

AN UPPER BOUND FOR THE L^1 POINCARÉ CONSTANT OF THE HAMMING CUBE

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ABSTRACT. Let C be the smallest constant such that $\mathbb{E}|f - \mathbb{E}f| \leq C \mathbb{E}|\nabla f|$ for every n and every real function f on the Hamming cube $\{-1, 1\}^n$, where $|\nabla f|$ is the Euclidean length of $(D_1 f, \dots, D_n f)$ and $D_j f(x) = (f(x) - f(x^{(j)}))/2$. Ben Efraim and Lust-Piquard proved $C \leq \pi/2$. Ivanisvili, Li, van Handel and Volberg showed $C < \pi/2$, and Ivanisvili and Stone obtained $C \leq \pi/2 - 1.3 \cdot 10^{-4}$. The Gaussian case gives $C \geq \sqrt{\pi/2}$. We prove $C \leq 1.42$. By a Pisier-type inequality of Ivanisvili, van Handel and Volberg, $\mathbb{E}|f - \mathbb{E}f|$ is at most the average over the cube of $J(\nabla f)$, where $J(\lambda) = \int_0^{\pi/2} \mathbb{E}|\sum_j \lambda_j \xi_j| d\theta$ and $\xi_1, \xi_2, \dots$ are independent standardized biased random signs that take the positive value with probability $(1 + \sin \theta)/2$. Hence $C \leq K$, the supremum of J over unit vectors. Bounding the integrand separately for each θ , as earlier arguments do, cannot give less than 1.4217. We bound $\mathbb{E}|\sum_j \lambda_j \xi_j|$ by $\mathbb{E}h(\sum_j \lambda_j \xi_j)$ for even polynomials $h \geq |\cdot|$ of degree six. The bound depends on λ only through $\sum_j \lambda_j^3$, $\sum_j \lambda_j^4$ and $\sum_j \lambda_j^6$, so it does not depend on the dimension. A certificate with 1761 polynomials, 302 squares in the plane of $(\sum_j \lambda_j^4, \sum_j \lambda_j^6)$ and 1024 intervals of θ is verified in exact rational and ball arithmetic. We also show that $J(\lambda) < \sqrt{\pi/2}|\lambda|$ when all λ_j have the same sign, while $K \geq \sqrt{\pi/2}$. Thus $C = \sqrt{\pi/2}$ would follow from the inequality $J(\lambda) \leq J(|\lambda_1|, \dots, |\lambda_n|)$. The bound $C \leq 1.42$ and the bound $K \leq 1.42$ are also verified in Lean 4 with Mathlib, by the Lean kernel and with the standard axioms only.

1. INTRODUCTION

Let x be uniformly distributed on the Hamming cube $\{-1, 1\}^n$. For $f: \{-1, 1\}^n \rightarrow \mathbb{R}$ and $1 \leq j \leq n$ put

$$D_j f(x) = \frac{f(x) - f(x^{(j)})}{2}, \quad |\nabla f|(x) = \left(\sum_{j=1}^n D_j f(x)^2 \right)^{1/2},$$

where $x^{(j)}$ is obtained from x by changing the sign of the j th coordinate. Let C be the smallest constant such that

$$(1.1) \quad \mathbb{E}|f(x) - \mathbb{E}f(x)| \leq C \mathbb{E}|\nabla f|(x)$$

for all $n \geq 1$ and all f . This is the L^1 Poincaré constant of the Hamming cube. It is constant 11a in the repository of optimization constants [12].

Ben Efraim and Lust-Piquard [1] proved $C \leq \pi/2$ by a non-commutative argument. (Their gradient omits the factor $1/2$, and the bound appears there as $\pi/4$.) Several other proofs give the same constant [2]. Ivanisvili, Li, van Handel and Volberg [2] showed that $C < \pi/2$, with a gap that was not made explicit. R. van Handel obtained an explicit gap of about $6 \cdot 10^{-6}$ (see [3, Section 1.3]), and Ivanisvili and Stone [3, Corollary 3] proved $C \leq \pi/2 - 1.3 \cdot 10^{-4}$. In the other direction, the central limit theorem carries (1.1) over to Gauss space, where the sharp constant

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is $\sqrt{\pi/2}$ [10]. Hence $C \geq \sqrt{\pi/2} \approx 1.2533$ [2]. Whether $C = \sqrt{\pi/2}$ is open [2]. Ivanisvili and Stone [3, Remark 1.3] suggest that it is.

Theorem 1.1. $C \leq 71/50 = 1.42$.

The proof has two parts. The first reduces the problem to sums of independent random variables. For $p \in (0, 1)$ let $\xi^{(p)}$ be the random variable with

$$(1.2) \quad \mathbb{P}\left(\xi^{(p)} = \sqrt{\frac{1-p}{p}}\right) = p, \quad \mathbb{P}\left(\xi^{(p)} = -\sqrt{\frac{p}{1-p}}\right) = 1-p,$$

so that $\mathbb{E}\xi^{(p)} = 0$ and $\mathbb{E}(\xi^{(p)})^2 = 1$. For $\lambda \in \mathbb{R}^n$ put

$$(1.3) \quad J(\lambda) = \int_0^{\pi/2} \mathbb{E} \left| \sum_{j=1}^n \lambda_j \xi_j^{(p(\theta))} \right| d\theta, \quad p(\theta) = \frac{1 + \sin \theta}{2},$$

where $\xi_1^{(p)}, \dots, \xi_n^{(p)}$ are independent copies of $\xi^{(p)}$ (the integrand is defined for $\theta < \pi/2$), and let K be the supremum of $J(\lambda)$ over all n and all unit vectors $\lambda \in \mathbb{R}^n$. Ivanisvili, van Handel and Volberg [4, Theorem 1.2] proved a Pisier-type inequality on the cube. For real functions and the convex function $|\cdot|$ it states, after a change of variables, that

$$(1.4) \quad \mathbb{E}|f - \mathbb{E}f| \leq \mathbb{E}_y J(\nabla f(y)),$$

where y is uniform on the cube and $\nabla f(y) = (D_1 f(y), \dots, D_n f(y))$. Earlier arguments bound the inner expectation, for each value of p separately, by $\mathbb{E}|\nabla f|$ times $R(p)$, the supremum of $\mathbb{E}|\sum_j \lambda_j \xi_j^{(p)}|$ over unit vectors λ . This gives $C \leq \int_0^{\pi/2} R(p(\theta)) d\theta$. The route goes back to [2, Sections 2–5], where it is used in a dual form to prove $C < \pi/2$. Ivanisvili and Stone [3, Section 2.4] made it explicit with an upper bound for $R(p)$. We take the supremum over directions only after the integration over p . Since $J(a) \leq K|a|$, (1.4) gives $C \leq K$ (Theorem 2.2).

The second part bounds K . Let $S = \sum_j \lambda_j \xi_j^{(p(\theta))}$. On each of finitely many intervals of θ we bound $\mathbb{E}|S|$ either by 1 or by $\mathbb{E}h(S)$ for an even polynomial $h \geq |\cdot|$ of degree six. The moments $\mathbb{E}S^4$ and $\mathbb{E}S^6$ depend on λ only through the power sums $P_r = \sum_j \lambda_j^r$, $r = 3, 4, 6$. (Ivanisvili and Stone also used fourth and sixth moments, in their bound for $R(p)$ [3, Section 2.3]. Here $\mathbb{E}S^6$ is computed exactly and the polynomials h are chosen by linear programming.) The power sums lie in an explicit region that does not depend on n (Section 3). A finite certificate covers this region and the interval $[0, \pi/2]$. It consists of 1761 polynomials, 302 squares in the (P_4, P_6) -plane and 1024 intervals of θ . Two programs, which use different methods for the main checks, verify it in exact rational and ball arithmetic (Section 4). Theorem 1.1, the first inequality of Theorem 2.2 and Theorem 4.2 are also verified in Lean 4 with Mathlib (Section 4).

The order of the supremum and the integral matters. With the supremum inside, the bound is $\int_0^{\pi/2} R(p(\theta)) d\theta$, and we show in Remark 5.2 that this integral is at least 1.42177. Hence no bound for $R(p)$ alone can reach 1.42. This agrees with the value $\pi/2 - 0.149$ that Ivanisvili and Stone [3, Remark 1.3] computed under the assumption that the supremum defining $R(p)$ is attained at equal weights. The constant K also lies between two quantities that appear in the dual approach of [2, Section 11], and $C \leq K$ can be derived from that approach as well (Remark 2.4). Integrals of the same kernel with a fixed vector appear in [5, Section 6], where they bound fractional spectral moments of bounded functions.

The method has two limitations. First, $K \geq \sqrt{\pi/2}$ (Remark 2.3), so the reduction cannot give a bound below $\sqrt{\pi/2}$. Second, numerical computations show that the moment bounds of Section 3 cannot prove $K < 1.4160$ (Remark 5.1). In Section 6 we find an exact formula for J on vectors with nonnegative coordinates and deduce that $J(\lambda) < \sqrt{\pi/2} |\lambda|$ for every nonzero

λ whose coordinates have one sign (Theorem 6.5 and Corollary 6.6). Thus $C = \sqrt{\pi/2}$ would follow from the inequality $J(\lambda) \leq J(|\lambda_1|, \dots, |\lambda_n|)$ for all λ .

