

A LOWER BOUND OF 0.836 FOR LEBESGUE’S UNIVERSAL COVERING PROBLEM

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ABSTRACT. A universal cover is a convex set in the plane that contains a congruent copy of every set of diameter 1. Lebesgue asked for the least area Λ of a universal cover. The best upper bound is $\Lambda \leq 0.8440935944$, due to Gibbs. The previous lower bounds were $\Lambda \geq 0.83474$, due to Hatchett, and $\Lambda \geq 0.8344$, due to Mishra. We prove $\Lambda \geq 0.836$. Like the earlier lower bounds, the proof uses the fact that a universal cover contains the convex hull of congruent copies of any finitely many sets of diameter 1. We use the disk and the regular Reuleaux triangle, pentagon and heptagon of width 1, whose relative placements form an eight-dimensional space. The main ingredient is a second-order lower bound for the hull area on a box of placements. We follow finitely many points of the bodies as they move. While these points stay in angular order around a point of the disk, the area of the polygon they span is at most the hull area. This area is a polynomial in the translations and in the sines and cosines of the angles of the placement. A Taylor expansion with a certified bound for the smallest eigenvalue of the Hessian bounds it on the whole box. A checker in ball arithmetic verifies the resulting certificate, which has 1,552,274 nodes.

1. INTRODUCTION

A planar set has diameter at most 1 if any two of its points are at distance at most 1. A *universal cover* is a convex set in the plane that contains a congruent copy of every planar set of diameter at most 1. In a letter of 1914 to Pál, Lebesgue asked for a universal cover of least area [12]. We write Λ for the infimum of the areas of universal covers. Its value is not known [2], and it is listed as constant 13b in the repository of optimization constants [14].

Upper bounds come from explicit covers. Pál [12] proved that the regular hexagon circumscribed about a disk of diameter 1 is a universal cover. He then cut off two of its corners, which shows $\Lambda \leq 2 - 2/\sqrt{3} < 0.8453$. Sprague [13], Hansen [7], and Baez, Bagdasaryan and Gibbs [1] removed further small pieces. The best upper bound, $\Lambda \leq 0.8440935944$, is due to Gibbs [6].

Lower bounds come from test sets. Let $T_1, \dots, T_m$ be sets of diameter at most 1. A universal cover contains a congruent copy of each T_i , and since it is convex, it contains the convex hull of these copies. Hence Λ is at least the least area $M(T_1, \dots, T_m)$ of such a hull (Lemma 2.2). Pál [12] took a disk and an equilateral triangle of diameter 1 and proved $\Lambda \geq \pi/8 + \sqrt{3}/4 > 0.8257$.¹ Elekes [4] took a disk and the regular polygons with 3^i sides and obtained 0.8271. Brass and Sharifi [3] took a disk, an equilateral triangle and a regular pentagon. They reduced the placements of these sets to a five-dimensional space, subdivided it, and proved $\Lambda \geq 0.832$. Xie [15] certified 0.833 for the same three sets with interval arithmetic. Gibbs [5] proposed sets of constant width as test sets. For the disk and the regular Reuleaux 3-, 5-, 7- and 9-gons, the smallest hull area found by his simulated annealing was 0.83699098. Hatchett [8], Mishra [11] and Zeng [16] used the disk with the Reuleaux triangle and pentagon of width 1, which

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¹The repository [14] credits the bound 0.8257 to Elekes [4]. We follow Mishra [11, Section 1] in crediting it to Pál.

contain the triangle and the pentagon. They proved $\Lambda \geq 0.83474$, $\Lambda \geq 0.8344$ and $\Lambda \geq 0.834$, respectively.

Let D be the disk of diameter 1, and let B_n be the regular Reuleaux n -gon of width 1 (Definition 2.1).

Theorem 1.1. *We have*

$$M(D, B_3, B_5) \geq 0.8347 \quad \text{and} \quad M(D, B_3, B_5, B_7) \geq 0.836.$$

Consequently $\Lambda \geq 0.836$.

The sets D and B_n are convex, have diameter 1 and have an axis of symmetry. Therefore the bound holds whether or not congruent copies may be reflected, and whether a universal cover must contain copies of all sets of diameter at most 1 or only of the convex sets of diameter 1. Although the first bound of Theorem 1.1 is below the bound 0.83474 of Hatchett [8] for the same three bodies, we include it as a test of the method in five dimensions.

The method. We fix the disk and normalize the placements of the Reuleaux polygons (Section 2). For D, B_3, B_5 a placement is a point $p \in \mathbb{R}^5$, and for D, B_3, B_5, B_7 a point $p \in \mathbb{R}^8$. Let $A(p)$ be the area of the convex hull of the placed bodies. A corner estimate confines the placements with $A(p) \leq \tau$ to a box X , so $\Lambda \geq \tau$ follows once $A > \tau$ on X . We prove this by a branch and bound over subboxes of X . The main question is then how to bound A from below on a box. Mishra [11, Proposition 4.9] bounds the hull area on a box by the hull area of eroded bodies that lie in every placement of the box. Xie [15], Hatchett [8] and Zeng [16] use polygons spanned by points of the bodies that move with the bodies. Xie bounds their areas by interval arithmetic [15, Section 6]. The bounds of Mishra and Xie use no derivatives of the area. They lose an amount proportional to the width of the box even where the area hardly varies on the box, so near a minimizing placement the boxes must be very small. For the four bodies D, B_3, B_5, B_7 , Mishra extrapolated from his eight-dimensional runs that his method would need about 8.6×10^{11} boxes to prove 0.8344. He adds that a correction for the flattening of the growth seen in five dimensions reduces this figure substantially, but not to a feasible size [11, Section 6.3]. Hatchett applies ball arithmetic to the area polynomial expanded about the centre of the box, and the mean-value method of Zeng [16, Section S13.4] subtracts an enclosure of the derivative over the box from the area at the centre. These two bounds use the first derivatives of the area, but they control the second-order terms through their absolute values.

