THE SHARP HESSIAN BOUND FOR CONVEX FOUR-BODY CENTRAL CONFIGURATIONS

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ABSTRACT. In a previous paper we proved that the planar four-body problem has exactly one convex central configuration for each choice of positive masses and each cyclic order of the bodies, by proving that these configurations are nondegenerate. Here we prove the sharp form of this nondegeneracy: at every convex central configuration, the Hessian of the potential on a level set of the moment of inertia exceeds two thirds of its radial part. The constant $\frac{2}{3}$ is optimal; it is approached by isosceles trapezoids in which the two masses on one of the parallel sides tend to zero. We give two independent computer-assisted proofs of an equivalent inequality for the largest eigenvalue of a 2×2 matrix that depends only on the shape of the configuration, and one of them is formalized in the Lean proof assistant.

1. INTRODUCTION

1.1. **The sharp bound.** A central configuration (CC) is a configuration of point masses in which the Newtonian acceleration of each body is $-\lambda$ times its position relative to the centre of mass, with the same $\lambda > 0$ for all bodies. A planar CC that rotates rigidly about its centre of mass with angular velocity $\sqrt{\lambda}$ solves the n-body problem; these solutions are the relative equilibria. MacMillan and Bartky [MB32] proved that for every choice of four positive masses and every cyclic ordering of the bodies there is a strictly convex planar CC with that ordering; see also [Xia04, Moe15]. Albouy, Fu and Sun conjectured that it is unique up to similarity [AFS08], and Albouy, Cabral and Santos included the question in their list of open problems [ACS12, Problem 10]. In [Tom26a] we proved this conjecture.

Theorem A ([Tom26a])**.** *Let $m_1, \dots, m_4 > 0$. For each cyclic order of the four bodies there is exactly one convex central configuration with that cyclic order, up to similarity. It is a non-degenerate local minimum of $UI^{1/2}$ on the shape space, and it depends real-analytically on the masses.*

The proof rests on nondegeneracy, by a route that is well known [San21a, San21b, Rob25, San26]. For a CC q let

$$Q(v) = D^2\Big(U + \frac{\lambda}{2} I\Big)(q)[v, v], \qquad K(v) = \sum_{i<j} \frac{3 m_i m_j}{r_{ij}^3} \Big(\frac{\langle q_i - q_j, v_i - v_j \rangle}{r_{ij}} \Big)^2,$$

where $\lambda = U(q)/I(q)$ is held fixed. Thus Q is the Hessian of the Lagrange function for U on a level set of I, and $K \geq 0$ is its radial part, which vanishes exactly on the infinitesimal translations and rotations. Here we prove the sharp lower bound for Q in terms of K.

Theorem B. *At every convex central configuration of four bodies,*

$$Q(v) \;>\; \tfrac{2}{3}\, K(v) \qquad \text{for all } v \in (\mathbb{R}^2)^4 \text{ with } K(v) > 0,$$

and the constant $\frac{2}{3}$ cannot be improved.

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Dziobek's relations give $Q = K - |\sigma|\,|L|^2$ with an explicit linear map L to $\mathbb{R}^2$, and the generalized eigenvalues of the pair (Q, K) are $1, 1, 1, 1-\mu_1, 1-\mu_2$, where $\mu_1 \geq \mu_2$ are the eigenvalues of a 2×2 matrix G built from the configuration alone (Section 2). Thus Theorem B says that $\lambda_{\max}(G) < \frac{1}{3}$ at every convex CC, and this is the form in which we prove it. In [Tom26a] we proved $\mu_1 < 1$ by hand, which gives nondegeneracy, and showed that μ_1 comes arbitrarily close to $\frac{1}{3}$.

At a convex CC, nondegeneracy says only that $Q(v) > 0$ whenever $K(v) > 0$; it gives no lower bound for $Q(v)/K(v)$ that holds for all masses. Theorem B gives the best such bound. Since also $Q \leq K$, the generalized eigenvalues of (Q, K) at a convex CC lie in $(\frac{2}{3}, 1]$ whatever the masses, while at a degenerate CC one of them vanishes (Proposition 2.5(c)). Degenerate planar four-body CCs exist, among them some concave kites [Rob25, LX25, LPRX26], and all of them are concave [Tom26a, Corollary B(iii)]. Theorem B thus measures how far the convex ones stay from degeneracy, uniformly in the masses. Uniformity is the point, since the set of masses is not compact: as some masses tend to zero, a lower bound for Q/K could tend to zero as well, and Theorem B shows that it stays above $\frac{2}{3}$. The constant is approached along isosceles trapezoids in which the two masses on one of the parallel sides tend to zero [Tom26a, Section 6], so any proof has to be sharp in that limit. Santoprete's numerical experiments had indicated that $\lambda_{\max}(G) < \frac{1}{3}$ (personal communication).

Computer-assisted proofs have a long history for central configurations. Hampton and Moeckel [HM06] proved with symbolic computation that the number of four-body CCs is finite, Moczurad and Zgliczyński [MZ19] listed the planar CCs of five, six and seven equal masses with interval arithmetic, and Sun, Xie and You [SXY23, SXY25] proved uniqueness and nondegeneracy in special cases with interval arithmetic.

1.2. **Two proofs.** We give two independent proofs, both computer-assisted; each can be read without the other, and each gives something that the other does not. A reader who wants only one proof may prefer the second, which is shorter.

The first treats by hand the configurations whose sides are all longer than three quarters of the radius $(M/\lambda)^{1/3}$, so that computation is needed only for configurations with a short side. For these it fixes a shortest side and describes the configurations by three parameters (Figure 1 shows the construction). A coverage lemma shows that five families of configurations contain them all, and certificates in exact rational arithmetic, deposited with this paper, bound μ_1 on these families. Along the way this proof gives a stronger bound for configurations whose shortest side is at most three fifths of the radius (Proposition 3.13).

The second works in the coordinates of Corbera, Cors and Roberts [CCR19], in which the convex CCs form the zero set of one function over an explicit three-dimensional domain. After a blow-up of the coordinates near the corner where two masses tend to zero, one computation with Taylor models, together with a separate theorem for a small disc around the line of the blown-up corner on which the margin vanishes, certifies the inequality on the whole domain. This proof is formalized in Lean (Section 9).

Section 9 also describes a formal proof of Theorem A of [Tom26a], which obtains nondegeneracy from the bound $Q \geq \frac{1}{4}K$, certified by a simpler computation, in place of the hand proof of [Tom26a]. This development also proves that the normalized mass map of Corbera, Cors and Roberts is injective (Remark 6.4).

Section 2 recalls the results of [Tom26a] that we use. Sections 3–5 and Appendix A give the first proof, Sections 6 and 7 and Appendix B the second, and Section 8 completes both.

2. Setting

We use the notation of [Tom26a] and recall the results that this paper needs; proofs are in [Tom26a, Sections 2 and 3]. Throughout, $m_1,\dots,m_4>0$, $M=\sum_i m_i$, and $q=(q_1,\dots,q_4)\in(\mathbb{R}^2)^4$ is a configuration. A configuration is a *central configuration* (CC) if

$$\nabla_{q_i}U(q)+\lambda\, m_i(q_i-c)=0,\qquad i=1,\dots,4, \tag{2.1}$$

where $U=\sum_{i<j}m_im_j/r_{ij}$, c is the centre of mass and $\lambda=U(q)/I(q)$. The six unordered pairs $e=ij$ are the *edges*; for an edge we write $q_e=q_i-q_j$, $v_e=v_i-v_j$, $r_e=|q_e|$, $R_e=r_e^2$ and $s_e=r_e^{-3}$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and write J for multiplication by $\sqrt{-1}$. Put

$$\lambda'=\frac{\lambda}{M},\qquad w_e=s_e-\lambda',\qquad \omega_e=m_im_jw_e\quad(e=ij).$$

Then (2.1) is equivalent to

$$\sum_{j\neq i}\omega_{ij}\,(q_j-q_i)=0,\qquad i=1,\dots,4. \tag{2.2}$$

Conversely, if a collision-free configuration satisfies (2.2) for some $\lambda'\in\mathbb{R}$, then it is a CC with $\lambda=M\lambda'$ [Tom26a, Section 2.1].

2.1. **The Hessian.** Let q be a CC, let $Q=D^2\big(U+\frac{\lambda}{2}I\big)(q)$ with λ frozen, let $\mathrm{d}r_e(v)=\langle q_e,v_e\rangle/r_e$ be the first variation of r_e, and put $\kappa_e=3m_im_js_e$ and

$$K(v)=\sum_e\kappa_e\,\mathrm{d}r_e(v)^2.$$

Then $Q(v)=K(v)-\sum_e\omega_e|v_e|^2$, and the space $\mathcal{T}$ of infinitesimal translations and rotations lies in $\ker Q$ [Tom26a, Lemma 2.1]. The class of q in the shape space is a nondegenerate local minimum of $UI^{1/2}$ if and only if $Q(v)>0$ for every $v\notin\mathcal{T}$, and the Morse index of q is the index of Q [Tom26a, Lemma 2.2].

2.2. **Areas and Dziobek's relations.** Let $[ijk]=\frac12\det(q_j-q_i,q_k-q_i)$ and $A_1=[234]$, $A_2=-[134]$, $A_3=[124]$, $A_4=-[123]$.

Lemma 2.1 ([Tom26a, Lemma 2.3])**.** (a) $\sum_l A_l=0$ *and* $\sum_l A_lq_l=0$.
(b) $A_l(\mu q+t\mathbf{1})=|\mu|^2A_l(q)$ *for* $\mu\in\mathbb{C}^*$, $t\in\mathbb{C}$, *and* $A_l(\bar q)=-A_l(q)$.
(c) *If* $q_i=q_j$ *with* $i\neq j$, *then* $A_k=0$ *for* $k\notin\{i,j\}$. *If the* q_l *are not all equal, then all* A_l *vanish if and only if the four points are collinear.*
(d) *If no three of the* q_l *are collinear, every* A_l *is nonzero, and* q *is a strictly convex quadrilateral with vertices* q_1,q_2,q_3,q_4 *in counterclockwise order if and only if the signs of* (A_1,A_2,A_3,A_4) *are* $(+,-,+,-)$.

Lemma 2.2 ([Tom26a, Lemma 2.4])**.** *Suppose that no three of* $q_1,\dots,q_4$ *are collinear. Then the kernel of* $v\mapsto(\mathrm{d}r_e(v))_e$ *is* $\mathcal{T}$, *this map has rank* 5, *and the solutions* (ω_e) *of* (2.2) *are the multiples of* $(A_iA_j)_{ij}$. *The image is the hyperplane* $\tau^\perp$, *where* $\tau_e=A_iA_j\,r_e$ *for* $e=ij$. *In particular, if* q *is a CC then* $K(v)=0$ *if and only if* $v\in\mathcal{T}$.

A planar CC of four bodies is either collinear or has no three bodies on a line [Tom26a, Lemma 2.5]. For a noncollinear planar CC there is a constant $\sigma<0$ such that

$$m_im_j\,w_{ij}=\sigma\,A_iA_j\qquad\text{for all } i<j \tag{2.3}$$

(Dziobek's relations; [Tom26a, Lemma 2.6]).

Proposition 2.3 ([Tom26a, Proposition 2.7])**.** *Let* q *be a convex CC whose cyclic order is* (1234). *Then:*

(a) $w_{12}, w_{23}, w_{34}, w_{14} > 0 > w_{13}, w_{24}$, *that is,* $\min(r_{13}, r_{24}) > \lambda'^{-1/3} > \max(r_{12}, r_{23}, r_{34}, r_{14})$*;*
(b) $w_{12}w_{34} = w_{13}w_{24} = w_{14}w_{23}$*;*
(c) *a longest side and a shortest side are opposite each other.*

In particular λ' *is determined by the configuration:*

$$\lambda' = \frac{s_{12}s_{34} - s_{13}s_{24}}{s_{12} + s_{34} - s_{13} - s_{24}}. \tag{2.4}$$

2.3. **The identity and the matrix** G**.**

Proposition 2.4 ([Tom26a, Proposition 3.1])**.** *Let* q *be a noncollinear planar CC of four bodies. Then*

$$Q(v) = K(v) - |\sigma|\,|L(v)|^2, \qquad L(v) = \sum_{l=1}^{4} A_l v_l \in \mathbb{R}^2, \tag{2.5}$$

for all $v \in (\mathbb{R}^2)^4$*.*

For an edge $e = ij$ put

$$D_e = -\frac{w_e}{3 s_e A_i A_j}, \qquad n_e = q_i + q_j, \qquad d_e = D_e \tau_e^2 = -\frac{w_e\, A_i A_j\, r_e^5}{3}, \tag{2.6}$$

so that $D_e = |\sigma|/\kappa_e > 0$, n_e is twice the midpoint of e and $d_e > 0$. Let

$$M_2 = \sum_l A_l\, q_l q_l^{\mathsf{T}}, \qquad W = \sum_e d_e, \qquad \bar{n} = \frac{1}{W}\sum_e d_e n_e, \qquad C = \sum_e d_e\,(n_e - \bar{n})(n_e - \bar{n})^{\mathsf{T}},$$

and $G = M_2^{-1} C M_2^{-1}$. On convex CCs, D_e, d_e and hence G are functions of the configuration alone, by (2.4); the masses do not enter. For $y \in \mathbb{R}^2$ let

$$S(y) = \sum_e D_e\,(\beta_e + \tau_e y)(\beta_e + \tau_e y)^{\mathsf{T}},$$

where the vectors $\beta_e \in \mathbb{R}^2$ are given explicitly in [Tom26a, Lemma 3.4]; the trace of $S(y)$ is a sum of squares with explicit coefficients.

Proposition 2.5 ([Tom26a, Propositions 3.5 and 3.10])**.** *Let* q *be a noncollinear planar CC of four bodies.*

(a) *For all* v *and* y, $|\sigma|\,|L(v)|^2 \le \lambda_{\max}(S(y))\,K(v) \le \operatorname{tr} S(y)\,K(v)$*; hence* $Q \ge (1 - \lambda_{\max}(S(y)))K \ge (1 - \operatorname{tr} S(y))K$*.*
(b) *There is* $y^* \in \mathbb{R}^2$ *with* $S(y) = G + c_0(y - y^*)(y - y^*)^{\mathsf{T}}$ *for all* y*, where* $c_0 = W > 0$*; in particular* $G \preceq S(y)$ *for every* y*.*
(c) *Let* $\mu_1 \ge \mu_2 \ge 0$ *be the eigenvalues of* G*. The generalized eigenvalues of the pair* (Q, K) *on* $(\mathbb{R}^2)^4/\mathcal{T}$ *are* $1, 1, 1, 1 - \mu_1, 1 - \mu_2$*. In particular* $Q \ge (1 - \mu_1)K$*, the constant is optimal, the Morse index of* q *is the number of* $\mu_i > 1$*, and* q *is degenerate if and only if* $\mu_1 = 1$ *or* $\mu_2 = 1$*.*

Corollary 2.6 ([Tom26a, Corollary 3.11])**.** *Let* q *be a noncollinear planar CC and* $c > 0$*. Then* $\mu_1 < c$ *if and only if* $cM_2^2 - C$ *is positive definite, that is, if and only if* $\det(cM_2^2 - C) > 0$ *and* $\operatorname{tr}(cM_2^2 - C) > 0$*. In particular* $\mu_1 < \frac{1}{3}$ *if and only if* $3C \prec M_2^2$*. The eigenvalues of* G *do not depend on the labelling of the bodies and do not change under similarities.*

The main theorem of [Tom26a] rests on the bound $\mu_1 < 1$ at every convex CC, proved there by hand [Tom26a, Theorem 4.1 and Corollary 5.1].

3. The sharp bound: reduction and coverage

We prove that $\mu_1 < \frac{1}{3}$ at every convex CC.

Theorem 3.1. *At every strictly convex central configuration of four positive masses, $\mu_1 < \frac{1}{3}$, that is, $3C \prec M_2^2$.*

The two forms are equivalent by Corollary 2.6. The proof is at the end of this section.

3.1. **The shortest side.** Let q be a convex CC and put $R_c = \lambda'^{-1/3}$. By Proposition 2.3(a), every side of q is shorter than R_c and every diagonal is longer.

Definition 3.2. The *roundness* $\varpi \in (0,1)$ of q is the length of a shortest side divided by R_c.

Lemma 3.3. (a) *After a cyclic relabelling, and a reflection if needed, the bodies are in counterclockwise order* (1234) *and q_1q_2 is a shortest side. This changes neither μ_1 (Corollary 2.6) nor ϖ.*

(b) *If $\varpi > r_0$ for some $r_0 \in (0,1)$, then every diagonal is longer than every side, and the shortest side is longer than r_0 times the longest side.*

Proof. (a) A cyclic relabelling keeps the cyclic order, and a reflection reverses the orientation. The roundness depends only on the lengths and on λ'. (b) By Proposition 2.3(a), every diagonal is longer than R_c and every side is shorter than R_c. Hence every diagonal is longer than every side. The shortest side has length $\varpi R_c > r_0 R_c$, and by Proposition 2.3(a) again, $r_0 R_c$ exceeds r_0 times the longest side. □

From now on q_1q_2 is a shortest side and the order is counterclockwise. If $\varpi > \frac{3}{4}$, then $\mu_1 < \frac{1}{3}$ by Theorem 4.1. The rest of this section treats $\varpi \le \frac{3}{4}$.

3.2. **The lens-line chart.** We scale so that $\lambda' = 1$. Then $R_c = 1$ and $w_e = r_e^{-3} - 1$, so $w_e > 0$ on the sides and $w_e < 0$ on the diagonals. Put $\rho = r_{12}$, which equals ϖ, and $\eta_l = A_l/m_l$. Dziobek's relations (2.3) read $w_{ij} = \sigma\eta_i\eta_j$.

Lemma 3.4. *Let q be a convex CC with $\lambda' = 1$, in counterclockwise order* (1234). *Put $\nu = \eta_2/(\eta_2 - \eta_1)$. Then $0 < \nu < 1$, $\nu = w_{23}/(w_{23} - w_{13})$, and*

$$\nu\,|z - q_1|^{-3} + (1-\nu)\,|z - q_2|^{-3} = 1 \qquad \text{for } z = q_3, q_4.$$

Moreover $F := w_{12}w_{34} - w_{13}w_{24} = 0$.

Proof. By Lemma 2.1(d), $A_1 > 0 > A_2$, so $\eta_1 > 0 > \eta_2$ and $0 < \nu < 1$. For $l \in \{3,4\}$,

$$\nu\, w_{1l} + (1-\nu)\, w_{2l} = \sigma\eta_l\big(\nu\eta_1 + (1-\nu)\eta_2\big) = 0,$$

and adding $\nu + (1-\nu) = 1$ gives the displayed identity. Also $w_{23}/(w_{23} - w_{13}) = \eta_2/(\eta_2 - \eta_1) = \nu$, and $F = \sigma^2(\eta_1\eta_2\eta_3\eta_4 - \eta_1\eta_3\eta_2\eta_4) = 0$. □

We place $q_1 = (-(1-\nu)\rho, 0)$ and $q_2 = (\nu\rho, 0)$, with q_3 and q_4 in the upper half-plane. The origin is $\nu q_1 + (1-\nu)q_2$, a point of the segment $[q_1q_2]$. Let

$$\Phi(z) = \nu\,|z - q_1|^{-3} + (1-\nu)\,|z - q_2|^{-3} \quad (z \in \mathbb{C}), \qquad \Lambda = \{\Phi = 1\}, \qquad \Lambda^+ = \Lambda \cap \{\operatorname{Im} z > 0\},$$

and let θ_z be the polar angle of z about the origin. We call Λ the *lens* (Figure 1). The *apex* $X = \big((2\nu-1)\rho/2, \sqrt{1 - \rho^2/4}\big)$ satisfies $|X - q_1| = |X - q_2| = 1$, so $X \in \Lambda^+$. It lies on the perpendicular bisector $\{x = X_x\}$ of q_1q_2, which we call the *bisector*.

Definition 3.5. For $0 < \rho < 1$ and $0 < \nu < 1$, the *line* $\mathcal{L}(\rho, \nu)$ is the set of pairs $(q_3, q_4) \in \Lambda^+ \times \Lambda^+$ with $\theta_{q_3} < \theta_{q_4}$ and $F = 0$. Let $\mathcal{R} = \{\theta_{q_3} < \theta_X < \theta_{q_4}\}$.

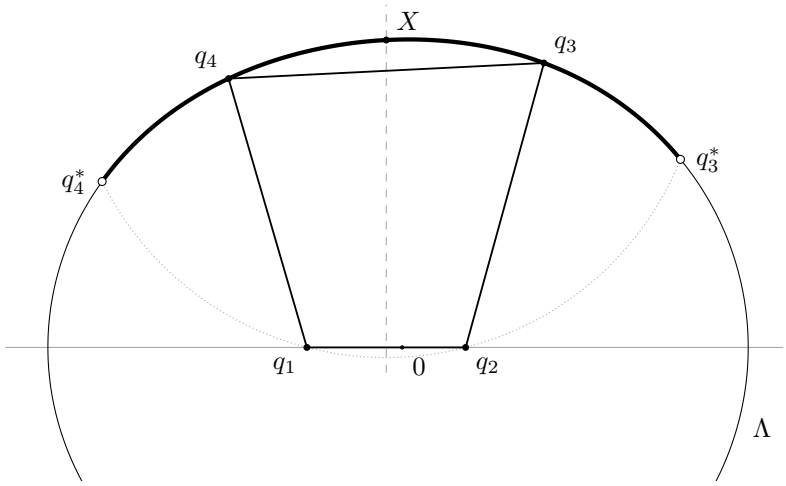


FIGURE 1. The lens-line chart for $\rho = \frac{1}{2}$ and $\nu = \frac{2}{5}$: the lens Λ, the bisector (dashed) through the apex X, and an arc of the circle of radius 1 about X (dotted), which passes through q_1, q_2, q_3^* and q_4^*. Along the arc $\mathcal{A}$ of Lemma 3.6, q_3 runs over the thick arc of Λ^+ from q_3^* to X, and q_4 over the thick arc from X to q_4^*. The quadrilateral is the convex CC at one point of $\mathcal{A}$.

By Lemma 3.4, every convex CC with $\lambda' = 1$, labelled as in Lemma 3.3, is a point of the line $\mathcal{L}(\rho, \nu)$ with $\rho = r_{12}$ and its own ν. At a CC, Dziobek's relations give the masses up to a common factor:

$$m_2 : m_3 : m_4 = \frac{A_2}{w_{12}} : \frac{A_3}{w_{13}} : \frac{A_4}{w_{14}}, \qquad \frac{m_1}{m_3} = \frac{A_1\, w_{23}}{A_3\, w_{12}}.$$

At a pair $(q_3, q_4) \in \Lambda^+ \times \Lambda^+$ with $\theta_{q_3} < \theta_{q_4}$ and $w_{13}w_{14} \neq 0$, we call the numbers given by these formulas its *Dziobek masses*. By (L2) below, a point of Λ is determined by its polar angle, so F is a function of θ_{q_3} and θ_{q_4} for $q_3, q_4 \in \Lambda$ with $0 \le \theta_{q_3} < \theta_{q_4} \le \pi$.

Lemma 3.6. *Let $0 < \rho < 1$ and $0 < \nu < 1$.*

(i) *If $q_3, q_4 \in \Lambda^+$ and $\theta_{q_3} < \theta_{q_4}$, then $q_1q_2q_3q_4$ is a strictly convex quadrilateral in counterclockwise order. Its Dziobek masses have one sign if and only if $(\theta_{q_3}, \theta_{q_4}) \in \mathcal{R}$.*

(ii) *There is $\theta_3^* \in (0, \theta_X)$ with the following property. For $\theta \in (0, \theta_X)$, the function $F(\theta, \cdot)$ has a zero in (θ_X, π) if and only if $\theta \in (\theta_3^*, \theta_X)$, and then exactly one, $\Theta(\theta)$. Hence the convex CCs of the line with positive masses lie on the arc $\mathcal{A} = \{(\theta, \Theta(\theta)) : \theta_3^* < \theta < \theta_X\}$.*

(iii) *Θ is C^1 and strictly increasing. Let q_3^* and q_4^* be the points of Λ^+ at distance 1 from X, with $\theta_{q_3^*} < \theta_X < \theta_{q_4^*}$, and put $\mathcal{A}_+ = (q_3^*, X)$ and $\mathcal{A}_- = (X, q_4^*)$. The zeros of F on the line with $\theta_{q_3} \le \theta_X \le \theta_{q_4}$ are the points of $\mathcal{A}$, $\mathcal{A}_+$ and $\mathcal{A}_-$, and they form the closure of $\mathcal{A}$.*

(iv) *Along $\mathcal{A}$, the abscissae x_3 and x_4 of q_3 and q_4 decrease strictly as θ_{q_3} increases. Hence $u = \frac{1}{2}\big((x_3 - X_x) + (x_4 - X_x)\big)$ decreases strictly along $\mathcal{A}$, and $u(\mathcal{A}_-) < 0 < u(\mathcal{A}_+)$.*

Proof. We first collect facts about Λ. Write $r_1 = |z - q_1|$, $r_2 = |z - q_2|$ and $w(r) = r^{-3} - 1$.

(L1) On Λ, $\nu w(r_1) + (1-\nu)w(r_2) = 0$, so $\min(r_1, r_2) \le 1 \le \max(r_1, r_2)$. Since $|r_1 - r_2| \le \rho$, both r_1 and r_2 lie in $[1-\rho, 1+\rho]$, and so does $|z|$.

(L2) *Star shape.* Fix $o \in [q_1q_2]$ and a unit vector e, and let $t_0 = \max\big(0, \langle e, q_1 - o\rangle, \langle e, q_2 - o\rangle\big)$. For $0 \le t \le t_0$ both distances from $o+te$ to q_1 and q_2 are at most ρ, so $\Phi(o+te) \ge \rho^{-3} > 1$. For $t > t_0$ both distances increase strictly with t, so $\Phi(o+te)$ decreases strictly to 0, with nonzero derivative. Hence every ray from o meets Λ exactly once, transversally. With o the origin this gives $\Lambda^+ = \{z(\theta) = R(\theta)(\cos\theta, \sin\theta) : 0 < \theta < \pi\}$ with R smooth.

(L3) *Orientation.* For $o \in [q_1q_2]$ the polar angle about o of $z(\theta)$ is a continuous injective function of θ on $[0,\pi]$, equal to 0 at $\theta = 0$ and to π at $\theta = \pi$. Hence it increases, and $\det(z-o, z'-o) > 0$ whenever $z, z' \in \Lambda^+$ and $\theta_z < \theta_{z'}$.

(L4) *Tangent.* Put $\alpha_1 = \nu r_1^{-5}$, $\alpha_2 = (1-\nu) r_2^{-5}$ and $o_z = (\alpha_1 q_1 + \alpha_2 q_2)/(\alpha_1+\alpha_2)$, a point of the open segment (q_1q_2). Then $\nabla\Phi(z) = -3(\alpha_1+\alpha_2)(z-o_z)$, so $\frac{d}{d\theta} z(\theta)$ is a multiple of $J(z-o_z)$. The multiple is positive, because $\det(z, \frac{d}{d\theta} z) = R^2 > 0$ and $\det\big(z, J(z-o_z)\big) = \langle z, z-o_z\rangle > 0$, as Φ decreases along the ray from the origin by (L2).

(L5) For a fixed point $Y \neq z$, $\frac{d}{d\theta}|z-Y|$ has the sign of $\det(Y-o_z, z-o_z)$.

(L6) With $Y = q_2$ this is positive, and with $Y = q_1$ negative. Thus r_2 increases and r_1 decreases along Λ^+. The apex X is the only point of Λ^+ with $r_1 = r_2 = 1$. For $\theta < \theta_X$ we have $r_2 < 1 < r_1$, and for $\theta > \theta_X$ the reverse.

(L7) $\frac{d}{d\theta} \operatorname{Re} z$ has the sign of $-\operatorname{Im} z < 0$, so the abscissa decreases strictly along Λ^+.

(i) The four orientations $\det(q_2-q_1, q_3-q_1)$, $\det(q_2-q_1, q_4-q_1)$, $\det(q_3-q_2, q_4-q_2)$ and $\det(q_3-q_1, q_4-q_1)$ are positive. The first two equal ρ times a height, and the last two are positive by (L3) with $o = q_2$ and $o = q_1$. Thus the quadrilateral is strictly convex and counterclockwise, and $A_1, A_3 > 0 > A_2, A_4$ by Lemma 2.1(d). On $\mathcal{R}$, (L6) gives $w_{13} < 0 < w_{23}$ and $w_{14} > 0 > w_{24}$, while $w_{12} = \rho^{-3} - 1 > 0$. Hence A_2/w_{12}, A_3/w_{13} and A_4/w_{14} are all negative, and $m_1/m_3 > 0$: the Dziobek masses have one sign. If $\theta_{q_3} > \theta_X$, then $w_{13} > 0$ and $m_3/m_2 < 0$. If $\theta_{q_4} < \theta_X$, then $w_{14} < 0$ and $m_4/m_2 < 0$. If $\theta_{q_3} = \theta_X$ or $\theta_{q_4} = \theta_X$, a denominator vanishes.

(ii) On $\mathcal{R}$, $\partial_{\theta_4} w_{34} < 0$ by (L5) and (L3), and $\partial_{\theta_4} w_{24} < 0$ by (L6). So $\partial_{\theta_4} F = w_{12}\, \partial_{\theta_4} w_{34} - w_{13}\, \partial_{\theta_4} w_{24} < 0$. In the same way $\partial_{\theta_3} F > 0$. At $q_4 = X$, $F = w_{12}\, w(|q_3 - X|)$, because $w_{24} = 0$ there. At $\theta_{q_4} = \pi$, $F < 0$. Indeed $w_{13} < 0$ and $w_{24} < 0$ there, and $|q_3 - q_4| > 1$: by (L5), $|q_3 - q_4|$ decreases in $\theta_{q_3} \in [0, \theta_X]$, and at $q_3 = X$ it equals $(r_{14}^2 + \rho\, r_{14} + 1)^{1/2} > 1$. The function $|q_3 - X|$ decreases strictly in $\theta_{q_3} \in [0, \theta_X)$, from a value larger than 1 to 0, so there is a unique θ_3^* with $|q_3 - X| = 1$. Hence $F(\theta, \theta_X) > 0$ exactly for $\theta > \theta_3^*$, and $F(\theta, \cdot)$ is strictly decreasing on $[\theta_X, \pi]$ with $F(\theta, \pi) < 0$. This proves the first claim of (ii). By (i), every convex CC of the line with positive masses is a zero of F in $\mathcal{R}$, which gives the second.

(iii) By the implicit function theorem, Θ is C^1 with $\Theta' = -\partial_{\theta_3} F / \partial_{\theta_4} F > 0$. The point q_3^* is unique by the proof of (ii). By (L5) and (L3), $|z - X|$ increases strictly for $\theta_z \in (\theta_X, \pi]$, from 0 to $(r_{14}^2 + \rho\, r_{14} + 1)^{1/2} > 1$, where $r_{14} = |z(\pi) - q_1|$; so q_4^* is unique too. If $q_4 = X$, then $F = w_{12}\, w(|q_3 - X|)$ vanishes only at $q_3 = q_3^*$. If $q_3 = X$, then $w_{13} = 0$, and $F = w_{12}\, w(|X - q_4|)$ vanishes only at $q_4 = q_4^*$. As $\theta \downarrow \theta_3^*$, $\Theta(\theta) \to \theta_X$, because $F(\theta_3^*, \cdot)$ vanishes at θ_X and decreases strictly; so $\mathcal{A}$ tends to $\mathcal{A}_+$. As $\theta \uparrow \theta_X$, the increasing function Θ has a limit, at which $F(\theta_X, \cdot)$ vanishes; so $\mathcal{A}$ tends to $\mathcal{A}_-$.

(iv) This follows from (iii) and (L7). At $\mathcal{A}_+$, $u = \frac{1}{2}(x_3 - X_x) > 0$, because q_3^* lies right of the bisector by (L7). In the same way $u(\mathcal{A}_-) = \frac{1}{2}(x_4 - X_x) < 0$. □

Lemma 3.7. *On $\mathcal{A}$, q_3 lies strictly right of the bisector and q_4 strictly left. $\mathcal{A}_+$ is the only point of the closure of $\mathcal{A}$ with q_4 on the bisector, and $\mathcal{A}_-$ the only one with q_3 on it. As a point of $\mathcal{A}$ tends to $\mathcal{A}_+$, w_{14} and w_{24} tend to 0, while w_{12}, w_{13} and the four A_l stay away from 0. So $m_4/m_l \to \infty$ for $l \neq 4$: one body carries almost all the mass. The same holds at $\mathcal{A}_-$ with q_3 in place of q_4.*

Proof. By (L7), X is the only point of Λ^+ on the bisector, and $\theta_{q_3} < \theta_X < \theta_{q_4}$ on $\mathcal{A}$. The rest follows from Lemma 3.6(i) and (iii) and the formulas for the Dziobek masses. □

The coordinate u of Lemma 3.6(iv) is a chart on the closure of $\mathcal{A}$. We use it around the curve $u = u_0(\rho, \nu)$, where

$$u_0(\rho, \nu) = \rho\,\big(\nu - \tfrac{1}{2}\big)\Big(\frac{35}{64} + \rho^2\Big(\frac{1}{3} - \frac{8}{5}\big(\nu - \tfrac{1}{2}\big)^2\Big)\Big). \tag{3.1}$$

Put $\zeta = u - u_0$. The *tube* is the set $\{|\zeta| \le \varepsilon\}$, where $\varepsilon = 1237/50000$. The polynomial u_0 is a convenient centre for the certificates; the arguments of this section use it only through Lemma 3.9.

