indices $-2, \dots, 2$) and $\varepsilon \geq 0$ let

$$U_p(\tau, \varepsilon) = \|T_\tau\|_1 + \frac{1}{2\pi} \int_0^{2\pi} R_p(|T_\tau(X)|, \varepsilon) dX.$$

For $s \geq 0$ and $\kappa \geq 1$ let $Z(s, \kappa)$ be the set of vectors $z = (z_{-2}, z_{-1}, 1, 1, z_2)$ with $|z_{-2}| \leq e^{-2\kappa s}$, $|z_{-1}| \leq e^{-\kappa s}$ and $|z_2| \leq 1$.

Proposition 3.3. *At a mixed event there are $s_1 \geq 0$ and $z \in Z(s_1, \frac{p}{p-1})$ such that for every $\tau \in \mathbb{C}^5$ and $\varepsilon \geq 0$ with $\operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1} > 0$,*

$$\min\left\{\frac{\mu(r_0)}{M(r_0)}, \frac{\mu(r_1)}{M(r_1)}\right\} \leq \frac{U_p(\tau, \varepsilon)}{\operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1}}.$$

Proof. Work on the progression $v_j + g\mathbb{Z}$ and write $w_k = c_{v_j+gk}(r_0)/c_{v_j}(r_0)$ and $w_k(\sigma) = c_{v_j+gk}(r_1)/c_{v_j}(r_1)$; by (E3), $w_k(\sigma) = w_k e^{k\sigma}$. By (E1) and (E2), $\mu(r_0) = |c_{v_j}(r_0)|$, $\mu(r_1) = |c_{v_j}(r_1)|$, $w_0 = w_0(\sigma) = 1$, $|w_{-a}| = 1$, $|w_b(\sigma)| = 1$, and $|w_k| \leq 1$, $|w_k(\sigma)| \leq 1$ for all k .

Case $a < b$ ($p = b$). We use the radius r_1 . The vector $(w_{-2b}(\sigma), w_{-b}(\sigma), 1, w_b(\sigma), w_{2b}(\sigma))$ satisfies $|w_{-lb}(\sigma)| = |w_{-lb}|e^{-lb\sigma} \leq e^{-lb\sigma}$ ($l = 1, 2$) and $|w_{2b}(\sigma)| \leq 1$; after a rotation (Lemma 2.4(b)) making $w_b(\sigma) = 1$ it is a vector $z \in Z(s_1, b/a) \subset Z(s_1, p/(p-1))$, where $s_1 = a\sigma$, because $b\sigma = (b/a)s_1$ and $b/a \geq b/(b-1)$. The off-lattice coefficient $w_{-a}(\sigma)$ has modulus $e^{-a\sigma} = e^{-s_1}$, and $-a$ is prime to b . Lemma 2.5 on the window $v_j + g(\{-2b, -b, 0, b, 2b\} \cup \{-a\})$, Lemma 2.4(a)–(c), (f), and Lemma 4.1 below with $S_L = \{-2b, \dots, 2b\}$ (step b) give $M(r_1) \geq \mu(r_1)(\operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1})/U_b(\tau, \varepsilon)$.

Case $a > b$ ($p = a$). We use the radius r_0 . Here $|w_{-2a}| \leq 1$, $|w_{la}| \leq e^{-la\sigma}$ ($l = 1, 2$; by (E1) at r_1), and the off-lattice coefficient w_b has modulus $e^{-b\sigma}$, with b prime to a . Reversing the window (Lemma 2.4(g)) and rotating so that the coefficient at a , originally w_{-a} , becomes 1 gives $z = (\cdot, \cdot, 1, 1, \cdot) \in Z(s_1, a/b) \subset Z(s_1, p/(p-1))$ with $s_1 = b\sigma$, and an off-lattice coefficient of modulus e^{-s_1} at $-b$. The rest is as before, with U_a . $\square$