We also use a moving polygon and expand its area about the centre of the box, but we bound the second-order term by the smallest eigenvalue of the Hessian. On a box with centre c we fix finitely many points of the bodies, the witnesses, and let each move with its body. Their positions $x_j(p)$ are polynomials in the translations and in the sines and cosines of the angles of p , and so is the fan area $F(p) = \frac{1}{2} \sum_j \det(x_j(p), x_{j+1}(p))$. If the witnesses stay in strict angular order around a point of the disk for all p in the box, winding once around it, then $F \leq A$ on the box (Lemma 3.3). We take the witnesses among the support points of the hull at c , so that $F(c)$ is close to $A(c)$. On the box we bound F from below by its Taylor expansion at c (Lemma 3.5). This bound uses the gradient of F at c , a certified lower bound for the smallest eigenvalue of the Hessian at c , and explicit bounds for the variation of the Hessian over the box (Lemmas 3.6 and 3.7). Consecutive witnesses on the same body contribute to F through a single term (Lemma 3.4), so the bound is cheap to evaluate. With this bound, the certificate of $M(D, B_3, B_5) \geq 0.8347$ has 53,752 nodes and 27,388 leaves. For the same bodies, Hatchett's certificate of the stronger bound 0.83474 has 922,408 nodes and 461,208 leaves. Its target is about twice as close as ours to the largest bound that these bodies can give, and certificates grow as this distance shrinks. Mishra's certificate of 0.8344 has 486,799,600 nodes

and 245,496,952 leaves. The certificate of $M(D, B_3, B_5, B_7) \geq 0.836$ has 1,552,274 nodes and 808,905 leaves.

Verification. A checker of about 400 lines of Python, independent of the search program, verifies the certificates in ball arithmetic with the Arb library [9] through python-flint. For each leaf it builds the witnesses again with a floating-point heuristic and then verifies every inequality that the proof uses: the witnesses lie in their bodies, they stay in angular order on the whole box, and the Taylor bound exceeds the target. It also checks the root box against the corner estimate, the structure of the trees and the presence of all parts. Two further programs decode the trees in exact rational arithmetic. A verifier written separately from the checker, which bounds F on a box without a Taylor remainder, re-verified samples of leaves of both certificates (Section 4.5 and Table 1).

Limits. Mishra exhibits a placement of D, B_3, B_5 whose hull has area at most 0.834781191 [11], so no lower bound obtained from these three bodies exceeds this value. Hatchett's bound 0.83474 and the first bound of Theorem 1.1 are within 4.2×10^{-5} and 8.2×10^{-5} of it, respectively. For D, B_3, B_5, B_7 we find numerically a placement with hull area 0.8364904560 (Section 5).

Section 2 reduces Theorem 1.1 to a statement about a box of placements. Section 3 proves the fan bound and the Taylor bound. Section 4 describes the certificates and the checker and proves Theorem 1.1. Section 5 records the limits of the method.

2. REDUCTION TO A BOX OF PLACEMENTS

We first fix notation. For $a, b \in \mathbb{R}^2$ put $\det(a, b) = a_1 b_2 - a_2 b_1$. Let $u(\varphi) = (\cos \varphi, \sin \varphi)$, let R_φ be the rotation of $\mathbb{R}^2$ about the origin by the angle φ , and let $J = R_{\pi/2}$. Rotations preserve $\det$, and J commutes with every R_φ . Let $D(y, r)$ be the closed disk with centre y and radius r , and let $D = D(0, 1/2)$. We write area for the Lebesgue measure on $\mathbb{R}^2$, conv for the convex hull, and $|\cdot|$ for the Euclidean norm.

Definition 2.1. For odd $n \geq 3$ let $R_n = 1/(2 \cos(\pi/(2n)))$, let $V_{n,l} = R_n u(2\pi l/n)$ for $l = 0, \dots, n-1$, and let $B_n = \bigcap_{l=0}^{n-1} D(V_{n,l}, 1)$.

The points $V_{n,l}$ are the vertices of a regular n -gon. The largest distance between two of them is $|V_{n,0} - V_{n,(n-1)/2}| = 2R_n \cos(\pi/(2n)) = 1$. So B_n is the regular Reuleaux n -gon of width 1. It has constant width 1, and hence diameter 1 [10]. We have $0.51 < R_7 < R_5 < R_3 < 0.58$.

We fix odd integers $n_1, \dots, n_k \geq 3$ with $k \geq 2$. Theorem 1.1 uses $(n_1, n_2) = (3, 5)$ and $(n_1, n_2, n_3) = (3, 5, 7)$. A *placement* is a vector

$$p = (t_1, \rho_2, t_2, \dots, \rho_k, t_k) \in \mathbb{R}^d, \quad d = 3k - 1,$$

with $t_i \in \mathbb{R}^2$ and $\rho_i \in \mathbb{R}$. We put $\rho_1 = 0$, and for the disk we put $\rho_0 = 0$ and $t_0 = 0$. Let $B^{(0)} = D$ and $B^{(i)} = B_{n_i}$ for $1 \leq i \leq k$. In the placement p , body i occupies $P_i(p) = R_{\rho_i} B^{(i)} + t_i$. Let

$$K(p) = \text{conv}(P_0(p) \cup P_1(p) \cup \dots \cup P_k(p)) \quad \text{and} \quad A(p) = \text{area } K(p).$$

For bounded sets $T_1, \dots, T_m \subset \mathbb{R}^2$ let $M(T_1, \dots, T_m)$ be the infimum of $\text{area conv}(g_1(T_1) \cup \dots \cup g_m(T_m))$ over all isometries $g_1, \dots, g_m$ of $\mathbb{R}^2$.

Lemma 2.2 below is the principle behind all the lower bounds above. Its normalization of the placements is that of Brass and Sharifi [3] (see also [11, Lemma 4.3]).

Lemma 2.2. *If $T_1, \dots, T_m$ are planar sets of diameter at most 1, then $\Lambda \geq M(T_1, \dots, T_m)$. Moreover, $M(D, B_{n_1}, \dots, B_{n_k}) = \inf_{p \in \mathbb{R}^d} A(p)$.*

Proof. Let Ω be a universal cover. For each i there is an isometry g_i with $g_i(T_i) \subseteq \Omega$. Since Ω is convex, it contains $\text{conv}(\bigcup_i g_i(T_i))$, so $\text{area } \Omega \geq M(T_1, \dots, T_m)$. Taking the infimum over Ω proves the first claim.