2. THE REDUCTION

We keep the notation of the introduction. The function J is defined by (1.3) on all of $\mathbb{R}^n$. It satisfies $J(c\lambda) = |c|J(\lambda)$, and it does not change if zero coordinates are appended to λ . Put

$$K_n = \sup\{J(\lambda) : \lambda \in \mathbb{R}^n, |\lambda| = 1\}, \quad K = \sup_{n \geq 1} K_n,$$

where $|\cdot|$ is the Euclidean norm. Since $J(\lambda) \leq \pi/2$ for unit vectors (Remark 2.3), $K \leq \pi/2$.

For $t > 0$ let $\rho = e^{-t}$, and let $\zeta = \zeta(t) = (\zeta_1, \dots, \zeta_n)$ be independent of x , with independent coordinates and $\mathbb{P}(\zeta_j = \pm 1) = (1 \pm \rho)/2$. Write $x\zeta = (x_1\zeta_1, \dots, x_n\zeta_n)$, $T_t f(x) = \mathbb{E}_\zeta f(x\zeta)$ and

$$\delta_j = \delta_j(t) = \frac{\zeta_j - \rho}{\sqrt{1 - \rho^2}}.$$

In [4] the vector $\zeta(t)$ is denoted $\xi(t)$, and $\delta_j(t)$ has the same name. The following identity is due to Ivanisvili, van Handel and Volberg [4, Lemma 2.1 and Remark 2.2], who credit the direct derivation through the derivative of $T_t f$ to J. Liu. Ivanisvili and Stone used it in [3, (2.5)]. We include the short proof.

Lemma 2.1. *For every $f: \{-1, 1\}^n \rightarrow \mathbb{R}$ and every $x \in \{-1, 1\}^n$,*

$$f(x) - \mathbb{E}f = \int_0^\infty \frac{1}{\sqrt{e^{2t} - 1}} \mathbb{E}_\zeta \left[\sum_{j=1}^n \delta_j(t) D_j f(x\zeta(t)) \right] dt,$$

and the integral converges absolutely.

Proof. Let $\pi_t(\pm 1) = (1 \pm e^{-t})/2$. Then $T_t f(x) = \sum_{z \in \{-1, 1\}^n} \prod_j \pi_t(z_j) f(xz)$ is a polynomial in e^{-t} , with $T_0 f = f$ and $T_t f(x) \rightarrow \mathbb{E}f$ as $t \rightarrow \infty$. Since $\frac{d}{dt} \pi_t(\pm 1) = \mp e^{-t}/2$,

$$\frac{d}{dt} T_t f(x) = -\frac{e^{-t}}{2} \sum_{j=1}^n \mathbb{E}[f(x\zeta^{j,+}) - f(x\zeta^{j,-})],$$

where $\zeta^{j,\pm}$ is ζ with ζ_j replaced by ± 1 . Checking the two values of ζ_j separately gives $f(x\zeta^{j,+}) - f(x\zeta^{j,-}) = 2\zeta_j D_j f(x\zeta)$. The left side does not depend on ζ_j . Since $\zeta_j^2 = 1$ and ζ_j is independent of the other coordinates,

$$\mathbb{E}[\delta_j D_j f(x\zeta)] = \mathbb{E}[\zeta_j \delta_j \cdot \zeta_j D_j f(x\zeta)] = \mathbb{E}[\zeta_j \delta_j] \cdot \frac{1}{2} \mathbb{E}[f(x\zeta^{j,+}) - f(x\zeta^{j,-})],$$

and $\mathbb{E}[\zeta_j \delta_j] = (1 - \rho^2)/\sqrt{1 - \rho^2} = \sqrt{1 - \rho^2}$. Hence

$$\frac{d}{dt} T_t f(x) = -\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \sum_{j=1}^n \mathbb{E}[\delta_j D_j f(x\zeta)] = -\frac{1}{\sqrt{e^{2t} - 1}} \sum_{j=1}^n \mathbb{E}[\delta_j D_j f(x\zeta)].$$

Integrating from 0 to τ and letting $\tau \rightarrow \infty$ gives the identity. For the absolute convergence, $\mathbb{E}|\delta_j| = (\frac{1+\rho}{2}(1 - \rho) + \frac{1-\rho}{2}(1 + \rho))/\sqrt{1 - \rho^2} = \sqrt{1 - \rho^2}$, so the integrand is at most $e^{-t} \sum_j \max |D_j f|$ in absolute value. $\square$

Theorem 2.2. *For every n and every $f: \{-1, 1\}^n \rightarrow \mathbb{R}$,*

$$\mathbb{E}|f - \mathbb{E}f| \leq \mathbb{E}_y J(\nabla f(y)) \leq K_n \mathbb{E}|\nabla f|,$$

where y is uniform on $\{-1, 1\}^n$. Consequently $C \leq K$.

The first inequality is the case $X = \mathbb{R}$, $\Phi = |\cdot|$ of the Pisier-type inequality [4, Theorem 1.2], after the substitution $e^{-t} = \sin \theta$. In the variable p it is the first step of [3, Section 2.4]. We include the short proof. The second inequality holds because $J(a) = |a| J(a/|a|) \leq K_n |a|$ for $a \neq 0$.

Proof. By Lemma 2.1, the triangle inequality and Tonelli's theorem,

$$\mathbb{E}|f - \mathbb{E}f| \leq \int_0^\infty \frac{1}{\sqrt{e^{2t} - 1}} \mathbb{E}_{x,\zeta} \left| \sum_{j=1}^n \delta_j(t) D_j f(x\zeta(t)) \right| dt.$$

Fix t . For each value of ζ the map $x \mapsto x\zeta$ preserves the uniform measure, so the pair $(x\zeta, \zeta)$ has the law of (y, ζ) with y uniform and independent of ζ . Since δ_j is a function of ζ_j ,

$$\mathbb{E}_{x,\zeta} \left| \sum_j \delta_j D_j f(x\zeta) \right| = \mathbb{E}_y \mathbb{E}_\zeta \left| \sum_j \delta_j a_j(y) \right|, \quad a(y) = \nabla f(y).$$

Put $p = (1 + \rho)/2$. Then $1 - \rho = 2(1 - p)$, $1 + \rho = 2p$ and $1 - \rho^2 = 4p(1 - p)$, so $\delta_j = \sqrt{(1 - p)/p}$ with probability p and $\delta_j = -\sqrt{p/(1 - p)}$ with probability $1 - p$. Thus $\delta_1, \dots, \delta_n$ are independent copies of $\xi^{(p)}$. By Tonelli's theorem again, now exchanging $\mathbb{E}_y$ with the integral over t ,

$$\mathbb{E}|f - \mathbb{E}f| \leq \mathbb{E}_y \int_0^\infty \frac{1}{\sqrt{e^{2t} - 1}} \mathbb{E} \left| \sum_j a_j(y) \xi_j^{((1+e^{-t})/2)} \right| dt.$$

Substitute $e^{-t} = \sin \theta$ with $\theta \in (0, \pi/2)$. Then $dt = -(\cos \theta / \sin \theta) d\theta$ and $(e^{2t} - 1)^{-1/2} = \sin \theta / \cos \theta$, so $(e^{2t} - 1)^{-1/2} dt = -d\theta$. Also $(1 + e^{-t})/2 = p(\theta)$, and $t \in (0, \infty)$ corresponds to $\theta \in (0, \pi/2)$. Hence the integral over t equals $J(a(y))$. This is the first inequality, and the second was explained above. Finally $C = \sup_n \sup_f \mathbb{E}|f - \mathbb{E}f| / \mathbb{E}|\nabla f| \leq \sup_n K_n = K$. $\square$

Remark 2.3. (i) $\mathbb{E}|\xi^{(p)}| = 2\sqrt{p(1-p)} = \cos \theta$ for $p = p(\theta)$, so $J(e_1) = \int_0^{\pi/2} \cos \theta d\theta = 1$. For a unit vector λ , $\mathbb{E}|S| \leq (\mathbb{E}S^2)^{1/2} = 1$, so $J(\lambda) \leq \pi/2$. This recovers $C \leq \pi/2$.

(ii) $K_n \leq K_{n+1}$, and K_n is attained, since J is continuous on the unit sphere by dominated convergence.

(iii) $K \geq \sqrt{\pi/2}$. Let $\lambda = (1, \dots, 1)/\sqrt{n}$ and fix $\theta \in [0, \pi/2)$. By the central limit theorem, S converges in law to a standard Gaussian g as $n \rightarrow \infty$. Since $\mathbb{E}S^2 = 1$, the family $\{S\}$ is uniformly integrable, and $\mathbb{E}|S| \rightarrow \mathbb{E}|g| = \sqrt{2/\pi}$. By dominated convergence (with the bound $\mathbb{E}|S| \leq 1$), $J(\lambda) \rightarrow (\pi/2)\sqrt{2/\pi} = \sqrt{\pi/2}$. Theorem 2.2 therefore cannot give a bound below $\sqrt{\pi/2}$, and $K = \sqrt{\pi/2}$ would imply $C = \sqrt{\pi/2}$.

