

Essential enhancements of site percolation in every dimension

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Abstract

In 1991 Aizenman and Grimmett gave an argument that an essential enhancement of percolation strictly lowers the critical probability. Balister, Bollobás and Riordan later found that a combinatorial lemma in that argument is false. They reduced the case of site percolation on $\mathbb{Z}^d$ to a deterministic statement about rerouting paths, proved it for $d = 2$ and $d = 3$, and left the cases $d \geq 4$ open. We prove their Conjecture 1 for site percolation on $\mathbb{Z}^d$ for every $d \geq 2$. The proof does not confine the enhancement gadget to the ball centred at the pivotal site. Instead it allows any ℓ^1 ball that lies in a large ball around that site, has radius bounded below by a fixed constant, and meets the pivotal path in at least three sites. A two-port theorem, valid in every dimension, shows that the gadget can be wired in unless the first and last sites of the path in the ball form a corner pair. A compactness and mass-transport argument shows that some admissible ball avoids this configuration. As in the cases $d = 2, 3$, the result rests on the probabilistic part of the Aizenman–Grimmett argument. The deterministic part of the argument, Claim 5 of Balister, Bollobás and Riordan, is also checked in the Lean proof assistant for every $d \geq 2$.

1. Introduction

1.1. Enhancements

In site percolation on $\mathbb{Z}^d$, $d \geq 2$, each site is open with probability p , independently of the others. We write ω for the random set of open sites. Following Balister, Bollobás and Riordan [BBR], an *enhancement* is a map E_0 from the set $\mathcal{P}(\mathbb{Z}^d)$ of configurations to itself for which there is an r such that $E_0(\omega)$ depends only on $\omega \cap B_r$ and $E_0(\omega) \subseteq B_r$. We then say that E_0 has *range* r . (All balls are ℓ^1 balls; see Section 1.5.) Let α be a second random set, containing each site with probability s , independently of everything else. The *enhanced configuration* is

$$E(\omega, \alpha) = \omega \cup \bigcup_{v \in \alpha} (E_0(\omega - v) + v).$$

The rule E_0 need not be monotone. An enhancement is *essential* if there is a configuration ω that contains no doubly infinite path while $\omega \cup E_0(\omega)$ does. Let $\theta(p, s)$ be the probability that the origin lies in an infinite cluster of $E(\omega, \alpha)$. The main result of Aizenman and Grimmett [AG] is stated in [BBR] as a conjecture, for the reason explained below.

Conjecture 1 (Aizenman–Grimmett; [BBR, Conjecture 1]). *Let E_0 be an essential enhancement of site or bond percolation on a lattice $\mathcal{L}$ with critical probability $p_c > 0$. Then for any $s > 0$ there is a $\pi(s) < p_c$ such that $\theta(p, s) > 0$ for all p satisfying $\pi(s) < p < p_c$.*

As [BBR] point out, the statement has many implications, such as strict inequalities between the critical probabilities of suitable pairs of lattices. Essential enhancements, and couplings that replace them, are also used to prove strict monotonicity of percolation thresholds under covering maps [MS, MPR]. For background on percolation see [BR, G].

1.2. The gap and what was known

The argument of [AG] has a probabilistic part and a combinatorial part. The probabilistic part reduces Conjecture 1 to a deterministic statement, Claim 5 below: if a site v is pivotal for a crossing event and no enhancement is activated near v , then a bounded modification near v makes some site near v pivotal for the activation of the enhancement. Balister, Bollobás and Riordan describe this part as correct and reasonably complete. For the combinatorial part, [AG] reroute the pivotal path inside a cube around v so that it passes through a copy of a fixed finite gadget, and they support the rerouting only by a figure. Balister, Bollobás and Riordan showed that such a rerouting need not exist: a path that first enters the cube at a corner cannot always be continued into the interior without becoming adjacent to the other half of the path [BBR, §2].