Every placement is given by isometries, so $M(D, B_{n_1}, \dots, B_{n_k}) \leq \inf A$. Conversely, let $g_0, \dots, g_k$ be isometries. The reflection $S(y_1, y_2) = (y_1, -y_2)$ maps D to itself and permutes the points $V_{n,l}$, so it maps each $B^{(i)}$ to itself. If g_i reverses orientation, we replace it by $g_i \circ S$, which preserves orientation and has the same image of $B^{(i)}$. So we may assume that $g_i(y) = R_{\alpha_i}y + b_i$ for all i . The rigid motion $h(y) = R_{-\alpha_1}(y - b_0)$ satisfies $h \circ g_i(y) = R_{\alpha_i - \alpha_1}y + R_{-\alpha_1}(b_i - b_0)$. Since D is invariant under rotations about the origin, h maps $g_i(B^{(i)})$ onto $P_i(p)$ for all i , where p is the placement with $\rho_i = \alpha_i - \alpha_1$ and $t_i = R_{-\alpha_1}(b_i - b_0)$. As h preserves areas, the hull of $g_0(B^{(0)}), \dots, g_k(B^{(k)})$ has area $A(p) \geq \inf A$. $\square$

Lemma 2.3 (Symmetry). *For a placement p let $\bar{p} = (St_1, -\rho_2, St_2, \dots, -\rho_k, St_k)$. Then $A(\bar{p}) = A(p)$. Moreover, A is periodic in ρ_i with period $2\pi/n_i$ for $2 \leq i \leq k$. Consequently, $\inf A$ equals the infimum of A over the placements with $0 \leq \rho_2 \leq \pi/n_2$ and $0 \leq \rho_i \leq 2\pi/n_i$ for $3 \leq i \leq k$, and the reduction does not change $|t_1|, \dots, |t_k|$.*

Proof. The reflection S of the previous proof satisfies $S \circ R_\varphi = R_{-\varphi} \circ S$ and $S(B^{(i)}) = B^{(i)}$. Hence $S(P_i(p)) = R_{-\rho_i}B^{(i)} + St_i = P_i(\bar{p})$ for all i . So $S(K(p)) = K(\bar{p})$, and S preserves area. The rotation $R_{2\pi/n}$ permutes the points $V_{n,l}$, so $R_{2\pi/n}B_n = B_n$, and this gives the periodicity. Given a placement, we first shift ρ_2 into $[0, 2\pi/n_2)$. If then $\rho_2 > \pi/n_2$, we pass to $\bar{p}$ and shift its second coordinate $-\rho_2$ by $2\pi/n_2$ into $(0, \pi/n_2)$. Finally we shift $\rho_3, \dots, \rho_k$ into the intervals $[0, 2\pi/n_i)$. None of these steps changes $|t_1|, \dots, |t_k|$. $\square$

The next lemma gives the root box used by Brass and Sharifi [3]. In this form, with the area f , it is in [11, Lemma 4.4 and Remark 4.6]. Zeng [16, Sections S10.2 and S10.3] and Hatchett [8] prove the same estimate for B_3 and B_5 .

Lemma 2.4 (Corner lemma). *For $s \geq 1/2$ and $m \geq 0$ let*

$$f(s) = \frac{\pi - \arccos(1/(2s))}{4} + \frac{\sqrt{s^2 - 1/4}}{2} \quad \text{and} \quad g_n(m) = \sqrt{R_n^2 + m^2 + 2R_n m \cos(\pi/n)}.$$

Then f and g_n are increasing, $g_n(m) \geq R_n > 1/2$, and $A(p) \geq f(g_{n_i}(|t_i|))$ for every placement p and every $1 \leq i \leq k$.

Proof. A direct computation gives $f'(s) = \sqrt{4s^2 - 1}/(4s) > 0$ for $s > 1/2$. Since $\cos(\pi/n) > 0$, g_n is increasing, and $g_n(0) = R_n > 1/2$ because $\cos(\pi/(2n)) < 1$. Fix $i \geq 1$ and write $n = n_i$, $\rho = \rho_i$ and $t = t_i$. The corners $y_l = R_\rho V_{n,l} + t$ lie in $P_i(p)$. If $t = 0$, then $|y_l| = R_n = g_n(0)$. Otherwise we write $t = |t|u(\psi)$. The directions $\rho + 2\pi l/n$ of the vectors $R_\rho V_{n,l}$ are spaced by $2\pi/n$, so for some l the angle β between $u(\rho + 2\pi l/n)$ and $u(\psi)$ satisfies $\beta \leq \pi/n$. For this l ,

$$|y_l|^2 = R_n^2 + |t|^2 + 2R_n|t|\cos\beta \geq g_n(|t|)^2.$$

So $K(p)$ contains D and a point y with $s = |y| \geq g_n(|t|) > 1/2$. The set $\text{conv}(D \cup \{y\})$ is the union of two pieces. The first is the quadrilateral bounded by the two tangent segments from y to D and the two radii to their points of contact, of area $\frac{1}{2}\sqrt{s^2 - 1/4}$. The second is the sector of D outside this quadrilateral, of angle $2\pi - 2\arccos(1/(2s))$ and area $(\pi - \arccos(1/(2s)))/4$. Hence $A(p) \geq f(s) \geq f(g_n(|t|))$. $\square$

Corollary 2.5. *Let $\tau > 0$ and $w_1, \dots, w_k > 0$ with $f(g_{n_i}(w_i)) > \tau$ for all i . Let $X \subset \mathbb{R}^d$ be a box that contains every placement with $t_i \in [-w_i, w_i]^2$ for $1 \leq i \leq k$, $\rho_2 \in [0, \pi/n_2]$ and $\rho_i \in [0, 2\pi/n_i]$ for $3 \leq i \leq k$. If $A(p) \geq \tau$ for all $p \in X$, then $M(D, B_{n_1}, \dots, B_{n_k}) \geq \tau$, and hence $\Lambda \geq \tau$.*

Proof. Let p be a placement. By Lemma 2.3 there is a placement p' with $A(p') = A(p)$, with rotations in the stated ranges, and with $|t'_i| = |t_i|$ for all i . If $t'_i \in [-w_i, w_i]^2$ for all i , then $p' \in X$ and $A(p) \geq \tau$. Otherwise $|t'_i| > w_i$ for some i , and Lemma 2.4 gives $A(p) \geq f(g_{n_i}(|t'_i|)) \geq f(g_{n_i}(w_i)) > \tau$. So $\inf A \geq \tau$, and Lemma 2.2 gives the claim. $\square$

3. THE BOX BOUND

In this section we bound A from below on a box of placements. A *box* is a set $B = \prod_{a=1}^d [c_a - h_a, c_a + h_a]$ with centre c and half-widths $h_a \geq 0$. The *angle* from a nonzero vector v to a nonzero vector v' with $\det(v, v') > 0$ is the number $\theta \in (0, \pi)$ with $v'/|v'| = R_\theta(v/|v|)$.