Remark 2.4. We compare Theorem 2.2 with earlier arguments.

(i) Let $R(p) = \sup_n \sup_{|\lambda|=1} \mathbb{E} \left| \sum_j \lambda_j \xi_j^{(p)} \right|$. Bounding $\mathbb{E}_{x,\zeta} \left| \sum_j \delta_j D_j f(x\zeta) \right|$ by $R((1 + e^{-t})/2) \mathbb{E}|\nabla f|$ for each t gives $C \leq \int_0^{\pi/2} R(p(\theta)) d\theta$. This is the route of [3, Section 2.4], which uses the explicit upper bound [3, (1.8)] for $R(p)$. In the dual form of (ii) below it is the route of [2], which shows that $R(p) < 1$ for p near $3/4$ [2, Theorem 3.1]. The integral is at least K , and at least 1.42177 by Remark 5.2.

(ii) The inequality $\mathbb{E}|f - \mathbb{E}f| \leq K_n \mathbb{E}|\nabla f|$ also follows from the duality argument of [2, Section 2]. By Lemma 2.1 and [4, Lemma 2.1], $f - \mathbb{E}f = \int_0^\infty \sum_j D_j T_t D_j f dt$ (see [4, Remark 2.2]). Since D_j and T_t are self-adjoint and commute, $\mathbb{E}[(f - \mathbb{E}f)g] = \mathbb{E} \sum_j D_j f \int_0^\infty D_j T_t g dt$ for every g . By [4, Lemma 2.1], $D_j T_t g(x) = (e^{2t} - 1)^{-1/2} \mathbb{E}[\delta_j(t) g(x\zeta(t))]$. Hence, if $|g| \leq 1$ and λ is a unit vector, then $\sum_j \lambda_j \int_0^\infty D_j T_t g(x) dt \leq J(\lambda) \leq K_n$ for every x , and so $\mathbb{E}[(f - \mathbb{E}f)g] \leq K_n \mathbb{E}|\nabla f|$. Taking the supremum over all g with $|g| \leq 1$ gives the inequality. The same argument shows that the dual constant $C_{\text{dual},n} = \sup_{|g| \leq 1} \max_x \left| \int_0^\infty \nabla T_t g(x) dt \right|$ of [2, (12.1)] is at most K_n . (The

operators ∂_k used there satisfy $|\partial_k g| = |D_k g|$, so the Euclidean norms agree.) It equals the quantity [2, (11.11)]. In dimension n , the quantity [2, (11.12)] equals $\int_0^{\pi/2} R_n(p(\theta)) d\theta$, where $R_n(p)$ is the supremum of $\mathbb{E}|\sum_j \lambda_j \xi_j^{(p)}|$ over unit vectors of $\mathbb{R}^n$. Indeed, in (11.12) the expectation over z uniform on $\{-1, 1\}^n$ carries the weight $(1+\rho)^{n-d(z)}(1-\rho)^{d(z)} = \prod_k (1+\rho z_k)$, where $d(z)$ is the number of coordinates equal to -1 . With this weight it is the expectation over z with independent coordinates and $\mathbb{P}(z_k = 1) = (1+\rho)/2$. For such z , $(z_i - \rho)/(1 - \rho^2) = \delta_i/\sqrt{1 - \rho^2}$, where $\delta_i = (z_i - \rho)/\sqrt{1 - \rho^2}$ has the law of $\xi^{(p)}$ with $p = (1+\rho)/2$. Finally $d\rho/\sqrt{1 - \rho^2} = d\theta$ for $\rho = \sin \theta$. Thus K_n is obtained from (11.12) by taking the maximum over λ outside the integral over ρ . In particular, Theorem 4.2 gives $C_{\text{dual},n} \leq 71/50$ for every n . Ivanisvili, Li, van Handel and Volberg computed $C_{\text{dual},n}$ numerically for small n and remarked that a linear extrapolation of their values would reach $\pi/2$ near $n = 60$ [2, Section 12]. By the bound just stated, the values stay below 1.42.

(iii) Ivanisvili, Xie and Zhang [5, Section 6] use the function $\Psi_s(a) = (e^{2s} - 1)^{-1/2} \mathbb{E}|\sum_i a_i \delta_i(s)|$ and the integrals $\Gamma_\alpha(a) = \int_0^\infty s^{-\alpha} \Psi_s(a) ds$ with the vector a held fixed, $0 < \alpha < 1$, to bound fractional spectral moments $\sum_A |A|^\alpha \hat{f}(A)^2$ of functions with values in $[-1, 1]$. In this notation $J(a) = \int_0^\infty \Psi_s(a) ds$ corresponds to $\alpha = 0$.

3. MOMENT BOUNDS

We fix $\theta \in [0, \pi/2)$, put $p = p(\theta)$ and

$$q = p(1-p) = \frac{\cos^2 \theta}{4} \in \left(0, \frac{1}{4}\right],$$

and write $\xi = \xi^{(p)}$ and $\mu_r = \mathbb{E}\xi^r$. For a unit vector $\lambda \in \mathbb{R}^n$ let $S = \sum_j \lambda_j \xi_j$ and $P_r = \sum_j \lambda_j^r$. All bounds below depend on λ only through P_3 , P_4 and P_6 , and not on n .

Lemma 3.1. $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3^2 = (1-4q)/q$, $\mu_4 = (1-3q)/q$ and $\mu_6 = (1-5q+5q^2)/q^2$.

Proof. Let $a = \sqrt{(1-p)/p}$ and $b = -\sqrt{p/(1-p)}$, so that $ab = -1$, $a^2 = (1-p)/p$ and $b^2 = p/(1-p)$. Then $\mu_1 = pa + (1-p)b = 0$, $\mu_2 = (1-p) + p = 1$,

$$\mu_3 = pa^3 + (1-p)b^3 = \frac{(1-p)^2 - p^2}{\sqrt{q}} = \frac{1-2p}{\sqrt{q}}, \quad \mu_4 = \frac{(1-p)^3 + p^3}{q}, \quad \mu_6 = \frac{(1-p)^5 + p^5}{q^2}.$$

With $p + (1-p) = 1$ and $p(1-p) = q$ we have $(1-2p)^2 = 1-4q$, $p^3 + (1-p)^3 = 1-3q$ and $p^5 + (1-p)^5 = 1-5q+5q^2$. $\square$

The formula for $\mathbb{E}S^4$ in the next lemma appears in [3, Section 2.3].

Lemma 3.2. $\mathbb{E}S^2 = 1$,

$$\mathbb{E}S^4 = 3 + \kappa_4 P_4, \quad \mathbb{E}S^6 = 15 + 15\kappa_4 P_4 + 10\mu_3^2 P_3^2 + \kappa_6 P_6,$$

where $\kappa_4 = \mu_4 - 3 = (1-6q)/q$ and $\kappa_6 = \mu_6 - 15\mu_4 + 10\mu_3^2 + 30 = (1-30q+120q^2)/q^2$.

Proof. The variables $X_j = \lambda_j \xi_j$ are independent with mean zero. When $(\sum_j X_j)^4$ and $(\sum_j X_j)^6$ are expanded, every term in which some index occurs exactly once has expectation zero. Grouping the other terms by the partition of the positions into blocks of equal indices, and using $P_2 = 1$,

$$\mathbb{E}\left(\sum_j X_j\right)^4 = \sum_j \mathbb{E}X_j^4 + 3 \sum_{i \neq j} \mathbb{E}X_i^2 \mathbb{E}X_j^2 = \mu_4 P_4 + 3(1 - P_4),$$

$$\begin{aligned}\mathbb{E}\left(\sum_j X_j\right)^6 &= \sum_j \mathbb{E}X_j^6 + 15 \sum_{i \neq j} \mathbb{E}X_i^4 \mathbb{E}X_j^2 + 10 \sum_{i \neq j} \mathbb{E}X_i^3 \mathbb{E}X_j^3 + 15 \sum_{i,j,k \text{ distinct}} \mathbb{E}X_i^2 \mathbb{E}X_j^2 \mathbb{E}X_k^2 \\ &= \mu_6 P_6 + 15\mu_4(P_4 - P_6) + 10\mu_3^2(P_3^2 - P_6) + 15(1 - 3P_4 + 2P_6).\end{aligned}$$

Collecting terms gives the two formulas, and the closed forms of κ_4 and κ_6 follow from Lemma 3.1. $\square$

Lemma 3.3. *Let $h(z) = a + bz^2 + cz^4 + dz^6$ satisfy $h(z) \geq |z|$ for all real z . Then*

$$\mathbb{E}|S| \leq a + b + c\mathbb{E}S^4 + d\mathbb{E}S^6.$$

Moreover $h(z) \geq |z|$ for all real z if and only if $H(u) = a - u + bu^2 + cu^4 + du^6 \geq 0$ for all $u \geq 0$.