They replaced the rerouting by Conjecture 7 of [BBR]. It states that for some $r = r(d, m)$ the following holds: two induced paths, one red and one green, that start outside B_{r+2} , end at neighbours of the origin and have no red site adjacent to a green site, can be modified inside B_r so that they end at $\pm me_1$, contain no other site of B_m , and still have no red site adjacent to a green site. They showed that Conjecture 7 implies Conjecture 1 for site percolation on $\mathbb{Z}^d$, and they proved Conjecture 7 for $d = 2$ and $d = 3$ by a case analysis at the corners of ℓ^1 balls. This gives Conjecture 1 for site percolation on $\mathbb{Z}^2$ and $\mathbb{Z}^3$ [BBR, Theorem 2]. They also outlined the much easier cases of bond percolation on $\mathbb{Z}^d$, $d \geq 2$, and site percolation on the triangular lattice [BBR, Theorem 3]. They wrote that the general case remains open, even for site percolation on $\mathbb{Z}^d$, $d \geq 4$. For the cube used in [AG] they noted that in three or more dimensions the situation “becomes much worse”, with problems along edges as well as at corners [BBR, p. 8]. We are not aware of later work that settles the cases $d \geq 4$.

1.3. Main result

Theorem 1.1. *Let $d \geq 2$. Conjecture 1 holds for site percolation on $\mathbb{Z}^d$. That is, for every essential enhancement E_0 of site percolation on $\mathbb{Z}^d$ and every $s > 0$ there is $\pi(s) < p_c^{\text{site}}(\mathbb{Z}^d)$ such that $\theta(p, s) > 0$ for all p with $\pi(s) < p < p_c^{\text{site}}(\mathbb{Z}^d)$.*

The cases $d = 2, 3$ are [BBR, Theorem 2]. The new cases are $d \geq 4$, but the proof below treats all $d \geq 2$ in the same way. Like the proof in [BBR], ours uses the probabilistic part of [AG] as stated there: Conjecture 1 follows from Claim 5 (Theorem 2.1). We prove Claim 5 for every essential enhancement of site percolation on $\mathbb{Z}^d$ and every $d \geq 2$ (Theorem 2.13 together with Theorem 4.8). We do not prove Conjecture 7 for $d \geq 4$, and the proof does not need it.

1.4. Idea of the proof

Let v be pivotal for the event R_L that the enhanced configuration contains a path from 0 to the sphere S_L , with no activation sites near v . The deletion step of [BBR] removes from a large ball

$\Omega = B_K(v)$ every open site that is not needed for such a path. Afterwards the open sites in Ω are exactly the sites of $P \cap \Omega$ for a single induced path P from 0 to S_L , and every site of $P \cap \Omega$ is pivotal. We combine three observations.

Recentring. Since every site of $P \cap \Omega$ is pivotal, any ℓ^1 ball $D = B_\rho(c)$ with $B_{\rho+1}(c) \subseteq \Omega$ and $|P \cap D| \geq 3$ is a candidate host for the gadget of [BBR, Lemma 6], whatever its centre c . Only the first and last sites a, g of P in D , the *ports*, matter. Outside D the pieces of P stay disconnected from each other, so it suffices to join a and g to the two terminals of a gadget centred at c by a red set and a green set inside D , with no red site adjacent to a green site (Theorem 2.10). [BBR] used only the ball $B_r(v)$.

Two ports. Two sites $a \neq g$ of a sphere $S_\rho(c)$ can be wired in this way unless they form a *corner pair* $\{c + \rho\pi, c + (\rho - 1)\pi + \delta\}$, where π and δ are orthogonal signed unit vectors (Corollary 3.3, for $\rho \geq \max(2d + 1, m + 13, 20)$). We continue the ports inward by at most three layers until a hyperplane separates them, and then finish with [BBR, Lemma 8], which holds in every dimension. Corner pairs do fail, because the corner's only inner neighbour is adjacent to the other port.