Lemma 3.8. *The relabelling* $(1\,2)(3\,4)$ *followed by the reflection* $(x, y) \mapsto (-x, y)$ *maps* $\mathcal{L}(\rho, \nu)$ *onto* $\mathcal{L}(\rho, 1-\nu)$. *It maps* u *to* $-u$ *and* u_0 *to* $-u_0$, *hence* ζ *to* $-\zeta$. *It exchanges* $\mathcal{A}_-$ *and* $\mathcal{A}_+$ *and leaves* μ_1 *unchanged (Corollary 2.6).*

Proof. The composite keeps the counterclockwise order. It sends q_1 to $(-\nu\rho, 0)$ and q_2 to $((1-\nu)\rho, 0)$, which is the placement above with $1-\nu$ in place of ν. It preserves $\Phi = 1$ with $1-\nu$ in place of ν, the relation $F = 0$ and the apex. It sends u to $-u$, and u_0 is odd in $\nu - \frac{1}{2}$. □

Lemma 3.9. *Let* $0 < \rho \le \frac{3}{4}$ *and* $0 < \nu \le \frac{1}{2}$. *Put* $\phi_2 = \arcsin(\rho/2)$ *and* $\phi_{\min}(\rho) = \arcsin(\rho/2) + 2\arcsin\big((1-\rho)/2\big)$.

(a) $u(\mathcal{A}_+) \ge \frac{1}{2}\sin\phi_{\min}(\rho)$ *and* $u(\mathcal{A}_-) \le -\frac{1}{2}\sin\phi_{\min}(\rho)$. *In particular* $u(\mathcal{A}_+) - u_0 \ge 0.2966$ *and* $u_0 - u(\mathcal{A}_-) \ge 0.1055$, *so the tube lies strictly inside* $(u(\mathcal{A}_-), u(\mathcal{A}_+))$.

(b) *The heights of* q_3^* *and* q_4^* *are at least* $\cos\phi_2 - \cos\phi_{\min}(\rho)$. *On* $\mathcal{A}$, θ_{q_3} *and* $\pi - \theta_{q_4}$ *are at least* $\arcsin\big((\cos\phi_2 - \cos\phi_{\min}(\rho))/(1+\rho)\big)$, *and*

$$r_{34} \;\ge\; \Big(1 + \frac{(1-(1+\rho)^{-3})^2}{\rho^{-3}-1}\Big)^{-1/3}.$$

The bounds in (b) decrease in ρ.

Proof. (a) Since $|q_3^* - X| = 1$, we can write $q_3^* = X + (\sin\phi, -\cos\phi)$. Also $q_2 - X = (\sin\phi_2, -\cos\phi_2)$. As q_3^* lies above the axis, $\cos\phi < X_y = \cos\phi_2$, and as it lies right of the bisector, $\sin\phi > 0$. Hence $\phi > \phi_2$. Now $|q_3^* - q_2|^2 = 2 - 2\cos(\phi - \phi_2)$, with $|q_3^* - q_2| \ge 1 - \rho$ by (L1) and $|q_3^* - q_2| < 1$ by (L6). So $\phi \in [\phi_{\min}(\rho), \phi_2 + 60^\circ)$. On $\rho \le \frac{3}{4}$ this interval lies in $(0^\circ, 90^\circ)$, and $\phi_{\min}$ is decreasing, because its derivative is at most $\frac{1}{2}(1-\frac{1}{4})^{-1/2} - 1 < 0$. Therefore $u(\mathcal{A}_+) = \frac{1}{2}(x_3 - X_x) = \frac{1}{2}\sin\phi \ge \frac{1}{2}\sin\phi_{\min}(\frac{3}{4}) > 0.2966$. The same argument at $\mathcal{A}_-$, with q_1 in place of q_2, gives the bound for $u(\mathcal{A}_-)$. For u_0 put $t = \frac{1}{2} - \nu \in [0, \frac{1}{2})$. Then $-u_0 = \rho\, t\,\big(\frac{35}{64} + \frac{\rho^2}{3} - \frac{8}{5}\rho^2 t^2\big)$. This is nonnegative and increasing in t, since its derivative in t is at least $\rho\,(\frac{35}{64} - \frac{13}{15}\rho^2) > 0$. Thus $0 \le -u_0 \le \frac{\rho}{2}(\frac{35}{64} - \frac{\rho^2}{15}) \le 0.19102$; hence $u(\mathcal{A}_+) - u_0 \ge u(\mathcal{A}_+) \ge 0.2966$ and $u_0 - u(\mathcal{A}_-) \ge 0.2966 - 0.19102 > 0.1055$.

(b) The height of q_3^* is $X_y - \cos\phi = \cos\phi_2 - \cos\phi \ge \cos\phi_2 - \cos\phi_{\min}(\rho)$, and $|q_3^*| \le 1 + \rho$ by (L1). Along $\mathcal{A}$ the angle θ_{q_3} increases from $\theta_{q_3^*}$, which yields the bound for θ_{q_3}, and in the same way θ_{q_4} increases to $\theta_{q_4^*}$. On $\mathcal{A}$, $F = 0$ gives $w_{34} = w_{13}w_{24}/w_{12}$, with $|w_{13}|, |w_{24}| \le 1 - (1+\rho)^{-3}$ by (L1) and $w_{12} = \rho^{-3} - 1$. The bound for r_{34} follows. □

Lemma 3.10. *Let* $0 < \rho \le \frac{3}{4}$ *and* $0 < \nu \le \frac{1}{2}$. *Let* $s \mapsto \mathbf{q}(s)$, $s \in [0,1]$, *be a continuous path of pairs* (q_3, q_4) *that satisfy* $\Phi(q_3) = \Phi(q_4) = 1$, $F = 0$ *and*

$$E_s := (1-s)\big(u - u_0 + \varepsilon\big) + s\,(x_3 - X_x) = 0,$$

and suppose that $\mathbf{q}(0) \in \mathcal{A}$. *Then:*

(i) $\mathbf{q}(s) \in \mathcal{A}$ *and* $u(\mathbf{q}(s)) < u_0 - \varepsilon$ *for* $s \in (0,1)$;
(ii) $\mathbf{q}(1) = \mathcal{A}_-$;
(iii) *every point of* $\mathcal{A}$ *with* $u \le u_0 - \varepsilon$ *equals* $\mathbf{q}(s)$ *for some* $s \in [0,1)$.

The same holds with $E_s := (1-s)(u - u_0 - \varepsilon) + s\,(x_4 - X_x)$, *with* $u > u_0 + \varepsilon$ *in (i),* $\mathcal{A}_+$ *in (ii) and* $u \ge u_0 + \varepsilon$ *in (iii).*

Proof. Among the solutions of $\Phi(q_3) = \Phi(q_4) = 1$ and $F = 0$, those in the open set $\mathcal{R}$ form exactly $\mathcal{A}$, by Lemma 3.6(ii). Let s_1 be the supremum of the $s' \in (0, 1]$ with $\mathbf{q}([0, s')) \subset \mathcal{A}$; then $s_1 > 0$. On $(0, s_1)$, $x_3 - X_x > 0$ by Lemma 3.7, so $E_s = 0$ forces $u - u_0 + \varepsilon < 0$. If $s_1 < 1$, the point $\mathbf{q}(s_1)$ lies in the closure of $\mathcal{A}$ but not in $\mathcal{A}$, so it is $\mathcal{A}_-$ or $\mathcal{A}_+$. It is not $\mathcal{A}_+$, since $u(\mathbf{q}(s_1)) \le u_0 - \varepsilon < u(\mathcal{A}_+)$ by Lemma 3.9. It is not $\mathcal{A}_-$ either, because there $x_3 = X_x$, and $E_{s_1} = 0$ would give $u(\mathcal{A}_-) = u_0 - \varepsilon$, against $u_0 - u(\mathcal{A}_-) \ge 0.1055 > \varepsilon$. So $s_1 = 1$, which proves (i). At $s = 1$, $x_3 = X_x$, so $\mathbf{q}(1) = \mathcal{A}_-$ by Lemma 3.7. Finally $u(\mathbf{q}(0)) = u_0 - \varepsilon$, and $u(\mathbf{q}(1)) = u(\mathcal{A}_-)$ is smaller than u at every point of $\mathcal{A}$. Part (iii) follows from the intermediate value theorem and the injectivity of u on $\mathcal{A}$. The other band is handled by the same argument, with q_4 and $\mathcal{A}_+$ in place of q_3 and $\mathcal{A}_-$. □

3.3. **Coverage.** Each certified family of Section 5 is a continuous family of solutions of $\Phi(q_3) = \Phi(q_4) = 1$, $F = 0$ and one more equation, enclosed over a range of ρ and over $0 \le \nu \le \frac{1}{2}$. The next lemma says when such a family meets every convex CC of the lines it crosses. In the lemma, the *minus band* and the *plus band* are the bands that start at the tube edges $\zeta = -\varepsilon$ and $\zeta = \varepsilon$, respectively. A band may end at a *cap*, a configuration at which one of the Dziobek mass ratios is negative.

Lemma 3.11 (coverage)**.** *Let $\mathcal{I} \subset (0, \frac{3}{4}]$ be an interval with closure $\bar{\mathcal{I}}$. Let Γ be a family of solutions (q_3, q_4) of $\Phi(q_3) = \Phi(q_4) = 1$ and $F = 0$ over $\bar{\mathcal{I}} \times [0, \frac{1}{2}]$. It consists of a tube, parametrized by $\zeta \in [-\varepsilon, \varepsilon]$ with $u - u_0 = \zeta$, and two bands that start at the tube edges $\zeta = \mp\varepsilon$. Assume:*

(C0) *Γ is continuous on the tube and on each band;*
(C1) *the tube lies strictly inside $(u(\mathcal{A}_-), u(\mathcal{A}_+))$ for every $(\rho, \nu) \in \mathcal{I} \times (0, \frac{1}{2}]$;*
(C2) *on $\mathcal{A}$, θ_{q_3}, $\pi - \theta_{q_4}$ and r_{34} are bounded below by positive constants, for all $(\rho, \nu) \in \mathcal{I} \times (0, \frac{1}{2}]$;*
(C3) *at one point $(\rho_0, \nu_0) \in \bar{\mathcal{I}} \times [0, \frac{1}{2}]$, the tube centre $\zeta = 0$ has $x_3 > X_x > x_4$, and q_3 and q_4 lie above the axis;*
(C4) *on the minus band, either $E_s = 0$ as in Lemma 3.10 for $s \in [0, 1]$, or $u - u_0$ decreases from $-\varepsilon$ to a cap where $m_3/m_2 < 0$; and the same on the plus band with $u - u_0$ increasing from ε and $m_4/m_2 < 0$.*

Then every convex CC with positive masses on a line $\mathcal{L}(\rho, \nu)$ with $\rho \in \mathcal{I}$ and $0 < \nu \le \frac{1}{2}$ is a point of Γ, on its tube or on a band.

Proof. Let $\mathcal{P}$ be the set of $(\rho, \nu) \in \mathcal{I} \times (0, \frac{1}{2}]$ at which the tube centre lies in $\mathcal{R}$. By (C3), (C0) and (L7), $\mathcal{P}$ is nonempty, and it is open because $\mathcal{R}$ is open. It is also closed. Indeed, at a limit of points of $\mathcal{P}$ the tube centre is a limit of points of arcs. By (C2) it is a zero of F on its line with $\theta_{q_3} \le \theta_X \le \theta_{q_4}$, so by Lemma 3.6(iii) it lies on $\mathcal{A}$ or equals $\mathcal{A}_-$ or $\mathcal{A}_+$. By (C1), $|u - u_0| > \varepsilon$ at $\mathcal{A}_-$ and $\mathcal{A}_+$, while $u = u_0$ at the centre. Therefore $\mathcal{P} = \mathcal{I} \times (0, \frac{1}{2}]$. The same open and closed argument in $\zeta \in [-\varepsilon, \varepsilon]$ shows that the whole tube lies on $\mathcal{A}$. By Lemma 3.6(iv), it covers the points of $\mathcal{A}$ with $|u - u_0| \le \varepsilon$.

Now take the minus band. If it satisfies $E_s = 0$, Lemma 3.10 covers the points of $\mathcal{A}$ with $u \le u_0 - \varepsilon$, because the band starts at a point of $\mathcal{A}$. Otherwise it starts on $\mathcal{A}$ and ends at a cap, which is not in $\mathcal{R}$ by Lemma 3.6(i). As in the proof of Lemma 3.10, it can leave $\mathcal{A}$ only through $\mathcal{A}_-$ or $\mathcal{A}_+$, and not through $\mathcal{A}_+$, since $u \le u_0 - \varepsilon < u(\mathcal{A}_+)$ on it by (C1). So it reaches $\mathcal{A}_-$, and by continuity and the injectivity of u on $\mathcal{A}$ it covers every point of $\mathcal{A}$ with $u \le u_0 - \varepsilon$. The plus band is handled in the same way. Hence Γ covers $\mathcal{A}$, and Lemma 3.6(ii) finishes the proof. □

The three instances are listed in Table 1. Condition (C1) holds for all $\rho \le \frac{3}{4}$ by Lemma 3.9(a). Lemma 3.9(b), evaluated at the right end of each range, gives (C2) with the constants of the table. (C3) is one exact evaluation of the family at a rational point. For the inner range it is the

configuration at $\rho = 0$, where $q_1 = q_2$ and $q_2q_3q_4$ is an equilateral triangle with $x_3 = \frac{1}{2} = -x_4$. (C0) and (C4) are part of the certified statements of Section 5.

range	families	bands	(C2): θ, r_{34} at least	(C3) at (ρ, ν)
inner, $0 < \rho \le \frac{1}{4}$	inner tube, inner band	caps	17.0°, 0.99874	$(0, \frac{1}{4})$, closed form
middle, $\frac{1}{4} \le \rho \le \frac{3}{5}$	middle	$E_s = 0$	6.96°, 0.9524	$(\frac{17}{40}, \frac{1}{4})$
outer, $\frac{3}{5} \le \rho \le \frac{3}{4}$	outer	$E_s = 0$	3.99°, 0.8769	$(\frac{27}{40}, \frac{1}{4})$

TABLE 1. The three instances of Lemma 3.11. At the two rational points, $x_3 - X_x$ equals 0.42906 and 0.36537, and $x_4 - X_x$ equals -0.55423 and -0.58582; both bodies lie above the axis. The enclosure radii are below 10^{-13}.

3.4. **Assembly.** Put

$$T(\rho) = \frac{1}{3} - \frac{253}{15360}\,\rho^2, \tag{3.2}$$

so that $T(\rho) < \frac{1}{3}$ for $\rho > 0$. The certified families bound μ_1 by $T(\rho)$ for $\rho \le \frac{3}{5}$, and by $\frac{1}{3}$ beyond. For $c > 0$ we use the matrix $\Pi(c) = W(cM_2^2 - C)$ of (5.1). At a convex CC, $W > 0$, so $\Pi(c) \succ 0$ if and only if $\mu_1 < c$ (Corollary 2.6).

Lemma 3.12. *Let $0 < \rho \le \frac{1}{4}$, $0 \le \nu \le 1$ and $|\zeta| \le \varepsilon$. At the configuration of the inner tube family, $W > 0$ and $\Pi(T(\rho)) \succ 0$. So $\mu_1 < T(\rho)$ at every convex CC of the inner tube.*

Proof. Let $f(\rho, \nu, \zeta) = \det\big(T(\rho)I - G\big)$ at the configuration of the family. By Proposition 5.4, $\partial_\nu^2 f < 0$ for $0 \le \nu \le \frac{1}{2}$. By Lemma 3.8, $f(\rho, 1-\nu, -\zeta) = f(\rho, \nu, \zeta)$, so $\partial_\nu^2 f < 0$ on all of $[0, 1]$, and $f(\rho, \nu, \zeta) \ge \min\big(f(\rho, 0, \zeta), f(\rho, 0, -\zeta)\big)$. At $\nu = 0$ the equations $\Phi(q_3) = \Phi(q_4) = 1$ reduce to $r_{23} = r_{24} = 1$, and $F = 0$ then forces $r_{34} = 1$. Moreover q_3 and q_4 lie above the axis with $\theta_{q_3} < \theta_{q_4}$ there. Indeed, for $0 < \nu \le \frac{1}{2}$ the tube lies on $\mathcal{A}$ (proof of Lemma 3.11), where Lemma 3.9(b) bounds θ_{q_3}, $\pi - \theta_{q_4}$ and r_{34} below, uniformly in ν, and these bounds pass to $\nu = 0$. Thus $q_2q_3q_4$ is the equilateral triangle with q_3, q_4 in counterclockwise order, which is the face configuration of Proposition 5.5 with the same u, and there $f > 0$; hence $f > 0$ on the whole tube. As W and $\det M_2$ do not vanish on the family (Proposition 5.4), $\det \Pi(T(\rho)) = W^2(\det M_2)^2 f > 0$ there. At $\nu = 0$, $W > 0$ and $\Pi(T(\rho)) \succ 0$ by Proposition 5.5. The tube is connected, so Proposition 5.2(b) gives the claim. □

Proposition 3.13. *Let q be a convex CC with roundness $\varpi \le \frac{3}{4}$. If $\varpi \le \frac{3}{5}$, then $\mu_1 < T(\varpi)$. If $\frac{3}{5} \le \varpi \le \frac{3}{4}$, then $\mu_1 < \frac{1}{3}$.*

Proof. Label q as in Lemma 3.3, scale to $\lambda' = 1$, and place it on its line $\mathcal{L}(\rho, \nu)$, with $\rho = \varpi$ (Lemma 3.4). If $\nu > \frac{1}{2}$, we replace q by its mirror image, which lies on $\mathcal{L}(\rho, 1-\nu)$ and has the same μ_1 (Lemma 3.8). By Lemma 3.11, with the instance of Table 1 for the range of ρ, q is a point of a certified family, on its tube or on a band.

- On the inner range, $\mu_1 < T(\rho)$ on the tube by Lemma 3.12. A band of the inner band family has $\det \Pi(T(\rho)) > 0$ by Proposition 5.6(a). By Proposition 5.2(a), it forms one continuous family with its tube edge, where $W > 0$ and $\Pi(T(\rho)) \succ 0$ by Lemma 3.12. Proposition 5.2(b) then gives $\mu_1 < T(\rho)$ on the band.
- On the middle range, $\mu_1 < T(\rho)$ by Proposition 5.7.
- On the outer range, $\mu_1 < \frac{1}{3}$ by Proposition 5.8. □

Proof of Theorem 3.1. Label q as in Lemma 3.3. If $\varpi > \frac{3}{4}$, then $\mu_1 < \frac{1}{3}$ by Theorem 4.1; otherwise the bound follows from Proposition 3.13. □

4. Round configurations

Let q be a convex CC with roundness ϖ (Definition 3.2). In this section we prove the following.

Theorem 4.1 (Round configurations)**.** *If $\varpi > \frac{3}{4}$, then $\mu_1 < \frac{1}{3}$.*

The proof works with the side lengths and angles of the quadrilateral. We write the inequality as the positivity of a quadratic form on the six edges, with one weight for each pair of edges that share a body (Section 4.1). The only negative weights sit at the obtuse angles of the quadrilateral. Each of them is paid for by two detours through the diagonals (Section 4.2). When $\varpi > \frac{3}{4}$, any two sides differ by a factor less than $\frac{4}{3}$, and this confines the obtuse angles so much that four inequalities, each in essentially one angle, suffice (Section 4.4). Their proofs are in Appendix A.

Conventions. Each statement below says whether q is a convex CC or only a convex quadrilateral. Vertices are named by letters, which stand for the bodies $1, \dots, 4$ in an order adapted to the case at hand (Corollary 2.6). For a vertex i, α_i is the interior angle at i, and $\chi_i = -\cos\alpha_i$ if $\alpha_i > 90^\circ$. When i is a vertex, j, k denote its two neighbours on the quadrilateral and o the opposite vertex, unless other names are given. Angles are displayed in degrees. For positive numbers $y_1, \dots, y_n$ we write

$$\operatorname{ser}(y_1, \dots, y_n) = \big(y_1^{-1} + \cdots + y_n^{-1}\big)^{-1}.$$

We also use the functions

$$f(\phi) = \cos\phi \sin^2\phi, \qquad g(\phi;\alpha) = \tan\phi\Big(\Big(\frac{\sin\alpha}{\sin\phi}\Big)^3 - 1\Big) \qquad (0 < \phi < 90^\circ).$$

4.1. **The signed Laplacian.** For three distinct bodies i, j, k put

$$\varkappa_{i;jk} = \frac{m_i m_j m_k \cos\angle jik}{r_{ij}^2 r_{ik}^2},$$

and for $x \in \mathbb{R}^6$, indexed by the edges, put

$$\mathcal{B}(x) = \sum_i \sum_{\{j,k\} \not\ni i} \varkappa_{i;jk}\,(x_{ij} - x_{ik})^2.$$

The inner sum runs over the three pairs of edges at i, so $\mathcal{B}$ has twelve terms, one for each *corner*, that is, each pair of edges with a common body. Let $\mathbb{1} = (1, \dots, 1) \in \mathbb{R}^6$; then $\mathcal{B}(\mathbb{1}) = 0$.

Lemma 4.2 (Signed Laplacian)**.** *Let q be a CC and $a_{ij} = m_j(q_j - q_i)/r_{ij}^3$ for $i \neq j$. For every $x \in \mathbb{R}^6$,*

$$M\lambda' \sum_e \frac{m_i m_j}{r_e}\, x_e^2 - \sum_i m_i \Big|\sum_{j \neq i} x_{ij} a_{ij}\Big|^2 = \mathcal{B}(x).$$

Proof. For vectors b_1, b_2, b_3 with sum b and numbers y_1, y_2, y_3, expanding the squares gives

$$\sum_{l<l'} \langle b_l, b_{l'}\rangle (y_l - y_{l'})^2 = \sum_l y_l^2 \langle b_l, b\rangle - \Big|\sum_l y_l b_l\Big|^2.$$

Fix i and apply this to the three vectors a_{ij} and the numbers x_{ij}, $j \neq i$. By (2.2), $\sum_{j\neq i} a_{ij} = \lambda' \sum_j m_j(q_j - q_i) = M\lambda'(c - q_i)$. Multiply by m_i and sum over i. Since $m_i\langle a_{ij}, a_{ik}\rangle = \varkappa_{i;jk}$, the left sides add up to $\mathcal{B}(x)$. In the first sum on the right, the edge ij collects $M\lambda' m_i m_j r_{ij}^{-3} \langle q_j - q_i, (c - q_i) - (c - q_j)\rangle x_{ij}^2 = M\lambda' m_i m_j r_{ij}^{-1} x_{ij}^2$. □

Lemma 4.3 (Reduction)**.** *Let q be a convex CC. If $\mathcal{B} \succeq 0$ and $\ker\mathcal{B} = \mathbb{R}\mathbb{1}$, then $\mu_1 < \frac{1}{3}$.*

Proof. For $v \in (\mathbb{R}^2)^4$ let $I(v) = \sum_i m_i|v_i - \bar{v}|^2$, with $\bar{v} = \frac{1}{M}\sum_i m_i w_i$, so that $MI(v) = \sum_{i<j} m_i m_j |v_i - v_j|^2$, and put

$$\mathcal{Y}(v) = \lambda I(v) - \tfrac{1}{3}K(v), \qquad \lambda = M\lambda'.$$

Step 1: $\frac{1}{3}K(v) - |\sigma||L(v)|^2 = \mathcal{Y}(Jv)$ *for every* v. For an edge e, $|v_e|^2 = \mathrm{d}r_e(v)^2 + \mathrm{d}r_e(Jv)^2$, because $\langle Jq_e, v_e\rangle = -\langle q_e, Jv_e\rangle$. Multiplying by $m_i m_j s_e$ and summing gives $\sum_e m_i m_j s_e |v_e|^2 = \frac{1}{3}K(v) + \frac{1}{3}K(Jv)$. By (2.3), $m_i m_j s_e = \lambda' m_i m_j + \sigma A_i A_j$, so by the computation in the proof of [Tom26a, Proposition 3.1] the same sum equals $\lambda I(v) + |\sigma||L(v)|^2$. Since $I(Jv) = I(v)$, Step 1 follows.

Step 2: $\mathcal{Y} \succeq 0$, *and* $\mathcal{Y}(v) = 0$ *only if* $v \in J\mathcal{T}$, *the span of the translations and* $q - c$. Fix v, put $x_e = \mathrm{d}r_e(v)/r_e$ and $b_i = \sum_{j\neq i} x_{ij} a_{ij}$. The terms of $\sum_i m_i b_i$ cancel in pairs, so $\sum_i m_i b_i = 0$, and

$$\tfrac{1}{3}K(v) = \sum_{i<j} m_i m_j s_{ij} x_{ij} \langle q_i - q_j, v_i - v_j\rangle = -\sum_i m_i \langle b_i, v_i - \bar{v}\rangle.$$

Since $\sum_e (m_i m_j / r_e) x_e^2 = \frac{1}{3}K(v)$, the Cauchy–Schwarz inequality and Lemma 4.2 give

$$\big(\tfrac{1}{3}K(v)\big)^2 \le I(v)\sum_i m_i |b_i|^2 = I(v)\big(\tfrac{1}{3}\lambda K(v) - \mathcal{B}(x)\big).$$

As $\mathcal{B} \succeq 0$, this yields $\frac{1}{3}K(v) \le \lambda I(v)$ if $K(v) > 0$, and trivially if $K(v) = 0$. Hence $\mathcal{Y} \succeq 0$. Let $\mathcal{Y}(v) = 0$. If $K(v) > 0$, equality holds throughout, so $\mathcal{B}(x) = 0$ and $x = t\mathbb{1}^{\perp}$ for some $t \in \mathbb{R}$. Then $\mathrm{d}r_e(v - tq) = 0$ for every e, because $\mathrm{d}r_e(q) = r_e$, and $v - tq \in \mathcal{T}$ by Lemma 2.2. If $K(v) = 0$, then $v \in \mathcal{T}$, by the same lemma. In both cases $v = v_0 + t' J(q - c)$ with $v_0 \in J\mathcal{T}$ and $t' \in \mathbb{R}$. Now $\mathcal{Y}$ vanishes on translations, and $\mathcal{Y}(q - c) = \lambda I(q) - U = 0$, because $K(q) = 3\sum_e m_i m_j / r_e = 3U$. The zero set of the positive semidefinite form $\mathcal{Y}$ is its kernel, a linear space; it contains v and v_0, hence $t' J(q - c)$. But $\mathcal{Y}(J(q - c)) = \lambda I(q) > 0$, since K vanishes on $\mathcal{T}$. So $t' = 0$.

Step 3. Let $v \notin \mathcal{T}$. Then $Jv \notin J\mathcal{T}$, and Steps 1 and 2 give $|\sigma||L(v)|^2 < \frac{1}{3}K(v)$, that is, $Q(v) > \frac{2}{3}K(v)$. By Proposition 2.5(c), $1 - \mu_1$ is the smallest generalized eigenvalue of the pair (Q, K) on $(\mathbb{R}^2)^4/\mathcal{T}$. It is attained, so $1 - \mu_1 > \frac{2}{3}$. □

4.2. **Detours.** We regard $\mathcal{B}$ as a weighted graph: its vertices are the six edges of q, and two of them are joined when they form a corner. This graph is an octahedron, in which the two diagonals of q are opposite vertices. If j is a neighbour of i and o the vertex opposite i, we call $\varkappa_{i;jo}$ a *spoke*.

Lemma 4.4 (Signs)**.** *At a convex CC every spoke is positive, and* $\varkappa_{i;jk}$ *has the sign of* $\cos\alpha_i$. *Each diagonal forms a spoke with each of the four sides.*

Proof. In the triangle ijo, io is a diagonal of q and ij, jo are sides, so by Proposition 2.3(a) io is its longest side. Hence $\angle jio$ is acute and $\varkappa_{i;jo} > 0$. The angle $\angle jik$ is α_i. The diagonal io forms spokes with ij, ik at i and with oj, ok at o. □

Thus the negative weights of $\mathcal{B}$ are the corners $\varkappa_{i;jk}$ at the obtuse vertices i. We compensate them along paths of positive weights.

Lemma 4.5 (Path lemma)**.** *Let* $\mathcal{N}$ *be a finite set, and let* E^+, E^- *be disjoint sets of two-element subsets of* $\mathcal{N}$. *For* $x \in \mathbb{R}^{\mathcal{N}}$ *and* $e = \{e_-, e_+\}$ *put* $\partial_e x = x_{e_+} - x_{e_-}$, *and let*

$$\mathcal{B}(x) = \sum_{e\in E^+} \varkappa_e (\partial_e x)^2 - \sum_{f\in E^-} |\varkappa_f| (\partial_f x)^2, \qquad \varkappa_e > 0 \ (e \in E^+).$$

For each $f \in E^-$ let $\mathcal{P}_f$ be a set of paths in the graph $(\mathcal{N}, E^+)$ joining the two elements of f, and let $\theta_{e,\Gamma} > 0$ for $e \in \Gamma \in \mathcal{P}_f$, such that $\sum_f \sum_{\Gamma\in\mathcal{P}_f,\,\Gamma\ni e} \theta_{e,\Gamma} \le 1$ for every $e \in E^+$. If

$$\sum_{\Gamma\in\mathcal{P}_f} \operatorname{ser}\big(\theta_{e,\Gamma}\varkappa_e : e \in \Gamma\big) \;\ge\; |\varkappa_f| \qquad \text{for every } f \in E^-,$$

then $\mathcal{B} \succeq 0$. If all these inequalities are strict and $(\mathcal{N}, E^+)$ is connected, then $\ker \mathcal{B} = \mathbb{R}\mathbb{1}$.

Proof. Let $\Gamma \in \mathcal{P}_f$ have the edges $e_1, \dots, e_n$. Then $\partial_f x = \sum_l \pm\partial_{e_l} x$, and the Cauchy–Schwarz inequality gives

$$\operatorname{ser}\big(\theta_{e,\Gamma}\varkappa_e : e \in \Gamma\big)(\partial_f x)^2 \le \sum_{e\in\Gamma} \theta_{e,\Gamma}\varkappa_e(\partial_e x)^2.$$

Summing over Γ and f and using the bound on the capacities,

$$\mathcal{B}(x) \;\ge\; \sum_{f\in E^-} \Big(\sum_{\Gamma\in\mathcal{P}_f} \operatorname{ser}\big(\theta_{e,\Gamma}\varkappa_e : e \in \Gamma\big) - |\varkappa_f| \Big)(\partial_f x)^2 \;\ge\; 0.$$

If the inequalities are strict and $\mathcal{B}(x) = 0$, then $\partial_f x = 0$ for every $f \in E^-$, so $\sum_{e\in E^+} \varkappa_e(\partial_e x)^2 = 0$ and x is constant on each connected component of $(\mathcal{N}, E^+)$. □

Let $\alpha_i > 90°$. In the octahedron, the ends ij and ik of the negative corner $\varkappa_{i;jk}$ are joined by exactly two paths of two spokes: the *star path* ij, io, ik, through $\varkappa_{i;jo}$ and $\varkappa_{i;ko}$, and the *triangle path* ij, jk, ik, through $\varkappa_{j;ik}$ and $\varkappa_{k;ij}$. The spoke $\varkappa_{i;jo}$ lies on the star path of i and on the triangle path of j, and on no other such path. Put

$$\mathcal{V}_1(i) = \frac{|\varkappa_{i;jk}|}{\operatorname{ser}(\varkappa_{i;jo}, \varkappa_{i;ko})}, \qquad \mathcal{V}_2(i) = \frac{|\varkappa_{i;jk}|}{\operatorname{ser}(\varkappa_{j;ik}, \varkappa_{k;ij})}, \qquad \mathcal{V}(i) = \operatorname{ser}\big(\mathcal{V}_1(i), \mathcal{V}_2(i)\big).$$

Thus $\mathcal{V}(i) < 1$ if and only if $\operatorname{ser}(\varkappa_{i;jo}, \varkappa_{i;ko}) + \operatorname{ser}(\varkappa_{j;ik}, \varkappa_{k;ij}) > |\varkappa_{i;jk}|$.

Lemma 4.6 (Ratios)**.** *Let q be a convex CC and $\alpha_i > 90°$. Put $\phi_j = \angle jio$, $\phi_k = \angle kio$, $\beta = \angle ijk$ and $\gamma = \angle ikj$. Then*

$$\frac{|\varkappa_{i;jk}|}{\varkappa_{i;jo}} = \frac{|\cot\alpha_i|\, g(\phi_j;\alpha_j)}{1 - (\sin\beta/\sin\alpha_i)^3}, \qquad \frac{|\varkappa_{i;jk}|}{\varkappa_{j;ik}} = \frac{f(180° - \alpha_i)}{f(\beta)},$$

and the same holds with (j, ϕ_j, β) replaced by (k, ϕ_k, γ).

Proof. We have $|\varkappa_{i;jk}|/\varkappa_{i;jo} = |\cos\alpha_i|\, m_k r_{io}^2/(m_o r_{ik}^2 \cos\phi_j)$. The area $|A_k|$ is that of the triangle ijo, namely $\frac{1}{2} r_{ij} r_{io} \sin\phi_j$, and $|A_o| = \frac{1}{2} r_{ij} r_{ik} \sin\alpha_i$. By (2.3) for the edges ik and io, $m_k/m_o = |A_k w_{io}|/|A_o w_{ik}|$. The pairs $\{ik, jo\}$ and $\{io, jk\}$ are complementary, so Proposition 2.3(b) gives $w_{ik} w_{jo} = w_{io} w_{jk}$, where $w_{ik}, w_{jo} > 0 > w_{io}, w_{jk}$. Taking the mediant of the two equal ratios,

$$\frac{|w_{io}|}{w_{ik}} = \frac{w_{jo}}{|w_{jk}|} = \frac{|w_{io}| + w_{jo}}{w_{ik} + |w_{jk}|} = \frac{s_{jo} - s_{io}}{s_{ik} - s_{jk}}.$$

Together,

$$\frac{|\varkappa_{i;jk}|}{\varkappa_{i;jo}} = |\cot\alpha_i| \tan\phi_j \Big(\frac{r_{io}}{r_{ik}}\Big)^3 \frac{s_{jo} - s_{io}}{s_{ik} - s_{jk}} = |\cot\alpha_i| \tan\phi_j \, \frac{(r_{io}/r_{jo})^3 - 1}{1 - (r_{ik}/r_{jk})^3}.$$

The law of sines in the triangles ijo and ijk gives $r_{io}/r_{jo} = \sin\alpha_j/\sin\phi_j$ and $r_{ik}/r_{jk} = \sin\beta/\sin\alpha_i$; the angle of ijo at j is α_j. For the second formula, note that $|\varkappa_{i;jk}|/\varkappa_{j;ik}$ equals $|\cos\alpha_i|\, r_{jk}^2/(r_{ik}^2 \cos\beta)$, and that $r_{jk}/r_{ik} = \sin\alpha_i/\sin\beta$. □

Consequently

$$\begin{aligned}\mathcal{V}_1(i) &= |\cot\alpha_i|\Big[\frac{g(\phi_j;\alpha_j)}{1-(\sin\beta/\sin\alpha_i)^3}+\frac{g(\phi_k;\alpha_k)}{1-(\sin\gamma/\sin\alpha_i)^3}\Big],\\ \mathcal{V}_2(i) &= f(180^\circ-\alpha_i)\Big[\frac{1}{f(\beta)}+\frac{1}{f(\gamma)}\Big].\end{aligned}\tag{4.1}$$

Like $\mathcal{V}_1$ and $\mathcal{V}_2$, every ratio of two corner weights that occurs below is a function of the shape of q alone, given by Lemma 4.6. For a convex quadrilateral whose diagonals are longer than its sides we define these ratios, and $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}$, by the same formulas. Then $\phi_j, \phi_k, \beta, \gamma$ are acute, and the denominators are positive, because $\sin\beta/\sin\alpha_i = r_{ik}/r_{jk} < 1$.