Proof. Take expectations in $|S| \leq h(S)$ and use $\mathbb{E}S^2 = 1$. The second statement holds because h is even. $\square$

Lemma 3.4. *(P_4, P_6) lies in $\mathcal{R} = \{(s, v) : 0 \leq s \leq 1, s^2 \leq v \leq s^{3/2}\}$, and $0 \leq P_3^2 \leq P_4$.*

Proof. $0 \leq P_4 \leq P_2^2 = 1$. By the Cauchy–Schwarz inequality, $P_4^2 = (\sum_j \lambda_j \cdot \lambda_j^3)^2 \leq P_2 P_6 = P_6$ and $P_3^2 = (\sum_j \lambda_j \cdot \lambda_j^2)^2 \leq P_2 P_4 = P_4$. Finally $\max_j \lambda_j^4 \leq P_4$, so $P_6 \leq \max_j \lambda_j^2 \cdot P_4 \leq P_4^{1/2} P_4$. $\square$

4. THE CERTIFICATE

We now describe the finite object that proves $K \leq 71/50$ and how it is checked. A *dyadic square* is a square $[i_1 2^{-l}, (i_1 + 1) 2^{-l}] \times [i_2 2^{-l}, (i_2 + 1) 2^{-l}] \subset [0, 1]^2$ with integers $l, i_1, i_2 \geq 0$, in the coordinates (P_4, P_6) . Finitely many squares *tile* $[0, 1]^2$ if their union is $[0, 1]^2$ and their interiors are disjoint. Fix $N \geq 1$, let $\theta_i = i\pi/(2N)$ and $I_i = [\theta_i, \theta_{i+1}]$ for $0 \leq i < N$, and let $Q_i = [\cos^2 \theta_{i+1}/4, \cos^2 \theta_i/4]$, which is the range of q on I_i . For a polynomial $h = (a, b, c, d)$ as in Lemma 3.3, a square $B = [s_0, s_1] \times [v_0, v_1]$ and an index $i < N - 1$, let $U(h, B, i)$ be the supremum of

$$a + b + c(3 + \kappa_4 s) + d(15 + 15\kappa_4 s + 10\mu_3^2 r + \kappa_6 v)$$

over $q \in Q_i$, $s \in [s_0, s_1]$, $v \in [\max(v_0, s_0^2), \min(v_1, s_1^{3/2})]$ and $r \in [0, s_1]$ (the supremum of the empty set is $-\infty$), where μ_3^2 , κ_4 and κ_6 are the functions of q in Lemmas 3.1 and 3.2. We also put $U(*, B, i) = 1$ for a symbol $*$, which stands for the trivial bound $\mathbb{E}|S| \leq 1$.

Proposition 4.1. *Let $\gamma > 0$. Suppose that there are a finite list $\mathcal{L}$ of polynomials h with $h(z) \geq |z|$ on $\mathbb{R}$, a finite set of leaf squares and a finite set of outside squares, and for each leaf B and each i an element $h_{B,i}$ of $\mathcal{L} \cup \{*\}$, with $h_{B,N-1} = *$, such that*

- (a) *the leaf and outside squares are dyadic squares whose union is $[0, 1]^2$;*
- (b) *every outside square is disjoint from $\mathcal{R}$;*
- (c) *for every leaf B , $\sum_{i=0}^{N-1} |I_i| \min(1, U(h_{B,i}, B, i)) \leq \gamma$.*

Then $K \leq \gamma$.

Proof. Let λ be a unit vector. By Lemma 3.4, $(P_4, P_6) \in \mathcal{R}$. By (a) and (b), some leaf $B = [s_0, s_1] \times [v_0, v_1]$ contains (P_4, P_6) . Then $s_0^2 \leq P_4^2 \leq P_6 \leq P_4^{3/2} \leq s_1^{3/2}$ and $0 \leq P_3^2 \leq P_4 \leq s_1$. Let $\theta \in I_i$. If $h_{B,i} = *$, then $\mathbb{E}|S| \leq 1$. Otherwise $i < N - 1$, so $q \in Q_i$, and Lemmas 3.2 and 3.3 give $\mathbb{E}|S| \leq U(h_{B,i}, B, i)$, while also $\mathbb{E}|S| \leq 1$. Hence $\int_{I_i} \mathbb{E}|S| d\theta \leq |I_i| \min(1, U(h_{B,i}, B, i))$, and summing over i gives $J(\lambda) \leq \gamma$ by (c). $\square$

Theorem 4.2. $K \leq 71/50$.

Proof. The certificate file `mom_c142.json` contains data satisfying (a)–(c) of Proposition 4.1 with $\gamma = 71/50$ and $N = 1024$. It has 1761 polynomials, 302 leaves and 149 outside squares. The checker `check11a_mom.py` verifies that every polynomial of the list satisfies $h(z) \geq |z|$, that $h_{B,N-1} = *$ for every leaf, and (a)–(c), as described below. $\square$

Theorem 1.1 follows from Theorems 2.2 and 4.2.

Verification. The checker reads the certificate and verifies (a)–(c) in exact rational arithmetic and in ball arithmetic with the Arb library [6] through python-flint, at a precision of 200 bits.

- *Polynomials.* For each h in the list the checker confirms $d > 0$ and $H(0) = a > 0$, and computes the Sturm sequence of H in rational arithmetic. Equal numbers of sign changes at 0^+ and at $+\infty$ mean that H has no root in $(0, \infty)$. Hence $H > 0$ on $[0, \infty)$, and $h(z) \geq |z|$ by Lemma 3.3.
- *Tiling.* Each square is given by its triple (l, i_1, i_2) . Starting from $[0, 1]^2$, the checker subdivides every square that is not listed, and confirms that the process stops and reaches every listed square. This proves (a).
- *Outside squares.* A square $[s_0, s_1] \times [v_0, v_1]$ is disjoint from $\mathcal{R}$ if $v_1 < s_0^2$ or $v_0^2 > s_1^3$. This is checked in rational arithmetic and proves (b).
- *Bounds.* For each leaf and each i with $h_{B,i} \neq *$, the checker encloses Q_i by the ball hull of its endpoints, confirms that this ball does not contain 0, and evaluates the expression defining U in ball arithmetic over the boxes of q, s, v and r . Since Q_{N-1} contains 0, this also forces $h_{B,N-1} = *$. The upper end of the resulting ball bounds $U(h_{B,i}, B, i)$ from above. The checker forms the sums in (c) with these upper bounds and confirms that each of them is at most γ .

The checker also confirms Lemma 3.1 at 39 rational values of p , and Lemma 3.2, in a form for vectors that are not normalized, on random rational vectors of dimension at most 4, by exact enumeration. It accepts the certificate in under 10 seconds on one core. The largest of the 302 upper bounds that it computes for the sums in (c) is 1.4198937....

A second program, `check11a_indep.py`, uses different methods for the three main checks. It confirms that each H is positive on $[0, \infty)$ by certified isolation of the complex roots of the integer-scaled polynomial, and it checks the tiling by testing that the listed squares are distinct dyadic squares in $[0, 1]^2$, that none contains another, and that their areas add up to 1. It bounds U differently. For fixed q the expression is affine in (s, r, v) , so its maximum over the box is attained at a vertex. Since the coefficients are known only up to balls, the program takes the larger of the values at the two ends of each range, and only the dependence on q is enclosed, on four subintervals of each I_i . Where the range of q reaches 0, it uses the trivial bound, which is valid there, so it does not need $h_{B,N-1} = *$. It also accepts the certificate. Its bounds for the sums are smaller, and the largest of them is 1.4184488.... As a test of the two programs we ran both on perturbed certificates: one claiming the value 1.418, one with a polynomial that takes a negative value on $[0, \infty)$, one with a leaf missing, one with a leaf declared outside although it meets $\mathcal{R}$, and one in which all polynomials of a leaf are replaced by the first polynomial of the list. Both programs rejected all of them. The certificate data are summarized in Table 1.

Formal verification. Theorem 1.1, in the form $\mathbb{E}|f - \mathbb{E}f| \leq 71/50 \mathbb{E}|\nabla f|$ for all n and all f , the first inequality of Theorem 2.2, and Theorem 4.2, in the form $J(\lambda) \leq 71/50$ for all unit vectors λ with J defined by the integral (1.3), are also proved in Lean 4 [9] with Mathlib [7] (Lean version 4.35.0-rc3, Mathlib commit c55e6e786f). The proofs are checked by the Lean kernel and use only the standard axioms (`propext`, `Classical.choice` and `Quot.sound`). All computations with the certificate are evaluated by the kernel.

The certificate enters the formal proofs with a few changes for exact arithmetic. The endpoints of the intervals of θ are $2 \arctan t$ with dyadic t , so that their sines and cosines are rational.

TABLE 1. The certificate for $K \leq 71/50$.