The Corner Lemma. It remains to show that inside a large ball $B_K(v)$ some admissible ball is not corner-paired (Theorem 4.8). Suppose this fails for arbitrarily large K . Compactness and averaging over translates give a translation-invariant random pair $(V, \prec)$, where $V \subseteq \mathbb{Z}^d$ is nonempty and $\prec$ is a linear order on V , such that every site of V has exactly two neighbours in V , and every ball of radius at least ρ_0 containing at least three sites of V has its $\prec$ -first and $\prec$ -last sites forming a corner pair. Shrinking a corner-paired ball towards its corner (the *apex*) keeps its first and last sites, and the shrinking map is injective. Mass transport turns this into a strong statement: almost surely every ball of every radius $r \geq 2$ is corner-paired, and its apex is also the apex of corner-paired balls of every larger radius (Proposition 4.4). Two corner-paired balls with different poles and suitably placed apexes would contain each other's ports, and stationarity along lines then shows that no site is the apex of two balls with different poles (Lemma 4.6). Hence the apex map $c \mapsto c + r\sigma_r(c)$ is injective, and mass transport forces $V = \mathbb{Z}^d$. This contradicts the degree condition. The degree condition cannot be dropped: a “parity double sweep” on $V = \mathbb{Z}^d$ satisfies all the other hypotheses (Example 5.1).

1.5. Organization and notation

Section 2 recalls the setting and reduces Theorem 1.1 to the Corner Lemma $\text{CL}(d)$. Section 3 proves the two-port theorem. Section 4 proves $\text{CL}(d)$, completes the proof of Theorem 1.1 and describes a Lean formalization of its deterministic part. Section 5 contains remarks on sharpness and constants. Appendix A lists computational cross-checks.

Throughout, $|x| = \sum_i |x_i|$ is the ℓ^1 norm, $B_r(c) = \{x : |x - c| \leq r\}$ and $S_r(c) = \{x : |x - c| = r\}$, with $B_r = B_r(0)$ and $S_r = S_r(0)$. Two sites are adjacent if $|x - y| = 1$. The *layer* of x relative to c is $|x - c|$, and adjacent sites lie in consecutive layers. We write $U = \{\pm e_1, \dots, \pm e_d\}$ for the signed unit vectors, and $\pi, \delta \in U$ are called orthogonal if $\delta \neq \pm\pi$. For $D \subseteq \mathbb{Z}^d$, $N[D]$ is the set of sites in D or adjacent to D , and $\text{int } D$ is the set of sites of D all of whose neighbours lie in D . Thus $N[B_\rho(c)] = B_{\rho+1}(c)$ and $\text{int } B_\rho(c) = B_{\rho-1}(c)$. A *path* is a finite

sequence of distinct sites in which consecutive sites are adjacent. It is *induced* if no two non-consecutive sites are adjacent. A set $A \subseteq \mathbb{Z}^d$ contains a path from x to y if some path from x to y has all its sites in A . The components of A are those of the subgraph of $\mathbb{Z}^d$ induced by A . Where [BBR] write e_m for $(m, 0, \dots, 0)$, we write me_1 .

2. Setting and the reduction to a corner lemma

2.1. Crossing events and pivotal sites

As in [BBR, §3], all balls, spheres and distances are taken in the ℓ^1 metric, including the sphere $S_L = S_L(0)$ in the crossing event below. Let E_0 be an enhancement with range r_E (we write r_E for the range, which [BBR] call r , to free the letter r for radii). Let R_L be the event that the enhanced configuration contains a path from 0 to S_L . A site v is *p-pivotal* (for R_L , in the state (ω, α)) if $E(\omega \cup \{v\}, \alpha) \in R_L$ but $E(\omega \setminus \{v\}, \alpha) \notin R_L$. It is *s-pivotal* if $E(\omega, \alpha \cup \{v\}) \in R_L$ but $E(\omega, \alpha \setminus \{v\}) \notin R_L$. The following are the two deterministic statements of [BBR, §2]. The dimension d is fixed throughout, and constants may depend on it.