4. AVERAGING OVER ROOTS OF UNITY

Lemma 4.1. *Let $p \geq 2$, let $e \in \mathbb{Z}$ with $\gcd(e, p) = 1$, let $S_L \subset p\mathbb{Z}$ be finite, $S = S_L \cup \{e\}$, and let $c \in \mathbb{C}^S$ with $|c_e| = \eta$. For $\tau \in \mathbb{C}^{S_L}$ and $\varepsilon \geq 0$ put $Q(X) = \sum_{k \in S_L} \tau_k e^{-i(k/p)X}$. Then*

$$N_S(c) \geq \frac{\operatorname{Re} \sum_{k \in S_L} c_k \tau_k + \varepsilon \eta}{\|Q\|_1 + \frac{1}{2\pi} \int_0^{2\pi} R_p(|Q(X)|, \varepsilon) dX}$$

whenever the numerator is positive.

Proof. Let $\chi = \arg c_e$ (any value if $\eta = 0$) and $T(\theta) = Q(p\theta) + \varepsilon e^{-i(e\theta + \chi)}$. This is $T_{\tau'}$ for the vector $\tau' \in \mathbb{C}^S$ that extends τ by $\tau'_e = \varepsilon e^{-i\chi}$, and $\langle c, \tau' \rangle = \sum_{k \in S_L} c_k \tau_k + \varepsilon \eta$, so by Lemma 2.2 it suffices to bound $\|T\|_1$. Split $[0, 2\pi)$ into the arcs $[2\pi j/p, 2\pi(j+1)/p)$ and substitute $\theta = (X + 2\pi j)/p$ in the j -th arc. Since $k/p \in \mathbb{Z}$ for $k \in S_L$, $Q(p\theta) = Q(X)$, while $e\theta = eX/p + 2\pi ej/p$. Hence

$$\|T\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{p} \sum_{j=0}^{p-1} |Q(X) + \varepsilon e^{-i(eX/p + \chi)} \omega^j| dX, \quad \omega = e^{-2\pi i e/p}.$$

As $\gcd(e, p) = 1$, ω^j runs over all p -th roots of unity. For fixed X write $Q(X) = W e^{i\alpha}$; the inner average equals $\frac{1}{p} \sum_j |W + \varepsilon e^{i(\phi + 2\pi j/p)}|$ for some ϕ , which is at most $W + R_p(W, \varepsilon)$. $\square$

No cancellation between the lattice part and the off-lattice term is needed. For $\varepsilon > 0$ we have $R_p(W, \varepsilon) = \varepsilon r_p(W/\varepsilon)$ with

$$r_p(u) := R_p(u, 1) = \max_{\varphi} \frac{1}{p} \sum_{j=0}^{p-1} |u + e^{i(\varphi + 2\pi j/p)}| - u,$$

and $R_p(W, 0) = 0$. Let also $r_{\infty}(u) = \frac{1}{2\pi} \int_0^{2\pi} |u + e^{it}| dt - u = \frac{2}{\pi}(u+1)E\left(\frac{4u}{(u+1)^2}\right) - u$, where $E(m) = \int_0^{\pi/2} (1 - m \sin^2 x)^{1/2} dx$; the formula follows from $|u + e^{it}|^2 = (u+1)^2 - 4u \sin^2(t/2)$.

Lemma 4.2. *Let $p \geq 2$.*

- (i) r_p is nonincreasing on $[0, \infty)$.
- (ii) $r_p(u) \leq \sqrt{u^2 + 1} - u \leq 1/(2u)$.

- (iii) For $p_0 \geq 2$, $p \geq p_0$ and $u \geq 0$, $r_p(u) \leq r_\infty(u) + \text{err}(p_0, u)$, where $\text{err}(p_0, 1) = 2\pi/(3p_0^2)$ and for $u \neq 1$

$$\text{err}(p_0, u) = \min \left\{ \frac{2\pi \min(u, 1)}{3p_0^2}, \frac{4 \max(u, 1) \left| \binom{1/2}{p_0} \right| q^{p_0}}{1 - q^{p_0}} \right\}, \quad q = u \ (u < 1), \quad q = 1/u \ (u > 1).$$

Proof. (i) Write $\psi_j = \varphi + 2\pi j/p$. For fixed φ the derivative in u of $\frac{1}{p} \sum_j |u + e^{i\psi_j}| - u$ is $\frac{1}{p} \sum_j \text{Re}(u + e^{i\psi_j})/|u + e^{i\psi_j}| - 1 \leq 0$ wherever it exists, and the function is Lipschitz; a maximum of nonincreasing functions is nonincreasing.