Lemma 3.1 (Fan lemma). *Let $K \subset \mathbb{R}^2$ be compact and convex, let $z \in K$, and let $x_1, \dots, x_m \in K$ with $m \geq 3$. Indices are taken modulo m . Suppose that $\det(x_j - z, x_{j+1} - z) > 0$ for all j , and let θ_j be the angle from $x_j - z$ to $x_{j+1} - z$. If $\theta_1 + \dots + \theta_m = 2\pi$, then*

$$\text{area } K \geq \frac{1}{2} \sum_{j=1}^m \det(x_j - z, x_{j+1} - z) = \frac{1}{2} \sum_{j=1}^m \det(x_j, x_{j+1}).$$

Proof. Write $x_1 - z = r_1 u(\varphi_1)$ with $r_1 > 0$, and put $\varphi_{j+1} = \varphi_j + \theta_j$. Then $x_j - z = r_j u(\varphi_j)$ with $r_j > 0$ for all j , and $\varphi_{m+1} = \varphi_1 + 2\pi$. The triangle $\Delta_j = \text{conv}\{z, x_j, x_{j+1}\}$ lies in K because K is convex, and its area is $\frac{1}{2} \det(x_j - z, x_{j+1} - z)$. Since $\theta_j < \pi$, the interior of Δ_j lies in the open sector $\{z + s u(\psi) : s > 0, \varphi_j < \psi < \varphi_{j+1}\}$. The intervals $(\varphi_j, \varphi_{j+1})$ are disjoint subintervals of $[\varphi_1, \varphi_1 + 2\pi)$, and u is injective on this interval. Hence the sectors are pairwise disjoint, the triangles have disjoint interiors, and $\text{area } K \geq \sum_j \text{area } \Delta_j$. Finally,

$$\sum_j \det(x_j - z, x_{j+1} - z) = \sum_j \det(x_j, x_{j+1}) + \sum_j \det(z, x_j) - \sum_j \det(z, x_{j+1}),$$

and the last two sums are equal. $\square$

Definition 3.2. A *witness fan* consists of a point $z \in \mathbb{R}^2$ with $|z| < 1/2$ and a cyclic sequence of *witnesses* $(i_1, q_1), \dots, (i_m, q_m)$ with $m \geq 3$, where $i_j \in \{0, \dots, k\}$ and $q_j \in B^{(i_j)}$. At a placement p , witness j is at $x_j(p) = R_{\rho_{i_j}} q_j + t_{i_j}$. The *fan area* is

$$F(p) = \frac{1}{2} \sum_{j=1}^m \det(x_j(p), x_{j+1}(p)).$$

Thus $x_j(p) \in P_{i_j}(p) \subseteq K(p)$ and $z \in D \subseteq K(p)$. Each $x_j(p)$ is affine in the coordinates of the t_i and in the numbers $\cos \rho_i$ and $\sin \rho_i$. So F is a polynomial in these quantities, and in particular F is smooth.

Lemma 3.3 (Fan bound). *Let B be a box with centre c , and let a witness fan satisfy $\det(x_j(p) - z, x_{j+1}(p) - z) > 0$ for all j and all $p \in B$. If the angles from $x_j(c) - z$ to $x_{j+1}(c) - z$ add up to 2π , then $A(p) \geq F(p)$ for all $p \in B$.*

Proof. For $p \in B$ let $\theta_j(p)$ be the angle from $x_j(p) - z$ to $x_{j+1}(p) - z$. It depends continuously on p , because the determinant is positive on B . The sum $\Theta(p) = \sum_j \theta_j(p)$ is an integer multiple of 2π , since rotating the direction of $x_1(p) - z$ successively by $\theta_1(p), \dots, \theta_m(p)$ returns it to itself. As B is connected, Θ is constant on B , so $\Theta(p) = \Theta(c) = 2\pi$. We apply Lemma 3.1 to $K(p)$, the point z and the points $x_j(p)$. $\square$

Lemma 3.4 (Run identity). *Let R be a rotation and $a, b, t \in \mathbb{R}^2$. Then $\det(Ra + t, Rb + t) = \det(a, b) + \det(t, R(b - a))$. Consequently, if the witnesses $\alpha, \alpha + 1, \dots, \beta$ lie on one body i , then for every placement*

$$\sum_{j=\alpha}^{\beta-1} \det(x_j(p), x_{j+1}(p)) = \sum_{j=\alpha}^{\beta-1} \det(q_j, q_{j+1}) + \det(t_i, R_{\rho_i}(q_\beta - q_\alpha)),$$

and $\det(x_\alpha(p) - z, x_{\alpha+1}(p) - z) = \det(q_\alpha, q_{\alpha+1}) + \det(t_i - z, R_{\rho_i}(q_{\alpha+1} - q_\alpha))$.

Proof. We have $\det(Ra, Rb) = \det(a, b)$ and $\det(Ra, t) + \det(t, Rb) = \det(t, Rb - Ra)$. This proves the first identity, and summing it over j gives the second, since the sum telescopes. The third is the first with t replaced by $t_i - z$. $\square$

We split the cyclic sequence of witnesses into maximal *runs* of consecutive witnesses on the same body. Suppose that the witnesses lie on at least two bodies. Then consecutive runs lie on different bodies, also cyclically. Let run r lie on body ι_r and consist of the witnesses $\alpha_r, \dots, \beta_r$. Lemma 3.4 gives

$$(3.1) \quad F(p) = \frac{1}{2} \sum_r \left(C_r + \det(t_{\iota_r}, R_{\rho_{\iota_r}} w_r) + \det(x_{\beta_r}(p), x_{\alpha_r+1}(p)) \right),$$

where $C_r = \sum_{j=\alpha_r}^{\beta_r-1} \det(q_j, q_{j+1})$ and $w_r = q_{\beta_r} - q_{\alpha_r}$. We call $\frac{1}{2} \det(t_{\iota_r}, R_{\rho_{\iota_r}} w_r)$ a *run term* and $\frac{1}{2} \det(x_{\beta_r}(p), x_{\alpha_r+1}(p))$ a *transition term*. A run term depends on the parameters of one body, and it vanishes for the disk. A transition term depends on the parameters of two different bodies.