We now state the criteria that we use. If exactly two angles are obtuse and adjacent, we call the larger one the *hub* h and the other the *partner* p (either choice if they are equal), and we label the vertices h, p, o, k in cyclic order. The negative corners are $\varkappa_{h;pk}$ and $\varkappa_{p;ho}$. Consider

$$\text{(P)}\qquad \mathcal{V}_1(p) < 1, \quad \text{that is,} \quad \operatorname{ser}(\varkappa_{p;hk}, \varkappa_{p;ok}) > |\varkappa_{p;ho}|.$$

Under (P), $\varkappa_{p;ok} > |\varkappa_{p;ho}|$, and the number

$$\theta_p = \frac{|\varkappa_{p;ho}|/\varkappa_{p;hk}}{1-|\varkappa_{p;ho}|/\varkappa_{p;ok}} \in (0,1)\tag{4.2}$$

satisfies $\operatorname{ser}(\theta_p\varkappa_{p;hk}, \varkappa_{p;ok}) = |\varkappa_{p;ho}|$. The partner keeps the share θ_p of the corner $\varkappa_{p;hk}$, and the hub receives the rest:

$$\text{(H)}\qquad H := \frac{1}{\mathcal{V}_1(h)} + \frac{\operatorname{ser}\big((1-\theta_p)\varkappa_{p;hk}, \varkappa_{k;hp}\big)}{|\varkappa_{h;pk}|} > 1.$$

If exactly three angles are obtuse, they are at consecutive vertices a, m, b, and we label the vertices a, m, b, d in cyclic order. The negative corners are $\varkappa_{a;md}$, $\varkappa_{m;ab}$ and $\varkappa_{b;md}$. Four spokes are demanded by two obtuse vertices: $\varkappa_{a;mb}$ and $\varkappa_{m;ad}$ by a and m, and $\varkappa_{m;bd}$ and $\varkappa_{b;ma}$ by m and b. Each of these is shared in proportion to χ: of a spoke demanded by i and j, the vertex i receives the fraction $\chi_i/(\chi_i+\chi_j)$. For $i \in \{a, m, b\}$, let ρ_i be the sum of the series conductances of the star path and the triangle path of i, computed with the received weights, divided by the absolute value of the negative corner at i. Explicitly, with $\theta_{ij} = \chi_i/(\chi_i+\chi_j)$,

$$\begin{aligned}\rho_a|\varkappa_{a;md}| &= \operatorname{ser}(\theta_{am}\varkappa_{a;mb}, \varkappa_{a;db}) + \operatorname{ser}(\theta_{am}\varkappa_{m;ad}, \varkappa_{d;am}),\\ \rho_m|\varkappa_{m;ab}| &= \operatorname{ser}(\theta_{ma}\varkappa_{m;ad}, \theta_{mb}\varkappa_{m;bd}) + \operatorname{ser}(\theta_{ma}\varkappa_{a;mb}, \theta_{mb}\varkappa_{b;ma}),\\ \rho_b|\varkappa_{b;md}| &= \operatorname{ser}(\theta_{bm}\varkappa_{b;ma}, \varkappa_{b;da}) + \operatorname{ser}(\theta_{bm}\varkappa_{m;bd}, \varkappa_{d;bm}).\end{aligned}$$

Proposition 4.7 (Criteria)**.** *Let q be a convex CC. Then $\mathcal{B} \succeq 0$ and $\ker\mathcal{B} = \mathbb{R}\mathbb{K}$ in each of the following cases:*

(a) *no angle of q is obtuse;*
(b) *no two obtuse vertices are adjacent, and $\mathcal{V}(i) < 1$ at every obtuse vertex i;*
(c) *exactly two angles are obtuse, they are adjacent, and* (P) *and* (H) *hold;*
(d) *exactly three angles are obtuse, and $\rho_a, \rho_m, \rho_b > 1$.*

Proof. We apply Lemma 4.5 with $\mathcal{N}$ the six edges, E^- the corners at the obtuse vertices, and E^+ the corners with positive weight; corners of weight 0 are omitted. By Lemma 4.4, E^+ contains the eight spokes, so $(\mathcal{N}, E^+)$ is connected. Every negative corner $\varkappa_{i;jk}$ uses its star path and its triangle path, and only spokes carry capacity.

(a) Here $E^- = \emptyset$.

(b) All capacities are 1. A spoke lies on paths of at most two vertices, and these are adjacent, so it is used at most once. The required inequality at i is $\mathcal{V}(i) < 1$.

(c) By (H), there is $\theta'_p \in (\theta_p, 1)$ for which (H) still holds with θ'_p in place of θ_p. The partner uses its star path, with capacity θ'_p on $\varkappa_{p;hk}$ and 1 on $\varkappa_{p;ok}$, and not its triangle path. Since $\theta'_p > \theta_p$, $\operatorname{ser}(\theta'_p\varkappa_{p;hk}, \varkappa_{p;ok}) > |\varkappa_{p;ho}|$. The hub uses its star path with capacities 1, and its triangle path with capacity $1-\theta'_p$ on $\varkappa_{p;hk}$ and 1 on $\varkappa_{k;hp}$. Then the total on $\varkappa_{p;hk}$ is 1. The spokes $\varkappa_{h;po}$, $\varkappa_{h;ko}$, $\varkappa_{k;hp}$ and $\varkappa_{p;ok}$ lie on paths of one obtuse vertex only, since o and k are not obtuse. The inequality at h is (H) with θ'_p.

(d) The fractions received from a shared spoke add up to 1, and every other spoke lies on paths of one obtuse vertex only. The inequality at i is $\rho_i > 1$. □

4.3. **Order relations and auxiliary functions.** For $0 < r_0 < 1$, let $\mathrm{Round}(r_0)$ be the set of convex quadrilaterals in which every diagonal is longer than every side and the shortest side is at least r_0 times the longest side. By Lemma 3.3(b), a convex CC with $\varpi > \frac{3}{4}$ lies in $\mathrm{Round}(\frac{3}{4})$. In $\mathrm{Round}(\frac{3}{4})$ the ratio of any two sides lies in $[\frac{3}{4}, \frac{4}{3}]$. The next two lemmas hold at every convex CC.

Lemma 4.8 (Two adjacent obtuse angles)**.** *Let q be a convex CC whose only obtuse angles are at adjacent vertices h, p, with the vertices h, p, o, k in cyclic order. Then hp is the strictly shortest side and ok the strictly longest side.*

Proof. Write $\ell_1 = r_{hp}$, $\ell_2 = r_{po}$, $\ell_3 = r_{ok}$, $\ell_4 = r_{kh}$. The law of cosines for the diagonal pk in the triangles khp and pok, and for ho in hpo and okh, gives

$$\ell_1^2 + \ell_4^2 < r_{pk}^2 \le \ell_2^2 + \ell_3^2, \qquad \ell_1^2 + \ell_2^2 < r_{ho}^2 \le \ell_3^2 + \ell_4^2, \tag{4.3}$$

since $\alpha_h, \alpha_p > 90^\circ \ge \alpha_o, \alpha_k$. By Proposition 2.3(a), (b), $w_{hp}w_{ok} = w_{po}w_{kh}$ with four positive factors, and w_e decreases with r_e. If $\ell_3 \le \ell_2$, then $w_{ok} \ge w_{po}$, so $w_{hp} \le w_{kh}$ and $\ell_1 \ge \ell_4$, which contradicts the second inequality of (4.3). If $\ell_3 \le \ell_4$, then $w_{hp} \le w_{po}$ and $\ell_1 \ge \ell_2$, which contradicts the first. So $\ell_3 > \ell_2, \ell_4$. Then $w_{hp} = w_{po}w_{kh}/w_{ok} > w_{kh}, w_{po}$, that is, $\ell_1 < \ell_2, \ell_4$. Adding the two inequalities of (4.3) gives $\ell_1 < \ell_3$. □

Lemma 4.9 (Opposite vertices)**.** *Let q be a convex CC, let i, o be opposite vertices and j, k the other two. If $\alpha_i \ge \alpha_o$, then $r_{ij} \le r_{oj}$ and $r_{ik} \le r_{ok}$, with strict inequalities if $\alpha_i > \alpha_o$.*

Proof. By Proposition 2.3(a), (b), $w_{ij}w_{ok} = w_{ik}w_{oj}$ with four positive factors, so $r_{ij} < r_{oj}$ if and only if $r_{ik} < r_{ok}$, and likewise with equality. Since jk is a diagonal, the angles at j and k of the triangles jik and jok are acute.

We show that $r_{oj} > r_{ij}$ implies $\alpha_o < \alpha_i$. Let $\mathcal{O}$ be the circle through j, i, k, let $\mathcal{K}$ be the closed region bounded by the chord jk and the arc of $\mathcal{O}$ through i, and let $\mathbb{D}_j$, $\mathbb{D}_k$ be the closed discs of radii r_{ij}, r_{ik} about j, k. Then $\mathcal{K} \subset \mathbb{D}_j \cup \mathbb{D}_k$. Indeed, the arc from j to i subtends the central angle $2\angle jki < 180^\circ$, so the distance to j increases along it, and the arc lies in $\mathbb{D}_j$. So does the circular segment between this arc and the chord ji, since each of its points lies on a segment from j to a point of the arc. In the same way the arc from i to k and its segment lie in $\mathbb{D}_k$. The foot i' of the altitude from i lies on the segment jk, and the triangles jii' and $i'ik$ lie in $\mathbb{D}_j$ and $\mathbb{D}_k$, because on a triangle the distance to a point is largest at a vertex. Let o' be the reflection of o in the line jk. It lies on the same side as i, and $\angle jo'k = \alpha_o$. If $r_{oj} > r_{ij}$, then also $r_{ok} > r_{ik}$, so $o' \notin \mathbb{D}_j \cup \mathbb{D}_k$. Hence $o' \notin \mathcal{K}$, o' lies outside $\mathcal{O}$, and $\alpha_o < \alpha_i$.

Exchanging i and o: if $r_{ij} > r_{oj}$, then $\alpha_i < \alpha_o$. Hence $\alpha_i \ge \alpha_o$ gives $r_{ij} \le r_{oj}$ and $r_{ik} \le r_{ok}$. If $r_{ij} = r_{oj}$, then $r_{ik} = r_{ok}$, so $o' = i$ and $\alpha_o = \alpha_i$. □

Lemma 4.10 (Auxiliary functions)**.** (a) *For fixed α, $g(\phi;\alpha)$ is decreasing in $\phi \in (0^\circ, 90^\circ)$; for fixed ϕ, it is increasing in $\sin\alpha$. Hence $g(\phi;\alpha) \le g_0(\phi) := g(\phi;90^\circ) = \tan\phi\,(\csc^3\phi - 1)$, and g_0 is decreasing.*

(b) $g_1(\phi) := g(\phi; 180^\circ - 2\phi) = \tan\phi\,(8\cos^3\phi - 1) = 2\sin\phi + 2\sin 3\phi - \tan\phi$ *is concave on* $(0^\circ, 60^\circ)$ *and decreasing on* $[35^\circ, 90^\circ)$.

(c) g_0 *is convex on* $(0^\circ, \arctan\sqrt{6}]$, *where* $\arctan\sqrt{6} = 67.79^\circ$. *Moreover* $-g_0'/g_0 \le \Lambda$, *where* $\Lambda(\phi) = \sec\phi + \cos\phi\csc^2\phi$ *is convex on* $(0^\circ, 90^\circ)$.

(d) f *is increasing on* $[0^\circ, \phi_f]$ *and decreasing on* $[\phi_f, 90^\circ]$, *where* $\phi_f = \arctan\sqrt{2} \approx 54.74^\circ$, *and* $1/f$ *is convex on* $(0^\circ, 90^\circ)$.

Proof. (a) With $w = \sin\alpha/\sin\phi$ and $x = \sin\phi$, $\partial_\phi g = [w^3(1 - 3\cos^2\phi) - 1]/\cos^2\phi$. This is negative if $3\cos^2\phi \ge 1$. Otherwise $w \le 1/x$, and the numerator is at most $(3x^2 - x^3 - 2)/x^3 = (x-1)(2 + 2x - x^2)/x^3 < 0$. The second statement is clear, and $g(\phi;\alpha) \le g(\phi; 90^\circ)$.

(b) The two expressions agree because $8\cos^2\phi\sin\phi = 2\sin\phi + 2\sin 3\phi$. Then $g_1'' = -2\sin\phi - 18\sin 3\phi - 2\sec^2\phi\tan\phi < 0$ on $(0^\circ, 60^\circ)$. Further $g_1' = 2\cos\phi + 6\cos 3\phi - \sec^2\phi = -p_5(\cos\phi)/\cos^2\phi$ with $p_5(t) = 16t^3 - 24t^5 + 1$. Since $p_5'(t) = 24t^2(2 - 5t^2)$, p_5 increases on $[0, \sqrt{2/5}]$ and decreases after; $p_5(0) = 1$ and $p_5(\cos 35^\circ) > 0.94$. So $p_5 > 0$ on $[0, \cos 35^\circ]$.

(c) With $x = \sin\phi$ and $y = \cos\phi$, $g_0'' = 2(1-x)/y^3 + 6y/x^4 - 1/(yx^2)$, and $6y/x^4 \ge 1/(yx^2)$ if and only if $\tan^2\phi \le 6$. Next, $(\Lambda + g_0'/g_0)\,yx^2(1 - x^3) = (1-x)^2(1-x^2) \ge 0$. Finally $\sec\phi$ is convex, and $(\cos\phi\csc^2\phi)'' = 5\csc^3\phi\cot\phi + \csc\phi\cot^3\phi > 0$.

(d) $f'(\phi) = \sin\phi\,(3\cos^2\phi - 1)$, and $(\log(1/f))'' = \sec^2\phi + 2\csc^2\phi > 0$. □

4.4. **The four inequalities.** By Proposition 4.7, Theorem 4.1 follows from four inequalities on $\mathrm{Round}(\frac{3}{4})$, one for each pattern of obtuse angles. Each is a verification in essentially one angle. We state them here; the proofs are in Appendix A.

Lemma 4.11. *Let* $q \in \mathrm{Round}(\frac{3}{4})$ *and* $\alpha_i > 90^\circ$. *Then* $\mathcal{V}(i) < 1$.

Sketch. A ratio of sides of at most $\frac{4}{3}$ forces $\angle jio, \angle kio < \arccos\frac{3}{8}$, so $\alpha_i < 136^\circ$. Each term of $\mathcal{V}_1(i)$ is at most an explicit function of α_i times a function of one angle, and $\mathcal{V}_2(i)$ is at most its value on the triangle with sides $\frac{3}{4}$ and 1. The constraint $\angle jio + \angle kio = \alpha_i$ is handled with tangent lines. What remains is the bound $\mathcal{V}_2(i) < 1$ for $\alpha_i \le 98.5^\circ$, and a verification of $\mathcal{V}(i) < 1$ on a subdivision of the remaining range of α_i (Appendix A.1).

Lemma 4.12. *Let* $q \in \mathrm{Round}(\frac{3}{4})$ *have exactly two obtuse angles, at adjacent vertices* h *and* p, *with* $\alpha_h \ge \alpha_p$. *Let the vertices* h, p, o, k *be in cyclic order, let* hp *be a shortest side and* ok *a longest side. Then* $\mathcal{V}_1(p) < 0.92759$. *In particular* (P) *holds.*

Sketch. Projecting $o - k$ onto the line hp gives $\chi_h + \chi_p \le \frac{1}{3}$. The two terms of $\mathcal{V}_1(p)$ are bounded by explicit functions of χ_p that increase on $(0, \frac{1}{6}]$, and at $\chi_p = \frac{1}{6}$ their sum is below 0.92759 (Appendix A.2).

Lemma 4.13. *Under the hypotheses of Lemma 4.12,* $H > 1$*; that is,* (H) *holds.*

Sketch. The two triangle corners at the hub are ordered, which bounds H below by a function of $\mathcal{V}_1(h)$, $\mathcal{V}_2(h)$ and θ_p. The hub star, the triangle detour and the share θ_p are bounded by monotone functions of χ_h; the inequality $H > 1$ is then verified cell by cell in χ_h (Appendix A.2).

Lemma 4.14. *Let* $q \in \mathrm{Round}(\frac{3}{4})$ *have obtuse angles exactly at three consecutive vertices* a, m, b, *with* $\alpha_a \ge \alpha_b$. *Let the vertices* a, m, b, d *be in cyclic order, and assume*

$$r_{am} \le r_{mb}, \qquad r_{ad} \le r_{bd}, \qquad r_{am} \le r_{ad}, \qquad r_{mb} \le r_{bd}. \tag{4.4}$$

Then $\rho_a > 1.086$, $\rho_b > 1.077$ *and* $\rho_m > 1.032$.

Sketch. Two projections give $\chi_a + \chi_m \le \frac{1}{3}$ and $\chi_b + \frac{3}{4}\chi_m \le \frac{1}{4}$. With the law of tangents and one concavity argument, each of the eight terms of ρ_a, ρ_b, ρ_m is bounded below by a function of one angle, α_a, α_b or $180^\circ - \alpha_m$; the stated bounds on ρ_a, ρ_b, ρ_m follow from a computation on cells in these angles (Appendix A.3).

4.5. **Proof of Theorem 4.1.** Let q be a convex CC with $\varpi > \frac{3}{4}$. By Lemma 3.3(b), $q \in \mathrm{Round}(\frac{3}{4})$. The four angles sum to 360°, so at most three of them are obtuse, and three obtuse vertices are consecutive. We verify the hypothesis of Proposition 4.7 in each case.

- If no angle is obtuse, (a) applies.
- If one angle is obtuse, or two at opposite vertices, Lemma 4.11 gives $\mathcal{V}(i) < 1$ at each obtuse vertex i, and (b) applies.
- If two angles are obtuse and adjacent, the hypotheses of Lemma 4.12 hold by Lemma 3.3(b) and Lemma 4.8. Lemmas 4.12 and 4.13 give (P) and (H), and (c) applies.
- If three angles are obtuse, name the ends a, b so that $\alpha_a \geq \alpha_b$. Lemma 4.9 gives (4.4), Lemma 4.14 gives $\rho_a, \rho_m, \rho_b > 1$, and (d) applies.

In every case $\mathcal{B} \succeq 0$ and $\ker \mathcal{B} = \mathbb{R}\mathbb{1}$, so $\mu_1 < \frac{1}{3}$ by Lemma 4.3. □

5. The certified families

This section proves the five propositions that Theorem 3.1 uses, one for each family:

- the inner tube family: $0 < \rho \leq \frac{1}{4}$, $|\zeta| \leq \varepsilon$ (Proposition 5.4);
- the inner face family: $\nu = 0$, $0 < \rho \leq \frac{1}{4}$, $|u - u_0(\rho, 0)| \leq \varepsilon$ (Proposition 5.5);
- the inner band family: $0 < \rho \leq \frac{1}{4}$, off the tube, out past the chart ends (Proposition 5.6);
- the middle family: $\frac{1}{4} \leq \rho \leq \frac{3}{5}$, the tube and both bands (Proposition 5.7);
- the outer family: $\frac{3}{5} \leq \rho \leq \frac{3}{4}$, the tube and both bands (Proposition 5.8).

Except for the inner face family, whose configurations are explicit, each family is a continuous zero Γ of the chart equations, enclosed by a computation. Lemma 3.11 shows that these zeros meet every convex CC of their ranges, and Proposition 3.13 combines the propositions. Four families are certified by one method, which we describe first. The inner face family needs only univariate certificates.

5.1. **The method.** Throughout, $\lambda' = 1$, so $w_e = r_e^{-3} - 1$ and $d_e = \frac{1}{3} A_i A_j \, r_e^2 (r_e^3 - 1)$ for $e = ij$. These formulas define d_e, W and M_2 for every planar configuration, and $\bar{n}$ and C wherever $W \neq 0$. For $c > 0$ put

$$\Pi(c) = W\,(cM_2^2 - C) = cWM_2^2 - W \sum_e d_e \, n_e n_e^{\mathsf{T}} + \Big(\sum_e d_e n_e\Big)\Big(\sum_e d_e n_e\Big)^{\mathsf{T}}. \tag{5.1}$$

The second form needs no inverse of M_2 and no division by W. At a convex CC we have $W > 0$, and Corollary 2.6 gives

$$\Pi(c) \succ 0 \iff \mu_1 < c, \qquad \det \Pi(c) = W^2 (\det M_2)^2 \, (c - \mu_1)(c - \mu_2).$$

We take $c = T(\rho)$, as in (3.2), for $\rho \leq \frac{3}{5}$, and $c = \frac{1}{3}$ for the outer family.

The computed matrix. Let $\Sigma = \mathrm{diag}(1, 3^{-1/2})$. The programs work in the coordinates $(x, y/\sqrt{3})$ of the placement of Section 3.2, in which the configurations at $\rho = 0$ have rational coordinates. In these coordinates M_2 and C become $\Sigma M_2 \Sigma$ and $\Sigma C \Sigma$, and the programs evaluate

$$\frac{16}{\rho^4}\,\Sigma\,\Pi(c)\,\Sigma = \frac{4W}{\rho^2}\Big(3\Big(\frac{2}{\sqrt{3}\,\rho}\Sigma M_2 \Sigma\Big)\mathrm{diag}(c, 3c)\Big(\frac{2}{\sqrt{3}\,\rho}\Sigma M_2 \Sigma\Big) - \frac{4}{\rho^2}\,\Sigma C \Sigma\Big).$$

Here $\mathrm{diag}(c, 3c)$ acts in the coordinates $(x, y/\sqrt{3})$. The identity holds because $\Sigma\,\mathrm{diag}(c, 3c)\,\Sigma$ is c times the identity matrix. The powers of ρ keep all entries bounded as $\rho \to 0$ on the charts. The computed matrix is congruent to a positive multiple of $\Pi(c)$. Its determinant therefore has

the sign of $\det \Pi(c)$, and it is positive definite exactly when $\Pi(c)$ is. We state everything for $\Pi(c)$.

Charts. In the placement of Section 3.2, $q_1 = (-(1-\nu)\rho, 0)$ and $q_2 = (\nu\rho, 0)$. A configuration of a family is given by q_3 and q_4, which solve the lens equations $\Phi(q_3) = \Phi(q_4) = 1$ of Lemma 3.4, the equation $F = 0$ in the form

$$\rho^3 F = (1-\rho^3)\, w_{34} - \rho^3 w_{13} w_{24} = 0,$$

and one equation that fixes the position on the arc. In the *tube chart* it is

$$u - u_0(\rho, \nu) - \zeta = 0, \tag{5.2}$$

with u_0 as in (3.1). The middle and outer families cover the bands by the *band equation*. For $0 \le s \le 1$ it reads

$$(1-s)\big(u - u_0 + \varepsilon\big) + s\,(x_3 - X_x) = 0 \quad \text{or} \quad (1-s)\big(u - u_0 - \varepsilon\big) + s\,(x_4 - X_x) = 0 \tag{5.3}$$

on the minus band and on the plus band, where x_k is the abscissa of q_k. This is the equation $E_s = 0$ of Lemma 3.10. At $s = 0$ the band starts at the tube edge $\zeta = \mp\varepsilon$, and at $s = 1$ it reaches the bisector crossing. The inner band family uses the tube chart with $\varepsilon \le |\zeta| \le \bar{\zeta}_{\pm}(\rho, \nu)$ (Section 5.4). We compute only $0 \le \nu \le \frac{1}{2}$; Lemma 3.8 gives $\frac{1}{2} \le \nu \le 1$.

The algebra. We split the parameter region of a family into a few pieces. On a piece, $\rho = \rho_0 + a$ with $|a| \le a_0$, the third parameter (ζ or s) is an affine function of $b \in [-\frac{1}{2}, \frac{1}{2}]$, and ν runs over an interval $\mathcal{I}$. Let $\mathfrak{B}$ be the algebra of series $g = \sum_{k,l \ge 0} g_{kl}\, a^k b^l$ whose coefficients are functions on $\mathcal{I}$ with absolutely convergent Chebyshev series [Tre19], normed by

$$\|g\| = \sum_{k,l} \mathcal{J}(g_{kl})\, a_0^k \big(\tfrac{1}{2}\big)^l,$$

where $\mathcal{J}(p)$ is the sum of the absolute values of the Chebyshev coefficients of p on $\mathcal{I}$. For a family with several pieces we take the product of the pieces' algebras, with the largest of the norms. This is a commutative Banach algebra with $\|1\| = 1$. Its norm bounds the supremum of $|g|$ over $|a| \le a_0$, $|b| \le \frac{1}{2}$, $\nu \in \mathcal{I}$.

Enclosure. Let $\mathbf{q} \in \mathfrak{B}^4$ collect the coordinates of q_3 and q_4, and write the four chart equations as $\mathcal{H}(\mathbf{q}) = 0$. The five unknown distances enter through $r^{-1} = (r^2)^{-1/2}$, which we expand in convergent series on the balls below. Thus $\mathcal{H}$ is analytic there, and its derivative is multiplication by a Jacobian matrix $D\mathcal{H}(\mathbf{q})$ over $\mathfrak{B}$. We use the Newton–Kantorovich theorem in the following form [KA82, DLM07].

Lemma 5.1. *Let $\bar{\mathbf{q}} \in \mathfrak{B}^4$, let $\mathbf{A}$ be a 4×4 matrix over $\mathfrak{B}$, and let $\hat{r} > 0$. Use the maximum norm on $\mathfrak{B}^4$ and the induced norm on matrices. Suppose that*

$$\|\mathbf{A}\mathcal{H}(\bar{\mathbf{q}})\| \le Y_0, \qquad \|I - \mathbf{A}\, D\mathcal{H}(\bar{\mathbf{q}})\| \le Z_0, \qquad \|\mathbf{A}\,(D\mathcal{H}(\mathbf{q}) - D\mathcal{H}(\bar{\mathbf{q}}))\| \le Z_1 \ \text{ for } \|\mathbf{q} - \bar{\mathbf{q}}\| \le \hat{r},$$

and that $Y_0 + (Z_0 + Z_1)\hat{r} < \hat{r}$. Then $\mathcal{H}$ has exactly one zero $\mathbf{q}^*$ with $\|\mathbf{q}^* - \bar{\mathbf{q}}\| \le \hat{r}$, and $\|\mathbf{q}^* - \bar{\mathbf{q}}\| \le Y_0/(1 - Z_0 - Z_1)$.

Proof. Since $Z_0 < 1$, the matrix $\mathbf{A}\, D\mathcal{H}(\bar{\mathbf{q}})$ is invertible. As $\mathfrak{B}$ is commutative, $\det \mathbf{A}$ is then a unit, and $\mathbf{A}$ is invertible. The map $N(\mathbf{q}) = \mathbf{q} - \mathbf{A}\mathcal{H}(\mathbf{q})$ has derivative $I - \mathbf{A}\, D\mathcal{H}(\mathbf{q})$, of norm at most $Z_0 + Z_1 < 1$ on the ball. So N is a contraction there, and $\|N(\mathbf{q}) - \bar{\mathbf{q}}\| \le (Z_0 + Z_1)\hat{r} + Y_0 < \hat{r}$. By Banach's fixed point theorem N has exactly one fixed point in the ball, and the fixed points of N are the zeros of $\mathcal{H}$. The bound follows from $\|\mathbf{q}^* - \bar{\mathbf{q}}\| \le (Z_0 + Z_1)\|\mathbf{q}^* - \bar{\mathbf{q}}\| + Y_0$. □

The head $\bar{\mathbf{q}}$ is a truncated series with rational coefficients. We compute it in exact arithmetic, for the inner tube family order by order in ρ and otherwise by Newton steps in $\mathfrak{B}$ from a few printed rational seeds. The matrix $\mathbf{A}$ is a low-order truncation of an approximate inverse of

$D\mathcal{H}(\bar{\mathbf{q}})$. Neither needs to be accurate, since Lemma 5.1 checks them. We take $\hat{r} = 2Y_0/(1 - Z_0)$ and $Z_1 \geq \|\mathbf{A}\|\,\hat{r}\,\sup\|D^2\mathcal{H}\|$, with the supremum over the ball of radius $\hat{r}$ about the head, bounded in closed form. Then the hypothesis of Lemma 5.1 reduces to $2Z_1 < 1 - Z_0$, and $\mathbf{q}^*$ lies within $\hat{r}$ of the head. The constants also bound the corresponding quantities at each parameter point, because the norm dominates the supremum and the bound for the second derivatives is a closed form in such norms. Therefore, at each parameter point p, $\mathbf{q}^*$ gives the only solution of the chart equations within $\hat{r}$ of the head. We call it $\Gamma(p)$. It depends continuously on p.

Proposition 5.2. *Let a family be computed on finitely many pieces as above.*

(a) *Let pieces* 1 *and* 2*, of this family or of a neighbouring one, meet in a connected face $\mathcal{F}$ of their parameter regions. Let Y_0, Z_0, Z_1 be the constants of piece* 2*, and let $\hat{r}_1$ be the radius of piece* 1*. If the two heads differ by d at some point of $\mathcal{F}$ and $d + Y_0/(1 - Z_0 - Z_1) < \hat{r}_1$, then the two zeros agree on $\mathcal{F}$. If this holds on every face between pieces, the zeros form a single continuous zero Γ on the union of the pieces.*

(b) *Let $c > 0$, and let $\mathcal{P}$ be a connected set of parameter points. Suppose that $\det\Pi(c) > 0$ at $\Gamma(p)$ for every $p \in \mathcal{P}$, and that $\Pi(c) \succ 0$ at $\Gamma(p_0)$ for some $p_0 \in \mathcal{P}$. Then $\Pi(c) \succ 0$ at $\Gamma(p)$ for every $p \in \mathcal{P}$, and $\mu_1 < c$ at every $\Gamma(p)$ that is a convex CC. If moreover $W > 0$ at $\Gamma(p_0)$, then $W > 0$ at every $\Gamma(p)$.*

Proof. (a) At the given point the zero of piece 2 lies within $d + Y_0/(1 - Z_0 - Z_1) < \hat{r}_1$ of the head of piece 1, so it equals the zero of piece 1 by uniqueness. The set of points of $\mathcal{F}$ where the two zeros agree is closed. It is open, because near a point of agreement the zero of piece 2 stays within $\hat{r}_1$ of the head of piece 1. As $\mathcal{F}$ is connected, the zeros agree on all of $\mathcal{F}$.

(b) The eigenvalues of the symmetric matrix $\Pi(c)$ depend continuously on p and never vanish, so both stay positive on $\mathcal{P}$. At a convex CC, $W > 0$, and Corollary 2.6 gives $\mu_1 < c$. Finally $\det\Pi(c) = W^2\det(cM_2^2 - C) > 0$ shows that W has no zero on $\mathcal{P}$, so its sign is constant. □

Chains. We carry each derived quantity g as an exact head h and a tail $e \geq \|g - h\|$. Sums and products follow from the triangle inequality and submultiplicativity. A truncated product drops the orders above fixed degrees, and we bound the dropped part exactly from the coefficient norms of the factors. For a reciprocal or an inverse square root we take a head, bound its defect, and sum the geometric or binomial series. The chain runs from the distances through the areas, the weights d_e, W, the edge sums n_e and M_2 to the entries of the computed matrix, its determinant and its trace. For the inner tube family it ends with $f = \det(T(\rho)I - G)$ instead, through exact reciprocals of $\det M_2$ and W.

Sign certificates. We decide signs by the following elementary lemma, which rests on the range bound of the Bernstein form [Gar86, Far12].

Lemma 5.3. *Let B be a box, let g be a function on B, and let $h = h_1 + h_2$ be a polynomial with $|g - h| \leq e$ and $|h_2| \leq e'$ on B. If every Bernstein coefficient of h_1 on B exceeds $e + e'$, then $g > 0$ on B.*

Proof. The Bernstein polynomials of B are nonnegative and sum to 1, so h_1 is at least its least Bernstein coefficient. Hence $g \geq h_1 - |h_2| - |g - h| > 0$. □

We apply Lemma 5.3 with h the head of the quantity and e its tail. Each piece is a single box. We expand h exactly in products $T_iT_jT_m$ of Chebyshev polynomials in the three box variables, scaled to $[-1, 1]$. The block h_1 consists of the terms with $i \leq i_0$, $j \leq j_0$, $m \leq m_0$ for a printed triple (i_0, j_0, m_0). We convert it exactly to Bernstein form. Since $|T_i| \leq 1$, we may take for e' the sum of the absolute values of the other coefficients. The inner tube family uses monomials in centred variables in place of Chebyshev polynomials, with e' the sum of the other terms' maxima on the box.