Claim 4 ([BBR, Claim 4]). *Let E_0 be an essential enhancement. There are constants L_0 and R , depending only on E_0 , such that whenever $L \geq L_0$ and a site v is p-pivotal for R_L in the state (ω, α) , there is a state (ω', α') differing from (ω, α) only within distance R of v such that some site w within distance R of v is s-pivotal for R_L in the state (ω', α') .*

Claim 5 ([BBR, Claim 5]). *The same statement, under the additional hypothesis $\alpha \cap B_R(v) = \emptyset$.*

2.2. The probabilistic reduction

Theorem 2.1 (Aizenman–Grimmett [AG, Theorem 1 and §3]; in the form of [BBR, pp. 4–6 and 9]). *Let $d \geq 2$ and let E_0 be an essential enhancement of site percolation on $\mathbb{Z}^d$. If Claim 5 holds for E_0 , then the conclusion of Conjecture 1 holds for E_0 .*

The formal theorem of [AG, Theorem 1] is stated for site percolation on $\mathbb{Z}^d$, which is exactly the case of Theorem 2.1 (see also [BBR, p. 3]). We recall the structure of the argument from [BBR, §2]. First, Claim 5 implies Claim 4 [BBR, pp. 5–6]. Let v be p-pivotal in (ω, α) and delete the sites of $\alpha \cap B_R(v)$ from α one at a time. The enhanced configuration is increasing in α . If at some stage $E(\omega \cup \{v\}, \alpha')$ no longer contains a path from 0 to S_L , the last deleted site w is s-pivotal in the state $(\omega \cup \{v\}, \alpha')$, which differs from (ω, α) only within $B_R(v)$. Otherwise v is still p-pivotal in $(\omega, \alpha \setminus B_R(v))$, and Claim 5 applies. Second, Claim 4 yields, through a finite-to-one modification map, an integer L_0 and a continuous positive function g on $(0, 1)^2$ with $\mathbb{E}_{p,s} N_s(L) \geq g(p, s) \mathbb{E}_{p,s} N_p(L)$ for all $L \geq L_0$, where $N_p(L)$ and $N_s(L)$ count the p-pivotal and s-pivotal sites [BBR, (2)]. Third, a version of the Margulis–Russo formula gives $\partial_s \tau_L \geq g \partial_p \tau_L$ for $\tau_L(p, s) = \mathbb{P}_{p,s}(E(\omega, \alpha) \in R_L)$ [BBR, (3)]. Comparing τ_L along curves that do not depend on L and letting $L \rightarrow \infty$ gives the conclusion of Conjecture 1 for $s \in (0, 1)$ [AG, pp. 822–823]. For $s = 1$ one may take $\pi(1) = \pi(1/2)$, since $\theta(p, s)$ is non-decreasing in s . The proof of the cases $d = 2, 3$ in [BBR] rests on Theorem 2.1 in the same way [BBR, p. 9].

Theorem 1.1 thus reduces to Claim 5 for every essential enhancement of site percolation on $\mathbb{Z}^d$.

2.3. The gadget

Lemma 2.2 (the gadget; [BBR, Lemma 6] in ℓ^1 form). *Let E_0 be an essential enhancement of site percolation on $\mathbb{Z}^d$ with range r_E . There are an integer $m > r_E$ and a finite set $\omega_g \subseteq \mathbb{Z}^d$ such that*

1. $\omega_g \subseteq B_m$ and $\omega_g \cap S_m = \{me_1, -me_1\}$;
2. ω_g contains no path from $-me_1$ to me_1 ;
3. $\omega_g \cup E_0(\omega_g)$ contains a path from $-me_1$ to me_1 .

Moreover $E_0(\omega_g) \subseteq B_{r_E} \subseteq B_{m-1}$.