Lemma 3.5 (Taylor bound). *Let B be a box with centre c and half-widths h , and let F be twice continuously differentiable near B . Put $g = \nabla F(c)$ and $H = \nabla^2 F(c)$. Let $\lambda \leq \min(0, \lambda_{\min}(H))$, and let W be a symmetric matrix with $|\partial_a \partial_b F(\xi) - H_{ab}| \leq W_{ab}$ for all $\xi \in B$ and all a, b . Then for all $p \in B$*

$$F(p) \geq L := F(c) - \sum_a |g_a| h_a + \frac{\lambda}{2} \sum_a h_a^2 - \frac{1}{2} \sum_{a,b} W_{ab} h_a h_b.$$

Proof. Let $p \in B$ and $v = p - c$, so $|v_a| \leq h_a$. Taylor's theorem for $s \mapsto F(c + sv)$ gives a point ξ on the segment from c to p with $F(p) = F(c) + g \cdot v + \frac{1}{2} v^T \nabla^2 F(\xi) v$. Here $g \cdot v \geq -\sum_a |g_a| h_a$ and $v^T H v \geq \lambda_{\min}(H) |v|^2 \geq \lambda \sum_a h_a^2$, since $\lambda \leq 0$. Moreover, $v^T (\nabla^2 F(\xi) - H) v \geq -\sum_{a,b} W_{ab} |v_a| |v_b| \geq -\sum_{a,b} W_{ab} h_a h_b$. $\square$

We now compute the derivatives of the terms of (3.1) and bound their variation on a box B with centre c . For a body $i \geq 1$ let r_i be the half-width of B in ρ_i , with $r_1 = 0$, and let $\delta_i = (h_{i,1}^2 + h_{i,2}^2)^{1/2}$, where $h_{i,1}$ and $h_{i,2}$ are the half-widths of B in the two coordinates of t_i . Put $r_0 = \delta_0 = 0$. For a witness (i, q) let $e(i, q) = |q| r_i + \delta_i$. For a function ϕ on B and $p \in B$ write $\Delta\phi = \phi(p) - \phi(c)$, and let e_1, e_2 be the standard basis of $\mathbb{R}^2$.

Lemma 3.6 (Derivatives and variation bounds). *Let B be a box with centre c , and let $p \in B$.*

- (a) *We have $\|R_{\rho_i(p)} - R_{\rho_i(c)}\| \leq r_i$ and $|\Delta t_i| \leq \delta_i$. The position x of a witness (i, q) satisfies $|\Delta x| \leq e(i, q)$.*
- (b) *Let (i, q) and (i', q') be consecutive witnesses with positions x and y , and let $\eta = \det(x - z, y - z)$. If $i = i' = 0$, then $\Delta\eta = 0$. If $i = i' \geq 1$, then $|\Delta\eta| \leq |q' - q|(\delta_i(1 + r_i) + |t_i(c) - z| r_i)$. If $i \neq i'$, then*

$$|\Delta\eta| \leq e(i, q)(|y(c) - z| + e(i', q')) + |x(c) - z| e(i', q').$$

- (c) Let $G = \frac{1}{2} \det(t_i, R_{\rho_i} w)$ be a run term. If $i = 1$, then G is linear in t_1 . If $i \geq 2$, the second derivatives of G that do not vanish identically are $\partial^2 G / \partial \rho_i^2 = -\frac{1}{2} \det(t_i, R_{\rho_i} w)$ and $\partial^2 G / \partial \rho_i \partial t_i = \frac{1}{2} R_{\rho_i} w$, and on B

$$\left| \Delta \frac{\partial^2 G}{\partial \rho_i^2} \right| \leq \frac{|w|}{2} (\delta_i(1 + r_i) + |t_i(c)| r_i), \quad \left| \Delta \frac{\partial^2 G}{\partial \rho_i \partial t_{i,a}} \right| \leq \frac{|w|}{2} r_i \quad (a = 1, 2).$$

- (d) Let $G = \frac{1}{2} \det(x, y)$ be a transition term, where x is the position of a witness (i, q) and y that of a witness (i', q') with $i \neq i'$, and put $e = e(i, q)$ and $e' = e(i', q')$. The second derivatives of G with respect to the coordinates of t_i and $t_{i'}$ are constant. The other second derivatives ϕ of G that do not vanish identically, and bounds for $|\Delta \phi|$ on B , are as follows, where $a, b \in \{1, 2\}$:

$$\begin{aligned} \phi &= \frac{\partial^2 G}{\partial \rho_i \partial t_{i',b}} = \frac{1}{2} \det(JR_{\rho_i} q, e_b), & |\Delta \phi| &\leq \frac{1}{2} |q| r_i, \\ \phi &= \frac{\partial^2 G}{\partial t_{i,a} \partial \rho_{i'}} = \frac{1}{2} \det(e_a, JR_{\rho_{i'}} q'), & |\Delta \phi| &\leq \frac{1}{2} |q'| r_{i'}, \\ \phi &= \frac{\partial^2 G}{\partial \rho_i \partial \rho_{i'}} = \frac{1}{2} \det(R_{\rho_i} q, R_{\rho_{i'}} q'), & |\Delta \phi| &\leq \frac{1}{2} |q| |q'| (r_i + r_{i'}), \\ \phi &= \frac{\partial^2 G}{\partial \rho_i^2} = -\frac{1}{2} \det(R_{\rho_i} q, y), & |\Delta \phi| &\leq \frac{1}{2} |q| (r_i(|y(c)| + e') + e'), \\ \phi &= \frac{\partial^2 G}{\partial \rho_{i'}^2} = -\frac{1}{2} \det(x, R_{\rho_{i'}} q'), & |\Delta \phi| &\leq \frac{1}{2} |q'| (r_{i'}(|x(c)| + e) + e). \end{aligned}$$

Only derivatives with respect to existing parameters occur: t_i exists for $i \geq 1$ and ρ_i for $i \geq 2$.

Proof. (a) For rotations, $\|R_\varphi - R_{\varphi'}\| = 2|\sin((\varphi - \varphi')/2)| \leq |\varphi - \varphi'|$. Hence $|\Delta x| \leq \|\Delta R_{\rho_i}\| |q| + |\Delta t_i| \leq e(i, q)$.

(b) If $i = i' = 0$, then x and y are constant. If $i = i' \geq 1$, Lemma 3.4 gives $\eta = \det(q, q') + \det(t_i - z, R_{\rho_i} w)$ with $w = q' - q$. Writing $R = R_{\rho_i}$ and $R_c = R_{\rho_i(c)}$, we get

$$\Delta \eta = \det(\Delta t_i, R_c w) + \det(t_i(c) - z, (R - R_c)w) + \det(\Delta t_i, (R - R_c)w),$$

and (a) bounds the three terms by $\delta_i |w|$, $|t_i(c) - z| r_i |w|$ and $\delta_i r_i |w|$. If $i \neq i'$, then $\Delta \eta = \det(\Delta x, y - z) + \det(x(c) - z, \Delta y)$, and $|y - z| \leq |y(c) - z| + e(i', q')$ by (a).