(ii) By the Cauchy–Schwarz inequality, $\frac{1}{p} \sum_j |u + e^{i\psi_j}| \leq (\frac{1}{p} \sum_j |u + e^{i\psi_j}|^2)^{1/2} = (u^2 + 1)^{1/2}$, because $\sum_j e^{i\psi_j} = 0$ for $p \geq 2$.

(iii) Let $F(t) = |u + e^{it}|$. Its Fourier series converges absolutely, so by Poisson summation $\frac{1}{p} \sum_j F(\varphi + 2\pi j/p) = \sum_{m \in \mathbb{Z}} \hat{F}(mp) e^{imp\varphi}$ and $r_p(u) - r_\infty(u) \leq \sum_{m \neq 0} |\hat{F}(mp)|$. First bound: $F'(t) = -u \sin t / F(t)$. For $u \neq 1$, the function $u \sin t / F(t)$ is unimodal on $[0, \pi]$; indeed its derivative has the sign of $\frac{B_u}{2} c^2 + A_u c + \frac{B_u}{2}$ with $c = \cos t$, $A_u = u^2 + 1$, $B_u = 2u$, which has exactly one root in $(-1, 0)$. Its maximum, attained at $c = -u$ for $u < 1$ and at $c = -1/u$ for $u > 1$, equals $\min(u, 1)$. So F' has total variation $4 \min(u, 1)$ over a period; this also holds for $u = 1$, where $F = 2|\cos(t/2)|$. Two integrations by parts give $|\hat{F}(n)| \leq 4 \min(u, 1)/(2\pi n^2)$, and $2 \sum_{m \geq 1} 4 \min(u, 1)/(2\pi m^2 p^2) = 2\pi \min(u, 1)/(3p^2)$. Second bound ($u \neq 1$): $F(t) = \max(u, 1) |1 + qe^{it}|$ (for $u < 1$ use $|u + e^{it}| = |1 + ue^{-it}|$ and the evenness of F). With $(1+w)^{1/2} = \sum_{j \geq 0} \gamma_j w^j$, $\gamma_j = \binom{1/2}{j}$, we get $\hat{F}(n) = \max(u, 1) \sum_{j \geq 0} \gamma_{j+n} \gamma_j q^{2j+n}$ for $n \geq 0$. Since $|\gamma_j|$ decreases for $j \geq 1$ and $\sum_{j \geq 1} |\gamma_j| x^j = 1 - \sqrt{1-x}$, this gives $|\hat{F}(n)| \leq \max(u, 1) |\gamma_n| q^n (2 - \sqrt{1-q^2}) \leq 2 \max(u, 1) |\gamma_n| q^n$, and then $\sum_{m \neq 0} |\hat{F}(mp)| \leq 4 \max(u, 1) |\gamma_p| q^p / (1 - q^p)$. Both bounds are nonincreasing in p . $\square$

5. THE TWO FINITE-DIMENSIONAL PROBLEMS

Throughout, $\beta_0 = 12601/20000 = 0.63005$.