Polynomials: in the list; used	1761; 638
Leaf squares; outside squares	302; 149
Levels l of the leaf squares	2–11
Intervals of θ	$N = 1024$
Intervals with $h_{B,i} = *$, per leaf	4–35
Largest computed bound for a leaf sum: checker; second program	1.4198937; 1.4184488

They differ from the points θ_i by less than 10^{-9} , and each interval keeps the polynomial that the certificate assigns to it. The lengths of the intervals enter only through $\sum_i |I_i| = \pi/2$, the inequality $\sin |I_i| \leq |I_i|$ and the bound $\pi < 3.141593$. For each square and interval the bound is maximized exactly over a box with dyadic corners that contains the ranges of $1/q$, P_3^2 , P_4 and P_6 . Since the bound is affine in P_3^2 , P_4 , P_6 and convex in $1/q$, the maximum is attained at a corner. Positivity of H is checked by subdivision of Bernstein coefficients instead of Sturm sequences.

In the formal proof, Theorem 1.1 is derived from Theorems 2.2 and 4.2, as in the paper. A second formal proof of Theorem 1.1 does not use Theorem 2.2. It applies the mean value theorem, on each interval of the partition, to the semigroup of Lemma 2.1 in the variable θ , and combines the result with the bounds of the certificate.

How the certificate was found. This part is not needed for the proof. For 56 values of m_4 and, for each of them, 45 values of m_6 we solved the linear program of minimizing $a+b+cm_4+dm_6$ subject to $H(u) \geq 0$ on a grid of 3601 points in $[0, 80]$. We rounded b, c, d to multiples of 2^{-40} . For a we took the maximum of $u - bu^2 - cu^4 - du^6$ over a grid in $[0, 200]$, added 10^{-10} , rounded up to a multiple of 2^{-40} , and added 10^{-8} up to five times until the exact Sturm test passed. This gave the list of 1761 polynomials. A quadtree search over $[0, 1]^2$, run in floating point, then chose for every square and every interval the polynomial with the smallest bound, and subdivided squares whose sum exceeded $\gamma - 2 \cdot 10^{-4}$. Squares disjoint from $\mathcal{R}$ were set aside as outside squares.

5. LIMITS OF THE METHOD

Remark 5.1 is numerical, and Remark 5.2 is a certified computation. Neither is used in the proof of Theorem 1.1.

Remark 5.1. Fix θ , m_4 and m_6 . If h is as in Lemma 3.3 and X is a symmetric random variable with $\mathbb{E}X^2 = 1$, $\mathbb{E}X^4 = m_4$ and $\mathbb{E}X^6 = m_6$, then $a + b + cm_4 + dm_6 = \mathbb{E}h(X) \geq \mathbb{E}|X|$. We take for X the symmetric variable such that X^2 takes the two values given by the nodes of the two-point Gauss quadrature for the moment sequence $1, 1, m_4, m_6$. It exists when $m_4 > 1$ and $m_6 > m_4^2$, which holds at the points used below. Then $\mathbb{E}|X|$ is a lower bound for every bound that Lemma 3.3 can give, and direct solutions of the linear program show numerically that it is the best such bound. Let $M(P_4, P_6)$ be the integral over θ of the minimum of 1 and $\mathbb{E}|X|$, with m_4 and m_6 given by Lemma 3.2 and $P_3^2 = P_4$, the largest value allowed by Lemma 3.4. On a grid over all of $\mathcal{R}$, the largest value of M is 1.4160, attained near $P_4 = 0.02$ on the boundary $P_6 = P_4^{3/2}$. Consequently the method of Section 4 cannot prove $K < 1.4160$, however fine the subdivision. With the list of polynomials and the 1024 intervals used here, the search succeeds for $\gamma = 1.42$. For $\gamma = 1.418$ it reaches its depth limit, level 22, at the square with triple $(22, 187218, 39554)$, near $(P_4, P_6) = (0.0446, 0.0094)$, where M is only 1.4137. (The search requires each leaf sum to be at most $\gamma - 2 \cdot 10^{-4}$ in floating point.) That failure therefore

comes from the finite list, the finite number of intervals and this margin. Near the boundary $P_6 = P_4^{3/2}$, unit vectors have one coordinate with λ_j^4 close to P_4 and the other fourth powers small, so P_3^2 is close to $P_4^{3/2}$ there and the bound $P_3^2 \leq P_4$ is far from tight. A joint description of the possible values of (P_3, P_4, P_6) might lower this ceiling.

Remark 5.2. The bound $\int_0^{\pi/2} R(p(\theta)) d\theta$ of Remark 2.4(i) cannot reach 1.42. Indeed, $R(p) \geq \mathbb{E}|n^{-1/2} \sum_{j \leq n} \xi_j^{(p)}|$ for every n . With $w = 1 - p$, the sum $\sum_{j \leq n} \xi_j^{(p)}$ equals $(nw - Y)/\sqrt{w(1 - w)}$, where Y is binomial with parameters n and w . If $nw \leq 1$, the only value of Y below its mean is 0, so

$$\mathbb{E} \left| n^{-1/2} \sum_{j \leq n} \xi_j^{(p)} \right| = \frac{2nw(1 - w)^n}{\sqrt{nw(1 - w)}} = 2\sqrt{nw} (1 - w)^{n-1/2}.$$

By a theorem of Stone [11, Theorem 4.3.1], $n = \lfloor 1/(2w - w^2) \rfloor$ maximizes the left side over n , but for a lower bound any n with $nw \leq 1$ may be used. We split $[0, \pi/2 - 10^{-6}]$ into 400,000 equal intervals and take on each interval this n at the left endpoint, where w is largest, so that $nw \leq 1$ on the whole interval. The right side is a unimodal function of w , so its minimum over an interval is attained at an endpoint. Evaluating at the endpoints in ball arithmetic gives the certified bound $\int_0^{\pi/2} R(p(\theta)) d\theta \geq 1.42177$. A midpoint rule for the integral of the equal-weight maximum gives 1.42178, in agreement with [3, Remark 1.3].

The same limit applies to any bound $\mathbb{E}_{x, \zeta} |\sum_j \delta_j(t) D_j f(x\zeta)| \leq c(t) \mathbb{E} |\nabla f|$ that holds for all n and all f , whether or not it is obtained through R . Let $n = 2m - 1$ and let f be the indicator of $\{x_1 + \dots + x_n \geq 1\}$. At a point y with $y_1 + \dots + y_n = \pm 1$, the gradient $\nabla f(y)$ equals $\pm \frac{1}{2}$ times the indicator vector of the m coordinates of y equal to ± 1 , and at all other points it vanishes. As in the proof of Theorem 2.2, the left side then equals $\mathbb{E} |\nabla f| \cdot \mathbb{E} |m^{-1/2} \sum_{j \leq m} \xi_j^{(p)}|$ with $p = (1 + e^{-t})/2$. Hence $c(t)$ is at least the equal-weight value for every m , and $\int_0^\infty c(t)(e^{2t} - 1)^{-1/2} dt \geq 1.42177$.

6. DIRECTIONS WITH COORDINATES OF ONE SIGN

By Remark 2.3(iii) and Theorem 4.2, $\sqrt{\pi/2} \leq K \leq 1.42$. In this section we give exact formulas for $J(\lambda)$ when all coordinates of λ are nonnegative (Theorem 6.5). We deduce that $J(\lambda) < \sqrt{\pi/2} |\lambda|$ for every nonzero λ whose coordinates are all nonnegative or all nonpositive, and that the constant $\sqrt{\pi/2}$ cannot be lowered (Corollary 6.6). Among unit vectors in $[0, \infty)^n$, the maximum of J is attained at equal weights (Corollary 6.7).

Throughout this section, $\varphi_X(t) = \mathbb{E} e^{itX}$ is the characteristic function of a real random variable X , and $\mathbf{1}_n = (1, \dots, 1) \in \mathbb{R}^n$. For $\alpha \in (0, \pi/2)$ let ξ^α be the variable $\xi^{(p)}$ of (1.2) with $p = \cos^2 \alpha$. The superscript α is a label, not an exponent. For this p we have $\sqrt{(1 - p)/p} = \tan \alpha$ and $\sqrt{p/(1 - p)} = \cot \alpha$, so

$$(6.1) \quad \mathbb{P}(\xi^\alpha = \tan \alpha) = \cos^2 \alpha, \quad \mathbb{P}(\xi^\alpha = -\cot \alpha) = \sin^2 \alpha.$$

As α runs over $(0, \pi/2)$, the parameter $p = \cos^2 \alpha$ runs over $(0, 1)$. For $\lambda \in \mathbb{R}^n$ let $\xi_1^\alpha, \dots, \xi_n^\alpha$ be independent copies of ξ^α , and put $S_\alpha = \sum_{j=1}^n \lambda_j \xi_j^\alpha$.