Proof. Balister, Bollobás and Riordan prove the ℓ^∞ form [BBR, §4] and note that either form implies the other. We derive the ℓ^1 form. Write B_M^∞ and S_M^∞ for ℓ^∞ balls and spheres. Since $B_{r_E} \subseteq B_{r_E}^\infty$, the map E_0 also has range r_E with respect to the ℓ^∞ metric. The ℓ^∞ form of the lemma gives an integer $M > r_E$ and a finite set $\omega_\infty \subseteq B_M^\infty$ with $\omega_\infty \cap S_M^\infty = \{\pm Me_1\}$, such that ω_∞ contains no path from $-Me_1$ to Me_1 while $\omega_\infty \cup E_0(\omega_\infty)$ does. Put $m = dM$, let $\Sigma_\pm = \{\pm te_1 : M < t \leq m\}$, and let $\omega_g = \omega_\infty \cup \Sigma_+ \cup \Sigma_-$.

(1) The sites of ω_∞ other than $\pm Me_1$ lie in $B_{M-1}^\infty \subseteq B_{d(M-1)} \subseteq B_{m-1}$. The sites $\pm te_1$ with $M \leq t \leq m$ have norm t , which equals m only for $t = m$. Hence $\omega_g \subseteq B_m$ and $\omega_g \cap S_m = \{\pm me_1\}$, and $m > M > r_E$.

(2) A site te_1 with $t > M$ has the neighbours $(t \pm 1)e_1$ and $te_1 \pm e_j$ ($j \geq 2$). The latter have ℓ^∞ norm $t > M$, so they are not in ω_∞ , and $(t - 1)e_1 \in \omega_\infty$ only if $t = M + 1$. Thus Σ_+ is a path attached to $\omega_\infty \cup \Sigma_-$ only through the edge between $(M + 1)e_1$ and Me_1 , and similarly for Σ_- . A path in ω_g from $-me_1$ to me_1 would therefore contain a path in ω_∞ from $-Me_1$ to Me_1 , which does not exist.

(3) $E_0(\omega_g)$ depends only on $\omega_g \cap B_{r_E}$. Since $B_{r_E} \subseteq B_{M-1}^\infty$ and $\Sigma_\pm \cap B_M^\infty = \emptyset$, we have $\omega_g \cap B_{r_E} = \omega_\infty \cap B_{r_E}$, so $E_0(\omega_g) = E_0(\omega_\infty) \subseteq B_{r_E}$. A path in $\omega_\infty \cup E_0(\omega_\infty)$ from $-Me_1$ to Me_1 , extended along Σ_- and Σ_+ , joins $-me_1$ to me_1 in $\omega_g \cup E_0(\omega_g)$. $\square$

From now on E_0 , r_E , m and ω_g are fixed as in Lemma 2.2. Note that $E_0(\omega_g) \subseteq B_{m-1}$: activating the enhancement at the centre of a copy of the gadget changes nothing on or outside S_m , and it joins the two terminals $\pm me_1$, which ω_g alone does not join.

2.4. Two consecutive layers

We need the following elementary fact in Lemma 3.1 and in the boundary case of Theorem 2.13.

Lemma 2.3. *Let $d \geq 2$, $n \geq 1$, $i \in \{1, \dots, d\}$, and let $s < n$ be an integer. Then the set $A = (S_n \cup S_{n+1}) \cap \{x : x_i > s\}$ contains ne_i and is connected. In particular $S_n \cup S_{n+1}$ is connected.*

Proof. Let $x \in A$. We walk from x to ne_i inside $S_n \cup S_{n+1}$, changing one coordinate by one unit at each step. Write y for the current site.

Phase 1, while $y_i < 0$. If $|y| = n$, move some coordinate y_j with $j \neq i$ one unit away from 0 (this is possible since $d \geq 2$). The new site lies in S_{n+1} . If $|y| = n + 1$, replace y_i by $y_i + 1$. As $y_i < 0$,

the new site lies in S_n . Every two steps raise y_i by one, so eventually $y_i = 0$.