5.1. Equal gaps.

Lemma 5.1 (symmetries). *For $\xi = (\sigma, x, y, \varphi_{-2}, \varphi_{-1}, \varphi_2) \in K$ let $\rho(\xi) = (\sigma, y, x, \varphi_2 + 2\varphi_{-1}, \varphi_{-1}, \varphi_{-2} - 2\varphi_{-1})$ and $\bar{\xi} = (\sigma, x, y, -\varphi_{-2}, -\varphi_{-1}, -\varphi_2)$. Then $N_5(C_0(\rho\xi)) = N_5(C_1(\xi))$, $N_5(C_1(\rho\xi)) = N_5(C_0(\xi))$ and $N_5(C_i(\bar{\xi})) = N_5(C_i(\xi))$; hence $\Lambda(\rho\xi) = \Lambda(\bar{\xi}) = \Lambda(\xi)$.*

Proof. Reversing $C_1(\xi)$ (Lemma 2.4(g)) gives $(ye^{i\varphi_2}, 1, 1, e^{-\sigma} e^{i\varphi_{-1}}, xe^{-2\sigma} e^{i\varphi_{-2}})$, and the rotation $c_k \mapsto c_k e^{-ik\varphi_{-1}}$ turns it into $C_0(\rho\xi)$. Reversing and rotating $C_0(\xi)$ in the same way gives $C_1(\rho\xi)$. Finally $C_i(\bar{\xi}) = \overline{C_i(\xi)}$. $\square$

Given $\xi \in K$, reduce φ_{-1} modulo 2π into $[-\pi, \pi]$, apply $\xi \mapsto \bar{\xi}$ if $\varphi_{-1} < 0$, then apply ρ if $x > y$ (it does not change φ_{-1}), and finally reduce φ_{-2} and φ_2 modulo 2π . By Lemma 5.1, Λ is unchanged, and the result lies in

$$D_K = [0, \infty) \times \{(x, y) \in [0, 1]^2 : x \leq y\} \times [-\pi, \pi] \times [0, \pi] \times [-\pi, \pi].$$

Lemma 5.2 (tail). *For all $x \in [0, 1]$ and $\psi \in \mathbb{R}$ there is $\tau \in \mathbb{C}^5$ with $T_\tau \neq 0$ and $\text{Re}\langle (xe^{i\psi}, 1, 1, 0, 0), \tau \rangle \geq 1.61 \|T_\tau\|_1$; in particular $N_5((xe^{i\psi}, 1, 1, 0, 0)) \geq 1.61$. Consequently $\Lambda(\xi) \leq 1/N_5(C_0(\xi)) < 0.6285$ whenever $\sigma \geq 4$.*

Proof. The first statement is a certified computation (Section 6); the second follows by Lemma 2.2. For the third, the rotation $c_k \mapsto c_k e^{ik\varphi_{-1}}$ maps $(xe^{i\varphi_{-2}}, e^{i\varphi_{-1}}, 1, 0, 0)$ to $(xe^{i\psi}, 1, 1, 0, 0)$ with $\psi = \varphi_{-2} - 2\varphi_{-1}$, and Lemma 2.4(e) gives $N_5(C_0(\xi)) \geq 1.61 - e^{-\sigma} - e^{-2\sigma} \geq 1.61 - e^{-4} - e^{-8} > 1.5913$. $\square$

Theorem 5.3 (certified). $\Lambda(\xi) \leq \beta_0$ for every $\xi \in K$.

Proof. By Lemma 5.1 it suffices to treat $\xi \in D_K$, and by Lemma 5.2 only $\sigma \leq 4$. The certificate of Section 6 splits $[0, 4] \times [0, 1]^2 \times [-P, P] \times [0, P] \times [-P, P]$ into finitely many boxes. The boxes that lie entirely in $\{x > y\}$ do not meet D_K ; every other box carries an index i and a vector τ with $T_\tau \neq 0$ and $\operatorname{Re}\langle C_i(\xi), \tau \rangle \geq \|T_\tau\|_1/\beta_0$ on the box, and Lemma 2.2 gives $N_5(C_i(\xi)) \geq 1/\beta_0$. $\square$

5.2. Mixed gaps. For $p \geq 2$, $s_1 \geq 0$ and $z \in Z(s_1, p/(p-1))$ let

$$\Phi_p(z, s_1) = \inf \left\{ \frac{U_p(\tau, \varepsilon)}{\operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1}} : \tau \in \mathbb{C}^5, \varepsilon \geq 0, \operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1} > 0 \right\}.$$

Theorem 5.4 (certified). $\Phi_p(z, s_1) \leq \beta_0$ for all $p \geq 2$, $s_1 \geq 0$ and $z \in Z(s_1, p/(p-1))$.