What is verified. All computations use exact rational arithmetic, with norms and tails rounded upward; in print, least coefficients are rounded down and the bounds e and e' up. Each program decides every inequality exactly, including the hypotheses of Lemma 5.1, the hypothesis of Proposition 5.2(a) on every face between pieces, and each certificate. Floating point appears only in printing and in numerical cross-checks, which the proof does not use. For each family the accompanying data contain the programs described in this section, checksums of every file, and one script that repeats the computation and confirms every inequality; every head $\bar{\mathbf{q}}$ is fixed there by a checksum of its file or of its content. Table 2 collects the data of the four families certified by this method.

	inner tube	inner band	middle	outer
ρ	$0 \le \rho \le \frac{1}{4}$	$0 \le \rho \le \frac{1}{4}$	$\frac{17}{40} \pm \frac{7}{40}$	$\frac{27}{40} \pm \frac{3}{40}$
orders in (a,b); degree in ν	$(48,24)$; exact	$(48,33)$; 24	$(56,42)$; 40	$(48,60)$; 68
$\|\mathbf{A}\|$	≤ 1.53	≤ 2.22	≤ 2.42	≤ 2.84
Y_0	$\le 2.68 \cdot 10^{-13}$	$\le 5.17 \cdot 10^{-15}$	$\le 3.65 \cdot 10^{-15}$	$\le 8.97 \cdot 10^{-19}$
Z_0	$\le 1.63 \cdot 10^{-10}$	$\le 2.04 \cdot 10^{-5}$	$\le 6.02 \cdot 10^{-5}$	$\le 9.67 \cdot 10^{-5}$
Z_1	$\le 1.42 \cdot 10^{-9}$	$\le 5.73 \cdot 10^{-11}$	$\le 9.41 \cdot 10^{-10}$	$\le 7.43 \cdot 10^{-12}$
tail of the determinant	$4.970 \cdot 10^{-7}$ (of f)	$3.791 \cdot 10^{-10}$	$1.522 \cdot 10^{-7}$	$3.684 \cdot 10^{-9}$
least margin	$0.1059\,\rho^2$	$4.329 \cdot 10^{-6}$	$2.482 \cdot 10^{-7}$	$3.545 \cdot 10^{-8}$

TABLE 2. The four families certified by the method of Section 5.1. The tail is that of the determinant of the computed matrix. The least margin is the least Bernstein coefficient minus $e + e'$, over all certificates of the family; for the inner tube family it is the bound for $-\partial_\nu^2 f$.

5.2. **The inner tube family.**

Proposition 5.4. *Let $0 < \rho \le \frac{1}{4}$, $0 \le \nu \le \frac{1}{2}$ and $|\zeta| \le \varepsilon$. Then $W \ne 0$ and $\det M_2 \ne 0$ at $\Gamma(\rho,\nu,\zeta)$. Let $f = \det(T(\rho)I - G)$ there. With ∂_ν taken at fixed ρ and ζ,*

$$-\partial_\nu^2 f > 0.105\,\rho^2.$$

The certificate. The family is enclosed on the polydisc $|\rho| \le \frac{1}{4}$, $|\zeta| \le \frac{1}{20}$, with $0 \le \nu \le \frac{1}{2}$. At $\rho = 0$ the bodies q_1 and q_2 coincide and $q_2q_3q_4$ is an equilateral triangle with unit sides, in the position fixed by ζ; this configuration is known in closed form. We compute Γ as a double series in (ρ,ζ) whose coefficients are polynomials in ν with rational coefficients, order by order in ρ. Every element that this inverts is a nonzero rational constant. The head is the series truncated at the orders given in Table 2, with no truncation in ν. In the norm, ρ and ζ take the places of a and b, with weights $\frac{1}{4}$ and $\frac{1}{20}$, and $\mathcal{J}$ is the jet norm

$$\mathcal{J}(p) = \sup|p| + \tfrac{1}{30}\sup|p'| + \tfrac{1}{1800}\sup|p''| \qquad \text{on } [0,\tfrac{1}{2}].$$

Leibniz' rule makes it submultiplicative. We bound it through the Chebyshev coefficients of p and the inequalities $|T_j'| \le j^2$ and $|T_j''| \le j^2(j^2-1)/3$ on $[-1,1]$. The orders dropped in products are bounded by the norm with the larger weights $\frac{1}{2}$ and $\frac{1}{5}$. The chain encloses f; Table 2 gives its tail.

Proof. The chain inverts $\det M_2$ and W, each as an exact head with a bounded defect, so neither vanishes on the family. At $\rho = 0$ the chart equations do not involve ν, and their derivative in ρ is affine in ν. In Γ, and in f, the coefficient of ρ^0 therefore does not depend on ν and the

coefficient of ρ^1 is affine in ν. Hence $\partial_\nu^2 f$ is divisible by ρ^2, and so is ∂_ν^2 of the head. Since $\mathcal{J}(p) \geq \frac{1}{1800} \sup |p''|$, the tail of $\partial_\nu^2 f/\rho^2$ is at most $1800 \cdot 16 \cdot 4.970 \cdot 10^{-7} \leq 1.432 \cdot 10^{-2}$; the factor 16 is a_0^{-2} with $a_0 = \frac{1}{4}$. We apply Lemma 5.3 to $-\partial_\nu^2 f/\rho^2$ on the box $[0, \frac{1}{4}] \times [0, \frac{1}{2}] \times [-\varepsilon, \varepsilon]$, with a block of monomials in ρ, $\nu - \frac{1}{4}$ and ζ. The least Bernstein coefficient is at least 0.12398, and $e' \leq 3.663 \cdot 10^{-3}$, so

$$-\partial_\nu^2 f/\rho^2 \;\geq\; 0.12398 - 0.003663 - 0.01432 \;>\; 0.105. \qquad \square$$

At $\rho = 0$ and $\zeta = 0$ the exact value of $-\partial_\nu^2 f/\rho^2$ is $\frac{899}{6912} \approx 0.130$, for every ν. Lemma 3.12 combines Proposition 5.4 with Proposition 5.5 and Lemma 3.8.

5.3. **The inner face family.** On the face $\nu = 0$ the chart equations give $r_{23} = r_{24} = r_{34} = 1$ (see the proof of Lemma 3.12). So the face configurations are $q_1 = (-\rho, 0)$, $q_2 = 0$ and

$$q_3 = \big(\cos(\tfrac{\pi}{3} + \phi), \sin(\tfrac{\pi}{3} + \phi)\big), \qquad q_4 = \big(\cos(\tfrac{2\pi}{3} + \phi), \sin(\tfrac{2\pi}{3} + \phi)\big), \qquad |\phi| < \tfrac{\pi}{2}.$$

Then $u = \frac{1}{2}(\rho - \sqrt{3}\sin\phi)$ and $u - u_0(\rho, 0) = \frac{1}{2}\big(g(\rho) - \sqrt{3}\sin\phi\big)$, with $g(\rho) = \rho\big(\frac{99}{64} - \frac{\rho^2}{15}\big)$. We use the variables (ρ, t) with $t = \tan(\phi/2)/\sqrt{3}$. Then $\sqrt{3}\sin\phi = 6t/(1+3t^2)$, and in the coordinates $(x, y/\sqrt{3})$ every coordinate is rational in (ρ, t). Let $\mathcal{D}$ be the set of (ρ, t) with $0 < \rho \leq \frac{1}{4}$, $|t| < 1/\sqrt{3}$ and $|u - u_0(\rho, 0)| \leq \varepsilon$. It is connected: $6t/(1+3t^2)$ increases on $|t| < 1/\sqrt{3}$, so each section of $\mathcal{D}$ at fixed ρ is an interval in t, and its ends move continuously with ρ.

Proposition 5.5. *Let $0 < \rho \leq \frac{1}{4}$. At every face configuration with $|u - u_0(\rho, 0)| \leq \varepsilon$, $W > 0$ and $\mu_1 < T(\rho) - \rho^2/100$. Hence $\Pi(T(\rho)) \succ 0$ and $f > \rho^4/10^4$ there.*

Proof. (i) The pairs 23, 24 and 34 have unit length, so $d_{23} = d_{24} = d_{34} = 0$. On $\mathcal{D}$ the weights d_{12}, d_{13} and d_{14} are positive, by the sign checks in (vi). Hence $W > 0$, and since $q_2 = 0$,

$$W\,C = d_{12}d_{13}\,q_3 q_3^{\mathsf{T}} + d_{12}d_{14}\,q_4 q_4^{\mathsf{T}} + d_{13}d_{14}\,(q_3 - q_4)(q_3 - q_4)^{\mathsf{T}}.$$

(ii) For a fixed configuration, $C = \min_{p \in \mathbb{R}^2} \sum_e d_e (n_e - p)(n_e - p)^{\mathsf{T}}$ in the Loewner order, the minimum being attained at $p = \bar{n}$. So C, and with it μ_1, is nondecreasing in each d_e.

(iii) On $\mathcal{D}$, $r_{13} > 1 > r_{14}$ and $A_1A_3 > 0 > A_1A_4$. With $z = r_{13}^2 - 1$ and $y = r_{14}^2$, Taylor's formula with Lagrange remainder gives

$$(1+z)^{5/2} \leq 1 + \tfrac{5}{2}z + \tfrac{15}{8}z^2 + \tfrac{5}{16}z^3, \qquad y^{5/2} \geq 1 + \tfrac{5}{2}(y-1) + \tfrac{15}{8}(y-1)^2 + \tfrac{5}{16}(y-1)^3 - \tfrac{1}{10}(y-1)^4,$$

the second because $y \geq (1-\rho)^2 \geq \frac{9}{16}$. Inserted into $d_{13} = \frac{1}{3}A_1A_3(r_{13}^5 - r_{13}^2)$ and $d_{14} = \frac{1}{3}|A_1A_4|(r_{14}^2 - r_{14}^5)$, they give polynomial upper bounds $d_{13}^+ \geq d_{13}$ and $d_{14}^+ \geq d_{14}$. Let W^+, C^+ and μ_1^+ be W, C and μ_1 computed with them. By (ii), $\mu_1 \leq \mu_1^+$. The bounds are exact to third order at the corner $(\rho, t) = (0, 0)$.

(iv) Let $\hat{T} = T(\rho) - \rho^2/100$ and $\Pi^+ = W^+(\hat{T}M_2^2 - C^+)$. On $\mathcal{D}$, $\det \Pi^+$ is $\rho^6 H(\rho, t)/(1+3t^2)^{11}$ times a positive function, where H is a polynomial with rational coefficients, of bidegree $(16, 22)$ with 353 terms. Its lowest part is a positive definite quadratic form in (ρ, t).

(v) Three charts blow up the corner. In each, $H^\sharp(x, k) = H/x^2$ is a polynomial:

- the sector $(\rho, t) = (x, kx)$ with $0 \leq x \leq \frac{1}{4}$ and $0 \leq k \leq \ell_3(x)$, cut into three strips by the lines $k = \ell_1(x)$ and $k = \ell_2(x)$;
- the upper wedge $(\rho, t) = (kx, x)$ with $0 \leq x \leq \frac{1}{30}$ and $0 \leq k \leq 2$;
- the lower wedge $(\rho, t) = \big(kx, -\frac{3}{10}(1-k)x\big)$ with $0 \leq x \leq \frac{1}{10}$ and $0 \leq k \leq 1$.

Here ℓ_1, ℓ_2 and ℓ_3 are linear in x, with values 0.155, 0.375, 0.55 at $x = 0$ and 0.1425, 0.3825, 0.45 at $x = \frac{1}{4}$. $\mathcal{D}$ lies in the union of the three charts, with $x > 0$. This follows from a few exact rational inequalities and one univariate certificate, whose Bernstein coefficients all exceed

$1.66 \cdot 10^{-2}$. On a strip $L(x) \le k \le U(x)$ put $k = \frac{1}{2}(L+U) + \frac{1}{2}(U-L)\vartheta$ with $|\vartheta| \le 1$, and write $H^\sharp = \sum_j h_j(x)\vartheta^j$. If $h_2 \ge 0$, then

$$H^\sharp \;\ge\; \min(h_0 + h_1,\, h_0 - h_1) - \sum_{j \ge 3} \max_i |b_i(h_j)|,$$

where $b_i(h)$ are the Bernstein coefficients of h on the range of x. Each strip therefore needs three univariate certificates: $h_2 \ge 0$, and positivity of $h_0 \pm h_1 - \sum_{j\ge3} \max_i |b_i(h_j)|$. In each of these certificates, the least Bernstein coefficient divided by the largest $|b_i(h_0)|$ is at least 0.1139 for the strict inequalities and at least 0.0029 for $h_2 \ge 0$. Hence $H > 0$ and $\det \Pi^+ > 0$ on $\mathcal{D}$.

(vi) On $\mathcal{D}$, $-\frac{1}{100} < t < \frac{1}{10}$. Exact checks at the ends of this range, with monotonicity or concavity in t, give the sign facts used in (i) and (iii), $\det M_2 < 0$, and $d_{13}^+, d_{14}^+ > 0$, so $W^+ > 0$. At $(\rho, t) = (\frac{1}{10}, \frac{1}{40})$, which lies in $\mathcal{D}$, $\Pi^+ \succ 0$ by exact evaluation.

The proof of Proposition 5.2(b) applies to Π^+ and W^+ on $\mathcal{D}$. Thus $\Pi^+ \succ 0$, that is, $\mu_1^+ < \hat{T}$, and $\mu_1 \le \mu_1^+ < T(\rho) - \rho^2/100$. Finally $f = (T-\mu_1)(T-\mu_2) \ge (T-\mu_1)^2 > \rho^4/10^4$. □

5.4. **The inner band family.** The bands of this family end at $\zeta = -\bar{\zeta}_-(\rho,\nu)$ and $\zeta = \bar{\zeta}_+(\rho,\nu)$, where

$$\bar{\zeta}_- = 0.446 - 0.008\nu + 0.015\nu^2 - 0.01\nu^3 + \rho\,(-0.423 + 1.004\nu - 0.357\nu^2 + 0.198\nu^3) + 0.051\rho^2,$$

$$\bar{\zeta}_+ = 0.446 - 0.006\nu + 0.008\nu^2 - 0.003\nu^3 + \rho\,(0.377 - 0.623\nu - 0.196\nu^2 + 0.045\nu^3) + 0.025\rho^2.$$

For a configuration of the family put $m_k/m_2 = A_k w_{12}/(A_2 w_{1k})$ for $k = 3, 4$. At a CC these are the mass ratios, by Dziobek's relations.

Proposition 5.6. *Let $0 < \rho \le \frac{1}{4}$ and $0 \le \nu \le \frac{1}{2}$.*

(a) *If $\varepsilon \le -\zeta \le \bar{\zeta}_-(\rho,\nu)$ or $\varepsilon \le \zeta \le \bar{\zeta}_+(\rho,\nu)$, then $\det \Pi(T(\rho)) > 0$ at $\Gamma(\rho,\nu,\zeta)$.*

(b) *At $\zeta = -\bar{\zeta}_-(\rho,\nu)$, $m_3/m_2 < 0$. At $\zeta = \bar{\zeta}_+(\rho,\nu)$, $m_4/m_2 < 0$.*

The certificate. The series in ρ is centred at $\rho = 0$, with $a_0 = \frac{1}{4}$. The family covers the minus band $\zeta = -(\varepsilon + (\bar{\zeta}_- - \varepsilon)s)$ and the plus band $\zeta = \varepsilon + (\bar{\zeta}_+ - \varepsilon)s$, with $0 \le s \le 1$ and $0 \le \nu \le \frac{1}{2}$, in the pieces counted in Table 3. The head comes from Newton steps that start at the closed-form configuration at $\rho = 0$. Quantities that vanish at $\rho = 0$ are divided by ρ exactly, so the chain is regular there, and the computed matrix extends continuously to $\rho = 0$. The tail of its determinant is in Table 2; the two mass-sign products, whose signs are those of m_3/m_2 and m_4/m_2, have tail $1.621 \cdot 10^{-12}$. Certificates on the box $[0, \frac{1}{4}] \times [0, \frac{1}{2}] \times [s_0, s_1]$ of each piece give the sign of the determinant, and certificates at $s = 1$ on both bands give the mass signs. All of them pass, and Table 2 gives their least margin. The determinant is therefore positive also at $\rho = 0$. Proposition 5.2(a) is also checked at $\zeta = \mp\varepsilon$, against the inner tube family.

Proof. The certificates give (a) and (b). □

5.5. **The middle family.** The middle family covers $\frac{1}{4} \le \rho \le \frac{3}{5}$, in the tube chart for $|\zeta| \le \varepsilon$ and by the band equation (5.3) for the two bands, out to $s = 1$.

Proposition 5.7. *Let $\frac{1}{4} \le \rho \le \frac{3}{5}$ and $0 \le \nu \le \frac{1}{2}$.*

(a) *At Γ on the tube $|\zeta| \le \varepsilon$ and on both bands $0 \le s \le 1$, $W > 0$ and $\Pi(T(\rho)) \succ 0$. Hence $\mu_1 < T(\rho)$ at every convex CC of the family.*

(b) *At $s = 0$, $x_3 > X_x$ on the minus band and $x_4 < X_x$ on the plus band.*

The certificate. Pieces, counted in Table 3, cover the tube and both bands out to $s = 1$. On the tube $\zeta = 2\varepsilon b$. Table 2 gives $\rho_0 \pm a_0$ and the tail of the determinant; the trace has tail $2.401 \cdot 10^{-7}$. Certificates give the sign of the determinant on every piece and of the trace

on the tube pieces. The least margin comes from the determinant on the tube piece $\nu \le \frac{1}{10}$. On the band pieces that meet $s = 0$, more certificates give (b), with tail $\hat{r}$; the least bound is $x_3 - X_x \ge 0.2339$. Further, $W > 0$ at $(\rho, \nu, \zeta) = (\frac{17}{40}, \frac{3}{10}, 0)$.

Proof. By Proposition 5.2(a), Γ is a single continuous zero on a connected parameter region, and $\det \Pi(T(\rho)) > 0$ on it. At $(\frac{17}{40}, \frac{3}{10}, 0)$, $W > 0$, and $\Pi(T(\rho)) \succ 0$ because its determinant and trace are positive. Proposition 5.2(b) gives (a), and the certificates at $s = 0$ give (b). □

5.6. **The outer family.** The outer family covers $\frac{3}{5} \le \rho \le \frac{3}{4}$ in the same way, with $c = \frac{1}{3}$ in place of $T(\rho)$. The outer family uses the target $\frac{1}{3}$ because, numerically, μ_1 exceeds $T(\rho)$ near the face once $\rho \gtrsim 0.72$.

Proposition 5.8. *Let $\frac{3}{5} \le \rho \le \frac{3}{4}$ and $0 \le \nu \le \frac{1}{2}$.*

(a) *At Γ on the tube $|\zeta| \le \varepsilon$ and on both bands $0 \le s \le 1$, $W > 0$ and $\Pi(\frac{1}{3}) \succ 0$. Hence $\mu_1 < \frac{1}{3}$ at every convex CC of the family.*
(b) *At $s = 0$, $x_3 > X_x$ on the minus band and $x_4 < X_x$ on the plus band.*

The certificate. As in the middle family, the pieces counted in Table 3 cover the tube and both bands. The tail of the trace is $4.158 \cdot 10^{-9}$; that of the determinant, and $\rho_0 \pm a_0$, are in Table 2. The determinant is certified on every piece and the trace on the tube pieces. The least margin is that of the determinant on the minus-band piece $\nu \le \frac{1}{100}$, near $\rho = \frac{3}{4}$, $\nu = 0$; there the least coefficient minus e' is 10.6 times the tail. Part (b) is certified as for the middle family; the least bound is $x_3 - X_x \ge 0.1498$. Further, $W > 0$ at $(\frac{27}{40}, \frac{3}{10}, 0)$.

Proof. As for Proposition 5.7, with $c = \frac{1}{3}$. □

6. Normalized coordinates

6.1. **The coordinates of Corbera, Cors and Roberts.** Following [CCR19], for $a, b, c > 0$ and $-1 < x < 1$ put $s = \sqrt{1 - x^2}$ and let $q(a, b, c, x)$ be the configuration

$$q_1 = (1, 0), \qquad q_2 = a\,(x, s), \qquad q_3 = (-b, 0), \qquad q_4 = -c\,(x, s).$$

Its diagonals q_1q_3 and q_2q_4 meet at the origin, which is an interior point of both, and q_2 lies in the upper half-plane. Hence $q(a, b, c, x)$ is a strictly convex quadrilateral whose vertices q_1, q_2, q_3, q_4 are in counterclockwise order. Its squared mutual distances and oriented areas are

$$R_{12} = 1 + a^2 - 2ax, \qquad R_{13} = (1 + b)^2, \qquad R_{14} = 1 + c^2 + 2cx,$$
$$R_{23} = a^2 + b^2 + 2abx, \qquad R_{24} = (a + c)^2, \qquad R_{34} = b^2 + c^2 - 2bcx,$$
$$(A_1, A_2, A_3, A_4) = \tfrac{s}{2}\big(b(a + c),\ -c(1 + b),\ a + c,\ -a(1 + b)\big).$$

Consider the six functions

$$g_1 = b^2 + 2b - a^2 + 2ax, \qquad g_2 = c^2 + 2ac - 1 + 2ax, \qquad g_3 = a - c - 2x,$$
$$g_4 = 1 - b - 2ax, \qquad g_5 = a - c + 2bx, \qquad g_6 = 1 - b + 2cx.$$

Expanding the squared distances gives

$$R_{13} - R_{12} = g_1, \qquad R_{24} - R_{12} = g_2, \qquad R_{12} - R_{14} = (a + c)\,g_3,$$
$$R_{12} - R_{23} = (1 + b)\,g_4, \qquad R_{23} - R_{34} = (a + c)\,g_5, \qquad R_{14} - R_{34} = (1 + b)\,g_6. \tag{6.1}$$

Let

$$\mathcal{C} = \big\{(a, b, c, x) \in (0, \infty)^3 \times (-1, 1) :\ g_1 > 0,\ g_2 > 0,\ g_3 \ge 0,\ g_4 \ge 0,\ g_5 \ge 0,\ g_6 \ge 0\big\}.$$

Family	Region	Method	Pieces	Certificates	Other checks
inner tube	$0 < \rho \le \frac{1}{4},\ \ \lvert\zeta\rvert \le \varepsilon$	NK on a polydisc; exact tensor certificates	1	1	–
inner face	$\nu = 0,\ \ 0 < \rho \le \frac{1}{4},$ $\lvert u - u_0(\rho, 0)\rvert \le \varepsilon$	three blow-up charts; exact univariate Bernstein steps	5 strips	16	–
inner band	$0 < \rho \le \frac{1}{4},\ \ \lvert\zeta\rvert \ge \varepsilon$ out past the chart ends	NK step and inverse-free chain; Chebyshev–Bernstein certificates, face checks	1+2	3+2	3 faces
middle	$\frac{1}{4} \le \rho \le \frac{3}{5}$: the tube and both bands	NK step and inverse-free chain over 8 pieces; certificates, bounds at $s = 0$, face checks	2+3+3	8+2	5 bounds at $s = 0$, 12 faces, 1 point
outer	$\frac{3}{5} \le \rho \le \frac{3}{4}$: the tube and both bands, with $c = \frac{1}{3}$	as for the middle family, over 9 pieces	3+3+3	9+3	5 bounds at $s = 0$, 16 faces, 1 point
anchors	$(\rho, \nu, \zeta) = (\frac{17}{40}, \frac{1}{4}, 0)$ and $(\frac{27}{40}, \frac{1}{4}, 0)$	exact rational evaluation of the tube heads of the middle and outer families	–	–	2 evaluations

TABLE 3. The certified computations of the proof (NK: Newton–Kantorovich). Sums count pieces on the tube, the minus band and the plus band, and certificates for the determinant and for the trace (for the inner band family: pieces on the two bands, and certificates for the determinant and for the two mass signs). Throughout $\nu \in [0, \frac{1}{2}]$; Lemma 3.8 ($\nu \mapsto 1 - \nu$, $\zeta \mapsto -\zeta$) gives $\nu \in [\frac{1}{2}, 1]$. Here $\zeta = u - u_0$, and the tube is $\lvert\zeta\rvert \le \varepsilon$ with $\varepsilon = \frac{1237}{50000}$, as in the text. For the inner tube, inner band, middle and outer families a certificate is a block of at most 120 coefficients (monomial for the inner tube family, Chebyshev otherwise) converted exactly to Bernstein form, whose least Bernstein coefficient exceeds the sum of the absolute values of the dropped coefficients plus the tail of the enclosure. For the inner face family the certificates are univariate polynomials of degree at most 57 (at most 58 coefficients) with positive Bernstein coefficients.

By (6.1), a point (a,b,c,x) with $a,b,c>0$ lies in $\mathcal{C}$ if and only if

$$r_{13}>r_{12},\quad r_{24}>r_{12},\quad r_{12}\ge r_{14}\ge r_{34},\quad r_{12}\ge r_{23}\ge r_{34}. \tag{6.2}$$

In words, q_1q_2 is a longest side, q_3q_4 is a shortest side, and both diagonals are longer than q_1q_2. This is the normalization used in [CCR19, Section 3].

6.2. **The Dziobek function.** For $X,Y>0$ let

$$\psi(X,Y)=\frac{X+\sqrt{XY}+Y}{(\sqrt{X}+\sqrt{Y})\,(XY)^{3/2}},$$

a positive real-analytic function with $X^{-3/2}-Y^{-3/2}=(Y-X)\,\psi(X,Y)$ and $\psi(1,1)=\frac{3}{2}$. For $(a,b,c,x)\in\mathcal{C}$ we compare the numbers $s_e=R_e^{-3/2}$ with s_{12} through the gaps

$$\eta_{13}=s_{12}-s_{13},\quad \eta_{24}=s_{12}-s_{24},\quad \eta_{14}=s_{14}-s_{12},\quad \eta_{23}=s_{23}-s_{12},\quad \eta_{34}=s_{34}-s_{12}.$$

By (6.1),

$$\begin{aligned}\eta_{13}&=g_1\,\psi(R_{12},R_{13}),\qquad \eta_{24}=g_2\,\psi(R_{12},R_{24}),\qquad \eta_{14}=(a+c)\,g_3\,\psi(R_{14},R_{12}),\\ \eta_{23}&=(1+b)\,g_4\,\psi(R_{23},R_{12}),\qquad \eta_{34}=\big((1+b)\,g_4+(a+c)\,g_5\big)\,\psi(R_{34},R_{12}).\end{aligned} \tag{6.3}$$

In particular $\eta_{13},\eta_{24}>0$ and $\eta_{14},\eta_{23},\eta_{34}\ge 0$ on $\mathcal{C}$. Put

$$\delta=\eta_{13}+\eta_{24}+\eta_{34}=s_{12}+s_{34}-s_{13}-s_{24}>0$$

and define the *Dziobek function*

$$P=\eta_{14}\eta_{23}\,\delta+\eta_{13}\eta_{24}\,(\eta_{14}+\eta_{23}-\eta_{34}). \tag{6.4}$$

The factored forms (6.3) are the ones used in the computation. They make the signs of the η_e visible and avoid cancellation near the boundary of $\mathcal{C}$.

Lemma 6.1. *Let $(a,b,c,x)\in\mathcal{C}$ and $q=q(a,b,c,x)$. Define λ' by (2.4), that is, $\lambda'=(s_{12}s_{34}-s_{13}s_{24})/\delta$, and put $w_e=s_e-\lambda'$.*

(a) *We have*

$$w_{12}=\frac{\eta_{13}\eta_{24}}{\delta},\qquad w_{34}=\frac{(\eta_{13}+\eta_{34})(\eta_{24}+\eta_{34})}{\delta},\qquad w_{13}=-\frac{\eta_{13}(\eta_{13}+\eta_{34})}{\delta},$$
$$w_{24}=-\frac{\eta_{24}(\eta_{24}+\eta_{34})}{\delta},\qquad w_{14}=\eta_{14}+w_{12},\qquad w_{23}=\eta_{23}+w_{12}.$$

Consequently $w_{12},w_{14},w_{23},w_{34}>0>w_{13},w_{24}$, $w_{12}w_{34}=w_{13}w_{24}$ and $w_{12}w_{34}-w_{14}w_{23}=-P/\delta$.

(b) *Let F_{CCR} be the polynomial of* [CCR19, Lemma 3.1],

$$F_{\mathrm{CCR}}=(r_{24}^3-r_{14}^3)(r_{13}^3-r_{12}^3)(r_{23}^3-r_{34}^3)-(r_{12}^3-r_{14}^3)(r_{24}^3-r_{34}^3)(r_{13}^3-r_{23}^3).$$

Then $F_{\mathrm{CCR}}\prod_e s_e=-P$.

(c) *q is a central configuration for some positive masses if and only if $P=0$. In that case the masses are unique up to a common positive factor. Normalizing $\sigma=-1$ in (2.3), they are determined by $m_im_j=-A_iA_j/w_{ij}$ for all $i<j$.*

Proof. (a) Since $\delta=s_{12}+s_{34}-s_{13}-s_{24}$, we get $\delta\,w_{12}=\delta s_{12}-s_{12}s_{34}+s_{13}s_{24}=(s_{12}-s_{13})(s_{12}-s_{24})=\eta_{13}\eta_{24}$. In the same way $\delta w_{34}=(s_{34}-s_{13})(s_{34}-s_{24})$ and $\delta w_{13}=(s_{12}-s_{13})(s_{13}-s_{34})$, $\delta w_{24}=(s_{12}-s_{24})(s_{24}-s_{34})$. Moreover $w_{14}-w_{12}=s_{14}-s_{12}$ and $w_{23}-w_{12}=s_{23}-s_{12}$. This proves the formulas, and the signs follow from the signs of the η_e. The identity $w_{12}w_{34}=w_{13}w_{24}$ is immediate from the formulas. Finally $w_{34}-w_{12}=\eta_{34}$, so

$$w_{12}w_{34}-w_{14}w_{23}=w_{12}(\eta_{34}-\eta_{14}-\eta_{23})-\eta_{14}\eta_{23}=-P/\delta.$$

(b) Writing $r_e^3 = 1/s_e$, each factor of F_{CCR} becomes a difference of two s_e divided by their product, and both products of three factors have denominator $\prod_e s_e$. Hence

$$F_{\mathrm{CCR}} \prod_e s_e = (s_{14} - s_{24})(s_{12} - s_{13})(s_{34} - s_{23}) - (s_{14} - s_{12})(s_{34} - s_{24})(s_{23} - s_{13}).$$

Substituting $s_{14} - s_{24} = \eta_{14} + \eta_{24}$, $s_{34} - s_{23} = \eta_{34} - \eta_{23}$, $s_{34} - s_{24} = \eta_{34} + \eta_{24}$, $s_{23} - s_{13} = \eta_{23} + \eta_{13}$ and expanding gives $-P$.

(c) Suppose that q is a CC for positive masses. The relation $w_{12}w_{34} = w_{13}w_{24}$ of Proposition 2.3(b) is linear in λ' and forces λ' to be given by (2.4). So the w_e of the CC are those of part (a), and $w_{12}w_{34} = w_{14}w_{23}$ gives $P = 0$. By (2.3), $m_i m_j = \sigma A_i A_j / w_{ij}$, so the six products $m_i m_j$, and hence the masses, are determined up to a common factor. Conversely, suppose $P = 0$ and put $\mu_{ij} = -A_i A_j / w_{ij}$. By Lemma 2.1(d), $A_i A_j < 0$ on the sides and $A_i A_j > 0$ on the diagonals, so part (a) gives $\mu_{ij} > 0$ for all six edges. By part (a) and $P = 0$, $\mu_{12}\mu_{34} = \mu_{13}\mu_{24} = \mu_{14}\mu_{23}$. Therefore $m_1 = (\mu_{12}\mu_{13}/\mu_{23})^{1/2}$, $m_2 = \mu_{12}/m_1$, $m_3 = \mu_{13}/m_1$, $m_4 = \mu_{14}/m_1$ are positive and satisfy $m_i m_j = \mu_{ij}$ for all $i < j$. With these masses $\omega_{ij} = m_i m_j w_{ij} = -A_i A_j$, which is a self-stress by Lemma 2.2. So (2.2) holds, and q is a CC. □

Let

$$\mathcal{E} = \{(a, b, c, x) \in \mathcal{C} : \ P(a, b, c, x) = 0\}.$$

By Lemma 6.1(b), $\mathcal{E}$ is the set denoted E in [CCR19], defined there by $F_{\mathrm{CCR}} = 0$. Corbera, Cors and Roberts show that the points of E are exactly the convex CCs, up to relabelling and similarity [CCR19, Lemma 3.1]. We record this in the form in which we use it.

Lemma 6.2. (a) *For every $(a, b, c, x) \in \mathcal{E}$, the configuration $q(a, b, c, x)$ is a counterclockwise convex CC with cyclic order* (1234) *for positive masses, which are unique up to a common factor.*

(b) *Let q be a convex CC for positive masses $m_1, \dots, m_4$. Then there are a permutation π of $\{1, 2, 3, 4\}$, a similarity ϕ of the plane and a point $(a, b, c, x) \in \mathcal{E}$ such that the configuration q' defined by $q'_{\pi(i)} = \phi(q_i)$ equals $q(a, b, c, x)$. It is a CC for the masses m' defined by $m'_{\pi(i)} = m_i$.*

(c) *The map $\mathcal{E} \to \mathcal{S}$, $(a, b, c, x) \mapsto [q(a, b, c, x)]$, is injective.*

Proof. (a) is Lemma 6.1(c). (b) Label the bodies so that the cyclic order is (1234) and $q_1 q_2$ is a longest side. By Proposition 2.3, $q_3 q_4$ is a shortest side and both diagonals are longer than every side, so (6.2) holds. These relations are preserved by similarities. If the vertices are in clockwise order, apply a reflection. Then apply the orientation-preserving similarity that moves the intersection point of the diagonals to the origin and q_1 to $(1, 0)$. Now $q_3 = (-b, 0)$ with $b > 0$, and $q_2 = a(\cos\theta, \sin\theta)$, $q_4 = -c(\cos\theta, \sin\theta)$ with $a, c > 0$. Since the order is counterclockwise, $0 < \theta < \pi$. So the configuration is $q(a, b, c, \cos\theta)$, and $(a, b, c, \cos\theta) \in \mathcal{C}$ by (6.2). It is a CC because the equations (2.1) are invariant under similarities and under a simultaneous permutation of bodies and masses. Lemma 6.1(c) gives $P = 0$. (c) If $q(a', b', c', x') = \mu\, q(a, b, c, x) + t\mathbf{1}$ with $\mu \in \mathbb{C}^*$ and $t \in \mathbb{C}$, the similarity maps the intersection point of the diagonals to itself, so $t = 0$. It fixes $q_1 = (1, 0)$, so $\mu = 1$. □

Remark 6.3. A point (a, b, c, x) as in (b) is a normal form of q. It is not unique: in the proof of (b), the two endpoints of a longest side can be labelled in either order. Exchanging the labels $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, reflecting, and renormalizing maps (a, b, c, x) to $(1/a, c/a, b/a, x)$. This map preserves $\mathcal{E}$ and permutes the masses as $(m_1, m_2, m_3, m_4) \mapsto (m_2, m_1, m_4, m_3)$. So an unlabelled similarity class of convex CCs can appear more than once in $\mathcal{E}$. This does not matter for the estimates of Section 7, which are properties of each point of $\mathcal{E}$, and Section 7.2 uses the map to halve the domain.