Phase 2, while $y_i \geq 0$. If $y = ne_i$, stop. If $y = (n+1)e_i$, step to ne_i and stop. If $|y| = n+1$ and $y \neq (n+1)e_i$, some coordinate y_j with $j \neq i$ is nonzero. Move it one unit towards 0, which leads to S_n . If $|y| = n$ and $y \neq ne_i$, replace y_i by $y_i + 1$, which leads to S_{n+1} . Two consecutive steps starting in S_n raise y_i by one and lower $\sum_{j \neq i} |y_j| = n - y_i$ by one, so the walk reaches ne_i .

Along the walk y_i never decreases, except in a final step from $(n+1)e_i$ to ne_i . Since $x_i > s$ and $n > s$, the walk stays in A . For the last assertion take $s = -n - 2$. $\square$

2.5. The deletion step

In this subsection and the next we fix integers $K \geq m + 1$ and L , and a state (ω, α) in which a site v is p-pivotal for R_L , such that

$$K + 10 < |v| < L - K - 10 \quad \text{and} \quad \alpha \cap B_{K+r_E}(v) = \emptyset. \quad (2.1)$$

Put $\Omega = B_K(v)$. Then $0 \notin \Omega$, and every site of Ω has norm less than L .

Lemma 2.4. *There is a set $Y \subseteq \mathbb{Z}^d \setminus \Omega$ such that $E(\tilde{\omega}, \alpha) = \tilde{\omega} \cup Y$ for every configuration $\tilde{\omega}$ that agrees with ω outside Ω . For such $\tilde{\omega}$ and every $c \in \Omega$ we have $c \notin \alpha$ and $E(\tilde{\omega}, \alpha \cup \{c\}) = \tilde{\omega} \cup Y \cup (E_0(\tilde{\omega} - c) + c)$.*

Proof. Let $a \in \alpha$. Then $|a - v| > K + r_E$, so $B_{r_E}(a) \cap \Omega = \emptyset$. The set $E_0(\tilde{\omega} - a) + a$ depends only on $\tilde{\omega} \cap B_{r_E}(a)$, where $\tilde{\omega}$ agrees with ω , and it lies in $B_{r_E}(a)$. Hence $Y = \bigcup_{a \in \alpha} (E_0(\omega - a) + a)$ has the required properties. Finally $c \notin \alpha$ because $\Omega \subseteq B_{K+r_E}(v)$. $\square$

Lemma 2.5 (the deletion step; [BBR, p. 9]). *There are a configuration ω' that agrees with ω outside Ω and an induced path $P = (p_0, \dots, p_n)$ with $p_0 = 0$, $p_n \in S_L$ and $|p_j| < L$ for $j < n$, such that:*

1. $P \subseteq E(\omega', \alpha)$ and $v \in P$;
2. $E(\omega', \alpha) \cap \Omega = \omega' \cap \Omega = P \cap \Omega$;
3. for every $x \in P \cap \Omega$, every path from 0 to S_L in $E(\omega', \alpha)$ passes through x ;
4. both ends p_0 and p_n lie outside $B_{K+10}(v)$.

Proof. Put $\omega_0 = \omega \cup \{v\}$. As long as some site $x \in \omega_t \cap \Omega$ satisfies $E(\omega_t \setminus \{x\}, \alpha) \in R_L$, delete one such site. The process stops at a configuration ω' that agrees with ω outside Ω , satisfies $E(\omega', \alpha) \in R_L$, and has $E(\omega' \setminus \{x\}, \alpha) \notin R_L$ for every $x \in \omega' \cap \Omega$. The site v is never deleted. Indeed, each $\omega_t \setminus \{v\}$ is contained in $\omega \setminus \{v\}$ and agrees with it outside Ω , so by Lemma 2.4

$$E(\omega_t \setminus \{v\}, \alpha) = (\omega_t \setminus \{v\}) \cup Y \subseteq (\omega \setminus \{v\}) \cup Y = E(\omega \setminus \{v\}, \alpha) \notin R_L.$$