Proof. *Tail* $s_1 \geq 4$. Take $\varepsilon = 0$, so that $U_p(\tau, 0) = \|T_\tau\|_1$. Let τ be given by Lemma 5.2 for $xe^{i\psi} = z_2$, and let $\tau'_k = \tau_{-k}$, so that $T_{\tau'}(\theta) = T_\tau(-\theta)$ and $\|T_{\tau'}\|_1 = \|T_\tau\|_1$. Then $\langle z, \tau' \rangle = \langle (z_2, 1, 1, 0, 0), \tau \rangle + z_{-1}\tau_1 + z_{-2}\tau_2$. Since $|\tau_k| \leq \|T_\tau\|_1$ (τ_k is a Fourier coefficient of T_τ) and $|z_{-1}| \leq e^{-s_1}$, $|z_{-2}| \leq e^{-2s_1}$ (as $p/(p-1) \geq 1$), we get $\operatorname{Re}\langle z, \tau' \rangle \geq (1.61 - e^{-4} - e^{-8})\|T_{\tau'}\|_1 > 1.5913\|T_{\tau'}\|_1$. Hence $\Phi_p(z, s_1) < 0.6285$.

Compact part $s_1 \leq 4$. The periods are split into the groups $\{2\}$, $\{3\}$, $\{4\}$, $\{5, 6\}$, $\{7, \dots, 10\}$, $\{11, \dots, 16\}$, $\{17, \dots, 24\}$ and $\{p \geq 25\}$. For a group G let $\kappa_G = \inf_{p \in G} p/(p-1)$ (so $\kappa_G = 1$ for the last group) and let $\bar{r}_G$ be the step function of Section 6, which satisfies $\bar{r}_G \geq r_p$ for all $p \in G$. Let $U_G(\tau, \varepsilon)$ be $U_p(\tau, \varepsilon)$ with $R_p(W, \varepsilon)$ replaced by $\varepsilon \bar{r}_G(W/\varepsilon)$ (and by 0 if $\varepsilon = 0$). For $p \in G$ we have $Z(s_1, p/(p-1)) \subset Z(s_1, \kappa_G)$ and $U_p \leq U_G$. The certificate covers, for each group, all pairs (s_1, z) with $s_1 \in [0, 4]$ and $z \in Z(s_1, \kappa_G)$, in the coordinates

$$z = (m_{-2}e^{-2\kappa_G s_1}e^{i\vartheta_{-2}}, m_{-1}e^{-\kappa_G s_1}e^{i\vartheta_{-1}}, 1, 1, m_2e^{i\vartheta_2}),$$

$$(s_1, m_{-2}, m_{-1}, m_2, \vartheta_{-2}, \vartheta_{-1}, \vartheta_2) \in [0, 4] \times [0, 1]^3 \times [-\pi, \pi] \times [0, \pi] \times [-\pi, \pi],$$

by boxes carrying (τ, ε) with $T_\tau \neq 0$ and $\operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1} \geq U_G(\tau, \varepsilon)/\beta_0$ on the box (so the left side is positive). The range $\vartheta_{-1} \in [-\pi, 0]$ follows by conjugation: $(\bar{\tau}, \varepsilon)$ works for $\bar{z}$, because $|T_{\bar{\tau}}(X)| = |T_\tau(-X)|$. $\square$

5.3. Proof of Theorem 1.1. Let f be transcendental. By Section 3 it has events at every $j \geq 1$, with $r_0 = e^{t_j} \rightarrow \infty$. At an equal-gap event, Proposition 3.2 and Theorem 5.3 give a radius $r \in \{r_0, r_1\}$ with $\mu(r)/M(r) \leq \beta_0$. At a mixed event, Proposition 3.3 and Theorem 5.4 give the same. Hence $\mu(r)/M(r) \leq \beta_0$ for arbitrarily large r . $\square$