Lemma 6.1. *Let $\lambda \in \mathbb{R}^n$.*

- (i) *For $\alpha \in (0, \pi/2)$, $S_{\pi/2 - \alpha}$ has the law of $-S_\alpha$.*
- (ii) *$J(\lambda) = \int_0^{\pi/2} \mathbb{E} |S_\alpha| d\alpha$.*

Proof. (i) By (6.1), $\xi^{\pi/2 - \alpha}$ takes the value $\tan(\pi/2 - \alpha) = \cot \alpha$ with probability $\cos^2(\pi/2 - \alpha) = \sin^2 \alpha$, and the value $-\cot(\pi/2 - \alpha) = -\tan \alpha$ with probability $\sin^2(\pi/2 - \alpha) = \cos^2 \alpha$, so

$\xi^{\pi/2-\alpha}$ has the law of $-\xi^\alpha$. By independence, the vector $(\xi_1^{\pi/2-\alpha}, \dots, \xi_n^{\pi/2-\alpha})$ has the law of $(-\xi_1^\alpha, \dots, -\xi_n^\alpha)$, and (i) follows.

(ii) We substitute $\theta = \pi/2 - 2\alpha$ in (1.3), with $\alpha \in (0, \pi/4)$. Then $p(\theta) = (1 + \cos 2\alpha)/2 = \cos^2 \alpha$, so $\xi^{(p(\theta))}$ is ξ^α , and $d\theta = -2 d\alpha$. Hence

$$J(\lambda) = 2 \int_0^{\pi/4} \mathbb{E}|S_\alpha| d\alpha.$$

By (i), $\mathbb{E}|S_\alpha| = \mathbb{E}|S_{\pi/2-\alpha}|$. Therefore $\int_0^{\pi/4} \mathbb{E}|S_\alpha| d\alpha = \int_{\pi/4}^{\pi/2} \mathbb{E}|S_\alpha| d\alpha$, and (ii) follows. $\square$

Lemma 6.2. *For every real random variable X ,*

$$\mathbb{E}|X| = \frac{2}{\pi} \int_0^\infty (1 - \operatorname{Re} \varphi_X(t)) \frac{dt}{t^2},$$

where both sides may be $+\infty$.

Proof. We first show that $c_0 = \int_0^\infty (1 - \cos s) s^{-2} ds$ equals $\pi/2$. For $0 < \varepsilon < b$, integration by parts gives

$$\int_\varepsilon^b \frac{1 - \cos s}{s^2} ds = \left[-\frac{1 - \cos s}{s} \right]_\varepsilon^b + \int_\varepsilon^b \frac{\sin s}{s} ds.$$

As $\varepsilon \rightarrow 0$ and $b \rightarrow \infty$, the boundary terms tend to 0 and the last integral tends to the Dirichlet integral $\pi/2$. The integrand on the left is nonnegative, so $c_0 = \pi/2$ by monotone convergence. For real $x \neq 0$ the substitution $s = t|x|$ gives $\int_0^\infty (1 - \cos tx) t^{-2} dt = c_0|x|$. Hence

$$|x| = \frac{2}{\pi} \int_0^\infty (1 - \cos tx) \frac{dt}{t^2}$$

for every real x , including $x = 0$. We apply this with $x = X$ and take expectations. Since the function $(t, x) \mapsto (1 - \cos tx) t^{-2}$ is nonnegative and continuous on $(0, \infty) \times \mathbb{R}$, Tonelli's theorem allows us to exchange $\mathbb{E}$ and the integral. Finally, $\mathbb{E} \cos tX = \operatorname{Re} \varphi_X(t)$. $\square$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\overline{\mathbb{D}}$ be its closure.

Lemma 6.3. *Fix $u \geq 0$. For $z \in \overline{\mathbb{D}} \setminus \{-1, 0, 1\}$ put*

$$\Phi_u(z) = \frac{1}{4z} \left[(1+z)^2 \exp\left(-u \frac{1-z}{1+z}\right) - (1-z)^2 \exp\left(-u \frac{1+z}{1-z}\right) \right],$$

and put $\Phi_u(0) = (1+u)e^{-u}$. Then the following hold.

- (a) Φ_u is holomorphic in $\mathbb{D}$.
- (b) $|\Phi_u(z)| \leq 4$ for $z \in \mathbb{D}$.
- (c) Φ_u is continuous on $\overline{\mathbb{D}} \setminus \{-1, 1\}$.
- (d) For $\alpha \in (0, \pi/2)$, $\Phi_u(e^{2i\alpha}) = \mathbb{E}e^{iu\xi^\alpha}$, and $\Phi_u(e^{-2i\alpha})$ is the complex conjugate of $\Phi_u(e^{2i\alpha})$.

Proof. Put $w_1(z) = (1-z)/(1+z)$ and $w_2(z) = (1+z)/(1-z)$. The function w_1 is holomorphic on $\mathbb{C} \setminus \{-1\}$, and w_2 is holomorphic on $\mathbb{C} \setminus \{1\}$. Since $(1-z)(1+\bar{z}) = 1 - |z|^2 + (\bar{z} - z)$ and $\bar{z} - z$ is purely imaginary,

$$\operatorname{Re} w_1(z) = \frac{1 - |z|^2}{|1+z|^2} \geq 0 \quad (z \in \overline{\mathbb{D}}, z \neq -1).$$

In the same way, $\operatorname{Re} w_2(z) = (1 - |z|^2)/|1-z|^2 \geq 0$ for $z \in \overline{\mathbb{D}}, z \neq 1$. Hence the functions $\beta_k = \exp(-uw_k)$, $k = 1, 2$, satisfy $|\beta_k| \leq 1$ at every point of $\overline{\mathbb{D}}$ where they are defined.

(a) The numerator $\nu(z) = (1+z)^2\beta_1(z) - (1-z)^2\beta_2(z)$ is holomorphic in $\mathbb{D}$, and $\nu(0) = e^{-u} - e^{-u} = 0$, so $\nu(z)/(4z)$ has a removable singularity at 0, with value $\nu'(0)/4$. From $w'_1(0) = -2$ and $w'_2(0) = 2$ we get $\beta'_1(0) = 2ue^{-u}$ and $\beta'_2(0) = -2ue^{-u}$. Therefore

$$\nu'(0) = (2e^{-u} + \beta'_1(0)) - (-2e^{-u} + \beta'_2(0)) = 4(1+u)e^{-u},$$

and the extension takes the value $\Phi_u(0)$ at 0.

(b) For $1/2 \leq |z| < 1$ we have $|\Phi_u(z)| \leq (|1+z|^2 + |1-z|^2)/(4|z|) \leq (4+4)/2 = 4$. By the maximum principle the same bound holds for $|z| \leq 1/2$.

(c) The defining formula is continuous on $\mathbb{D} \setminus \{-1, 0, 1\}$. At 0, continuity follows from (a).

(d) Let $z = e^{i\psi}$ with $0 < |\psi| < \pi$. Then $1+z = 2\cos(\psi/2)e^{i\psi/2}$ and $1-z = -2i\sin(\psi/2)e^{i\psi/2}$. Hence

$$\frac{(1+z)^2}{4z} = \cos^2 \frac{\psi}{2}, \quad \frac{(1-z)^2}{4z} = -\sin^2 \frac{\psi}{2}, \quad w_1(z) = -i \tan \frac{\psi}{2}, \quad w_2(z) = i \cot \frac{\psi}{2},$$

and so

$$\Phi_u(e^{i\psi}) = \cos^2 \frac{\psi}{2} e^{iu \tan(\psi/2)} + \sin^2 \frac{\psi}{2} e^{-iu \cot(\psi/2)}.$$

For $\psi = 2\alpha$ with $\alpha \in (0, \pi/2)$, this is $\mathbb{E}e^{iu\xi^\alpha}$ by (6.1). Replacing ψ by $-\psi$ conjugates each term, because $\cos^2$ and $\sin^2$ are even and $\tan$ and $\cot$ are odd. $\square$

Lemma 6.4. For $u_1, \dots, u_n \geq 0$,

$$\frac{2}{\pi} \int_0^{\pi/2} \operatorname{Re} \prod_{j=1}^n \mathbb{E} e^{iu_j \xi^\alpha} d\alpha = \prod_{j=1}^n (1 + u_j) e^{-u_j}.$$

Proof. Let $F = \prod_{j=1}^n \Phi_{u_j}$. By Lemma 6.3, F is holomorphic in $\mathbb{D}$, $|F| \leq 4^n$ in $\mathbb{D}$, and F is continuous on $\mathbb{D} \setminus \{-1, 1\}$. Moreover $F(0) = \prod_j (1 + u_j) e^{-u_j}$, $F(e^{2i\alpha}) = \prod_j \mathbb{E} e^{iu_j \xi^\alpha}$ for $\alpha \in (0, \pi/2)$, and $F(e^{-i\psi}) = \overline{F(e^{i\psi})}$ for $\psi \in (0, \pi)$. For $0 < r < 1$ the mean-value property of holomorphic functions gives

$$F(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(re^{i\psi}) d\psi.$$

As $r \rightarrow 1$, $F(re^{i\psi}) \rightarrow F(e^{i\psi})$ for every $\psi \notin \{0, \pm\pi\}$ by continuity, and the integrands are bounded by 4^n . By dominated convergence,

$$F(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\psi}) d\psi = \frac{1}{\pi} \int_0^{\pi} \operatorname{Re} F(e^{i\psi}) d\psi.$$