(c) We have $\partial_\varphi R_\varphi = R_\varphi J$ and $J^2 = -I$. Hence $\partial G / \partial \rho_i = \frac{1}{2} \det(t_i, R_{\rho_i} Jw)$, $\partial^2 G / \partial \rho_i^2 = -\frac{1}{2} \det(t_i, R_{\rho_i} w)$, and $\partial^2 G / \partial \rho_i \partial t_{i,a} = \frac{1}{2} \det(e_a, JR_{\rho_i} w) = \frac{1}{2} (R_{\rho_i} w)_a$, while G is linear in t_i . The variation of $\det(t_i, R_{\rho_i} w)$ is bounded as in (b), with $t_i(c)$ in place of $t_i(c) - z$, and $|\Delta (R_{\rho_i} w)_a| \leq r_i |w|$ by (a).

(d) The point x depends only on the parameters of body i , and y only on those of body i' . We have $\partial x / \partial t_{i,a} = e_a$, $\partial x / \partial \rho_i = JR_{\rho_i} q$ and $\partial^2 x / \partial \rho_i^2 = -R_{\rho_i} q$, and all other second derivatives of x vanish. The same holds for y . Hence the mixed derivatives are $\frac{1}{2} \det(\partial x, \partial y)$ for one parameter of each body, the second derivatives in ρ_i and in $\rho_{i'}$ are as stated, and the remaining second derivatives vanish. The derivatives in two coordinates of t_i and $t_{i'}$ are $\frac{1}{2} \det(e_a, e_b)$. Using $\det(Ja, Jb) = \det(a, b)$ and (a), we get $|\det(JR_{\rho_i} q, e_b) - \det(JR_{\rho_i(c)} q, e_b)| \leq r_i |q|$, the same bound for the second line, and

$$|\Delta \det(R_{\rho_i} q, R_{\rho_{i'}} q')| \leq |\det(\Delta(R_{\rho_i} q), R_{\rho_{i'}} q')| + |\det(R_{\rho_i(c)} q, \Delta(R_{\rho_{i'}} q'))| \leq |q| |q'| (r_i + r_{i'}).$$

Finally, $|\Delta \det(R_{\rho_i} q, y)| \leq r_i |q| |y| + |q| |\Delta y| \leq |q| (r_i(|y(c)| + e') + e')$, and the last line is symmetric. $\square$

Let W be the symmetric $d \times d$ matrix obtained by adding, over all run terms and transition terms of (3.1), the bounds of Lemma 3.6(c) and (d) at the corresponding entries, symmetrically. Since $\nabla^2 F$ is the sum of the Hessians of these terms, W satisfies the hypothesis of Lemma 3.5. All entries of W are nonnegative.

Lemma 3.7 (Eigenvalue certificate). *Let H be a real symmetric $d \times d$ matrix, let $\lambda \in \mathbb{R}$ and $N = H - \lambda I$. For $a = 1, \dots, d$ define recursively*

$$L_{ab} = \frac{1}{D_b} \left(N_{ab} - \sum_{\gamma < b} L_{a\gamma} L_{b\gamma} D_\gamma \right) \quad (b < a), \quad D_a = N_{aa} - \sum_{\gamma < a} L_{a\gamma}^2 D_\gamma.$$

If the recursion yields $D_1, \dots, D_d > 0$, then $\lambda_{\min}(H) > \lambda$.

Proof. The recursion gives $N = LDL^T$, where L is unit lower triangular with entries L_{ab} below the diagonal and $D = \text{diag}(D_1, \dots, D_d)$. For $v \neq 0$ we have $L^T v \neq 0$, so $v^T N v = \sum_a D_a ((L^T v)_a)^2 > 0$. $\square$

If the entries of H are known only up to enclosing intervals and the recursion is evaluated in interval arithmetic, then the computed intervals contain the numbers D_a of the exact matrix, so positive lower endpoints of these intervals certify $\lambda_{\min}(H) > \lambda$.

4. THE CERTIFICATES AND THEIR VERIFICATION

4.1. Certificates. We now describe the certificates. A certificate for the bodies $D, B_{n_1}, \dots, B_{n_k}$ and a target τ consists of P parts. Each part has a header and an array of codes. The header records $(n_1, \dots, n_k)$, τ , a root box X given by its lower and upper corners, a grid $(s_1, \dots, s_d)$, the number P of parts and the index of the part. The root box is cut into $s_1 \cdots s_d$ equal cells, listed in lexicographic order of their grid positions. Part j holds the cells whose position in this list is congruent to j modulo P . The array of a part describes, for each of its cells in turn, a binary tree in depth-first preorder. A code $a \in \{0, \dots, d-1\}$ splits the current box at the midpoint of coordinate $a+1$, and the trees of the lower and the upper half follow. The code 254 marks a leaf closed by the fan bound, and the code 255 a leaf closed by the corner lemma. The header also records the parameters N and ε of the finer witness set of Section 4.2, which the checker reuses in step (5) below, and further parameters and statistics of the search, which it ignores.

4.2. The search. The search program is a depth-first branch and bound in double precision. At a box with centre c it builds a witness fan in three steps. First, it takes the support points of $K(c)$ in N equally spaced directions. Each of them, a point of the disk, a corner of a placed Reuleaux polygon or a point of one of its arcs, is recorded as a witness on the corresponding body. Second, it repeatedly inserts, for each edge of the current polygon, the support point of $K(c)$ in the direction of the outer normal of the edge if this point lies more than ε outside the edge. Third, it chooses the fan centre z among nine points of D and drops witnesses until a floating-point version of the inequalities of Lemma 3.6(b) holds. It then evaluates the bound L of Lemma 3.5, with W from Lemma 3.6 and the numerically computed $\lambda_{\min}(H)$. The witness sets use $N = 1024$ directions and $\varepsilon = 10^{-6}$. For the certificate with B_7 , the search first tries $N = 256$ and $\varepsilon = 10^{-5}$ and uses the finer set only when the first bound falls short of $\tau + 10^{-6}$ by at most 3×10^{-5} . A box is closed when $L \geq \tau + 10^{-6}$. Otherwise the search splits it in the coordinate a with the largest contribution $|g_a| h_a + \frac{1}{2} \max(0, -\lambda_{\min}(H)) h_a^2 + \frac{1}{2} (Wh)_a h_a$ to the loss in L , or in its widest coordinate if no witness fan passes the floating-point test. The released code contains one version of the search for each certificate. Rerunning it reproduced parts of that certificate exactly.