6. THE CERTIFICATES

6.1. Branch and bound. Each certificate is a list of boxes, products of closed intervals with rational endpoints, obtained by repeated bisection of a fixed initial grid. A box for Theorem 5.3 carries an index $i \in \{0, 1\}$ and a vector $\tau \in \mathbb{Q}(i)^5$; a box for Theorem 5.4 carries $\tau \in \mathbb{Q}(i)^5$ and a rational $\varepsilon \geq 0$; a box for Lemma 5.2 carries τ . The phases range over $[-P, P]$ (or $[0, P]$) with the rational $P = 3.14159265358980 > \pi$. The vectors come from a floating-point optimization of the dual problem at the centre of the box, rounded to multiples of 2^{-24} ; a box that cannot be certified is bisected, and the vectors of its parent are tried first. A box of the equal-gap domain lying entirely in $\{x > y\}$ is recorded as covered by the symmetry ρ .

A leaf is valid if a lower bound of the linear form $\operatorname{Re}\langle C_i(\xi), \tau \rangle$ (respectively $\operatorname{Re}\langle z, \tau \rangle + \varepsilon e^{-s_1}$) over the box is positive and at least $1/\beta_0$ times an upper bound of $\|T_\tau\|_1$ (respectively of $U_G(\tau, \varepsilon)$). A leaf for Lemma 5.2 is valid if a lower bound of $\operatorname{Re}\langle (xe^{i\psi}, 1, 1, 0, 0), \tau \rangle$ over the box is positive and at least 1.61 times an upper bound of $\|T_\tau\|_1$. Upper bounds for $\|T_\tau\|_1$ come from an adaptive subdivision of $[0, 2\pi]$. For the correction term in U_G , on each piece the integrand

$\varepsilon \bar{r}_G(|T_\tau(X)|/\varepsilon)$ is at most $\varepsilon \bar{r}_G(w/\varepsilon)$ for a rigorous lower bound w of $|T_\tau|$ on the piece, because $\bar{r}_G$ is nonincreasing; the first program takes w from a ball enclosure of T_τ ; the second uses a chord lower bound of $|T_\tau|^2$ along the piece, integrates the step function exactly, and uses the last table value beyond the grid.

6.2. The tables. For $p = 2, \dots, 24$ there is a table of integers n_j with $n_j 2^{-40} \geq r_p(u_j)$, on a grid of 1421 points ($u_j = j/256$ for $u_j \leq 4$, then geometric with ratio about 1.02 up to $u_N \approx 1.02 \cdot 10^4$, rounded down to multiples of $1/256$). For all $p \geq 25$ a single table holds rounded-up upper bounds $n_j 2^{-40} \geq r_\infty(u_j) + \text{err}(25, u_j)$ (Lemma 4.2(iii)). For a group G let n_j^G be the maximum of the entries of its tables and

$$\bar{r}_G(u) = n_j^G 2^{-40} \quad (u_j \leq u < u_{j+1}), \quad \bar{r}_G(u) = \min(n_N^G 2^{-40}, 1/(2u)) \quad (u \geq u_N).$$

Every table is nonincreasing in j , so $\bar{r}_G$ is nonincreasing, and $\bar{r}_G \geq r_p$ for $p \in G$ by Lemma 4.2(i),(ii). The values $r_p(u_j)$, $p \leq 24$, are bounded by an interval branch and bound over $\varphi \in [0, \pi/p]$ (the function of φ is even and $2\pi/p$ -periodic) with mean-value bounds, or by $r_\infty(u_j) + \text{err}(p, u_j)$ alone when $\text{err}(p, u_j) < 2 \cdot 10^{-8}$; otherwise the smaller of the two bounds is used. Some entries exceed the quantity they bound by less than 10^{-15} , so checking them needs more than double precision.