The second equality holds because the integrals over $(-\pi, 0)$ and $(0, \pi)$ are complex conjugates of each other. The substitution $\psi = 2\alpha$ gives the lemma. $\square$

Theorem 6.5. Let $\lambda \in [0, \infty)^n$. Then

$$(A1) \quad J(\lambda) = \int_0^\infty \left(1 - \prod_{j=1}^n (1 + \lambda_j t) e^{-\lambda_j t}\right) \frac{dt}{t^2}.$$

Let $Z_1, \dots, Z_n$ be independent random variables with density $f_Z(x) = \frac{2}{\pi}(1+x^2)^{-2}$ on $\mathbb{R}$. Then

$$(A2) \quad J(\lambda) = \frac{\pi}{2} \mathbb{E} \left| \sum_{j=1}^n \lambda_j Z_j \right|.$$

Let $\chi_1, \dots, \chi_n$ be independent random variables with density $\sqrt{2/\pi} r^2 e^{-r^2/2}$ on $(0, \infty)$, and let $V_j = \chi_j^{-2}$. Then $\mathbb{E}V_j = 1$ and

$$(A3) \quad J(\lambda) = \sqrt{\frac{\pi}{2}} \mathbb{E} \left(\sum_{j=1}^n \lambda_j^2 V_j \right)^{1/2}.$$

If $\lambda \neq 0$ and $L = \lambda_1 + \dots + \lambda_n$, then

$$(A4) \quad J(\lambda) = L - \sum_{k=2}^n (k-2)! \sigma_k(\lambda) L^{1-k},$$

where σ_k is the k th elementary symmetric polynomial.

Proof. We begin with (A1). By Lemma 6.1(ii) and Lemma 6.2 with $X = S_\alpha$,

$$J(\lambda) = \frac{2}{\pi} \int_0^{\pi/2} \int_0^\infty (1 - \operatorname{Re} \varphi_{S_\alpha}(t)) \frac{dt}{t^2} d\alpha.$$

The integrand is nonnegative, since $\operatorname{Re} \varphi_{S_\alpha} \leq 1$. By (6.1) and independence,

$$\varphi_{S_\alpha}(t) = \prod_{j=1}^n (\cos^2 \alpha e^{i\lambda_j t \tan \alpha} + \sin^2 \alpha e^{-i\lambda_j t \cot \alpha}),$$

which is continuous on $(0, \pi/2) \times (0, \infty)$. By Tonelli's theorem we may therefore exchange the two integrals. For $t > 0$ we have $\varphi_{S_\alpha}(t) = \prod_j \mathbb{E} e^{iu_j \xi^\alpha}$ with $u_j = \lambda_j t \geq 0$. Lemma 6.4 gives

$$\int_0^{\pi/2} \operatorname{Re} \varphi_{S_\alpha}(t) d\alpha = \frac{\pi}{2} \prod_{j=1}^n (1 + \lambda_j t) e^{-\lambda_j t}.$$

Substituting this into the integral over t gives (A1).

Next we prove (A2). We have

$$\int_{\mathbb{R}} e^{-|t|} e^{-itx} dt = 2 \operatorname{Re} \int_0^\infty e^{-(1+ix)t} dt = \frac{2}{1+x^2}$$

and

$$\int_{\mathbb{R}} |t| e^{-|t|} e^{-itx} dt = 2 \operatorname{Re} \int_0^\infty t e^{-(1+ix)t} dt = 2 \operatorname{Re} \frac{1}{(1+ix)^2} = \frac{2(1-x^2)}{(1+x^2)^2}.$$

Adding, $\int_{\mathbb{R}} (1+|t|) e^{-|t|} e^{-itx} dt = 4/(1+x^2)^2 = 2\pi f_Z(x)$. The functions $(1+|t|)e^{-|t|}$ and f_Z are continuous and integrable. By the Fourier inversion theorem, $\int_{\mathbb{R}} e^{itx} f_Z(x) dx = (1+|t|)e^{-|t|}$ for every real t . At $t = 0$ this gives $\int_{\mathbb{R}} f_Z = 1$. Hence f_Z is a probability density, and $\varphi_{Z_1}(t) = (1+|t|)e^{-|t|}$. Since $\lambda_j \geq 0$, the characteristic function of $\sum_j \lambda_j Z_j$ is $\prod_j (1 + \lambda_j |t|) e^{-\lambda_j |t|}$. Lemma 6.2 with $X = \sum_j \lambda_j Z_j$ now gives

$$\mathbb{E} \left| \sum_{j=1}^n \lambda_j Z_j \right| = \frac{2}{\pi} \int_0^\infty \left(1 - \prod_{j=1}^n (1 + \lambda_j t) e^{-\lambda_j t} \right) \frac{dt}{t^2},$$

and (A2) follows from (A1).

We turn to (A3). The density $\sqrt{2/\pi} r^2 e^{-r^2/2}$ is that of the chi distribution with three degrees of freedom. Let g be a standard Gaussian variable, and let χ be independent of g with this density. For $x \in \mathbb{R}$, the density of g/χ at x is

$$\int_0^\infty r \cdot \frac{e^{-x^2 r^2/2}}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} r^2 e^{-r^2/2} dr = \frac{1}{\pi} \int_0^\infty r^3 e^{-(1+x^2)r^2/2} dr = f_Z(x),$$

because $\int_0^\infty r^3 e^{-br^2/2} dr = 2/b^2$ for $b > 0$. Hence we may take $Z_j = g_j/\chi_j$. Here $g_1, \dots, g_n$ are standard Gaussian variables, and all the variables $g_1, \dots, g_n$ and $\chi_1, \dots, \chi_n$ are independent. Conditionally on $\chi_1, \dots, \chi_n$, the sum $\sum_j \lambda_j g_j/\chi_j$ is a centered Gaussian variable with variance $\sum_j \lambda_j^2 V_j$. Its conditional expected absolute value is therefore $\sqrt{2/\pi} (\sum_j \lambda_j^2 V_j)^{1/2}$. We take expectations and use (A2). This gives $J(\lambda) = \frac{\pi}{2} \sqrt{\frac{2}{\pi}} \mathbb{E}(\sum_j \lambda_j^2 V_j)^{1/2}$, which is (A3). Finally,

$$\mathbb{E}V_j = \int_0^\infty r^{-2} \sqrt{\frac{2}{\pi}} r^2 e^{-r^2/2} dr = 1.$$

Finally we prove (A4). We put $x_j = \lambda_j/L$ and $t = s/L$ in (A1). Since $\sum_j x_j = 1$,

$$\prod_{j=1}^n (1 + \lambda_j t) e^{-\lambda_j t} = e^{-s} \prod_{j=1}^n (1 + x_j s) = e^{-s} \sum_{k=0}^n \sigma_k(x) s^k,$$

with $\sigma_0 = 1$ and $\sigma_1(x) = 1$. Also $dt/t^2 = L ds/s^2$. Hence

$$\frac{J(\lambda)}{L} = \int_0^\infty (1 - (1+s)e^{-s}) \frac{ds}{s^2} - \sum_{k=2}^n \sigma_k(x) \int_0^\infty s^{k-2} e^{-s} ds.$$

Splitting the integral is allowed because each piece is integrable. Indeed, $(1 - (1+s)e^{-s})s^{-2}$ tends to $1/2$ as $s \rightarrow 0$ and is $O(s^{-2})$ as $s \rightarrow \infty$. The integrals in the sum equal $(k-2)!$. For the first integral we use $\frac{d}{ds}(1+s)e^{-s} = -se^{-s}$. Integration by parts on $[\varepsilon, b]$ gives

$$\int_\varepsilon^b (1 - (1+s)e^{-s}) \frac{ds}{s^2} = \left[-\frac{1 - (1+s)e^{-s}}{s} \right]_\varepsilon^b + \int_\varepsilon^b e^{-s} ds,$$

which tends to $0 + 1 = 1$ as $\varepsilon \rightarrow 0$ and $b \rightarrow \infty$. Since $\sigma_k(x) = \sigma_k(\lambda)L^{-k}$, this proves (A4). $\square$

Corollary 6.6. *Let $\lambda \in \mathbb{R}^n$ be nonzero, with $\lambda \in [0, \infty)^n$ or $\lambda \in (-\infty, 0]^n$. Then*

$$J(\lambda) < \sqrt{\pi/2} |\lambda|.$$

Moreover, $J(\mathbf{1}_n/\sqrt{n}) \rightarrow \sqrt{\pi/2}$ as $n \rightarrow \infty$. Hence the supremum of $J(\lambda)$ over all n and all unit vectors $\lambda \in [0, \infty)^n \cup (-\infty, 0]^n$ is $\sqrt{\pi/2}$, and it is not attained.