Remark 6.4. Corbera, Cors and Roberts hoped that the normalized mass map on $\mathcal{E}$ is injective, which would prove uniqueness [CCR19]. Theorem A of [Tom26a] gives this injectivity. For $z \in \mathcal{E}$ let $\mathfrak{m}(z)$ be the masses of Lemma 6.1(c), normalized so that their sum is 1. If $\mathfrak{m}(z) = \mathfrak{m}(z') = m$, then $q(z)$ and $q(z')$ are counterclockwise convex CCs with cyclic order (1234) for the masses m by Lemma 6.2(a). By Theorem A of [Tom26a] and its proof [Tom26a, Section 5], they have the same class, and $z = z'$ by Lemma 6.2(c). As a consistency check, suppose that the masses of $q(z)$ satisfy $m_1 = m_2$ and $m_3 = m_4$. The involution of Remark 6.3 maps z to a point of $\mathcal{E}$ with the same normalized masses, so it fixes z. Hence $a = 1$ and $b = c$, so that $q_3 = -b\,q_1$ and $q_4 = -b\,q_2$ with $|q_1| = |q_2| = 1$. The reflection in the perpendicular bisector of q_1q_2 is then linear and exchanges $q_1 \leftrightarrow q_2$ and $q_3 \leftrightarrow q_4$, so $q(z)$ is an isosceles trapezoid with $q_1q_2 \parallel q_3q_4$, in accordance with [Tom26a, Corollary C(b)].

6.3. **A priori bounds.**

Lemma 6.5 ([CCR19, Lemma 3.2])**.** *On $\mathcal{C}$,*

$$c \le a, \qquad b \le 1, \qquad ac < b^2 + 2b, \qquad a^2 < b^2 + b + 1, \qquad a^2 + ac + c^2 > 1.$$

Consequently $\frac{1}{\sqrt{3}} < a < \sqrt{3}$, $0 < c \le a$ and $0 < b \le 1$.

Proof. If $x \ge 0$ then $g_3 \ge 0$ gives $a - c \ge 2x \ge 0$, and $g_4 \ge 0$ gives $1 - b \ge 2ax \ge 0$. If $x < 0$ then $g_5 \ge 0$ gives $a - c \ge -2bx > 0$, and $g_6 \ge 0$ gives $1 - b \ge -2cx > 0$. The remaining inequalities are $g_1 + a\,g_3 = b^2 + 2b - ac > 0$, $g_1 + g_4 = b^2 + b + 1 - a^2 > 0$ and $g_2 + a\,g_3 = c^2 + ac + a^2 - 1 > 0$. Finally $a^2 < b^2 + b + 1 \le 3$, and $1 < a^2 + ac + c^2 \le 3a^2$. □

Hence the projection of $\mathcal{E} \cap \{b \ge \frac{1}{8}\}$ to (a, b, c) is contained in the box

$$\Omega_{\mathrm{dir}} = [\tfrac{1}{2}, \tfrac{7}{4}] \times [\tfrac{1}{8}, 1] \times [0, \tfrac{7}{4}].$$

6.4. **The blow-up chart.** As $b \to 0$ on $\mathcal{E}$, Lemma 6.5 forces $c \to 0$ (because $ac < b^2 + 2b$ and $a > 1/\sqrt{3}$), and $a \to 1$, $x \to \frac{1}{2}$ by Lemma 6.6 and the bounds on ξ below. So the configuration degenerates to $q_1 = (1, 0)$, $q_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $q_3 = q_4 = 0$: the bodies 3 and 4 collide at a point that forms an equilateral triangle with q_1 and q_2. To resolve this corner we use the coordinates (b, α, γ, ξ) defined by

$$a = 1 + b\alpha, \qquad c = b\gamma, \qquad x = \tfrac{1}{2} + b\xi.$$

Lemma 6.6. *Let $(a, b, c, x) \in \mathcal{C}$ with $0 < b \le \frac{1}{8}$. Then*

$$-2 < \alpha < \frac{1 + b}{2} \le \frac{9}{16}, \qquad 0 < \gamma < \frac{2 + b}{1 - 2b} \le \frac{17}{6}.$$

In particular $a > 1 - 2b \ge \frac{3}{4}$.

Proof. By Lemma 6.5, $a^2 < 1 + b + b^2$, so $a < 1 + \frac{1}{2}(b + b^2)$ and $\alpha < \frac{1}{2}(1 + b)$.

For the lower bound put $u = b^2 + 2b \le \frac{17}{64} < \frac{1}{3}$. Lemma 6.5 gives $c < u/a$ and $1 < a^2 + ac + c^2$. Since $c \mapsto ac + c^2$ is increasing for $c > 0$, we get $1 < a^2 + u + u^2/a^2$, that is, $p(a^2) > 0$, where

$$p(t) = t^2 - (1 - u)\,t + u^2.$$

The roots of p are $t_\pm = \frac{1}{2}\big(1 - u \pm \sqrt{(1 - 3u)(1 + u)}\big)$, which are real. Since $p(\frac{1}{3}) = (u + \frac{2}{3})(u - \frac{1}{3}) < 0$, we have $t_- < \frac{1}{3} < t_+$. As $a^2 > \frac{1}{3}$ and $p(a^2) > 0$, it follows that $a^2 > t_+$. Next, $p(t) = t\,(f(t) - 1)$ with $f(t) = t + u + u^2/t$, and a direct computation gives

$$f\big((1 - 2b)^2\big) - 1 = \frac{b\,(7b - 1)\,(3b^2 - 3b + 2)}{(1 - 2b)^2} < 0 \qquad (0 < b < \tfrac{1}{7}).$$

Hence $p((1 - 2b)^2) < 0$, so $(1 - 2b)^2 < t_+ < a^2$. This gives $a > 1 - 2b$, that is, $\alpha > -2$.

Finally $c < u/a < u/(1-2b)$, so $\gamma = c/b < (2+b)/(1-2b)$. □

Thus the projection of $\mathcal{E} \cap \{0 < b \le \frac{1}{8}\}$ to (b, α, γ) is contained in

$$\Omega_{\mathrm{ch}} = [0, \tfrac{1}{8}] \times [-2, 1] \times [0, 3].$$

We now rewrite all quantities in the chart; the complete list is in Appendix B. The functions $g_1, \dots, g_4$ are divisible by b, and we put $h_i = g_i/b$ for $i \le 4$ and $h_i = g_i$ for $i = 5, 6$:

$$\begin{aligned} h_1 &= 2 - \alpha + 2\xi + b\,(1 - \alpha^2 + 2\alpha\xi), & h_2 &= 2\gamma + 2\xi + \alpha + b\,(\gamma^2 + 2\alpha\gamma + 2\alpha\xi),\\ h_3 &= \alpha - \gamma - 2\xi, & h_4 &= -(1 + \alpha + 2\xi + 2b\alpha\xi),\\ h_5 &= 1 + b\,(\alpha - \gamma + 1 + 2b\xi), & h_6 &= 1 - b + b\gamma + 2b^2\gamma\xi. \end{aligned}$$

The squared lengths are $R_e = 1 + b\,\Delta_e$ for $e \neq 34$ and $R_{34} = b^2\varrho^2$, where

$$\Delta_{12} = \alpha - 2\xi + b(\alpha^2 - 2\alpha\xi), \qquad \Delta_{13} = 2 + b, \qquad \Delta_{14} = \gamma(b\gamma + 2x),$$
$$\Delta_{23} = 2\alpha + b\alpha^2 + 2x(1 + b\alpha) + b, \qquad \Delta_{24} = 2(\alpha + \gamma) + b(\alpha + \gamma)^2, \qquad \varrho^2 = 1 - 2\gamma x + \gamma^2.$$

The gaps η_e are rescaled as

$$\begin{aligned} \hat\eta_{13} = \frac{\eta_{13}}{b} = h_1\psi(R_{12}, R_{13}), \qquad & \hat\eta_{24} = \frac{\eta_{24}}{b} = h_2\psi(R_{12}, R_{24}),\\ \hat\eta_{14} = \frac{\eta_{14}}{b} = (a+c)\,h_3\,\psi(R_{14}, R_{12}), \qquad & \hat\eta_{23} = \frac{\eta_{23}}{b} = (1+b)\,h_4\,\psi(R_{23}, R_{12}),\\ \hat\eta_{34} = b^3\eta_{34} = \varrho^{-3} - b^3 s_{12}, \qquad & \hat\delta = b^3\delta = \hat\eta_{34} + b^4(\hat\eta_{13} + \hat\eta_{24}), \end{aligned}$$

and then

$$\hat P = b\,P = \hat\eta_{14}\hat\eta_{23}\,\hat\delta + \hat\eta_{13}\hat\eta_{24}\big(b^4(\hat\eta_{14} + \hat\eta_{23}) - \hat\eta_{34}\big). \tag{6.5}$$

In the chart, α, γ and b are the independent coordinates, and ξ plays the role of x. The inequalities $h_1, h_2 > 0$ and $h_3, h_4 \ge 0$ are linear in ξ, with coefficients $2a$, $2a$, -2 and $-2a$ in front of ξ, where $a > 0$. So they confine ξ to an explicit bounded interval (Section 7).

The areas scale as $A_1 = b\widehat{A}_1$ and $A_2 = b\widehat{A}_2$, with $\widehat{A}_1 = \frac{s}{2}(a+c)$ and $\widehat{A}_2 = -\frac{s}{2}\gamma(1+b)$, while A_3 and A_4 are unchanged. The quantities w_e scale as $w_{12} = b^5\hat w_{12}$, $w_e = b\,\hat w_e$ for $e \in \{13, 14, 23, 24\}$ and $w_{34} = b^{-3}\hat w_{34}$. Inserting these into the definitions of D_e and β_e, all negative powers of b cancel, and every τ_e is divisible by b. We put $\hat\tau_e = \tau_e/b$ and

$$\widehat{S}(Y) = \sum_e D_e\,(\beta_e + \hat\tau_e Y)(\beta_e + \hat\tau_e Y)^{\mathsf T}, \qquad \text{so that} \qquad \widehat{S}(Y) = S(Y/b) \quad (b > 0).$$

Appendix B gives D_e, $\hat\tau_e$ and β_e as explicit expressions in (b, α, γ, ξ), built from h_i, Δ_e, ϱ, s, ψ and the $\hat\eta_e$ by rational operations and square roots. These expressions are real-analytic wherever the square roots have positive arguments and the denominators do not vanish, in particular at points with $b = 0$ and $\gamma > 0$.

Lemma 6.7. *At $b = 0$ we have $x = \frac{1}{2}$, $\varrho^2 = \gamma^2 - \gamma + 1$, and*

$$\hat P(0, \alpha, \gamma, \xi) = -\frac{9}{4}\,\varrho^{-3}\big(3\alpha + 3\gamma - 3\alpha\gamma + 2(1+\gamma)\xi\big), \qquad \frac{\partial \hat P}{\partial \xi}(0, \alpha, \gamma, \xi) = -\frac{9}{2}\,(1+\gamma)\,\varrho^{-3} < 0.$$

Hence, near every point of $\{b = 0,\ \gamma > 0\}$ where $\hat P = 0$, the zero set of $\hat P$ is the graph of a real-analytic function $\xi(b, \alpha, \gamma)$ with $\xi(0, \alpha, \gamma) = \frac{3(\alpha\gamma - \alpha - \gamma)}{2(1+\gamma)}$.

Proof. At $b=0$, $R_e=1$ for $e\neq 34$, so every ψ equals $\frac{3}{2}$, $a+c=1+b=1$ and $\hat\delta=\hat\eta_{34}=\varrho^{-3}$. Thus $\hat P=\frac{9}{4}\varrho^{-3}(h_3h_4-h_1h_2)$ with $h_1=2-\alpha+2\xi$, $h_2=2\gamma+\alpha+2\xi$, $h_3=\alpha-\gamma-2\xi$ and $h_4=-(1+\alpha+2\xi)$. Expanding, the terms quadratic in ξ cancel, and $h_3h_4-h_1h_2=-(3\alpha+3\gamma-3\alpha\gamma+2(1+\gamma)\xi)$. The last statement is the analytic implicit function theorem. □

Thus in the chart the closure of $\mathcal{E}$ is, near $b=0$, an analytic graph over (b,α,γ) that crosses $b=0$ transversally. This is why a uniform estimate can hold up to the corner.

Remark 6.8. Near the corner the masses degenerate. By Lemma 6.1(c), $m_1m_2=-A_1A_2/w_{12}$ is of order b^{-3}, $m_3m_4=-A_3A_4/w_{34}$ is of order b^3, and the other four products are of order 1. Hence m_3/m_1 and m_4/m_2 are of order b^3. For example, along $\alpha=-\frac12$, $\gamma=\frac65$ one finds $m_3/m_1\approx 0.1506\,b^3$ and $m_4/m_2\approx 2.197\,b^3$ as $b\to 0$ (Proposition 6.10(b)). This end of $\mathcal{E}$ is thus the regime of two small masses near an equilateral Lagrangian point of the two large ones.

The inequalities that define $\mathcal{C}$ confine the dependent coordinate to explicit intervals.

Lemma 6.9. (a) *Let $(a,b,c,x)\in\mathcal{C}$. Then $\ell\le x\le u$, where*

$$\ell=\max\Big\{\frac{a^2-b^2-2b}{2a},\ \frac{1-c^2-2ac}{2a},\ \frac{c-a}{2b},\ \frac{b-1}{2c}\Big\},\qquad u=\min\Big\{\frac{a-c}{2},\ \frac{1-b}{2a}\Big\}.$$

(b) *Let $(a,b,c,x)\in\mathcal{C}$, and let (b,α,γ,ξ) be its chart coordinates. Then $\hat\ell\le\xi\le\hat u$, where $a=1+b\alpha$ and*

$$\hat\ell=\max\Big\{\frac{\alpha-2-b(1-\alpha^2)}{2a},\ -\frac{2\gamma+\alpha+b(\gamma^2+2\alpha\gamma)}{2a}\Big\},\qquad \hat u=\min\Big\{-\frac{1+\alpha}{2a},\ \frac{\alpha-\gamma}{2}\Big\}.$$

Proof. (a) Solve $g_1>0$, $g_2>0$, $g_5\ge0$, $g_6\ge0$, $g_3\ge0$ and $g_4\ge0$ for x, using $a,b,c>0$. (b) Write $h_1=2-\alpha+b(1-\alpha^2)+2a\xi$, $h_2=2\gamma+\alpha+b(\gamma^2+2\alpha\gamma)+2a\xi$ and $h_4=-(1+\alpha+2a\xi)$. Since $b>0$, the conditions $g_1>0$, $g_2>0$, $g_4\ge0$ and $g_3\ge0$ are equivalent to $h_1>0$, $h_2>0$, $h_4\ge0$ and $h_3=\alpha-\gamma-2\xi\ge0$. Solve these for ξ, using $a>0$. □

6.5. **The corner.** By Proposition 2.5(c), the largest constant θ^* such that $Q\ge\theta^*K$ at every convex CC is $\theta^*=1-\sup\lambda_{\max}(G)$, where the supremum is taken over all convex CCs. The next proposition describes the convex CCs near the face $b=0$ of the chart, which is the blown-up corner, and the limits of G there.

Proposition 6.10. *Let $\frac12<\gamma_*<2$ and $-\frac{\gamma_*}{2}<\alpha_*<\frac12$, put $\varrho_*=(\gamma_*^2-\gamma_*+1)^{1/2}$, and*

$$h_1^*=\frac{(1-2\alpha_*)(2-\gamma_*)}{1+\gamma_*},\qquad h_2^*=\frac{(2\alpha_*+\gamma_*)(2\gamma_*-1)}{1+\gamma_*},$$
$$h_3^*=\frac{(2\alpha_*+\gamma_*)(2-\gamma_*)}{1+\gamma_*},\qquad h_4^*=\frac{(1-2\alpha_*)(2\gamma_*-1)}{1+\gamma_*}.$$

(a) *Let $\xi(b,\alpha,\gamma)$ be the function of Lemma 6.7 near $(0,\alpha_*,\gamma_*)$. For all sufficiently small $b>0$ the point $z_b=\big(1+b\alpha_*,\,b,\,b\gamma_*,\,\frac12+b\,\xi(b,\alpha_*,\gamma_*)\big)$ lies in $\mathcal{E}$.*

(b) *Normalize the masses of $q(z_b)$ by $\sigma=-1$. As $b\to0$,*

$$m_1m_3\to\frac{1}{8h_1^*},\qquad m_1m_4\to\frac{1}{8h_3^*},\qquad m_2m_3\to\frac{\gamma_*}{8h_4^*},\qquad m_2m_4\to\frac{\gamma_*}{8h_2^*},$$
$$b^3m_1m_2\to\frac{\gamma_*}{12h_1^*h_2^*\varrho_*^3},\qquad \frac{m_3m_4}{b^3}\to\frac{3\varrho_*^3}{16},$$

and hence

$$\frac{m_1}{m_2}\to\frac{2\gamma_*-1}{\gamma_*(2-\gamma_*)},\qquad \frac{m_3}{m_4}\to\frac{\gamma_*+2\alpha_*}{1-2\alpha_*},\qquad \frac{m_3}{b^3m_1}\to\frac32\varrho_*^3h_3^*,\qquad \frac{m_4}{b^3m_2}\to\frac{3\varrho_*^3h_4^*}{2\gamma_*}.$$

The map from (α_*, γ_*) to the limits of m_1/m_2 and m_3/m_4 is a bijection onto $(0,\infty)^2$.

(c) *The matrix G of $q(z_b)$ converges as $b \to 0$ to a matrix G_0 that depends only on γ_* and has the eigenvalues $\frac{1}{4}$ and*

$$\mu(\gamma_*) = \frac{(2\gamma_* - 1)(2 - \gamma_*)}{3\gamma_*} = \frac{1}{3} - \frac{2(\gamma_* - 1)^2}{3\gamma_*}.$$

Hence $\lambda_{\max}(G) \to \max\big(\frac{1}{4}, \mu(\gamma_*)\big) \le \frac{1}{3}$, with equality if and only if $\gamma_* = 1$.

(d) *$\theta^* \le \frac{2}{3}$. More precisely, if $\gamma_* = 1$, the smallest generalized eigenvalue of (Q, K) at $q(z_b)$ tends to $\frac{2}{3}$. For $\alpha_* = 0$ and $\gamma_* = 1$ the configurations $q(z_b) = q(1, b, b, x_b)$, $x_b = \frac{1}{2} + b\,\xi(b, 0, 1)$, are isosceles trapezoids, with $m_1 = m_2$, $m_3 = m_4$ and $m_3/m_1 \sim \frac{3}{4}b^3$.*

Thus every pair of limiting ratios m_1/m_2, m_3/m_4 occurs on the face $b = 0$, and there the two masses m_3, m_4 are of order b^3 relative to m_1, m_2.

Proof. We drop the asterisks. At $b = 0$ and $\xi = \xi(0, \alpha, \gamma) = \frac{3(\alpha\gamma - \alpha - \gamma)}{2(1+\gamma)}$, the formulas of Section 6.4 give $h_i = h_i^*$ for $i \le 4$ and $h_5 = h_6 = 1$. These values are positive under our hypotheses, and they satisfy

$$h_1 + h_3 = 2 - \gamma, \qquad h_2 + h_4 = 2\gamma - 1, \qquad h_1 h_2 = h_3 h_4.$$

All quantities below are given by the formulas of Section 6.4 and Appendix B. These are real-analytic near $(0, \alpha, \gamma, \xi(0, \alpha, \gamma))$, so their limits along z_b as $b \to 0$ are their values at $b = 0$.

(a) By continuity, $h_1, \dots, h_6 > 0$ at $(b, \alpha, \gamma, \xi(b, \alpha, \gamma))$ for all small $b > 0$, and $\hat{P} = 0$ there by Lemma 6.7. Hence $g_i = b\,h_i > 0$ for $i \le 4$, $g_5, g_6 > 0$ and $P = \hat{P}/b = 0$. Moreover $a = 1 + b\alpha > 0$, $c = b\gamma > 0$ and $|x| < 1$ for small b. So $z_b \in \mathcal{E}$.

(b) At $b = 0$ we have $x = \frac{1}{2}$, $s = \frac{\sqrt{3}}{2}$, $R_e = 1$ for $e \ne 34$, $\psi = \frac{3}{2}$ and $a + c = 1$. Hence

$$\hat{\eta}_{13} = \tfrac{3}{2}h_1, \qquad \hat{\eta}_{24} = \tfrac{3}{2}h_2, \qquad \hat{\eta}_{14} = \tfrac{3}{2}h_3, \qquad \hat{\eta}_{23} = \tfrac{3}{2}h_4, \qquad \hat{\eta}_{34} = \hat{\delta} = \varrho^{-3},$$

and the formulas of Appendix B give

$$\hat{w}_{12} = \tfrac{9}{4}h_1 h_2 \varrho^3, \quad \hat{w}_{13} = -\tfrac{3}{2}h_1, \quad \hat{w}_{24} = -\tfrac{3}{2}h_2, \quad \hat{w}_{14} = \tfrac{3}{2}h_3, \quad \hat{w}_{23} = \tfrac{3}{2}h_4, \quad \hat{w}_{34} = \varrho^{-3},$$
$$\widehat{A}_1 = A_3 = \tfrac{\sqrt{3}}{4}, \qquad \widehat{A}_2 = -\tfrac{\sqrt{3}}{4}\gamma, \qquad A_4 = -\tfrac{\sqrt{3}}{4}.$$

By Lemma 6.1(c) and the scalings of Section 6.4,

$$m_1 m_2 = -b^{-3}\frac{\widehat{A}_1 \widehat{A}_2}{\hat{w}_{12}}, \qquad m_3 m_4 = -b^3 \frac{A_3 A_4}{\hat{w}_{34}}, \qquad m_1 m_3 = -\frac{\widehat{A}_1 A_3}{\hat{w}_{13}},$$
$$m_1 m_4 = -\frac{\widehat{A}_1 A_4}{\hat{w}_{14}}, \qquad m_2 m_3 = -\frac{\widehat{A}_2 A_3}{\hat{w}_{23}}, \qquad m_2 m_4 = -\frac{\widehat{A}_2 A_4}{\hat{w}_{24}}.$$

Inserting the values at $b = 0$ gives the six limits. The four ratios follow from

$$\frac{m_1}{m_2} = \frac{m_1 m_3}{m_2 m_3}, \qquad \frac{m_3}{m_4} = \frac{m_1 m_3}{m_1 m_4}, \qquad \frac{m_3}{m_1} = \frac{m_2 m_3}{m_1 m_2}, \qquad \frac{m_4}{m_2} = \frac{m_1 m_4}{m_1 m_2},$$

together with $h_1 h_2 = h_3 h_4$ and the explicit form of the h_i^*. Finally, $\gamma \mapsto (2\gamma - 1)/(\gamma(2 - \gamma))$ maps $(\frac{1}{2}, 2)$ onto $(0, \infty)$ and has the derivative $2(\gamma^2 - \gamma + 1)/(2\gamma - \gamma^2)^2 > 0$. For fixed γ, $\alpha \mapsto (\gamma + 2\alpha)/(1 - 2\alpha)$ maps $(-\frac{\gamma}{2}, \frac{1}{2})$ onto $(0, \infty)$ and has the derivative $2(1 + \gamma)/(1 - 2\alpha)^2 > 0$.

(c) We have $c_0 = b^2\hat{c}_0$ and $b_0 = b\,\hat{b}_0$, where $\hat{c}_0 = \sum_e D_e\hat{\tau}_e^2$ and $\hat{b}_0 = \sum_e D_e\hat{\tau}_e\beta_e$. Hence $G = \sum_e D_e\beta_e\beta_e^{\mathsf{T}} - \hat{c}_0^{-1}\hat{b}_0\hat{b}_0^{\mathsf{T}}$. At $b = 0$, Appendix B gives

$$(D_{12}, D_{13}, D_{14}, D_{23}, D_{24}, D_{34}) = \Big(0,\ \tfrac{8}{3}h_1,\ \tfrac{8}{3}h_3,\ \tfrac{8h_4}{3\gamma},\ \tfrac{8h_2}{3\gamma},\ \tfrac{16}{9}\Big),$$

$$(\hat{\tau}_{12}, \hat{\tau}_{13}, \hat{\tau}_{14}, \hat{\tau}_{23}, \hat{\tau}_{24}, \hat{\tau}_{34}) = \Big(0,\ \tfrac{3}{16},\ -\tfrac{3}{16},\ -\tfrac{3\gamma}{16},\ \tfrac{3\gamma}{16},\ -\tfrac{3\varrho}{16}\Big),$$

and, since $q_1 = (1, 0)$, $q_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $q_3 = q_4 = 0$ there,

$$\beta_{12} = 0, \qquad \beta_{13} = -\beta_{14} = \frac{\gamma - 2}{4\gamma}\Big(\frac{1}{\sqrt{3}}, 1\Big),$$

$$\beta_{23} = -\beta_{24} = \Big(-\frac{2\gamma - 1}{2\sqrt{3}}, 0\Big), \qquad \beta_{34} = \frac{\varrho}{2\gamma}\Big(\frac{1 - 2\gamma}{\sqrt{3}}, 1\Big).$$

Using $h_1 + h_3 = 2 - \gamma$ and $h_2 + h_4 = 2\gamma - 1$ we obtain

$$\hat{c}_0 = \tfrac{3}{32}(h_1 + h_3) + \tfrac{3\gamma}{32}(h_2 + h_4) + \tfrac{\varrho^2}{16} = \tfrac{\varrho^2}{4}, \qquad \hat{b}_0 = \tfrac{1}{2}(2 - \gamma)\beta_{13} - \tfrac{1}{2}(2\gamma - 1)\beta_{23} - \tfrac{\varrho}{3}\beta_{34},$$

and

$$G_0 = \tfrac{8}{3}(2 - \gamma)\,\beta_{13}\beta_{13}^{\mathsf{T}} + \tfrac{8(2\gamma-1)}{3\gamma}\,\beta_{23}\beta_{23}^{\mathsf{T}} + \tfrac{16}{9}\,\beta_{34}\beta_{34}^{\mathsf{T}} - \tfrac{4}{\varrho^2}\,\hat{b}_0\hat{b}_0^{\mathsf{T}}.$$

This depends only on γ. Explicitly,

$$G_0 = \frac{1}{48\varrho^2}\begin{pmatrix} 3(-8\gamma^3 + 21\gamma^2 - 12\gamma + 4) & \sqrt{3}\,(8\gamma^3 - 33\gamma^2 + 42\gamma - 16) \\ \sqrt{3}\,(8\gamma^3 - 33\gamma^2 + 42\gamma - 16) & \gamma^{-1}(-8\gamma^4 + 61\gamma^3 - 120\gamma^2 + 112\gamma - 32) \end{pmatrix},$$

and $\operatorname{tr} G_0 = \frac{1}{4} + \mu(\gamma)$, $\det G_0 = \frac{1}{4}\mu(\gamma)$. For example, at $\gamma = 1$ we have $\hat{b}_0 = \frac{7}{24}(\frac{1}{\sqrt{3}}, -1)$ and

$$G_0 = \begin{pmatrix} \frac{5}{16} & \frac{\sqrt{3}}{48} \\ \frac{\sqrt{3}}{48} & \frac{13}{48} \end{pmatrix},$$

with the eigenvalues $\frac{1}{4}$ and $\frac{1}{3}$. These computations were also carried out in computer algebra, and compared with the chart formulas evaluated at $b = 10^{-15}$. The eigenvalues depend continuously on the matrix, so $\lambda_{\max}(G) \to \max(\frac{1}{4}, \mu(\gamma))$. Finally $\mu(\gamma) \le \frac{1}{3}$, with equality if and only if $\gamma = 1$, and $\mu(1) = \frac{1}{3} > \frac{1}{4}$.

(d) By Proposition 2.5(c), the smallest generalized eigenvalue of (Q, K) at $q(z_b)$ is $1 - \lambda_{\max}(G)$. If $\gamma = 1$, it tends to $\frac{2}{3}$ by (c), and hence $\theta^* \le \frac{2}{3}$. For $\alpha = 0$ and $\gamma = 1$ we have $a = 1$ and $c = b$, so $q_3 = -b\,q_1$ and $q_4 = -b\,q_2$ with $|q_1| = |q_2| = 1$. As in Remark 6.4, the reflection in the perpendicular bisector of q_1q_2 exchanges $q_1 \leftrightarrow q_2$ and $q_3 \leftrightarrow q_4$, so $q(z_b)$ is an isosceles trapezoid. The reflected configuration with the labels $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ exchanged is again $q(z_b)$, and it is a CC for the masses (m_2, m_1, m_4, m_3). By the uniqueness of the masses in Lemma 6.1(c), $(m_2, m_1, m_4, m_3) = t\,m$ for some $t > 0$. Then $t^2 = 1$, so $m_1 = m_2$ and $m_3 = m_4$. Finally $\varrho = 1$ and $h_3^* = \frac{1}{2}$, so $m_3/(b^3 m_1) \to \frac{3}{4}$ by (b). □

7. The computer-assisted proof

This section gives a second proof of the inequality $3C \prec M_2^2$. For $(a, b, c, x) \in \mathcal{C}$ let w_e and δ be as in Lemma 6.1(a) and (6.4), put $d_e = -w_eA_iA_j\,r_e^5/3$ for $e = ij$, and let

$$T = W\,M_2^2 - 3\Big(W\sum_e d_e\,n_en_e^{\mathsf{T}} - tt^{\mathsf{T}}\Big), \qquad W = \sum_e d_e, \qquad t = \sum_e d_e\,n_e, \tag{7.1}$$

for the configuration $q(a, b, c, x)$. At a point of $\mathcal{E}$ these d_e are those of [Tom26a, Section 3.4], by the proof of Lemma 6.1(c), and $T = 3W\big(\frac{1}{3}M_2^2 - C\big)$ with $W > 0$. By Corollary 2.6, $\lambda_{\max}(G) < \frac{1}{3}$ there if and only if $T \succ 0$. We call T the test.

Proposition 7.1. *At every point of $\mathcal{E}$ the matrix T is positive definite. Consequently $\lambda_{\max}(G) < \frac{1}{3}$ at every point of $\mathcal{E}$.*

The proof is a computation, which Section 7.6 assembles.

7.1. **What is computed.** The certificates evaluate a normalized form of T, in which no square root of $1-x^2$ occurs and no area is inverted. Given weights d'_e and vectors $\tilde{n}_e \in \mathbb{R}^2$ for the six edges, a symmetric 2×2 matrix N, a number $F_0 > 0$ and a number s_2, put

$$W' = \sum_e d'_e, \qquad \bar{n}' = \frac{1}{W'}\sum_e d'_e \tilde{n}_e, \qquad z_e = (\operatorname{adj} N)\,(\tilde{n}_e - \bar{n}'),$$

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & s_2 \end{pmatrix} - \frac{1}{F_0}\sum_e d'_e\, z_e z_e^{\mathsf{T}}. \tag{7.2}$$

In the coordinates (a,b,c,x) we take $s_2 = 1-x^2$,

$$N = \begin{pmatrix} b - px^2 & -px \\ -px & -p \end{pmatrix}, \qquad p = ac, \qquad F_0 = \delta\,(kpb)^2, \qquad k = (1+b)(a+c),$$

with δ as in (6.4), the weights d'_e listed in Appendix B, which are built from the η_e, the R_e, the r_e and a, b, c by sums and products, and the vectors

$$\tilde{n}_e = \tilde{q}_i + \tilde{q}_j, \qquad \tilde{q}_1 = (1,0), \quad \tilde{q}_2 = (ax, a), \quad \tilde{q}_3 = (-b, 0), \quad \tilde{q}_4 = -(cx, c),$$

which are the vectors n_e in the frame scaled by $\operatorname{diag}(1, 1/s)$, where $s = \sqrt{1-x^2}$. Here $\operatorname{adj} N = \begin{pmatrix} -p & px \\ px & b - px^2 \end{pmatrix}$, and $\frac{s}{2} kDND = M_2$ with $D = \operatorname{diag}(1, s)$.

Lemma 7.2. *Let $(a,b,c,x) \in \mathcal{E}$. Then $D_e \tau_e^2 = \lambda_0\, d'_e$ for every edge, with $\lambda_0 = s^2/(12\delta) > 0$, and for every $u \in \mathbb{R}^2$,*

$$u^{\mathsf{T}} T u = 3\lambda_0^2\, \delta\, W'\, Y\big[kN(u_1, s\,u_2)\big],$$

where $Y[w] = w^{\mathsf{T}} Y w$ and $W' = \sum_e d'_e > 0$. In particular, if $\det Y > 0$ and $\operatorname{tr} Y > 0$, then $T \succ 0$.