4.3. The checker. The checker `1bcheck.py` works in ball arithmetic at 80 bits of precision. The box coordinates are double-precision numbers, which are exact in Arb. The checker reads τ as the shortest decimal number that represents the header value, here 0.8347 and 0.836. For each part it performs the following steps.

- (1) *Presence.* It verifies that all P parts exist and that their headers agree in the bodies, τ , X , the grid and P .
- (2) *Root box.* For each body $i \geq 1$ let w_i be the smallest of the numbers $-\ell$ and u over the two coordinate ranges $[\ell, u]$ of t_i in X . The checker verifies $w_i > 0$ and $f(g_{n_i}(w_i)) > \tau$. It also verifies that the range of ρ_2 in X contains $[0, \pi/n_2]$ and that the range of ρ_i contains $[0, 2\pi/n_i]$ for $i \geq 3$. So X satisfies the hypotheses of Corollary 2.5.
- (3) *Tree.* It decodes the codes and computes the boxes, splitting each box at the double-precision midpoint of the split coordinate. The codes are bytes. The checker rejects codes other than $0, \dots, d-1, 254$ and 255 , an array that ends inside a tree, and data after the last tree.
- (4) *Far leaves.* At a leaf with code 255 it verifies $f(g_{n_i}(\mu_i)) > \tau$ for some i , where μ_i is the smallest value of $|t_i|$ on the leaf. Then $A > \tau$ on the leaf by Lemma 2.4.
- (5) *Bound leaves.* At a leaf with code 254 it computes the centre c and the half-widths h in double precision and verifies, in rational arithmetic, that $c \pm h$ are exactly the endpoints of the leaf box. It builds a witness fan, which the certificate does not store, with its own implementation of the heuristic of Section 4.2, first with $N = 256$ and $\varepsilon = 10^{-5}$ and, if the verification below fails, with $N = 1024$ and $\varepsilon = 10^{-6}$. It moves the witnesses towards the centres of their bodies by the factor $1 - 2^{-40}$. It then verifies in ball arithmetic that $|z|^2 < 1/4$, that each witness lies in its body ($|q|^2 < 1/4$ for the disk, $|q - V_{n,l}|^2 < 1$ for all l for B_n), that $\det(x_j(c) - z, x_{j+1}(c) - z)$ exceeds the bound of Lemma 3.6(b) for all j , and that the computed sum of the angles at c differs from 2π by less than 1. It evaluates $F(c)$, $\nabla F(c)$ and $\nabla^2 F(c)$ through (3.1) and computes W . It puts $\lambda = \min(0, \lambda_f) - 10^{-9}(1 + |\lambda_f|)$, evaluated in double precision, where λ_f is a floating-point estimate of $\lambda_{\min}(\nabla^2 F(c))$, and certifies $\lambda_{\min}(\nabla^2 F(c)) > \lambda$ by Lemma 3.7. Finally it verifies $L > \tau$.

If the factorization in step (5) fails, the checker puts $\lambda = \min(0, \gamma)$, where γ is a lower endpoint, computed in ball arithmetic, of $\min_a(H_{aa} - \sum_{b \neq a} |H_{ab}|)$ for $H = \nabla^2 F(c)$. By Gershgorin's theorem, $\lambda_{\min}(H) \geq \gamma$, so again $\lambda \leq \min(0, \lambda_{\min}(H))$. The search of Section 4.2 is not part of the proof, and neither are the witnesses of step (5): only the inequalities verified in ball arithmetic enter it. The checker also reports the smallest lower endpoint of the bounds L of each part.

4.4. Proof of Theorem 1.1.

Proposition 4.1. *For the two certificates of Table 1, the checker accepts all parts. All box coordinates in the trees are exact in double precision, and the leaves tile the root box.*

Proposition 4.1 is verified by computation: the checker verifies its first sentence, and the tree decoders of Section 4.5 verify the second.

Proof of Theorem 1.1. Consider the certificate for $(n_1, n_2) = (3, 5)$ and $\tau = 0.8347$, or the certificate for $(n_1, n_2, n_3) = (3, 5, 7)$ and $\tau = 0.836$, and let X be its root box. By step (2), X satisfies the hypotheses of Corollary 2.5. By step (1) the cells of all parts together form the grid of X , and by Proposition 4.1 the leaves of their trees tile X . On a far leaf, $A > \tau$ by step (4) and Lemma 2.4 (the certificates have none). Let B be a bound leaf with centre c .

We show that $A > \tau$ on B . The checker verified step (5) on B itself. By Lemma 3.6(b) the determinants $\det(x_j(p) - z, x_{j+1}(p) - z)$ are positive for all $p \in B$. In particular each angle at c lies in $(0, \pi)$, so the sum of the angles at c is a multiple of 2π , and the check of step (5) shows

that it equals 2π . Lemma 3.3 gives $A \geq F$ on B . The witnesses lie on at least two bodies. Indeed, the points $x_j(c) - z$ do not lie in an open half-plane bounded by a line through the origin, since their angles lie in $(0, \pi)$ and add up to 2π . So z lies in the convex hull of the points $x_j(c)$. If all witnesses lay on one body i , then $z \in P_i(c)$, and Lemma 3.1 with $K = P_i(c)$ and the isodiametric inequality would give $F(c) \leq \text{area } P_i(c) \leq \pi/4 < \tau$. But $L \leq F(c)$, because $\lambda \leq 0$ and W has nonnegative entries, and the checker verified $L > \tau$. Here the checker evaluates $F(c)$ by grouping the witnesses into runs, and by Lemma 3.4 this gives $F(c)$ also when there is only one run. So (3.1) applies, and the matrix W computed by the checker bounds the variation of $\nabla^2 F$ on B by Lemma 3.6. Lemma 3.7, or Gershgorin's theorem in the fallback, gives $\lambda \leq \min(0, \lambda_{\min}(\nabla^2 F(c)))$. Now Lemma 3.5 gives $F \geq L > \tau$ on B .