6.3. Sizes and checking. The equal-gap certificate has 176,240 leaves and 15,276 boxes covered by symmetry; Lemma 5.2 has 18 leaves; the mixed certificate has 37,198 leaves in eight groups (from 2,820 to 7,707 per group). Two programs check the three certificates and the tables.

The first program evaluates the linear forms in ball arithmetic [7] at 80 bits, with a mean-value (centred) form intersected with the naive enclosure (the naive enclosure alone for the tail forms). It bounds $\|T_\tau\|_1$ by a midpoint rule with a Taylor remainder where $|T_\tau|$ is bounded away from 0, and by $2h \sup |T_\tau|$ on the remaining pieces, verifies coverage by reconstructing the bisection tree of each initial cell, and recomputes every table entry with the routine that produced it (the tables were built at 96 bits; the check runs at 80).

The second program was written separately, without access to the first, from the mathematical specification and the data format only; a misreading of the specification shared by both programs would not be detected. It computes the minimum of each linear form over a box from its separable structure (exactly up to ball enclosures for the equal-gap forms; with bisection in s_1 for the mixed forms), bounds $\|T_\tau\|_1$ through $|T| \leq \frac{1}{2}(|T|^2 w + w^{-1})$ with piecewise-linear weights $w > 0$, which gives closed-form integrals, checks coverage by a guillotine recursion proving that the boxes tile the domain, and bounds the table entries by Fourier estimates or a branch and bound at 128 bits. Both programs also check that the groups cover all $p \geq 2$ with admissible κ_G . For a leaf, call $\beta_0 L/U - 1$ its relative margin, where L is the lower bound of the linear form and U the upper bound of $\|T_\tau\|_1$ (or of U_G). The smallest relative margin among the equal-gap leaves, as bounded from below by the second program, is $2.1 \cdot 10^{-7}$.

Both programs report every leaf, every table entry and the coverage as verified (python-flint 0.9, FLINT 3.6). On a heavily loaded shared x86-64 server, the first program needed about 18 minutes and the second about 7 minutes, each with seven processes. The certificates, both programs, their outputs and SHA-256 checksums are available at <https://github.com/tejasvi/maximum-term-bound> and archived at doi:10.5281/zenodo.23521675.

7. REMARKS

Remark 7.1 (the optimal configuration). The value 0.630041 of $\sup_K \Lambda$ was obtained by numerical optimization; at the optimum a primal-dual computation gives 0.6300408 to seven digits. Since $\beta_0 - 0.630041 < 10^{-5}$, Theorem 5.3 is within 10^{-5} of the best bound that the five-term

equal-gap problem can give, and most leaves of the equal-gap certificate lie near the optimal configuration, where $\beta_0 - \Lambda$ is only about $9 \cdot 10^{-6}$. This configuration is real: $\sigma \approx 1.536$, $x = y \approx 0.417$ and $\varphi_{-2} = \varphi_{-1} = \varphi_2 = \pi$. The signs $(-, -, +, +, -)$ are those of the coefficients $(-1)^{q_n}$, $n = -2, \dots, 2$, of the Laurent series used by Clunie and Hayman for $B > 4/7$ [2, pp. 156–158] ($q_n = 0$ for $n \equiv 0, 1$ and $q_n = 1$ for $n \equiv 2, 3 \pmod{4}$), but the moduli differ from those of their series.

Remark 7.2 (wider windows). With a window of seven coefficients centred at the middle of five consecutive vertices with equal gaps, tested at the four jump radii between them, the best configuration we know for the equal-gap problem has value about 0.6059 (numerically); it is real, with the sign pattern of the Clunie–Hayman construction. A proof along these lines would also need a different treatment of mixed events: numerically, the five-coefficient single-radius bound of Proposition 3.3 tends to $(\sqrt{5} - 1)/2 \approx 0.618$ as $s_1 \rightarrow \infty$ (the off-lattice coefficient becomes negligible), so that argument alone cannot go below 0.618.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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