Proof. The first claim follows from Lemma 6.1(a) and the formulas for the d'_e in Appendix B; in particular $W = \lambda_0 W'$, so $W' > 0$ because $W > 0$. Since the weights d_e and d'_e are proportional and $n_e = D\tilde{n}_e$, we have $\bar{n} = D\bar{n}'$ and $C = \lambda_0\, DC'D$ with $C' = \sum_e d'_e(\tilde{n}_e - \bar{n}')(\tilde{n}_e - \bar{n}')^{\mathsf{T}}$. Moreover $T = W(M_2^2 - 3C)$. Put $\tilde{u} = Du$ and $w = kN\tilde{u}$. Since $M_2 = \frac{s}{2} kDND$, we get $u^{\mathsf{T}} M_2^2 u = |M_2 u|^2 = \frac{s^2}{4}|Dw|^2$, and $u^{\mathsf{T}} C u = \lambda_0\, \tilde{u}^{\mathsf{T}} C' \tilde{u}$. On the other hand $N \operatorname{adj} N = \det N \cdot I$ with $\det N = -pb$, so $w^{\mathsf{T}} z_e = k \det N\ \tilde{u}^{\mathsf{T}}(\tilde{n}_e - \bar{n}')$, and with $F_0 = \delta(kpb)^2$ this gives $Y[w] = |Dw|^2 - \frac{1}{\delta}\,\tilde{u}^{\mathsf{T}} C' \tilde{u}$. Hence

$$u^{\mathsf{T}} T u = \lambda_0 W' \Big(\frac{s^2}{4}|Dw|^2 - 3\lambda_0\, \tilde{u}^{\mathsf{T}} C' \tilde{u} \Big) = \frac{\lambda_0 s^2}{4}\, W'\, Y[w] = 3\lambda_0^2\, \delta\, W'\, Y[w],$$

because $3\lambda_0 = s^2/(4\delta)$. If $\det Y > 0$ and $\operatorname{tr} Y > 0$, then $Y \succ 0$, and $T \succ 0$ because kND is invertible. □

One can also check that $\det Y = 9s^2 \det(\frac{1}{3} I - G)$ and $\operatorname{tr} Y = 1 + s^2 - 3\operatorname{tr}(DGD)$, which shows how the test measures the margin $\frac{1}{3} - \lambda_{\max}(G)$. In the blow-up chart of Section 6.4 the same test is computed with every power of b cancelled (Appendix B); we write $\widehat{Y}$ for the matrix so obtained. The certificates evaluate Y directly where $b \geq \frac{1}{2}$ and $\widehat{Y}$ where $b \leq \frac{1}{2}$ (Section 7.3). At every point of $\mathcal{C}$ with $b > 0$, $\det \widehat{Y}$ and $\operatorname{tr} \widehat{Y}$ equal $\det Y$ and $\operatorname{tr} Y$, and $\widehat{Y}$ extends to $b = 0$ (Appendix B). On the line $b = 0$, $\gamma = 1$, every divisor in its formulas has a value independent of α, and every square root is taken of a quantity whose value there is 1.

7.2. **The symmetry.** The map $\iota(a,b,c,x) = (1/a, c/a, b/a, x)$ of Remark 6.3 is the relabelling $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ followed by an orientation-reversing similarity of ratio $1/a$. It maps $\mathcal{C}$ into $\mathcal{C}$ and central configurations to central configurations, with the masses relabelled, and it multiplies Q and K by the same positive factor, as in Lemma 8.1. Hence $\lambda_{\max}(G)$ takes the same value at z and $\iota(z)$, and it suffices to prove Proposition 7.1 on $\mathcal{E} \cap \{a \le 1\}$. In the chart, ι acts by $(b,\alpha,\gamma) \mapsto (b\gamma/a,\ -\alpha/\gamma,\ 1/\gamma)$, so it maps the line $b=0$, $\gamma=1$ to itself, with $\alpha \mapsto -\alpha$.

7.3. **The cover.** Every point of $\mathcal{E}$ with $a \le 1$ lies in one of the following regions. We cover the first two by grids of cells, the root grids; the certificates split the cells further.

- The direct region $b \ge \frac{1}{2}$, in the coordinates (a,b,c): here $a,b \in [\frac{1}{2},1]$ and $c \in [0,\frac{7}{4}]$, covered by 28 cells of side $\frac{1}{4}$.
- The blow-up chart $0 < b \le \frac{1}{2}$, in the coordinates (b,α,γ): 20 cells for $b \in [\frac{1}{4},\frac{1}{2}]$, $\alpha \in [-2,0]$, $\gamma \in [0,5]$; 20 cells for $b \in [\frac{1}{8},\frac{1}{4}]$, $\alpha \in [-4,0]$, $\gamma \in [0,5]$; and 24 cells for $b \in [0,\frac{1}{8}]$, $\alpha \in [-2,0]$, $\gamma \in [0,3]$.
- The corner disc $b^2 + (\gamma-1)^2 \le (\frac{1}{128})^2$, treated in Section 7.5.

The bounds on a, b, c, α and γ follow from Lemmas 6.5 and 6.6 and from $a \le 1$; for $0 < b \le \frac{1}{2}$, for example, $a \le 1$ and Lemma 6.5 give $-\frac{1}{2b} \le \alpha \le 0$ and $0 \le \gamma \le 5$.

A certificate splits a cell into boxes and gives each box a hint, which consists of the shear and the multiplier described below. On a box, the range of the dependent coordinate x, or ξ in the chart, is first bounded by Lemma 6.9, and the feasible zeros of P, or of $\hat{P}$ in the chart, are localized by interval Newton steps and bisection; intervals of x without a feasible zero are excluded. Boxes of the chart that lie in the corner disc are recognized exactly and left to Section 7.5. On each kept interval the hint gives a quadratic shear: the dependent coordinate is written as $x = x_0(z) + e\,t_3$, where z are the three box coordinates, x_0 is a quadratic polynomial in z with rational coefficients, $e > 0$ is the half-width of a slab and $|t_3| \le 1$. The check then has two phases.

(i) *Root phase.* At most two localization steps show that every zero of P (or $\hat{P}$) over the box lies in the slab, and shrink the slab.
(ii) *Test phase.* On the slab, the lower bounds of $\operatorname{tr} Y$ and of $\det Y - \mu g$ are positive, where $\mu \ge 0$ is part of the hint and $g \ge 0$ is one of the six functions that define $\mathcal{C}$ (or $\mu = 0$).

The multiplier μ admits boxes that meet the boundary of $\mathcal{C}$, where one of the masses tends to zero: zeros of P beyond the face $g = 0$ are not central configurations, and there the inequality need not hold.

7.4. **Taylor models.** The enclosures in both phases are fixed-point Taylor models in the four variables of the box and the slab: polynomials of total degree at most 3, or of degree less than 4 when the slab variable is given weight 2, with fixed-point coefficients and an interval remainder that bounds the difference from the true function on the box. Products are truncated by degree, and the truncated terms are bounded on the box and added to the remainder, as is the error of each rounding of a coefficient to a fixed-point number. Quotients and square roots use the identities $1/v = Y + (1-vY)/v$ and $\sqrt{v} = Y + (v - Y^2)/(\sqrt{v} + Y)$, where Y is a truncated series and the second term goes into the remainder. A box may also be checked with first-order slope enclosures, after Newton steps on them, when that suffices.

7.5. **The corner disc.** On the line $b = 0$, $\gamma = 1$ of the chart the margin of the test vanishes. By Proposition 6.10(c), $\lambda_{\max}(G) \to \frac{1}{3}$ along $\gamma_* = 1$, and $\det \widehat{Y}$ vanishes to second order on the line $b = 0$, $\gamma = 1$, for every α. An enclosure on boxes carries an error of fixed size and cannot reach this line, so on the corner disc we use Taylor forms with exactly computed coefficients.

Proposition 7.3. *Let $b > 0$, $\alpha \le 0$ and $b^2+(\gamma-1)^2 \le (\frac{1}{128})^2$, and suppose that $(1+b\alpha,\ b,\ b\gamma,\ \frac{1}{2}+b\xi) \in \mathcal{E}$. Then $\det \widehat{Y} > 0$ and $\operatorname{tr} \widehat{Y} > 0$ at (b, α, γ, ξ).*

The constraints of $\mathcal{C}$ give $\alpha \ge -\frac{17}{32}$ on the disc. Three certificates cover $\alpha \in [-\frac{17}{32}, -\frac{3}{8}]$, $[-\frac{3}{8}, -\frac{1}{8}]$ and $[-\frac{1}{8}, 0]$. Each computes with Taylor forms of order 5 in $(b, \gamma-1)$, whose coefficients are polynomials in α with rational coefficients, and with interval remainders. In each, the root ξ of $\hat{P}$ is first localized in an interval. A slope of $\hat{P}$ in ξ then expresses it as $\xi = \Xi - \hat{P}(\Xi)/\varsigma$, where Ξ is a polynomial in $(b, \gamma-1, \alpha)$ with rational coefficients and ς is enclosed by the slope, and the Taylor form of $\hat{P}(\Xi)$ has zero polynomial part. Finally the Taylor-form evaluation of $\widehat{Y}$ at this ξ shows that $(1 + p\,b + p'(\gamma-1)) \det \widehat{Y} > 0$ on the disc for rational numbers p, p' given in the certificate, and that $\operatorname{tr} \widehat{Y} > 0$ and $1 + p\,b + p'(\gamma-1) > 0$ there.

7.6. **The certificates.** Each certificate is a Boolean statement. The certificates of the cells assemble into one cover per root grid, and the three certificates of Section 7.5 cover the corner disc.

Proof of Proposition 7.1. By Section 7.2 we may assume $a \le 1$. By Section 7.3 the point lies in a cell of one of the root grids or in the corner disc. In the first case the certificate of the cell gives a box and a slab that contain the point, and the test phase gives $\operatorname{tr} Y > 0$ and $\det Y > \mu g \ge 0$ there. In the second case Proposition 7.3 gives $\operatorname{tr} \widehat{Y} > 0$ and $\det \widehat{Y} > 0$, hence $\operatorname{tr} Y > 0$ and $\det Y > 0$. In both cases $T \succ 0$ by Lemma 7.2. □

8. Proof of Theorem B

Proposition 7.1 holds on the normal forms of Section 6. To transfer it to all convex CCs we record how Q and K behave under the normalization of Lemma 6.2(b).

Lemma 8.1. *Let q be a convex CC for the masses m, and let π, ϕ, $(a, b, c, x) \in \mathcal{E}$, $q' = q(a, b, c, x)$ and m' be as in Lemma 6.2(b). There are a linear isomorphism $\hat{\phi}$ of $(\mathbb{R}^2)^4$ and a number $k > 0$ such that $Q'(\hat{\phi}v) = k^{-1}Q(v)$ and $K'(\hat{\phi}v) = k^{-1}K(v)$ for all v, where Q' and K' are the forms Q and K of the CC q' for the masses m'. In particular the pairs (Q, K) and (Q', K') have the same generalized eigenvalues, and $\lambda_{\max}(G)$ takes the same value at q and at q'.*

Proof. Write $\phi(z) = \phi_0 z + t$, where ϕ_0 is a real-linear map with $|\phi_0 u| = k|u|$ for some $k > 0$, so that $\langle \phi_0 u, \phi_0 u' \rangle = k^2 \langle u, u' \rangle$. Define $\hat{\phi}$ by $(\hat{\phi}v)_{\pi(i)} = \phi_0 v_i$. We have $r'_{\pi(i)\pi(j)} = k\, r_{ij}$ and $M' = M$, hence $U(q') = k^{-1}U(q)$ and $I(q') = k^2 I(q)$, and the constant λ' of q' equals k^{-3} times that of q. The corresponding pair potential $\tilde{\varphi}(r) = \frac{1}{r} + \frac{k^{-3}\lambda'}{2} r^2$ satisfies $\tilde{\varphi}(kr) = k^{-1}\varphi(r)$. Let U'_λ be the function U_λ of q', formed with its own multiplier, and let $p \mapsto \tilde{p}$, $\tilde{p}_{\pi(i)} = \phi(p_i)$, be the affine map with linear part $\hat{\phi}$ that sends q to q'. Then $U'_\lambda(\tilde{p}) = k^{-1}U_\lambda(p)$, and differentiating twice at q gives $Q'(\hat{\phi}v) = k^{-1}Q(v)$. Similarly $\kappa'_{\pi(i)\pi(j)} = k^{-3}\kappa_{ij}$ and $\mathrm{d}r'_{\pi(i)\pi(j)}(\hat{\phi}v) = \langle \phi_0 q_{ij}, \phi_0 v_{ij} \rangle / (k\, r_{ij}) = k\, \mathrm{d}r_{ij}(v)$, where we used that $\mathrm{d}r_e$ does not depend on the orientation of the edge e. Hence $K'(\hat{\phi}v) = k^{-1}K(v)$. The last statement follows from Proposition 2.5. □

Proof of Theorem B. Let q be a convex CC. By Theorem 3.1, $\mu_1 < \frac{1}{3}$ at q. Alternatively, by Lemma 6.2(b) there is a point of $\mathcal{E}$ whose configuration q' arises from q by a relabelling and a similarity, by Proposition 7.1 $\lambda_{\max}(G) < \frac{1}{3}$ at q', and by Lemma 8.1 $\lambda_{\max}(G)$ takes the same value at q. In either case Proposition 2.5 gives $Q(v) \ge (1 - \mu_1)K(v) > \frac{2}{3}K(v)$ whenever $K(v) > 0$, and $Q \ge \frac{2}{3}K$ everywhere. The constant $\frac{2}{3}$ cannot be improved by [Tom26a, Lemma 6.1]. □

9. Formal proofs

Two developments in the proof assistant Lean 4 with the Mathlib library formalize results of [Tom26a] and of this paper; see the statement on data and code availability. Neither contains unproved statements, except the statement files against which the registry's comparator checks the proofs, which state the theorems without proof by design. Both rely only on the kernel of Lean, the small program that checks every proof, and on the three standard axioms of Lean (propositional extensionality, the axiom of choice and the soundness of quotients). Their computations are Boolean statements that the kernel evaluates with its built-in arithmetic of natural numbers, not with compiled code.

The first development. It formalizes Theorem A of [Tom26a], with nondegeneracy and Morse index 0 at every convex CC, Corollary C and most of Corollary B of [Tom26a], and the injectivity of Remark 6.4, with a computer-assisted bound in place of the hand proof of [Tom26a, Theorem 4.1]. By Proposition 2.5(a), a bound $\operatorname{tr} S(y) < \frac{3}{4}$ for one vector y gives $Q \geq \frac{1}{4}K$, and a branch-and-bound computation, the trace test, certifies such a bound at every point of $\mathcal{E}$, in the coordinates of Section 6. With this bound the development follows [Tom26a, Sections 2, 3 (without Section 3.4) and 5]. It also contains most of the results of [Tom26a, Sections 2 and 3] on which this proof rests and the reduction to the normal form of Section 6. Its definitions, the trace test, the checker and the certificates are described in detail in [Tom26b].

The second development. It formalizes the inequality of Theorem B, in the form of the statements

$$\tfrac{2}{3}K(v) \leq Q(v) \quad \text{for all } v, \qquad \tfrac{2}{3}K(v) < Q(v) \quad \text{whenever } K(v) > 0,$$

at every configuration q that is a CC for positive masses and convex, where convexity means that two of the oriented areas are positive and two are negative. It takes the definitions and the reduction to the normal form from the registered first development. From Corollary 2.6 it takes only the bound $\sum_e D_e \langle g_e, w\rangle^2 < \frac{1}{3}|w|^2$ for the representation $g_e = \tau_e M_2^{-1}(z - n_e)$ of L with the best z, which it derives from $T \succ 0$. Its proof of Proposition 7.1 consists of the identification of the test with the normalized matrix Y of Section 7.1 at the normal form and in the chart, the symmetry of Section 7.2, the cover of Section 7.3, the theorem for the corner disc in Section 7.5 and the assembly. The formulas of Section 7.1 are written once, generically over a type class of operations: the checker evaluates this text in Taylor models, and the proofs evaluate the same text over $\mathbb{R}$, so there is no transcription between the checked formulas and the theorem. The soundness of the Taylor-model operations and of the checker, and the fact that every point of $\mathcal{E}$ with $a \leq 1$ lies in a cell of one of the root grids or in the corner disc, are proved in Lean as well. The theorems are stated through Q and K, and G does not occur. A statement file that imports only Mathlib and copies the definitions is compared with the proof by the registry's comparator, which checks that the statements and the definitions they use agree. The development is registered in the Palomar registry of formal proofs (see the statement on data and code availability).

Neither development formalizes the hand proofs of [Tom26a, Proposition 3.10 and Theorem 4.1] or of Sections 3–5, Proposition 6.10, the statements about $\lambda_{\max}(G)$, or the optimality of $\frac{2}{3}$.

Appendix A. Verification of the round-case inequalities

This appendix proves Lemmas 4.11, 4.12, 4.13 and 4.14, in the notation of Section 4. Every numerical step evaluates elementary functions at a few displayed points, and every displayed constant is rounded outward: upper bounds up and lower bounds down, in the last printed digit; table entries derived from other entries are computed from the printed values. Throughout we

use

$$\Omega_1(t)=\big(1-(\tfrac{25}{16}+\tfrac{3}{2}t)^{-3/2}\big)^{-1},\qquad \Omega_2(t)=\big(1-(2+2t)^{-3/2}\big)^{-1},\qquad \bar\beta(t)=\arctan\frac{\sqrt{1-t^2}}{\frac34+t}.$$

A.1. **One obtuse vertex.** In this subsection we prove Lemma 4.11; throughout, $q\in\mathrm{Round}(\frac34)$ and $\alpha_i>90^\circ$. We write $\alpha=\alpha_i$, $\chi=\chi_i$, and $\phi_j,\phi_k,\beta,\gamma$ are as in Lemma 4.6, so that $\phi_j+\phi_k=\alpha$ and $\beta+\gamma=180^\circ-\alpha$. Put

$$\phi_-=\arctan\tfrac34=36.870^\circ,\qquad \phi_+=\arctan\tfrac43=53.130^\circ,\qquad \phi_*=\arccos\tfrac38=67.976^\circ.$$

Since $\mathcal{V}(i)=\mathrm{ser}(\mathcal{V}_1(i),\mathcal{V}_2(i))$,

$$\mathcal{V}(i)<1\iff \mathcal{V}_2(i)\le1,\ \text{or } \mathcal{V}_2(i)>1 \text{ and } \mathcal{V}_1(i)<\frac{\mathcal{V}_2(i)}{\mathcal{V}_2(i)-1}. \tag{A.1}$$

Lemma A.1. *$\phi_j,\phi_k<\phi_*$. Hence $\alpha<2\phi_*=135.95^\circ$ and $\chi<\frac{23}{32}$.*

Proof. Suppose $\cos\phi_j\le\frac38$. By the law of cosines in the triangle ijo, $r_{jo}^2\ge r_{ij}^2+r_{io}^2-\frac34 r_{ij}r_{io}$. Since $r_{io}>r_{jo}$, this gives $r_{io}>\frac43 r_{ij}$, and then $r_{jo}^2\ge r_{ij}^2+r_{io}(r_{io}-\frac34 r_{ij})>r_{ij}^2+\frac43\cdot\frac{7}{12}r_{ij}^2=\frac{16}{9}r_{ij}^2$. Thus $r_{jo}/r_{ij}>\frac43$, contrary to $q\in\mathrm{Round}(\frac34)$. The same argument applies to ϕ_k. Finally $-\cos2\phi_*=1-2\cdot\frac{9}{64}=\frac{23}{32}$. □

Lemma A.2. *For $0^\circ<\phi<90^\circ$ and $t\ge\sin\phi$ let $h(\phi,t)=\big(\cos\phi+(t^2-\sin^2\phi)^{1/2}\big)/t$, and let*

$$\Psi(\phi)=\max\big\{\tan\phi\,(h(\phi,t)^3-1):\ \tfrac34\le t\le\tfrac43,\ t\ge\sin\phi\big\}.$$

Then $g(\phi_j;\alpha_j)\le\Psi(\phi_j)$ and $g(\phi_k;\alpha_k)\le\Psi(\phi_k)$. Moreover $\Psi=\Psi_1$ on $(0^\circ,\phi_-]$, $\Psi=\Psi_2$ on $[\phi_-,\phi_+]$ and $\Psi=\Psi_3$ on $[\phi_+,90^\circ)$, where

$$\Psi_1(\phi)=\tan\phi\,\big(h(\phi,\tfrac34)^3-1\big),\qquad \Psi_2(\phi)=g_0(\phi),\qquad \Psi_3(\phi)=\tan\phi\,\big(h(\phi,\tfrac43)^3-1\big),$$

and $\Psi(\phi_-)=\frac{49}{18}$, $\Psi(\phi_+)=\frac{61}{48}$, $\Psi(\phi_*)=0$.

Proof. The law of cosines $r_{jo}^2=r_{ij}^2+r_{io}^2-2r_{ij}r_{io}\cos\phi_j$ is a quadratic equation for r_{io}. Its smaller root is less than $r_{ij}<r_{io}$, so r_{io} is the larger root, and $r_{io}/r_{jo}=h(\phi_j,t)$ with $t=r_{jo}/r_{ij}\in[\frac34,\frac43]$. By the law of sines $\sin\alpha_j/\sin\phi_j=r_{io}/r_{jo}$, so $g(\phi_j;\alpha_j)=\tan\phi_j(h(\phi_j,t)^3-1)\le\Psi(\phi_j)$.

With $w=(t^2-\sin^2\phi)^{1/2}$ we have $\partial_t h=(\sin^2\phi-w\cos\phi)/(w\,t^2)$, which is positive for $t<\tan\phi$ and negative for $t>\tan\phi$; and $h(\phi,\tan\phi)=1/\sin\phi$, since $w=\sin\phi\tan\phi$ there. So the maximum is attained at $t=\frac34$, $\tan\phi$ or $\frac43$, according as $\tan\phi\le\frac34$, $\frac34\le\tan\phi\le\frac43$ or $\tan\phi\ge\frac43$; each of these points is at least $\sin\phi$. The values at the junctions follow from $\sin\phi_-=\frac35$, $\sin\phi_+=\frac45$, and $h(\phi_*,\frac43)=(\frac38+\frac{23}{24})\cdot\frac34=1$. □

Lemma A.3 (The triangle ijk)**.** *Let $\Omega(\alpha)=\Omega_1(\chi)$ and*

$$\overline{\mathcal{V}}_2(\alpha)=\chi\big(\tfrac{25}{16}+\tfrac32\chi\big)^{3/2}\Big[\frac{4}{3+4\chi}+\frac{64}{9(4+3\chi)}\Big].$$

(a) $1/(1-(\sin\beta/\sin\alpha)^3)\le\Omega(\alpha)$ *and* $1/(1-(\sin\gamma/\sin\alpha)^3)\le\Omega(\alpha)$.
(b) $\mathcal{V}_2(i)\le\overline{\mathcal{V}}_2(\alpha)$.
(c) *On $(90^\circ,2\phi_*)$, Ω is decreasing and $\overline{\mathcal{V}}_2$ is increasing.*

Proof. (a) Let $t=r_{ij}/r_{ik}\in[\frac34,\frac43]$. Then $(r_{jk}/r_{ik})^2=t^2+2t\chi+1$ and $(r_{jk}/r_{ij})^2=t^{-2}+2t^{-1}\chi+1$. Both are at least $\frac{9}{16}+\frac32\chi+1=\frac{25}{16}+\frac32\chi$, since $y^2+2y\chi+1$ increases for $y>0$. Now use $\sin\beta/\sin\alpha=r_{ik}/r_{jk}$ and $\sin\gamma/\sin\alpha=r_{ij}/r_{jk}$.

(b) The ratio $r_{ij}/r_{ik}=\sin\gamma/\sin\beta=\sin(\alpha+\beta)/\sin\beta$ has derivative $-\sin\alpha/\sin^2\beta$ in β. So the bound $\frac34\le r_{ij}/r_{ik}\le\frac43$ confines β to an interval $[\beta_-,\beta_+]$, with $\beta_-+\beta_+=180^\circ-\alpha$ by the

symmetry $j \leftrightarrow k$. By Lemma 4.10(d), $1/f(\beta)+1/f(180^\circ-\alpha-\beta)$ is convex in β, so it is largest at $\beta_\pm$, where its two values agree. At β_+ we have $r_{ij}/r_{ik}=\frac34$; scale so that $r_{ij}=\frac34$ and $r_{ik}=1$. Then $r_{jk}^2=\frac{25}{16}+\frac32\chi$, $\cos\beta=(r_{ij}+r_{ik}\chi)/r_{jk}$ and $\sin\beta=r_{ik}\sin\alpha/r_{jk}$, so

$$\frac{\sin^2\alpha}{f(\beta)}=\frac{r_{jk}^3}{r_{ik}^2(r_{ij}+r_{ik}\chi)}=\frac{4r_{jk}^3}{3+4\chi},\qquad \frac{\sin^2\alpha}{f(\gamma)}=\frac{r_{jk}^3}{r_{ij}^2(r_{ik}+r_{ij}\chi)}=\frac{64r_{jk}^3}{9(4+3\chi)}.$$

Since $f(180^\circ-\alpha)=\chi\sin^2\alpha$, (b) follows from (4.1).

(c) Ω_1 is decreasing and χ increases with α, so Ω decreases. Write $\overline{\mathcal{V}}_2=\chi(\frac{25}{16}+\frac32\chi)^{3/2}Y(\chi)$. Then $-Y'/Y\le\max\bigl(4/(3+4\chi),\,3/(4+3\chi)\bigr)\le\frac43$, by comparing the two terms of Y and of Y'. Hence $(\log\overline{\mathcal{V}}_2)'(\chi)\ge 1/\chi-\frac43>0$, because $\chi<\frac{23}{32}$ (Lemma A.1). □

Corollary A.4. *Let* $\widehat{\Psi}(\alpha)=\max\{\Psi(\phi)+\Psi(\alpha-\phi)\,:\,\alpha-\phi_*\le\phi\le\phi_*\}$, *and* $\widehat{\mathcal{V}}_1(\alpha)=\overline{\mathcal{V}}_2(\alpha)/(\overline{\mathcal{V}}_2(\alpha)-1)$ *when* $\overline{\mathcal{V}}_2(\alpha)>1$.

(a) $\mathcal{V}_1(i)\le|\cot\alpha|\,\Omega(\alpha)\,\widehat{\Psi}(\alpha)$.

(b) *If* $\alpha\le\alpha_2$, $\overline{\mathcal{V}}_2(\alpha_2)>1$ *and* $\mathcal{V}_1(i)<\widehat{\mathcal{V}}_1(\alpha_2)$, *then* $\mathcal{V}(i)<1$.

Proof. (a) follows from (4.1), Lemma A.2, Lemma A.3(a), and Lemma A.1, which puts ϕ_j and $\phi_k=\alpha-\phi_j$ in $(\alpha-\phi_*,\phi_*)$. (b) If $\mathcal{V}_2(i)\le1$, then $\mathcal{V}(i)<1$. Otherwise $1<\mathcal{V}_2(i)\le\overline{\mathcal{V}}_2(\alpha)\le\overline{\mathcal{V}}_2(\alpha_2)$ by Lemma A.3, and since $y\mapsto y/(y-1)$ decreases, $\mathcal{V}_2(i)/(\mathcal{V}_2(i)-1)\ge\widehat{\mathcal{V}}_1(\alpha_2)>\mathcal{V}_1(i)$. Now apply (A.1). □

Next we determine the shape of Ψ. Let $p_4(z)=576z^2-(16z^2-7)^2=-256z^4+800z^2-49$.

Lemma A.5. Ψ_3 *is decreasing and concave on* $[\phi_+,\phi_*]$.

Proof. Put $z=h(\phi,\frac43)$. Then $\cos\phi=(16z^2-7)/(24z)$, so z decreases from $\frac54$ to 1 as ϕ increases from ϕ_+ to ϕ_*. Moreover $\tan\phi=p_4^{1/2}/(16z^2-7)$, $\Psi_3=(z^3-1)p_4^{1/2}/(16z^2-7)$ and $d\phi/dz=-(16z^2+7)/(zp_4^{1/2})$. Differentiating,

$$\Psi_3'(\phi)=\frac{3z^2p_7(z)}{(16z^2-7)^2(16z^2+7)},\qquad \Psi_3''(\phi)=-\frac{3z^2p_4^{1/2}p_{11}(z)}{(16z^2-7)^3(16z^2+7)^3},$$

where

$$\begin{aligned}p_7&=4096z^7-11520z^5+7728z^3-3072z^2-343z-1344,\\ p_{11}&=3145728z^{11}-3866624z^9-1204224z^7+1572864z^6+2483712z^5+2064384z^4\\&\quad-1816528z^3+903168z^2+50421z+131712.\end{aligned}$$

Let $1\le z\le\frac54$. First, $p_7=z^3q_7(z^2)-3072z^2-343z-1344$, where $q_7(y)=4096y^2-11520y+7728$ is convex with $q_7(1)=304$ and $q_7(\frac{25}{16})=-272$, so $p_7\le304\cdot(\frac54)^3-4759=-\frac{16661}{4}<0$. Second, $q_{11}(y)=3145728y^2-3866624y-1204224$ increases for $y\ge1$, with $q_{11}(1)=-1925120$, and $p_{11}-z^7q_{11}(z^2)=1572864z^6+2483712z^5+z^3(2064384z-1816528)+903168z^2+50421z+131712$. Hence $p_{11}>z^5\underline{p}_{11}(z)$, where $\underline{p}_{11}(z)=-1925120z^2+1572864z+2483712$ is concave with $\underline{p}_{11}(1)=2131456$ and $\underline{p}_{11}(\frac54)=1441792$, so $p_{11}>0$. □

Lemma A.6. Ψ_2 *is decreasing and convex on* $[\phi_-,\phi_+]$.

Proof. Write $\Psi_2=1/f-\tan\phi$. With $\bar f=1/f$ we have $\bar f'=\bar f(\tan\phi-2\cot\phi)$ and $\bar f''=\bar f[(\tan\phi-2\cot\phi)^2+\sec^2\phi+2\csc^2\phi]$. Since $\tan^2\phi\le\frac{16}{9}<2$, $\Psi_2'=\bar f(\tan\phi-2\cot\phi)-\sec^2\phi<0$. Further,

$$\Psi_2''=\bar f''-2\sec^2\phi\tan\phi\ge\bar f(\sec^2\phi+2\csc^2\phi)-2\sec^2\phi\tan\phi=\frac{2-\sin^2\phi-2\sin^5\phi}{\cos^3\phi\sin^4\phi},$$

and as $\sin\phi \le \frac{4}{5}$, the numerator is at least $2 - \frac{16}{25} - 2(\frac{4}{5})^5 = \frac{2202}{3125}$. □

Lemma A.7. *Ψ_1 is decreasing on $[30.5°, \phi_-]$.*

Proof. Put $z = \frac{3}{4}h(\phi, \frac{3}{4})$. Then $\cos\phi = (16z^2+7)/(32z)$, which increases with z; $z = \frac{5}{4}$ at ϕ_-, and $z = \cos 30.5° + (\frac{9}{16} - \sin^2 30.5°)^{1/2} < 1.4139 < \frac{71}{50}$ at $30.5°$. Here $\tan\phi = p_4^{1/2}/(16z^2+7)$ and $\Psi_1 = p_4^{1/2}(64z^3-27)/(27(16z^2+7))$, so on $[\frac{5}{4}, \frac{71}{50}]$

$$(\log\Psi_1)'(z) = \frac{p_4'}{2p_4} + \frac{192z^2}{64z^3-27} - \frac{32z}{16z^2+7}.$$

On this interval $p_4' = -64z(16z^2-25) \le 0$ is decreasing and p_4 is concave, so the first term is at least $p_4'(\frac{71}{50})/(2\min(p_4(\frac{5}{4}), p_4(\frac{71}{50}))) = -660.007/(2\cdot 523.257) > -0.6307$. The second term is decreasing (the numerator of its derivative is $-12288z^4 - 10368z$), so it is at least its value 2.4777 at $\frac{71}{50}$. The third is decreasing, so it is at most its value $\frac{5}{4}$ at $\frac{5}{4}$. The sum is at least $0.597 > 0$, so Ψ_1 increases with z, that is, decreases with ϕ. □

Corollary A.8. *Ψ is continuous and decreasing on $[30.5°, \phi_*]$, with $\Psi(\phi_*) = 0$. It is convex on $[\phi_-, \phi_+]$ and concave on $[\phi_+, \phi_*]$, where $-\Psi'$ is positive and nondecreasing. At $60°$, $z_0 = h(60°, \frac{4}{3}) = (3+\sqrt{37})/8$ satisfies $16z_0^2 - 7 = 12z_0$, and*

$$\Psi(60°) = \sqrt{3}\,(z_0^3 - 1) = 0.802754\ldots, \qquad -\Psi'(60°) = -\frac{p_7(z_0)}{96(6z_0+7)} = 4.642130\ldots.$$

Lemma A.9. (a) *If $98.5° \le \alpha \le 106.27°$, then $\widehat{\Psi}(\alpha) \le \Psi(\alpha-\phi_*) + \Psi(\alpha/2)$.*

(b) *Let $\ell_{60}(\phi) = \Psi(60°) + \Psi'(60°)\,(\phi - 60°)$, with angles in radians. Then $\Psi \le \ell_{60}$ on $[38.2°, \phi_*]$. Hence, if $x = \alpha/2 \in [\phi_+, 60°]$, then $\widehat{\Psi}(\alpha) \le 2\Psi(x) + \varepsilon(x)$, where $\varepsilon(x) = 2(\ell_{60}(x) - \Psi(x))$ is nonincreasing on $[\phi_+, 60°]$.*

(c) *If $x = \alpha/2 \in [60°, \phi_*)$, then $\widehat{\Psi}(\alpha) \le 2\Psi(x)$.*

Proof. (a) By symmetry we may take $\phi \ge \alpha/2$. Then $\phi \in [\alpha/2, \phi_*]$ and $\alpha - \phi \in [\alpha - \phi_*, \alpha/2]$. As $\cos 68° = 0.37461 < \frac{3}{8}$, we have $\phi_* < 68°$ and $\alpha - \phi_* > 30.5°$. Now use Corollary A.8.

(b) On $[\phi_+, \phi_*]$, Ψ is concave and lies below its tangent ℓ_{60}. On $[38.2°, \phi_+] \subset [\phi_-, \phi_+]$, $\Psi_2 - \ell_{60}$ is convex; it is ≤ 0 at ϕ_+, and negative at $38.2°$, since $\Psi_2(38.2°) < 2.54049 < 2.56899 < \ell_{60}(38.2°)$. Thus $\Psi \le \ell_{60}$ on $[38.2°, \phi_*]$. If $\alpha \ge 2\phi_+$, then $\alpha - \phi_* \ge 2\phi_+ - \phi_* = 38.28°$, so ϕ and $\alpha - \phi$ lie in $[38.2°, \phi_*]$, and $\Psi(\phi) + \Psi(\alpha-\phi) \le \ell_{60}(\phi) + \ell_{60}(\alpha-\phi) = 2\ell_{60}(x)$. Finally $\varepsilon'(x) = 2(\Psi'(60°) - \Psi'(x)) \le 0$ for $\phi_+ \le x \le 60°$, by Corollary A.8.