Hence $A > \tau$ on X , and Corollary 2.5 gives $M(D, B_{n_1}, \dots, B_{n_k}) \geq \tau$ and $\Lambda \geq \tau$. $\square$

4.5. Further checks. We ran further programs on the certificates. Table 1 lists the results together with the data of the certificates and of the checker runs. Except for the tree decoder `treecheck.py`, these programs were written separately from the checker and the search. None of them uses code from either program, except the negative tests, which call the checker.

The tree decoder `treecheck.py` and a second, independent decoder rebuild every box of the trees in exact rational arithmetic. They repeat the double-precision computations of the checker and confirm that the box verified at each leaf equals the exact leaf box and that the leaf volumes add up to the volume of X . The checker also has a self-test mode, in which it compares $F(c)$ computed through (3.1) with the direct sum of Definition 3.2 at every leaf.

An independent verifier uses a different bound. Its witnesses are support points of the hull at the centre of the box, with the arcs subdivided finely. For fixed rotations, every determinant $\det(x_j - z, x_{j+1} - z)$ and the fan area F are affine in each translation coordinate separately, so their minima over a box are attained at vertices of the box of translations. At a vertex they are trigonometric polynomials of low degree in the rotations. The verifier bounds their minima over the box of rotations from below in ball arithmetic. For one rotation it computes the minimum exactly. For two rotations it computes the minima on the edges of the box of rotations exactly and excludes interior minima by the signs of partial derivatives, after up to four rounds of subdivision of the box of rotations. If this test still fails, it bounds the minimum on 16 subintervals of the first angle in ball arithmetic, minimizing exactly in the second angle. It splits a leaf when the resulting bound for F does not exceed τ . It re-verified stratified samples of leaves, including the leaves with the smallest checker bounds and the leaves that contain numerical minimizers.

We also searched numerically for placements with small hull area. We evaluated A at all leaf centres and minimized A locally from many starting points, with the hull area computed from the support functions of the bodies and checked against inner and outer polygons. Finally, we tested the checker on inputs it must reject. At a leaf that contains a placement near a numerical minimizer, its bound passes τ but fails a target above the numerical minimum of the family. Root boxes with a shortened range, headers with a larger τ and modified trees are rejected.

5. LIMITS OF THE METHOD

We record some numerical observations, not used in the proofs, on how far the method can go.

A lower bound obtained from a family of test sets cannot exceed the least hull area of the family. Mishra [11, Section 3] exhibits a placement of D, B_3, B_5 and bounds its hull area from above by the area of a circumscribed polygon. This gives $M(D, B_3, B_5) \leq 0.834781191$. The bound 0.83474 of Hatchett [8] is therefore within 4.2×10^{-5} of the largest bound that these

TABLE 1. The certificates of Theorem 1.1 and their verification. Times are wall-clock times per process on a shared x86-64 server, and “total” is their sum over the parts.

	D, B_3, B_5	D, B_3, B_5, B_7
Parameters d ; target τ	5; 0.8347	8; 0.836
Half-widths $w_1, \dots, w_k$ of X	0.193613, 0.196856	0.198977, 0.201023, 0.199964
Grid cells; parts	4^5 ; 16	4^8 ; 24
Nodes; leaves (far leaves)	53,752; 27,388 (0)	1,552,274; 808,905 (0)
Largest depth below a grid cell	43	50
Smallest leaf bound L	0.834700004923	0.836000001625
Smallest root margin $f(g_{n_i}(w_i)) - \tau$	4.0×10^{-9}	4.5×10^{-9}
Search: processes; longest part; total	16; 308 s; 531 s	24; 987 s; 8,728 s
Checker: longest part; total	78 s; 141 s	810 s; 7,119 s
Tree decoder: boxes exact; leaves tile X	yes; yes	yes; yes
Self-test of (3.1) at all leaves	passed	passed
Independent verifier: leaves (split)	451 (7)	350 (7)
its margin at the leaf of smallest L	2.1×10^{-5}	5.7×10^{-4}
Smallest A : leaf centres; local search	0.8347836763; 0.8347809458	0.8365614; 0.8364904560
Negative tests: leaf near minimizer	passes 0.8347, fails 0.8348	passes 0.836, fails 0.8365
root, header and tree modifications	8 of 8 rejected	2 of 2 (root) rejected

three bodies can give, and the first bound of Theorem 1.1 within 8.2×10^{-5} . Local minimization found no placement with hull area below 0.8347809458 (Table 1).

For D, B_3, B_5, B_7 , Mishra located a placement with hull area 0.836494901 numerically [11, Section 6.3]. Local minimization finds a placement whose hull area, computed in floating point by integrating the support function of the hull in closed form, is 0.83649045598. Rounded to ten decimals, this placement is

$$t_1 = (-0.0025482897, -0.0019350278), \quad t_2 = (0.0250869718, -0.0032906511),$$

$$t_3 = (0.0028934171, -0.0127181545), \quad \rho_2 = 0.0287183302, \quad \rho_3 = 0.7016242528,$$

and the full values are in the file `opt8.point.json` of the archive. At the rounded placement the same computation gives 0.83649045600. There the polygon spanned by the support points in 2^{24} equally spaced directions has area 0.836490455978, and the polygon cut out by the corresponding support lines has area 0.836490465924. Numerically, therefore, the four bodies cannot give more than 0.8364904560. The second bound of Theorem 1.1 is within 5×10^{-4} of this value. For the disk and the regular Reuleaux 3-, 5-, 7- and 9-gons, which have eleven placement parameters, Gibbs’s annealing found the hull area 0.83699098, with the arcs replaced by polygons of diameter 1 [5]. This suggests that these five bodies cannot give much more than 0.837.

DATA AVAILABILITY

The certificates, the checker `lbcheck.py`, the tree decoder `treecheck.py`, the search programs, the placement and the program for the polygon bounds of Section 5, and the independent verifier, decoder, minimization and negative tests of Section 4.5 are archived at doi:10.5281/zenodo.23509974 and available at <https://github.com/tejasvi/lebesgue-cover-bound>. The two certificates are xz-compressed tar files of 17,308 and 487,316 bytes, which also contain lists of the SHA-256 hashes of their parts. Their own SHA-256 hashes are

2026ebe7e5339b89542e0429e5ad704b1a956278e3ae455b6d667b4c11892846 (D, B_3, B_5),

d8a7f09dc236a7c647147bacc3e903f3c9e2bd2cadb601d000d1a67a2a237681 (D, B_3, B_5, B_7) .

The checker needs Python 3 with NumPy and python-flint.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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