Proof. For $u \geq 0$ let $\omega(u) = \log(1+u) - u + u^2/2$. Then $\omega(0) = 0$ and $\omega'(u) = u^2/(1+u) > 0$ for $u > 0$. So

$$(1+u)e^{-u} = e^{-u^2/2} e^{\omega(u)} > e^{-u^2/2} \quad (u > 0).$$

Let $\lambda \in [0, \infty)^n$ be nonzero. For every $t > 0$ some $\lambda_j t$ is positive, and therefore

$$\prod_{j=1}^n (1 + \lambda_j t) e^{-\lambda_j t} = e^{-|\lambda|^2 t^2/2} \exp\left(\sum_{j=1}^n \omega(\lambda_j t)\right) > e^{-|\lambda|^2 t^2/2}.$$

The substitution $s = |\lambda|t$ and integration by parts give

$$\int_0^\infty (1 - e^{-|\lambda|^2 t^2/2}) \frac{dt}{t^2} = |\lambda| \left(\left[-\frac{1 - e^{-s^2/2}}{s} \right]_0^\infty + \int_0^\infty e^{-s^2/2} ds \right) = \sqrt{\frac{\pi}{2}} |\lambda|.$$

By (A1), $\sqrt{\pi/2} |\lambda| - J(\lambda)$ is the integral of $(\prod_j (1 + \lambda_j t) e^{-\lambda_j t} - e^{-|\lambda|^2 t^2/2}) t^{-2}$ over $(0, \infty)$. Both integrals involved are finite, and this integrand is continuous and positive on $(0, \infty)$. Hence $J(\lambda) < \sqrt{\pi/2} |\lambda|$. For $\lambda \in (-\infty, 0]^n$ we use $J(-\lambda) = J(\lambda)$.

The limit was shown in Remark 2.3(iii). It also follows from (A3). Let $V_1, V_2, \dots$ be independent copies of the variable V_1 of Theorem 6.5, and put $\bar{V}_n = (V_1 + \dots + V_n)/n$. Then $J(\mathbf{1}_n/\sqrt{n}) = \sqrt{\pi/2} \mathbb{E}\bar{V}_n^{1/2}$. By the strong law of large numbers, $\bar{V}_n \rightarrow \mathbb{E}V_1 = 1$ almost surely.

The family $\{\bar{V}_n^{1/2}\}$ is uniformly integrable. Indeed, for $c > 0$ we have $\bar{V}_n^{1/2} < \bar{V}_n/c$ on the event $\{\bar{V}_n^{1/2} > c\}$, so the expectation of $\bar{V}_n^{1/2}$ over this event is at most $\mathbb{E}\bar{V}_n/c = 1/c$. Hence $\mathbb{E}\bar{V}_n^{1/2} \rightarrow 1$. The last statement of the corollary follows from the strict inequality and the limit. $\square$

Corollary 6.7. *The function $G(x) = J(\sqrt{x_1}, \dots, \sqrt{x_n})$ is symmetric and concave on $[0, \infty)^n$. In particular, G is Schur-concave there, that is, $G(x) \geq G(y)$ whenever x is majorized by y . Consequently:*

- (i) $J(\lambda) \leq J(\mathbf{1}_n/\sqrt{n})$ for every unit vector $\lambda \in [0, \infty)^n$.
- (ii) $J(\mathbf{1}_n/\sqrt{n}) \leq J(\mathbf{1}_{n+1}/\sqrt{n+1})$ for every $n \geq 1$. Thus $J(\mathbf{1}_n/\sqrt{n})$ is nondecreasing in n and converges to $\sqrt{\pi/2}$.

Proof. By (A3), $G(x) = \sqrt{\pi/2} \mathbb{E}(\sum_j x_j V_j)^{1/2}$. This is finite, since $\mathbb{E}(\sum_j x_j V_j)^{1/2} \leq (\sum_j x_j \mathbb{E}V_j)^{1/2} = (\sum_j x_j)^{1/2}$ by Jensen's inequality. The function G is symmetric because $V_1, \dots, V_n$ are independent and identically distributed. For every outcome, $x \mapsto (\sum_j x_j V_j)^{1/2}$ is a linear function with values in $[0, \infty)$ composed with the concave square root, so it is concave on $[0, \infty)^n$. Taking expectations preserves concavity. If x is majorized by y , then x is a convex combination of coordinate permutations of y , by theorems of Hardy, Littlewood and Pólya and of Birkhoff [8]. Concavity and symmetry then give $G(x) \geq G(y)$.

For (i), let $x = (\lambda_1^2, \dots, \lambda_n^2)$ and let $x^{(1)}, \dots, x^{(n)}$ be its cyclic shifts. Their average is $\mathbf{1}_n/n$, so concavity and symmetry give $J(\mathbf{1}_n/\sqrt{n}) = G(\mathbf{1}_n/n) \geq \frac{1}{n} \sum_k G(x^{(k)}) = G(x) = J(\lambda)$. For (ii), apply the same argument in dimension $n+1$ to $x = (1/n, \dots, 1/n, 0)$, whose cyclic shifts average to $\mathbf{1}_{n+1}/(n+1)$. This gives $J(\mathbf{1}_{n+1}/\sqrt{n+1}) \geq J(\mathbf{1}_n/\sqrt{n}, 0) = J(\mathbf{1}_n/\sqrt{n})$, because appending a zero coordinate does not change J . The limit is part of Corollary 6.6. $\square$

Remark 6.8. The bound of Corollary 6.6 is approached at the rate $n^{-1/2}$:

$$\sqrt{n} (\sqrt{\pi/2} - J(\mathbf{1}_n/\sqrt{n})) \rightarrow \frac{1}{3} \quad (n \rightarrow \infty).$$

To see this, let ω be as in the proof of Corollary 6.6, put $u = t/\sqrt{n}$ and $y = n\omega(u)$. Then $((1+u)e^{-u})^n = e^{-t^2/2}e^y$, and (A1) together with the integral computed in that proof gives

$$\sqrt{\pi/2} - J(\mathbf{1}_n/\sqrt{n}) = \int_0^\infty e^{-t^2/2} (e^y - 1) \frac{dt}{t^2}.$$

For $u \geq 0$ we have $0 \leq \omega(u) \leq u^3/3$, because $u^3/3 - \omega(u)$ vanishes at 0 and has derivative $u^3/(1+u) \geq 0$. Let $t \leq \sqrt{n}$. Then $0 \leq y \leq t^3/(3\sqrt{n}) \leq t^2/3$ and $e^y - 1 \leq ye^y$, so

$$0 \leq \sqrt{n} e^{-t^2/2} (e^y - 1) t^{-2} \leq \frac{t}{3} e^{-t^2/6}.$$

The right side is integrable on $(0, \infty)$. For fixed t , $\omega(u)/u^3 \rightarrow 1/3$ as $u \rightarrow 0$, so $\sqrt{n}y \rightarrow t^3/3$ and $\sqrt{n}(e^y - 1) \rightarrow t^3/3$. For $t > \sqrt{n}$ we have $u > 1$, and $(1+u)e^{-u}$ is decreasing on $[0, \infty)$. This gives $0 \leq e^{-t^2/2}e^y \leq (2/e)^n$. Therefore the part of the integral over $(\sqrt{n}, \infty)$, multiplied by $\sqrt{n}$, is at most

$$\sqrt{n} \int_{\sqrt{n}}^\infty ((2/e)^n + e^{-t^2/2}) \frac{dt}{t^2} \leq (2/e)^n + e^{-n/2},$$

which tends to 0. By dominated convergence, the limit equals $\int_0^\infty \frac{t}{3} e^{-t^2/2} dt = \frac{1}{3}$.

The case of coordinates of both signs remains open. Suppose that $J(\lambda) \leq J(|\lambda_1|, \dots, |\lambda_n|)$ for all n and all $\lambda \in \mathbb{R}^n$. Then Corollaries 6.6 and 6.7 give $K_n = J(\mathbf{1}_n/\sqrt{n})$ for every n and $K = \sqrt{\pi/2}$, and so $C = \sqrt{\pi/2}$ by Remark 2.3(iii). In a numerical test with 2000 random unit

vectors of mixed signs in each dimension $n = 2, \dots, 8$, the inequality $J(\lambda) \leq J(|\lambda_1|, \dots, |\lambda_n|)$ held in every case. We conjecture that $K = \sqrt{\pi/2}$. By (A4),

$$J(\mathbf{1}_n/\sqrt{n}) = \sqrt{n} \left(1 - \sum_{k=2}^n (k-2)! \binom{n}{k} n^{-k} \right).$$

This equals 1, 1.0606602, 1.0905505, 1.1574618, 1.2209961 and 1.2428767 for $n = 1, 2, 3, 10, 100$ and 1000, and $\sqrt{n}(\sqrt{\pi/2} - J(\mathbf{1}_n/\sqrt{n}))$ equals 0.330 at $n = 1000$.

DATA AVAILABILITY

The certificate `mom_c142.json`, the checker `check11a_mom.py`, the second program `check11a_indep.py`, the search program `mom_bb.py` and the programs for the numerical remarks of Sections 5 and 6 are archived at doi:10.5281/zenodo.23524866 and available at <https://github.com/tejasvi/hamming-cube-poincare>. The certificate has SHA-256 hash

931895626b499b932af4ecf332f5491e06a0ae94e2e031ef6e8ee45496fb0758.

The checkers need Python 3.9 or later and python-flint. The Lean project is in the directory `lean` of the archive.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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