(c) Let $\ell_x(\phi) = \Psi(x) + \Psi'(x)(\phi - x)$. On $[\phi_+, \phi_*]$, $\Psi \le \ell_x$ by concavity. If $2x - \phi_* < \phi_+$, consider $[2x - \phi_*, \phi_+]$, which lies in $[52°, \phi_+]$ because $120° - \phi_* = 52.02°$. There $(\Psi_2 - \ell_x)' = \Psi_2'(\phi) - \Psi'(x) \ge \Psi_2'(52°) - \Psi'(60°) > 0$, because Ψ_2' is increasing, $-\Psi'(x) \ge -\Psi'(60°) > 4.64213$, and $\Psi_2'(52°) > -3.37754$. So $\Psi_2 - \ell_x \le (\Psi - \ell_x)(\phi_+) \le 0$ on this interval; hence $\Psi \le \ell_x$ on $[\alpha - \phi_*, \phi_*]$, and $\widehat{\Psi}(\alpha) \le 2\ell_x(x) = 2\Psi(x)$. □

Lemma A.10. *For $x \in [\phi_+, \phi_*)$ let $\widetilde{\Psi}(x) = 2|\cot 2x|\,\Psi(x) = (\tan^2 x - 1)(h(x, \frac{4}{3})^3 - 1)$. Then $\widetilde{\Psi}$ is increasing on $[\phi_+, 58.5°]$, decreasing on $[60°, \phi_*)$, and $\widetilde{\Psi} < 0.9394$ on $[\phi_+, \phi_*)$.*

Proof. In the variable $z = h(x, \frac{4}{3})$ of Lemma A.5, $\tan^2 x - 1 = N_1(z)/(16z^2-7)^2$ with $N_1 = -512z^4 + 1024z^2 - 98$, so

$$\widetilde{\Psi} = \frac{N_1(z)(z^3-1)}{(16z^2-7)^2}, \qquad \frac{d\widetilde{\Psi}}{dz} = -\frac{6z\,m_7(z)}{(16z^2-7)^3},$$

where $m_7 = 4096z^7 - 6912z^5 + 5712z^3 - 3072z^2 - 343z - 1344$. On $[1, \frac{5}{4}]$, $m_7' = z^4(28672z^2 - 34560) + 17136z^2 - 6144z - 343 \ge \underline{m}_7(z) := -5888z^4 + 17136z^2 - 6144z - 343$, and $\underline{m}_7$

is concave with $\underline{m}_7(1) = 4761$ and $\underline{m}_7(\frac{5}{4}) = 4377$, so m_7 is increasing. As $m_7(1.1555) < -2.2$ and $m_7(1.1557) > 2.2$, $\widetilde{\Psi}$ increases in z on $[1, z_p]$ and decreases on $[z_p, \frac{5}{4}]$, for some $z_p \in (1.1555, 1.1557)$. Since N_1 is positive and decreasing on $[1, \frac{5}{4}]$ ($N_1' = 2048z(1-z^2)$ and $N_1(\frac{5}{4}) = 252$),

$$\widetilde{\Psi} \le \frac{N_1(1.1555)\,(1.1557^3 - 1)}{(16 \cdot 1.1555^2 - 7)^2} < 0.939360 < 0.9394.$$

Finally z decreases as x increases. The formula $\cos x = (16z^2 - 7)/(24z)$ gives a value below $0.5219 < \cos 58.5^\circ$ at $z = 1.16$, and a value above $0.5005 > \cos 60^\circ$ at $z = 1.136$. So $z(58.5^\circ) > 1.16 > z_p > 1.136 > z(60^\circ)$. □

Proof of Lemma 4.11. By Lemma A.1, $\alpha \in (90^\circ, 2\phi_*)$. We cover this interval by three ranges.

Range I: $90^\circ < \alpha \le 98.5^\circ$. By Lemma A.3, $\mathcal{V}_2(i) \le \overline{\mathcal{V}}_2(98.5^\circ) < 0.95613 < 1$, so $\mathcal{V}(i) < 1$.

Range II: $98.5^\circ \le \alpha \le 106.27^\circ$. By Corollary A.4(a) and Lemma A.9(a), $\mathcal{V}_1(i) \le |\cot\alpha|\,\Omega(\alpha)\,(\Psi(\alpha - \phi_*) + \Psi(\alpha/2))$. On a cell $[\alpha_1, \alpha_2]$ every factor is monotone, by Lemma A.3(c) and Corollary A.8: $|\cot\alpha| \le |\cot\alpha_2|$, $\Omega(\alpha) \le \Omega(\alpha_1)$, $\Psi(\alpha - \phi_*) \le \Psi(\alpha_1 - 68^\circ)$ and $\Psi(\alpha/2) \le \Psi(\alpha_1/2)$. Table 4 shows that the resulting bound for $\mathcal{V}_1(i)$ is less than $\widehat{\mathcal{V}}_1(\alpha_2)$ on both cells, so $\mathcal{V}(i) < 1$ by Corollary A.4(b).

$[\alpha_1, \alpha_2]$	$\lvert\cot\alpha_2\rvert$	$\Omega(\alpha_1)$	$\Psi(\alpha_1 - 68^\circ)$	$\Psi(\alpha_1/2)$	bound for $\mathcal{V}_1$	$\widehat{\mathcal{V}}_1(\alpha_2)$
$[98.5^\circ, 104^\circ]$	0.24933	1.72294	3.35679	1.50881	2.09017	2.60218
$[104^\circ, 106.27^\circ]$	0.29186	1.59822	2.84178	1.33580	1.94866	2.10084

TABLE 4. Range II: the factor bounds on each cell, the resulting bound for $\mathcal{V}_1(i)$, and $\widehat{\mathcal{V}}_1(\alpha_2)$. $\Psi(30.5^\circ)$ and $\Psi(36^\circ)$ are on the branch Ψ_1, and $\Psi(49.25^\circ)$ and $\Psi(52^\circ)$ on Ψ_2.

Range III: $x = \alpha/2 \in [\phi_+, \phi_*)$. Since $2\phi_+ = 106.260^\circ$, the three ranges cover $(90^\circ, 2\phi_*)$. By Corollary A.4(a) and Lemma A.9(b), (c),

$$\mathcal{V}_1(i) \le \Omega(\alpha)\big[\widetilde{\Psi}(x) + |\cot\alpha|\,\varepsilon(x)\big] \quad (x \le 60^\circ), \qquad \mathcal{V}_1(i) \le \Omega(\alpha)\,\widetilde{\Psi}(x) \quad (x \ge 60^\circ).$$

On a cell $[x_1, x_2]$ we use $\Omega(\alpha) \le \Omega(2x_1)$, $|\cot\alpha| \le |\cot 2x_2|$ and $\varepsilon(x) \le \varepsilon(x_1)$, and Lemma A.10: $\widetilde{\Psi}(x) \le \widetilde{\Psi}(x_2)$ if $x_2 \le 58.5^\circ$, $\widetilde{\Psi}(x) < 0.9394$ on $[58.5^\circ, 60^\circ]$, and $\widetilde{\Psi}(x) \le \widetilde{\Psi}(x_1)$ if $x_1 \ge 60^\circ$. Table 5 shows that the resulting bound is less than $\widehat{\mathcal{V}}_1(2x_2)$ on every cell, and Corollary A.4(b) applies. For the last cell, $\widehat{\mathcal{V}}_1(2\phi_*)$ is computed with $\chi = \frac{23}{32}$, where $\frac{25}{16} + \frac{3}{2}\chi = (\frac{13}{8})^2$.

$[x_1, x_2]$	$\Omega(2x_1)$	bound for $\widetilde{\Psi}$	$\lvert\cot 2x_2\rvert$	$\varepsilon(x_1)$	bound for $\mathcal{V}_1$	$\widehat{\mathcal{V}}_1(2x_2)$
$[\phi_+, 56.5^\circ]$	1.55823	0.90202	0.42448	0.17705	1.52267	1.56476
$[56.5^\circ, 58.5^\circ]$	1.46524	0.93732	0.50953	0.04764	1.40897	1.43662
$[58.5^\circ, 60^\circ]$	1.42367	0.9394	0.57736	0.00899	1.34479	1.37313
$[60^\circ, 61.6^\circ]$	1.39737	0.92695			1.29530	1.32331
$[61.6^\circ, \phi_*)$	1.37306	0.87392			1.19995	1.21448

TABLE 5. Range III: the factor bounds on each cell, the resulting bound for $\mathcal{V}_1(i)$, and $\widehat{\mathcal{V}}_1(2x_2)$. The bounds for $\widetilde{\Psi}$ are $\widetilde{\Psi}(56.5^\circ)$, $\widetilde{\Psi}(58.5^\circ)$, 0.9394, $\widetilde{\Psi}(60^\circ)$ and $\widetilde{\Psi}(61.6^\circ)$; $\varepsilon(\phi_+)$ uses $\Psi(\phi_+) = \frac{61}{48}$.

□

A.2. **Two adjacent obtuse angles.** In this subsection we prove Lemmas 4.12 and 4.13; throughout, q satisfies their hypotheses. We write

$$x = \chi_h, \quad y = \chi_p, \quad \vartheta = 180^\circ - \alpha_h, \quad \beta = \angle hpk, \quad \gamma = \angle hkp, \quad \psi = \angle kpo = \alpha_p - \beta, \quad \phi = \angle pho.$$

Thus $0 < y \le x$, $\cos\vartheta = x$ and $\beta + \gamma = \vartheta$.

Lemma A.11. (a) $x + y \le \frac{1}{3}$. *Hence* $y \le \frac{1}{6}$, $\alpha_p \le 90^\circ + \arcsin\frac{1}{6} < 99.595^\circ$ *and* $\alpha_h \le 90^\circ + \arcsin\frac{1}{3} < 109.472^\circ$.
(b) $\vartheta/2 \le \beta \le \bar\beta(x) \le \phi_+$, *and* $\bar\beta$ *is decreasing.*
(c) $r_{po}/r_{ho} \le (\frac{25}{16} + \frac{3}{2}y)^{-1/2}$ *and* $r_{hp}/r_{ho} \le (2 + 2y)^{-1/2}$.

Proof. (a) Project $o - k = (h - k) + (p - h) + (o - p)$ onto the unit vector $(p - h)/r_{hp}$. The three terms give $r_{kh}x$, r_{hp} and $r_{po}y$, so $r_{kh}x + r_{hp} + r_{po}y \le r_{ok}$. Every side is at least $\frac{3}{4}r_{ok}$, so $\frac{3}{4}(1 + x + y) \le 1$. As $y \le x$, $y \le \frac{1}{6}$.

(b) In the triangle hpk, $\sin\beta/\sin\gamma = r_{kh}/r_{hp} \in [1, \frac{4}{3}]$. The lower bound gives $\beta \ge \gamma$. The upper bound gives $\sin\beta \le \frac{4}{3}\sin(\vartheta - \beta)$, that is, $\tan\beta \le \sin\vartheta/(\frac{3}{4} + \cos\vartheta)$. The derivative of $\arctan(\sin\vartheta/(\frac{3}{4} + \cos\vartheta))$ in ϑ is $(1 + \frac{3}{4}\cos\vartheta)/(\frac{25}{16} + \frac{3}{2}\cos\vartheta) > 0$, and ϑ decreases as x increases; $\bar\beta(0) = \phi_+$.

(c) $r_{ho}^2 = r_{hp}^2 + r_{po}^2 + 2r_{hp}r_{po}y$, with $r_{hp} \ge \frac{3}{4}r_{po}$ and $r_{hp} \le r_{po}$. □

By Lemma 4.6 at p, $\mathcal{V}_1(p) = a + b'$, where

$$a = \frac{|\varkappa_{p;ho}|}{\varkappa_{p;hk}} = \frac{|\cot\alpha_p|\, g(\beta;\alpha_h)}{1 - (r_{po}/r_{ho})^3}, \qquad b' = \frac{|\varkappa_{p;ho}|}{\varkappa_{p;ok}} = \frac{|\cot\alpha_p|\, g(\psi;\alpha_o)}{1 - (r_{hp}/r_{ho})^3}; \tag{A.2}$$

here $\sin\phi/\sin\alpha_p = r_{po}/r_{ho}$ and $\sin\angle hop/\sin\alpha_p = r_{hp}/r_{ho}$. Note that $\theta_p = a/(1 - b')$ in (4.2).

Proof of Lemma 4.12. Let $\alpha_p(t) = 90^\circ + \arcsin t$, so that $\alpha_p = \alpha_p(y)$ and $|\cot\alpha_p(t)| = t/\sqrt{1 - t^2}$, and let

$$a^+(t) = |\cot\alpha_p(t)|\, \Omega_1(t)\, g_1\big(\tfrac{1}{2}\arccos(\tfrac{1}{3} - t)\big), \qquad b^+(t) = |\cot\alpha_p(t)|\, \Omega_2(t)\, g_0\big(\alpha_p(t) - \bar\beta(t)\big).$$

Step 1: $a \le a^+(y)$. By Lemma 4.10(a) and $\beta \ge \vartheta/2$, $g(\beta;\alpha_h) \le g(\vartheta/2; 180^\circ - \vartheta) = g_1(\vartheta/2)$. With Lemma A.11(c), $a \le |\cot\alpha_p|\Omega_1(y)g_1(\vartheta/2)$. As $x \le \frac{1}{3}$, $\vartheta/2 \in [35.26^\circ, 45^\circ]$, and $\vartheta/2$ decreases as x increases. Since g_1 is decreasing (Lemma 4.10(b)) and $x \le \frac{1}{3} - y$, $g_1(\vartheta/2) \le g_1(\frac{1}{2}\arccos(\frac{1}{3} - y))$.

Step 2: $b' \le b^+(y)$. The angle ψ is acute, since pk is the longest side of the triangle pok. As $\sin\alpha_o \le 1$, $g(\psi;\alpha_o) \le g_0(\psi)$. Since $x \ge y$ and $\bar\beta$ is decreasing, $\psi = \alpha_p - \beta \ge \alpha_p - \bar\beta(x) \ge \alpha_p - \bar\beta(y)$, and g_0 is decreasing. Now use Lemma A.11(c).

Step 3: a^+ *is increasing on* $(0, \frac{1}{6}]$. First, $(\log|\cot\alpha_p(t)|)' = 1/t + t/(1 - t^2) \ge 6$. Second, $|(\log\Omega_1)'(t)| = \frac{9}{4}\xi^{-5/2}/(1 - \xi^{-3/2})$ with $\xi = \frac{25}{16} + \frac{3}{2}t$; this decreases in t, and its value at 0 is $1.51082 < 1.5109$. Third, let $\hat\vartheta = \frac{1}{2}\arccos(\frac{1}{3} - t) \in [35.26^\circ, 45^\circ]$. Then $d\hat\vartheta/dt = 1/(2\sin 2\hat\vartheta) \le 1/(2 \cdot 0.9428)$, because $\sin 2\hat\vartheta \ge \sin(\arccos\frac{1}{3}) = \frac{2\sqrt{2}}{3}$. By Lemma 4.10(b), g_1' is decreasing on $(0^\circ, 60^\circ)$ and negative on $[35^\circ, 60^\circ)$, so $|g_1'(\hat\vartheta)| \le |g_1'(45^\circ)| = 2 + 2\sqrt{2} < 4.8285$, and $g_1(\hat\vartheta) \ge g_1(45^\circ) = 2\sqrt{2} - 1 > 1.8284$. So $|(\log g_1(\hat\vartheta))'(t)| \le 4.8285/(1.8284 \cdot 2 \cdot 0.9428) < 1.4006$. As $6 - 1.5109 - 1.4006 > 0$, a^+ is increasing.

Step 4: b^+ *is increasing on* $(0, \frac{1}{6}]$. Let $\psi(t) = \alpha_p(t) - \bar\beta(t)$. As in Step 3, $|(\log\Omega_2)'(t)|$ is at most its value $0.82038 < 0.8204$ at 0. By Lemma 4.10(c), $-g_0'/g_0 \le \Lambda$, so

$$(\log b^+)'(t) \;\ge\; \frac{1}{t} - 0.8204 - \Lambda(\psi(t))\,\psi'(t), \qquad \psi'(t) = \frac{1}{\sqrt{1 - t^2}}\Big(1 + \frac{1 + \frac{3}{4}t}{\frac{25}{16} + \frac{3}{2}t}\Big).$$

The function $(1+\frac{3}{4}t)/(\frac{25}{16}+\frac{3}{2}t)$ decreases, from 0.64 at $t=0$. Since ψ is increasing and Λ is convex, on an interval of t the value $\Lambda(\psi(t))$ is at most the larger of its values at the ends. We have $\psi(0)=90^\circ-\phi_+$ with $\Lambda(\psi(0))=\frac{125}{36}<3.4723$, $\Lambda(\psi(0.12))<2.7022$ and $\Lambda(\psi(\frac{1}{6}))<2.6099$. On $(0,0.12]$ this gives $(\log b^+)'\ge\frac{25}{3}-0.8204-3.4723\cdot 1.64/\sqrt{1-0.12^2}>1.77$. On $[0.12,\frac{1}{6}]$, $\psi'(t)\le 1.64860$, and $(\log b^+)'\ge 6-0.8204-2.7022\cdot 1.64860>0.72$.

Step 5. By Steps 1–4 and Lemma A.11(a), $\mathcal{V}_1(p)=a+b'\le a^+(y)+b^+(y)\le a^+(\frac{1}{6})+b^+(\frac{1}{6})$, and $a^+(\frac{1}{6})<0.62068$, $b^+(\frac{1}{6})<0.30691$. □

Proof of Lemma 4.13. Step 1: reduction. By Lemma 4.12, $\theta_p\in(0,1)$. By Lemma 4.6 at h, $z_1:=\varkappa_{p;hk}/|\varkappa_{h;pk}|=f(\beta)/f(\vartheta)$ and $z_2:=\varkappa_{k;hp}/|\varkappa_{h;pk}|=f(\gamma)/f(\vartheta)$, and $1/\mathcal{V}_2(h)=\operatorname{ser}(z_1,z_2)$. Let $\zeta(\theta)=(2-2\theta)/(2-\theta)$, which is decreasing on $[0,1)$. Then

$$\frac{\operatorname{ser}((1-\theta_p)z_1,z_2)}{\operatorname{ser}(z_1,z_2)}-\zeta(\theta_p)=\frac{\theta_p(1-\theta_p)(z_1-z_2)}{(2-\theta_p)\big((1-\theta_p)z_1+z_2\big)}.$$

Since $\gamma\le\beta\le\phi_+<\phi_f$, $f(\gamma)\le f(\beta)$ by Lemma 4.10(d), so $z_1\ge z_2$. Hence

$$H\ \ge\ \frac{1}{\mathcal{V}_1(h)}+\frac{\zeta(\theta_p)}{\mathcal{V}_2(h)}\ >\ \frac{1}{\mathcal{V}_1(h)}. \tag{A.3}$$

Step 2: a bound for $\mathcal{V}_1(h)$. By Lemma 4.6 at h, $\mathcal{V}_1(h)=\varsigma_1 g(\phi;\alpha_p)+\varsigma_2 g(\alpha_h-\phi;\alpha_k)$ with $\varsigma_1=|\cot\alpha_h|/(1-(\sin\beta/\sin\alpha_h)^3)$ and $\varsigma_2=|\cot\alpha_h|/(1-(\sin\gamma/\sin\alpha_h)^3)$. Here $\sin\beta/\sin\alpha_h=r_{hk}/r_{pk}$, $\sin\gamma/\sin\alpha_h=r_{hp}/r_{pk}$, $r_{pk}^2=r_{hp}^2+r_{hk}^2+2r_{hp}r_{hk}x$ and $r_{hp}\le r_{hk}\le\frac{4}{3}r_{hp}$. Consequently $\varsigma_2\le\varsigma_1\le\bar\varsigma_1(x):=|\cot\alpha_h|\Omega_1(x)$ and $\varsigma_2\le\bar\varsigma_2(x):=|\cot\alpha_h|\Omega_2(x)$. If $\phi<45^\circ$, then $r_{hp}\le r_{po}$ gives $\angle hop\le\phi$ in the triangle hpo, so $\alpha_p\ge 180^\circ-2\phi>90^\circ$ and $g(\phi;\alpha_p)\le g_1(\phi)$ by Lemma 4.10(a). Always $g(\phi;\alpha_p)\le g_0(\phi)$ and $g(\alpha_h-\phi;\alpha_k)\le g_0(\alpha_h-\phi)$. Hence $\mathcal{V}_1(h)\le\Xi(\phi):=\varsigma_1\hat g(\phi)+\varsigma_2 g_0(\alpha_h-\phi)$, where $\hat g=g_1$ on $(0^\circ,45^\circ)$ and $\hat g=g_0$ on $[45^\circ,90^\circ)$.

Next we locate ϕ. In the triangle hpo, $\tan\phi=\sqrt{1-y^2}/(t+y)$ with $t=r_{hp}/r_{po}\in[\frac{3}{4},1]$. Hence $\phi\ge\phi_{\rm lo}(y):=\arctan(\sqrt{1-y^2}/(1+y))=45^\circ-\frac{1}{2}\arcsin y\ge\phi_{\rm lo}(\frac{1}{6})=40.20^\circ$ and $\phi\le\phi_{\rm hi}(y):=\arctan(\sqrt{1-y^2}/(\frac{3}{4}+y))\le\phi_+$. In the triangle hko, $r_{hk}\le r_{ok}$ gives $\angle hok\le\alpha_h-\phi$, and $\alpha_k\le 90^\circ$ gives $\angle hok\ge 90^\circ-(\alpha_h-\phi)$; so $\alpha_h-\phi\ge 45^\circ$. Thus $\phi\in[\phi_{\rm lo}(y),\phi_{\rm end}]$ with $\phi_{\rm end}=\min(\phi_{\rm hi}(y),\alpha_h-45^\circ)$.

On $[45^\circ,\phi_{\rm end}]$ both terms of Ξ are convex by Lemma 4.10(c), since $\phi\le\phi_+$ and $45^\circ\le\alpha_h-\phi\le 64.48^\circ$. There Ξ is therefore at most $\max(\Xi(45^\circ),E_2)$, where $E_2=\varsigma_1 g_0(\phi_{\rm end})+\varsigma_2 g_0(\alpha_h-\phi_{\rm end})$. On $[\phi_{\rm lo}(y),45^\circ]$ we use one of two bounds.

- Since g_1 and g_0 are decreasing, $E_1':=\varsigma_1 g_1(\phi_{\rm lo}(y))+\varsigma_2 g_0(\alpha_h-45^\circ)$ bounds Ξ there.
- Let $x\ge 0.2$. Then $\alpha_h\ge 90^\circ+\arcsin 0.2>101.536^\circ$, so $\alpha_h-\phi\in[56.536^\circ,69.28^\circ]$ for $\phi\in[40.2^\circ,45^\circ]$. On this range $|g_0'(\alpha_h-\phi)|<2.7917$: on $[56.536^\circ,\arctan\sqrt{6}]$ because g_0' increases and $g_0'(56.536^\circ)>-2.7914$, and on $[\arctan\sqrt{6},70^\circ]\subset[67.79^\circ,70^\circ]$ because $|g_0'(\alpha_h-\phi)|=3\csc^3(\alpha_h-\phi)-\sec^2(\alpha_h-\phi)(\csc^3(\alpha_h-\phi)-1)\le 3\csc^3 67.79^\circ-\sec^2 67.79^\circ(\csc^3 70^\circ-1)<2.345$. Also $g_1'(\phi)\le g_1'(40.2^\circ)<-3.24$. As $\varsigma_2\le\varsigma_1$, $\Xi'(\phi)\le\varsigma_1(g_1'(\phi)+|g_0'(\alpha_h-\phi)|)<0$. Thus Ξ is decreasing, and $E_1'':=\Xi(\phi_{\rm lo}(y))$ bounds Ξ on $[\phi_{\rm lo}(y),45^\circ]$.

Both bounds are at least $\Xi(45^\circ)$. So $\mathcal{V}_1(h)\le\max(E_1,E_2)$ with $E_1=E_1'$, or with $E_1=E_1''$ when $x\ge 0.2$; the term E_2 is needed only if $\phi_{\rm end}\ge 45^\circ$.

Step 3: bounds for $\mathcal{V}_2(h)$ and θ_p. By (4.1), $\mathcal{V}_2(h)=f(\vartheta)Y(\vartheta,\beta)$ with $Y(\vartheta,\beta)=1/f(\beta)+1/f(\vartheta-\beta)$. By Lemma 4.10(d), Y is increasing in $\beta\ge\vartheta/2$ and decreasing in ϑ when $\vartheta-\beta\le\phi_f$. Hence $\mathcal{V}_2(h)\le f(\vartheta)Y(\vartheta,\bar\beta(x))$. The number $\theta_p=a/(1-b')$ increases with a and with $b'<1$, and Steps 1 and 2 of the proof of Lemma 4.12 give $a\le\bar a:=|\cot\alpha_p|\Omega_1(y)g_1(\vartheta/2)$ and $b'\le\bar b:=|\cot\alpha_p|\Omega_2(y)g_0(\alpha_p-\bar\beta(x))$. Thus $\theta_p\le\bar a/(1-\bar b)$.

Step 4: the cell computation. We cover $x \in (0, \frac{1}{3}]$ by the six cells $[x_a, x_b]$ of Table 6, and put $y_c = \min(x_b, \frac{1}{3} - x_a, \frac{1}{6})$, so that $y \le y_c$ by Lemma A.11(a). On a cell we replace every factor by its worst value, using the following monotonicities.

- $\bar{\varsigma}_1$ and $\bar{\varsigma}_2$ increase with x, because $1/x \ge 3$ exceeds $|(\log \Omega_1)'| \le 1.5109$ and $|(\log \Omega_2)'| \le 0.8204$. In the same way $|\cot \alpha_p(t)|\Omega_1(t)$ increases with t. Also $f(\vartheta) = x(1-x^2)$ and $g_1(\vartheta/2)$ increase with x, while $\bar{\beta}$ and ϕ_{lo}, ϕ_{hi} decrease.
- $g_0(\alpha_h - \phi) \le g_0(\alpha_h(x_a) - \phi)$ for fixed ϕ, where $\alpha_h(x_a) = 90^\circ + \arcsin x_a$. Therefore $E_1' \le \bar{\varsigma}_1(x_b) g_1(\phi_{\mathrm{lo}}(y_c)) + \bar{\varsigma}_2(x_b) g_0(\alpha_h(x_a) - 45^\circ)$. If $x_a \ge 0.2$, then $E_1'' \le \overline{\Xi}(\phi_{\mathrm{lo}}(y))$ with $\overline{\Xi}(\phi) = \bar{\varsigma}_1(x_b) g_1(\phi) + \bar{\varsigma}_2(x_b) g_0(\alpha_h(x_a) - \phi)$, and $\overline{\Xi}$ is decreasing on $[40.2^\circ, 45^\circ]$ by the argument of Step 2, because $\bar{\varsigma}_2 \le \bar{\varsigma}_1$. Hence $E_1'' \le \overline{\Xi}(\phi_{\mathrm{lo}}(y_c))$.
- $\phi_{\mathrm{end}} \ge \phi_c := \min(\phi_{\mathrm{hi}}(y_c), \alpha_h(x_a) - 45^\circ)$ and $\alpha_h - \phi_{\mathrm{end}} \ge \max(\alpha_h(x_a) - \phi_+, 45^\circ)$. When $\phi_{\mathrm{end}} \ge 45^\circ$, $g_0(\phi_{\mathrm{end}}) \le g_0(\max(\phi_c, 45^\circ))$.
- $\mathcal{V}_2(h) \le x_b(1 - x_b^2)\, Y\big(\arccos x_b, \bar{\beta}(x_a)\big)$; here $\vartheta - \bar{\beta}(x_a) \in (17^\circ, 49^\circ)$ for every $\vartheta \in [\arccos x_b, 90^\circ]$.
- For fixed $\beta_0 \in [\bar{\beta}(\frac{1}{3}), \phi_+]$, $t \mapsto |\cot \alpha_p(t)|\Omega_2(t) g_0(\alpha_p(t) - \beta_0)$ is increasing on $(0, \frac{1}{6}]$: as in Step 4 of the proof of Lemma 4.12, its logarithmic derivative is at least $6 - 0.8204 - 3.4723/\sqrt{1 - 1/36} > 1.6$, since $\alpha_p(t) - \beta_0 \in [36.87^\circ, 58.57^\circ]$, where $\Lambda \le \frac{125}{36}$. Hence $\bar{b} \le |\cot \alpha_p(y_c)|\Omega_2(y_c) g_0(\alpha_p(y_c) - \bar{\beta}(x_a))$, and $\bar{a} \le |\cot \alpha_p(y_c)|\Omega_1(y_c) g_1(\frac{1}{2}\arccos x_b)$.

Table 6 lists the resulting bounds $\overline{E} \ge \max(E_1, E_2) \ge \mathcal{V}_1(h)$, $\overline{\mathcal{V}}_2 \ge \mathcal{V}_2(h)$ and $\bar{\theta}_p \ge \theta_p$; on the last four cells $\bar{b} < 0.31$. On the first two cells $\overline{E} < 1$, so $H > 1$ by (A.3). On the other four, (A.3) gives $H \ge 1/\overline{E} + \zeta(\bar{\theta}_p)/\overline{\mathcal{V}}_2$, which is at least 1.08697.

$[x_a, x_b]$	y_c	$\overline{E}$	$\overline{\mathcal{V}}_2$	$\bar{\theta}_p$	$1/\overline{E} + \zeta(\bar{\theta}_p)/\overline{\mathcal{V}}_2$
$[0, 0.12]$	0.12	0.76088			
$[0.12, 0.17]$	1/6	0.95295			
$[0.17, 0.2]$	49/300	1.05724	1.35157	0.89458	1.08697
$[0.2, 0.25]$	2/15	1.18778	1.75590	0.72297	1.08899
$[0.25, 0.3]$	1/12	1.32861	2.15222	0.43522	1.08807
$[0.3, 1/3]$	1/30	1.38283	2.38478	0.16708	1.10425

TABLE 6. The hub: the cell bounds of Step 4 and, on the last four cells, the resulting lower bound for H. The first three cells use E_1', the last three E_1''.

□

A.3. **Three obtuse angles.** In this subsection we prove Lemma 4.14; throughout, q satisfies its hypotheses. By (4.4), am is a shortest and bd a longest side. We write

$$\xi_a = \angle mab, \quad \xi_b = \angle mba, \quad \eta_a = \angle dab, \quad \eta_b = \angle dba, \quad \vartheta = 180^\circ - \alpha_m = \xi_a + \xi_b,$$

$p = \chi_m = \cos\vartheta$, and $\phi_a = \angle amd$, $\delta_a = \angle adm$, $\phi_b = \angle bmd$, $\delta_b = \angle bdm$. Thus $\alpha_a = \xi_a + \eta_a$, $\alpha_b = \xi_b + \eta_b$ and $\alpha_a + \phi_a + \delta_a = \alpha_b + \phi_b + \delta_b = 180^\circ$. The angles $\xi_a, \xi_b, \eta_a, \eta_b, \phi_a, \delta_a, \phi_b, \delta_b$ are acute, because in each of the triangles amb, abd, amd, bmd the longest side is a diagonal. Put $t_m = r_{mb}/r_{am}$, $t_a = r_{ad}/r_{am}$ and $t_d = r_{bd}/r_{am}$, so that $1 \le t_m \le t_d \le \frac{4}{3}$ and $1 \le t_a \le t_d$.

Lemma A.12. (a) $t_m p + 1 + t_a \chi_a \le t_d$ *and* $t_d \chi_b + t_m + p \le t_a$.
(b) $\chi_a + p \le \frac{1}{3}$ *and* $\chi_a \le (\frac{1}{3} - t_m p)/(t_m + p) \le \bar{\chi}(p) := (\frac{1}{3} - p)/(1 + p)$.
(c) $\chi_b \le 1 - \frac{3}{4}(t_m + p)$. *Hence* $p + \chi_b \le \frac{1}{4}(1 + p)$, $p \le \frac{1}{3} - \frac{4}{3}\chi_b$ *and* $p + \chi_b \le \frac{1}{3}(1 - \chi_b)$.
(d) $\alpha_a \le 90^\circ + \arcsin\frac{1}{3} = 109.4712^\circ$, $\alpha_b \le 90^\circ + \arcsin\frac{1}{4} = 104.4775^\circ$ *and* $\vartheta \ge \arccos\frac{1}{3} = 70.5288^\circ$.

Proof. (a) Project $d-b=(m-b)+(a-m)+(d-a)$ onto $(a-m)/r_{am}$; the terms give $r_{mb}p$, r_{am} and $r_{ad}\chi_a$, and the sum is at most r_{bd}. Project $a-d=(b-d)+(m-b)+(a-m)$ onto $(m-b)/r_{mb}$; the terms give $r_{bd}\chi_b$, r_{mb} and $r_{am}p$, and the sum is at most r_{ad}. Divide by r_{am}.

(b) The first inequality of (a) with $t_m, t_a \ge 1$ and $t_d \le \frac{4}{3}$ gives $p+1+\chi_a \le \frac{4}{3}$. The second gives $t_a \ge t_m+p$, and then the first gives $t_m p+1+(t_m+p)\chi_a \le \frac{4}{3}$. The bound $(\frac{1}{3}-t_m p)/(t_m+p)$ decreases in t_m, with derivative $-(\frac{1}{3}+p^2)/(t_m+p)^2$.

(c) By (a) and $t_a \le t_d$, $t_d\chi_b \le t_d - t_m - p$, so $\chi_b \le 1-(t_m+p)/t_d \le 1-\frac{3}{4}(t_m+p)$. With $t_m \ge 1$ this gives $\chi_b \le \frac{1}{4}-\frac{3}{4}p$; the other three inequalities are rearrangements of this one.

(d) follows from (b) and (c). □

By Lemma 4.6 at a, m and b, with $\theta_{ij}=\chi_i/(\chi_i+\chi_j)$, $|\cot\alpha_a| = \chi_a/\sin\alpha_a$ and $f(180^\circ-\alpha_a) = \chi_a \sin^2\alpha_a$,

$$
\begin{aligned}
&\rho_a = \operatorname{ser}(a_P, a_Q) + \operatorname{ser}(a_U, a_W), \qquad \rho_m = \operatorname{ser}(z_1, z_2) + \operatorname{ser}(z_3, z_4),\\
&a_P = \frac{\varkappa_{a;db}}{|\varkappa_{a;md}|} = \frac{1-(r_{am}/r_{md})^3}{|\cot\alpha_a|\, g(\eta_a;\alpha_d)}, \qquad a_Q = \frac{\theta_{am}\varkappa_{a;mb}}{|\varkappa_{a;md}|} = \frac{\sin\alpha_a\,(1-(r_{ad}/r_{md})^3)}{(\chi_a+p)\, g(\xi_a;\alpha_m)},\\
&a_U = \frac{\varkappa_{d;am}}{|\varkappa_{a;md}|} = \frac{f(\delta_a)}{f(180^\circ-\alpha_a)}, \qquad a_W = \frac{\theta_{am}\varkappa_{m;ad}}{|\varkappa_{a;md}|} = \frac{f(\phi_a)}{(\chi_a+p)\sin^2\alpha_a},\\
&z_1 = \frac{\theta_{ma}\varkappa_{m;ad}}{|\varkappa_{m;ab}|} = \frac{\sin\vartheta\,(1-(r_{mb}/r_{ab})^3)}{(p+\chi_a)\, g(\phi_a;\alpha_a)}, \qquad z_2 = \frac{\theta_{mb}\varkappa_{m;bd}}{|\varkappa_{m;ab}|} = \frac{\sin\vartheta\,(1-(r_{am}/r_{ab})^3)}{(p+\chi_b)\, g(\phi_b;\alpha_b)},\\
&z_3 = \frac{\theta_{ma}\varkappa_{a;mb}}{|\varkappa_{m;ab}|} = \frac{f(\xi_a)}{(p+\chi_a)\sin^2\vartheta}, \qquad z_4 = \frac{\theta_{mb}\varkappa_{b;ma}}{|\varkappa_{m;ab}|} = \frac{f(\xi_b)}{(p+\chi_b)\sin^2\vartheta},
\end{aligned}
\tag{A.4}
$$

and $\rho_b = \operatorname{ser}(b_P, b_Q) + \operatorname{ser}(b_U, b_W)$, where b_P, b_Q, b_U, b_W are obtained from a_P, a_Q, a_U, a_W by replacing $(a, \xi_a, \eta_a, \phi_a, \delta_a)$ with $(b, \xi_b, \eta_b, \phi_b, \delta_b)$. Here we used the law of sines: $\sin\delta_a/\sin\alpha_a = r_{am}/r_{md}$, $\sin\phi_a/\sin\alpha_a = r_{ad}/r_{md}$, $\sin\xi_a/\sin\alpha_m = r_{mb}/r_{ab}$ and $\sin\xi_b/\sin\alpha_m = r_{am}/r_{ab}$.

We bound each ratio by a function of one angle. For $90^\circ < \alpha < 180^\circ$ and $\chi = -\cos\alpha$ let

$$
\underline{\delta}(\alpha) = \frac{180^\circ-\alpha}{2} - \arctan\Big(\frac{1}{7}\cot\frac{\alpha}{2}\Big), \qquad \bar{\phi}(\alpha) = 180^\circ - \alpha - \underline{\delta}(\alpha),
$$

and for $0^\circ < \vartheta < 90^\circ$ let $\underline{\gamma}(\vartheta) = \vartheta - \bar{\beta}(\cos\vartheta)$, with $\bar{\beta}$, Ω_1 and Ω_2 as at the start of this appendix. Define

$$
\begin{aligned}
\underline{a}_P(\alpha) &= \frac{1}{\Omega_2(\chi)\,|\cot\alpha|\, g_1(\alpha/2)}, \qquad \underline{a}_Q(\alpha) = \frac{3\sin\alpha}{\Omega_1(\chi)\, g_1\big(\frac{1}{2}\arccos(\frac{1}{3}-\chi)\big)},\\
\underline{a}_{UW}(\alpha) &= \min_{\delta\in\{\underline{\delta}(\alpha),\,(180^\circ-\alpha)/2\}} \operatorname{ser}\Big(\frac{f(\delta)}{f(180^\circ-\alpha)}, \frac{3f(180^\circ-\alpha-\delta)}{\sin^2\alpha}\Big),\\
\underline{b}_P(\alpha) &= \frac{\sin\alpha}{\Omega_2(\chi)\,\chi\, g_0(45^\circ)}, \qquad \underline{b}_Q(\alpha) = \frac{\sin\alpha}{\Omega_1(\chi)\,\frac{1-\chi}{3}\, g_0\big(\underline{\gamma}(\arccos(\frac{1}{3}-\frac{4}{3}\chi))\big)},\\
\underline{b}_{UW}(\alpha) &= \min_{\delta\in\{\underline{\delta}(\alpha),\,(180^\circ-\alpha)/2\}} \operatorname{ser}\Big(\frac{f(\delta)}{f(180^\circ-\alpha)}, \frac{f(180^\circ-\alpha-\delta)}{\frac{1}{3}(1-\chi)\sin^2\alpha}\Big),
\end{aligned}
$$

and, with $p=\cos\vartheta$ and $\Upsilon(t,p) = (t+p)\big(1-t^3(1+t^2+2tp)^{-3/2}\big)$,

$$
\begin{aligned}
\underline{z}_1(\vartheta) &= \frac{\sin\vartheta\,\min\big(\Upsilon(1,p), \Upsilon(\frac{4}{3},p)\big)}{(\frac{1}{3}+p^2)\, g_1\big(45^\circ - \frac{1}{2}\arcsin\bar{\chi}(p)\big)}, \qquad \underline{z}_2(\vartheta) = \frac{4\sin\vartheta}{\Omega_2(p)\,(1+p)\, g_0(90^\circ-\vartheta/2)},\\
\underline{z}_{34}(\vartheta) &= \Big[\sin^2\vartheta\Big(\frac{\frac{1}{3}+p^2}{(1+p)\, f(\vartheta/2)} + \frac{1+p}{4f(\underline{\gamma}(\vartheta))}\Big)\Big]^{-1}.
\end{aligned}
$$

Proposition A.13. $\rho_a \geq \operatorname{ser}(\underline{a}_P, \underline{a}_Q)(\alpha_a) + \underline{a}_{UW}(\alpha_a)$, $\rho_b \geq \operatorname{ser}(\underline{b}_P, \underline{b}_Q)(\alpha_b) + \underline{b}_{UW}(\alpha_b)$ *and* $\rho_m \geq \operatorname{ser}(\underline{z}_1, \underline{z}_2)(\vartheta) + \underline{z}_{34}(\vartheta)$.

Proof. We first collect four facts.

(i) In the triangle amb, $\sin\xi_a/\sin\xi_b = t_m \in [1, \frac{4}{3}]$. As in Lemma A.11(b), $\vartheta/2 \leq \xi_a \leq \bar{\beta}(p) \leq \phi_+$ and $\underline{\gamma}(\vartheta) \leq \xi_b \leq \vartheta/2$; moreover $\underline{\gamma}$ is increasing, since $\underline{\gamma}'(\vartheta) = (\frac{9}{16} + \frac{3}{4}\cos\vartheta)/(\frac{25}{16} + \frac{3}{2}\cos\vartheta)$. Also $\xi_b < 45°$, and $\eta_b > 90° - \xi_b > 45°$ because $\alpha_b > 90°$.

(ii) By the law of tangents in the triangle amd, $\tan\frac{\phi_a - \delta_a}{2} = \frac{t_a - 1}{t_a + 1}\cot\frac{\alpha_a}{2}$, and $\frac{t_a-1}{t_a+1} \in [0, \frac{1}{7}]$. Hence $\underline{\delta}(\alpha_a) \leq \delta_a \leq (180° - \alpha_a)/2 \leq \phi_a \leq \bar{\phi}(\alpha_a)$. The same holds for b, with $r_{bd}/r_{mb} \in [1, \frac{4}{3}]$. Note $\bar{\phi}(\alpha) \leq 45° + \arctan\frac{1}{7} = \phi_+$.

(iii) By the law of sines in amd and bmd, $\sin\phi_a/\sin\phi_b = r_{ad}\sin\alpha_a/(r_{bd}\sin\alpha_b) \leq 1$, because $r_{ad} \leq r_{bd}$ and $90° < \alpha_b \leq \alpha_a$. Thus $\phi_b \geq \phi_a$, and $\phi_b \geq \alpha_m/2 = 90° - \vartheta/2$.

(iv) $r_{md}^2 = r_{am}^2 + r_{ad}^2 + 2r_{am}r_{ad}\chi_a$ is at least $r_{am}^2(2+2\chi_a)$ and at least $r_{ad}^2(\frac{25}{16} + \frac{3}{2}\chi_a)$. Consequently $1 - (r_{am}/r_{md})^3 \geq 1/\Omega_2(\chi_a)$ and $1 - (r_{ad}/r_{md})^3 \geq 1/\Omega_1(\chi_a)$, and likewise with (mb, bd, χ_b). In units $r_{am} = 1$, $r_{ab}^2 = 1 + t_m^2 + 2t_m p$, so $1 - (r_{mb}/r_{ab})^3 = 1 - t_m^3(1 + t_m^2 + 2t_m p)^{-3/2}$ and $1 - (r_{am}/r_{ab})^3 \geq 1/\Omega_2(p)$.

The ratio ρ_a. Since $\vartheta < 90°$, $\xi_a < 90° - \xi_b$, so $\delta := \eta_a - \xi_b > \alpha_a - 90° > 0$. Since $\eta_b > 90° - \xi_b$, $\alpha_d = 180° - \eta_a - \eta_b < 90° - \delta$, so $\sin\alpha_d < \cos\delta$ and $g(\eta_a; \alpha_d) \leq g(\eta_a; 90° - \delta)$ by Lemma 4.10(a). As $\eta_a \geq \eta_b > 90° - \eta_a + \delta$, we have $\eta_a > 45° + \delta/2$, so $g(\eta_a; 90° - \delta) \leq g(45° + \delta/2; 90° - \delta) = g_1(45° + \delta/2) \leq g_1(\alpha_a/2)$. With (iv), $a_P \geq \underline{a}_P(\alpha_a)$. By Lemma A.12(b), $\chi_a + p \leq \frac{1}{3}$ and $\vartheta \geq \arccos(\frac{1}{3} - \chi_a)$; by (i), $g(\xi_a; \alpha_m) \leq g(\vartheta/2; 180° - \vartheta) = g_1(\vartheta/2) \leq g_1(\frac{1}{2}\arccos(\frac{1}{3} - \chi_a))$. With (iv), $a_Q \geq \underline{a}_Q(\alpha_a)$. Finally $a_U = f(\delta_a)/f(180° - \alpha_a)$ and $a_W \geq 3f(180° - \alpha_a - \delta_a)/\sin^2\alpha_a$. The reciprocal of ser of these two expressions is a convex function of δ_a, by Lemma 4.10(d), so on the interval of (ii) the minimum of ser is attained at an end. Hence $\operatorname{ser}(a_U, a_W) \geq \underline{a}_{UW}(\alpha_a)$.

The ratio ρ_b. By (i), $g(\eta_b; \alpha_d) \leq g_0(\eta_b) \leq g_0(45°)$, and with (iv) $b_P \geq \underline{b}_P(\alpha_b)$. By Lemma A.12(c), $\chi_b + p \leq \frac{1}{3}(1 - \chi_b)$ and $\vartheta \geq \arccos(\frac{1}{3} - \frac{4}{3}\chi_b)$; by (i), $g(\xi_b; \alpha_m) \leq g_0(\xi_b) \leq g_0(\underline{\gamma}(\vartheta)) \leq g_0(\underline{\gamma}(\arccos(\frac{1}{3} - \frac{4}{3}\chi_b)))$. With (iv), $b_Q \geq \underline{b}_Q(\alpha_b)$. The bound $\operatorname{ser}(b_U, b_W) \geq \underline{b}_{UW}(\alpha_b)$ follows as for a.

The ratio ρ_m. By Lemma A.12(b), $p + \chi_a \leq (\frac{1}{3} + p^2)/(t_m + p)$, so by (iv) $(1 - (r_{mb}/r_{ab})^3)/(p + \chi_a) \geq \Upsilon(t_m, p)/(\frac{1}{3} + p^2)$. Then $\Upsilon(t, p) = t + p - (t + p)t^3(1 + t^2 + 2tp)^{-3/2}$, and

$$\frac{d^2}{dt^2}\big((t+p)t^3(1+t^2+2tp)^{-3/2}\big) = t(1+t^2+2tp)^{-7/2}\big[3t(4-t^2)+3p^3t^2+9p^2t^3+6p^2t+15pt^2+6p\big] > 0$$

for $0 < t < 2$. So Υ is concave in t, and $\Upsilon(t_m, p) \geq \min(\Upsilon(1, p), \Upsilon(\frac{4}{3}, p))$. By (ii) and Lemma 4.10(a), $g(\phi_a; \alpha_a) \leq g((180° - \alpha_a)/2; \alpha_a) = g_1((180° - \alpha_a)/2)$, and $\alpha_a \leq 90° + \arcsin\bar{\chi}(p)$ by Lemma A.12(b); so $g(\phi_a; \alpha_a) \leq g_1(45° - \frac{1}{2}\arcsin\bar{\chi}(p))$. Hence $z_1 \geq \underline{z}_1(\vartheta)$. Next, by (iv), Lemma A.12(c) and (iii), $z_2 \geq \sin\vartheta/(\Omega_2(p) \cdot \frac{1}{4}(1 + p) \cdot g_0(90° - \vartheta/2)) = \underline{z}_2(\vartheta)$. Finally $1/\operatorname{ser}(z_3, z_4) = \sin^2\vartheta\,[(p + \chi_a)/f(\xi_a) + (p + \chi_b)/f(\xi_b)]$, where $p + \chi_a \leq (\frac{1}{3} + p^2)/(1 + p)$, $p + \chi_b \leq \frac{1}{4}(1 + p)$, and $f(\xi_a) \geq f(\vartheta/2)$, $f(\xi_b) \geq f(\underline{\gamma}(\vartheta))$ by (i) and Lemma 4.10(d). Thus $\operatorname{ser}(z_3, z_4) \geq \underline{z}_{34}(\vartheta)$. □

Proof of Lemma 4.14. By Lemma A.12(d), $\alpha_a \in (90°, 90° + \arcsin\frac{1}{3}]$, $\alpha_b \in (90°, 90° + \arcsin\frac{1}{4}]$ and $\vartheta \in [\arccos\frac{1}{3}, 90°)$. We cover these intervals by the cells of Tables 7–9. On a cell $[\alpha_0, \alpha_1]$, with $\chi_0 = -\cos\alpha_0$ and $\chi_1 = -\cos\alpha_1$, we replace each factor of $\underline{a}_P, \underline{a}_Q, \underline{a}_{UW}$ and $\underline{b}_P, \underline{b}_Q, \underline{b}_{UW}$ by its worst value:

- $1/\Omega_1$, $1/\Omega_2$, χ and $|\cot\alpha|$ increase with α, and $\sin\alpha$ decreases; g_0 and g_1 decrease (Lemma 4.10); $\arccos(\frac{1}{3} - \chi)$ and $\arccos(\frac{1}{3} - \frac{4}{3}\chi)$ increase with χ; $\underline{\gamma}$ increases;

- $\underline{\delta}$ is decreasing, because $\underline{\delta}'(\alpha) = -\frac{1}{2} + \frac{1}{14}\csc^2\frac{\alpha}{2}/(1+\frac{1}{49}\cot^2\frac{\alpha}{2}) \le -\frac{1}{2}+\frac{1}{7}$; $\bar{\phi}$ is decreasing; both stay below ϕ_f, where f increases; and $f(180^\circ-\alpha) = \chi\sin^2\alpha$ increases with α, as $180^\circ - \alpha \ge \phi_f$.

Thus $\underline{a}_P \ge 1/(\Omega_2(\chi_0)|\cot\alpha_1|g_1(\alpha_0/2))$, $\underline{a}_Q \ge 3\sin\alpha_1/(\Omega_1(\chi_0)g_1(\frac{1}{2}\arccos(\frac{1}{3}-\chi_0)))$, and at the end $\delta = \underline{\delta}(\alpha)$ of $\underline{a}_{UW}$ the two arguments of ser are at least $f(\underline{\delta}(\alpha_1))/f(180^\circ-\alpha_1)$ and $3f(\bar{\phi}(\alpha_1))/\sin^2\alpha_0$; similarly at the other end, and for the bounds at b. On a cell $[\vartheta_0,\vartheta_1]$, with $p_0 = \cos\vartheta_0 \ge p \ge p_1 = \cos\vartheta_1$, we use that Υ increases with p, $\bar{\chi}$ decreases, $(\frac{1}{3}+p^2)/(1+p)$ is convex, $\sin\vartheta$ increases, and f increases on $[0^\circ, 45^\circ]$:

$$\underline{z}_1 \ge \frac{\sin\vartheta_0 \min(\Upsilon(1,p_1), \Upsilon(\frac{4}{3},p_1))}{(\frac{1}{3}+p_0^2)\, g_1(45^\circ - \frac{1}{2}\arcsin\bar{\chi}(p_1))}, \quad \underline{z}_2 \ge \frac{4\sin\vartheta_0}{\Omega_2(p_1)(1+p_0)g_0(90^\circ-\vartheta_1/2)},$$

$$\underline{z}_{34} \ge \Big[\sin^2\vartheta_1\Big(\max_{p\in\{p_0,p_1\}} \frac{\frac{1}{3}+p^2}{(1+p)f(\vartheta_0/2)} + \frac{1+p_0}{4f(\underline{\gamma}(\vartheta_0))}\Big)\Big]^{-1}.$$

The tables list these cell bounds and the resulting lower bound for ρ_a, ρ_b, ρ_m. By Proposition A.13, $\rho_a \ge 1.08683$, $\rho_b \ge 1.07744$ and $\rho_m \ge 1.03235$. □

$[\alpha_0,\alpha_1]$	$\underline{a}_P \ge$	$\underline{a}_Q \ge$	$\underline{a}_{UW} \ge$	$\rho_a \ge$
$[90^\circ, 98^\circ]$	2.51566	0.61116	0.69142	1.18312
$[98^\circ, 103^\circ]$	2.12103	0.76139	0.54916	1.10942
$[103^\circ, 106.5^\circ]$	2.13466	0.86175	0.47602	1.08993
$[106.5^\circ, 90^\circ + \arcsin\frac{1}{3}]$	2.24066	0.93706	0.42610	1.08683

TABLE 7. The end a: the cell bounds for $\underline{a}_P$, $\underline{a}_Q$, $\underline{a}_{UW}$ and the resulting lower bound for ρ_a.

$[\alpha_0,\alpha_1]$	$\underline{b}_P \ge$	$\underline{b}_Q \ge$	$\underline{b}_{UW} \ge$	$\rho_b \ge$
$[90^\circ, 94^\circ]$	5.05604	0.34992	0.84461	1.17188
$[94^\circ, 98^\circ]$	2.64802	0.46898	0.72988	1.12829
$[98^\circ, 101.5^\circ]$	1.90650	0.60852	0.63813	1.09941
$[101.5^\circ, 90^\circ + \arcsin\frac{1}{4}]$	1.54804	0.74926	0.57255	1.07744

TABLE 8. The end b: the cell bounds for $\underline{b}_P$, $\underline{b}_Q$, $\underline{b}_{UW}$ and the resulting lower bound for ρ_b.

APPENDIX B. FORMULAS OF THE BLOW-UP CHART

This appendix lists the formulas used in the chart (b,α,γ,ξ) of Section 6.4. Recall that $a = 1+b\alpha$, $c = b\gamma$, $x = \frac{1}{2}+b\xi$ and $s = \sqrt{1-x^2}$, and let h_i, Δ_e, ϱ^2, $\hat{\eta}_e$, $\hat{\delta}$ and $\hat{P}$ be as in Section 6.4. For $e \ne 34$ put $R_e = 1 + b\,\Delta_e$, $r_e = \sqrt{R_e}$ and $s_e = 1/(R_e r_e)$. Put $\varrho = \sqrt{\varrho^2}$ and $R_{34} = b^2\varrho^2$, so that $r_{34} = b\varrho$ and $s_{34} = b^{-3}\varrho^{-3}$ for $b > 0$.

$[\vartheta_0, \vartheta_1]$	$\underline{z}_1 \geq$	$\underline{z}_2 \geq$	$\underline{z}_{34} \geq$	$\rho_m \geq$
$[\arccos\frac{1}{3}, 74^\circ]$	1.05639	1.66987	0.38588	1.03293
$[74^\circ, 78^\circ]$	1.01301	1.58401	0.41449	1.03235
$[78^\circ, 82^\circ]$	0.96563	1.50569	0.45151	1.03983
$[82^\circ, 85^\circ]$	0.93067	1.46630	0.48738	1.05669
$[85^\circ, 87.5^\circ]$	0.87875	1.41878	0.51078	1.05342
$[87.5^\circ, 89.2^\circ]$	0.83824	1.39447	0.53006	1.05359
$[89.2^\circ, 90^\circ]$	0.81700	1.39460	0.54396	1.05914

TABLE 9. The middle vertex m: the cell bounds for $\underline{z}_1$, $\underline{z}_2$, $\underline{z}_{34}$ and the resulting lower bound for ρ_m; $\vartheta = 180^\circ - \alpha_m$.

The quantities w_e. Lemma 6.1(a) gives $w_{12} = b^5\hat{w}_{12}$, $w_e = b\,\hat{w}_e$ for $e \in \{13, 14, 23, 24\}$ and $w_{34} = b^{-3}\hat{w}_{34}$, where

$$\hat{w}_{12} = \frac{\hat{\eta}_{13}\hat{\eta}_{24}}{\hat{\delta}}, \qquad \hat{w}_{13} = -\frac{\hat{\eta}_{13}\,(\hat{\eta}_{34} + b^4\hat{\eta}_{13})}{\hat{\delta}}, \qquad \hat{w}_{24} = -\frac{\hat{\eta}_{24}\,(\hat{\eta}_{34} + b^4\hat{\eta}_{24})}{\hat{\delta}},$$
$$\hat{w}_{14} = \hat{\eta}_{14} + b^4\hat{w}_{12}, \qquad \hat{w}_{23} = \hat{\eta}_{23} + b^4\hat{w}_{12}, \qquad \hat{w}_{34} = \frac{(\hat{\eta}_{34} + b^4\hat{\eta}_{13})(\hat{\eta}_{34} + b^4\hat{\eta}_{24})}{\hat{\delta}}.$$

Areas. We have $A_1 = b\widehat{A}_1$ and $A_2 = b\widehat{A}_2$, where

$$\widehat{A}_1 = \tfrac{s}{2}\,(a + c), \qquad \widehat{A}_2 = -\tfrac{s}{2}\,\gamma\,(1 + b), \qquad \text{and} \qquad A_3 = \tfrac{s}{2}\,(a + c), \qquad A_4 = -\tfrac{s}{2}\,a\,(1 + b).$$

The coefficients D_e. Inserting these expressions into (2.6), and using $w_{34}/s_{34} = \varrho^3\hat{w}_{34}$, we get

$$D_{12} = -\frac{b^3\hat{w}_{12}}{3s_{12}\widehat{A}_1\widehat{A}_2}, \qquad D_{13} = -\frac{\hat{w}_{13}}{3s_{13}\widehat{A}_1 A_3}, \qquad D_{14} = -\frac{\hat{w}_{14}}{3s_{14}\widehat{A}_1 A_4},$$
$$D_{23} = -\frac{\hat{w}_{23}}{3s_{23}\widehat{A}_2 A_3}, \qquad D_{24} = -\frac{\hat{w}_{24}}{3s_{24}\widehat{A}_2 A_4}, \qquad D_{34} = -\frac{\varrho^3\hat{w}_{34}}{3A_3 A_4}.$$

The self-stress. The rescaled self-stress $\hat{\tau}_e = \tau_e/b$, with $\tau_e = A_i A_j r_e$ as in Lemma 2.2, is

$$\hat{\tau}_{12} = b\,\widehat{A}_1\widehat{A}_2\,r_{12}, \qquad \hat{\tau}_{13} = \widehat{A}_1 A_3\,r_{13}, \qquad \hat{\tau}_{14} = \widehat{A}_1 A_4\,r_{14},$$
$$\hat{\tau}_{23} = \widehat{A}_2 A_3\,r_{23}, \qquad \hat{\tau}_{24} = \widehat{A}_2 A_4\,r_{24}, \qquad \hat{\tau}_{34} = A_3 A_4\,\varrho.$$

The vectors β_e. The vectors β_e of [Tom26a, Lemma 3.4] are not rescaled. With $q_1 = (1,0)$, $q_2 = a\,(x,s)$, $q_3 = (-b,0)$ and $q_4 = -c\,(x,s)$, they are $\beta_e = -\sum_{l\notin e} C_{e,l}\,q_l$, where

$$C_{12,3} = \frac{r_{12}\,(R_{14}+R_{24}-R_{12})}{8A_3}, \qquad C_{12,4} = \frac{r_{12}\,(R_{13}+R_{23}-R_{12})}{8A_4},$$
$$C_{13,2} = \frac{r_{13}\,(\Delta_{14}-\Delta_{13}+b\varrho^2)}{8\widehat{A}_2}, \qquad C_{13,4} = \frac{r_{13}\,(R_{12}+R_{23}-R_{13})}{8A_4},$$
$$C_{14,2} = \frac{r_{14}\,(\Delta_{13}-\Delta_{14}+b\varrho^2)}{8\widehat{A}_2}, \qquad C_{14,3} = \frac{r_{14}\,(R_{12}+R_{24}-R_{14})}{8A_3},$$
$$C_{23,1} = \frac{r_{23}\,(\Delta_{24}-\Delta_{23}+b\varrho^2)}{8\widehat{A}_1}, \qquad C_{23,4} = \frac{r_{23}\,(R_{12}+R_{13}-R_{23})}{8A_4},$$
$$C_{24,1} = \frac{r_{24}\,(\Delta_{23}-\Delta_{24}+b\varrho^2)}{8\widehat{A}_1}, \qquad C_{24,3} = \frac{r_{24}\,(R_{12}+R_{14}-R_{24})}{8A_3},$$
$$C_{34,1} = \frac{\varrho\,(R_{23}+R_{24}-R_{34})}{8\widehat{A}_1}, \qquad C_{34,2} = \frac{\varrho\,(R_{13}+R_{14}-R_{34})}{8\widehat{A}_2}.$$

These are the coefficients $C_{e,l} = r_e N_{e,l}/(8A_l)$ of [Tom26a, Lemma 3.4]. For the four coefficients with $l \in \{1,2\}$ and $e \neq 34$ we used

$$N_{13,2} = R_{14}+R_{34}-R_{13} = b\,(\Delta_{14}-\Delta_{13}+b\varrho^2), \qquad N_{14,2} = b\,(\Delta_{13}-\Delta_{14}+b\varrho^2),$$
$$N_{23,1} = b\,(\Delta_{24}-\Delta_{23}+b\varrho^2), \qquad N_{24,1} = b\,(\Delta_{23}-\Delta_{24}+b\varrho^2),$$

and for $e = 34$ we used $r_{34} = b\varrho$. In each case the factor b cancels the factor b of A_1 or A_2.

Analyticity at $b = 0$. At a point with $b = 0$ and $\gamma > 0$ we have $x = \frac{1}{2}$, $s = \frac{\sqrt{3}}{2}$, $R_e = 1$ for $e \neq 34$, $\varrho^2 = \gamma^2 - \gamma + 1 > 0$, $\widehat{A}_1 = \frac{\sqrt{3}}{4}$, $\widehat{A}_2 = -\frac{\sqrt{3}}{4}\gamma$, $A_3 = -A_4 = \frac{\sqrt{3}}{4}$ and $\hat{\delta} = \varrho^{-3}$. So every denominator above is nonzero and every square root has a positive argument, and all these quantities are real-analytic in a neighbourhood of the point.

These are the formulas used in the chart computation of the trace test and in the proof of Proposition 6.10. A script in the repository of the first development compares them with the direct formulas in computer algebra.

The test. The certificates of Section 7 compute the quantities of (7.1) as follows. In the coordinates (a,b,c,x), with the η_e and δ of Section 6, $k = (1+b)(a+c)$, $R_{13} = (1+b)^2$ and $R_{24} = (a+c)^2$, the weights are

$$d'_{12} = \eta_{13}\eta_{24}\,bck\,R_{12}^2 r_{12}, \qquad d'_{13} = \eta_{13}(\eta_{34}+\eta_{13})\,bR_{24}\,R_{13}^2 r_{13},$$
$$d'_{14} = (\delta\eta_{14}+\eta_{13}\eta_{24})\,abk\,R_{14}^2 r_{14}, \qquad d'_{23} = (\delta\eta_{23}+\eta_{13}\eta_{24})\,ck\,R_{23}^2 r_{23},$$
$$d'_{24} = \eta_{24}(\eta_{34}+\eta_{24})\,acR_{13}\,R_{24}^2 r_{24}, \qquad d'_{34} = (\eta_{34}+\eta_{13})(\eta_{34}+\eta_{24})\,ak\,R_{34}^2 r_{34}.$$

By Lemma 6.1(a) these are $d'_e = -\delta w_e \tilde{A}_i \tilde{A}_j\, r_e^5$, where $(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,\tilde{A}_4) = \big(b(a+c), -c(1+b), a+c, -a(1+b)\big)$ are the doubled areas of the scaled frame, $A_l = \frac{s}{2}\tilde{A}_l$. The vectors are

$$\tilde{n}_{12} = (1+ax,\,a), \qquad \tilde{n}_{13} = (1-b,\,0), \qquad \tilde{n}_{14} = (1-cx,\,-c),$$
$$\tilde{n}_{23} = (ax-b,\,a), \qquad \tilde{n}_{24} = (ax-cx,\,a-c), \qquad \tilde{n}_{34} = (-b-cx,\,-c).$$

In the chart, with the rescaled gaps $\hat{\eta}_e$ and $\hat{\delta}$ above, $\hat{\delta}_1 = \hat{\eta}_{34} + b^4\hat{\eta}_{13}$, $\hat{\delta}_2 = \hat{\eta}_{34} + b^4\hat{\eta}_{24}$ and $\hat{p} = a\gamma$, the test uses $s_2 = 1 - x^2$,

$$\widehat{N} = \begin{pmatrix} 1-\hat{p}x^2 & -\hat{p}x \\ -\hat{p}x & -\hat{p} \end{pmatrix}, \qquad \widehat{F}_0 = \hat{\delta}\,(k\hat{p})^2,$$

the vectors $\tilde{n}_e$ above with $a = 1 + b\alpha$, $c = b\gamma$, $x = \frac{1}{2} + b\xi$, and the weights

$$\hat{d}_{12} = b^5 \hat{\eta}_{13}\hat{\eta}_{24}\,\gamma k\, R_{12}^2 r_{12}, \qquad \hat{d}_{13} = \hat{\eta}_{13}\hat{\delta}_1\, R_{24}\, R_{13}^2 r_{13},$$
$$\hat{d}_{14} = (\hat{\delta}\hat{\eta}_{14} + b^4\hat{\eta}_{13}\hat{\eta}_{24})\, ak\, R_{14}^2 r_{14}, \qquad \hat{d}_{23} = (\hat{\delta}\hat{\eta}_{23} + b^4\hat{\eta}_{13}\hat{\eta}_{24})\,\gamma k\, R_{23}^2 r_{23},$$
$$\hat{d}_{24} = \hat{\eta}_{24}\hat{\delta}_2\, a\gamma R_{13}\, R_{24}^2 r_{24}, \qquad \hat{d}_{34} = \hat{\delta}_1\hat{\delta}_2\, ak\, \varrho^5.$$

For $b > 0$ these differ from the direct weights by a common positive power of b, and $\widehat{N}$, $\widehat{F}_0$ differ from N, F_0 by compensating powers, so that $\det \widehat{Y} = \det Y$ and $\operatorname{tr} \widehat{Y} = \operatorname{tr} Y$ on $\mathcal{C}$. Every quantity is defined at $b = 0$.

Data and code availability

The programs, certificates, logs and formal proofs are available as follows.

- The certificates of Sections 3 and 5: `https://doi.org/10.5281/zenodo.23727138`. For each item of Table 3 the deposit contains its programs and data and a script that reproduces its certificates; its `README.md` describes the files, the commands and the software.
- The trace test and the first formal development: `https://github.com/tejasvi/lean4body`, release `v1.0`, commit `58ac0fbf4d66`, archived at `https://doi.org/10.5281/zenodo.23004297`. The repository also contains two independent ball-arithmetic implementations of the trace test with their logs, and the script of the computer-algebra computation in the proof of Proposition 6.10. Its registered version is `https://github.com/tejasvi/lean4body-uniqueness`, commit `353653eceec4`, entry PALOMAR-2026-10-02-000011 (version 2) of the Palomar registry of formal proofs, which checks it with Lean's kernel and the independent type checkers nanoda and con-ron.
- The second formal development (Section 7): `https://github.com/tejasvi/lean4body-optimal-bound`, commit `4967a62ec6f4`, entry PALOMAR-2026-10-27-000009 (version 1) of the Palomar registry, which checks it with Lean's kernel and nanoda. The search program that produced the hints of its certificates is not published; it is not needed to check the proof.

The certificates of Sections 3 and 5 are computed with the Python library `python-flint` [JB26], a wrapper of FLINT [FLI26], and, for the inner face family, with `sympy` [Meu17]; the two evaluations for condition (C3) of Lemma 3.11 use only the Python standard library. The ball-arithmetic implementations of the trace test use `python-flint` (Arb [Joh17]). Both formal developments use Lean 4.35.0-rc3 and Mathlib at the tag `v4.35.0-rc3`.

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