Let P be a shortest path from 0 to S_L in $E(\omega', \alpha)$. A chord would shorten it, so P is induced. Since the norm changes by one at each step, P meets S_L only at its last site and all its other sites have norm less than L . By Lemma 2.4, $E(\omega' \setminus \{x\}, \alpha) = E(\omega', \alpha) \setminus \{x\}$ for $x \in \Omega$. Hence every path from 0 to S_L in $E(\omega', \alpha)$ passes through every site of $\omega' \cap \Omega$, and in particular $\omega' \cap \Omega \subseteq P$

and $v \in P$. As $Y \cap \Omega = \emptyset$ and $P \subseteq E(\omega', \alpha)$, this gives (2), and (3) follows. Finally $|p_0 - v| = |v| > K + 10$ and $|p_n - v| \geq L - |v| > K + 10$. $\square$

From now on ω' and P are as in Lemma 2.5. We order P from $p_0 = 0$ to $z := p_n$, and for $x \in P \setminus \{0, z\}$ we write $\text{pred}(x)$ and $\text{succ}(x)$ for the sites before and after x on P .

Lemma 2.6 (the class property; [BBR, pp. 10–11]). *Let $D \subseteq \Omega$ meet P . Write $P = O_0 I_1 O_1 \cdots I_k O_k$ with $k \geq 1$, where $I_1, \dots, I_k$ are the maximal runs of consecutive sites of P in D and $O_0, \dots, O_k$ are the runs between them. ($O_0 \ni 0$ and $O_k \ni z$ are nonempty because $0, z \notin \Omega$.) Put $O_D = E(\omega', \alpha) \setminus D$, and let H_j be the component of O_D that contains O_j . Then $H_0, \dots, H_k$ are pairwise distinct, and $H_j \cap S_L = \emptyset$ for every $j < k$.*

Proof. By Lemma 2.5(3), every path from 0 to S_L in $E(\omega', \alpha)$ meets each I_j . Suppose that $H_i = H_j$ with $i < j$, and let Q be a path in O_D from O_i to O_j . Follow P from 0 to O_i , then Q , then P from O_j to z . This walk lies in $E(\omega', \alpha)$ and avoids I_j , and it contains a path from 0 to S_L that avoids I_j , a contradiction. If H_j met S_L for some $j < k$, then P from 0 to O_j followed by a path in O_D from O_j to S_L would avoid I_k . $\square$

For $D = B_r(v)$ the set O_D is the configuration ω^* of [BBR, (6)], but Lemma 2.6 needs only $D \subseteq \Omega$.

2.6. Admissible balls, ports and designs

Definition 2.7. A set $D \subseteq \mathbb{Z}^d$ is *admissible* if $N[D] \subseteq \Omega$ and $|P \cap D| \geq 3$. Its *ports* are the first site $a(D)$ and the last site $g(D)$ of P in D .

Recentring. A ball $D = B_\rho(c)$ is admissible if and only if $B_{\rho+1}(c) \subseteq \Omega$ and $|P \cap B_\rho(c)| \geq 3$. Here c is an arbitrary site of $\mathbb{Z}^d$: it need not lie on P , and D need not contain v . The Corner Lemma uses this freedom. For a ball $D = B_\rho(c)$ both ports lie on $S_\rho(c)$: the site $\text{pred}(a(D))$ lies outside D , and the distance to c changes by exactly one at each step of P . The same holds for $g(D)$.

Proposition 2.8. *Let D be admissible, put $a = a(D)$ and $g = g(D)$, and use the notation of Lemma 2.6.*

1. $a \neq g$, and a and g are not adjacent.
2. The sites of O_D adjacent to a are $\text{pred}(a) \in O_0$ and, if $I_1 = \{a\}$, also $\text{succ}(a) \in O_1$. The sites of O_D adjacent to g are $\text{succ}(g) \in O_k$ and, if $I_k = \{g\}$, also $\text{pred}(g) \in O_{k-1}$.
3. Let $\mathcal{A}$ consist of H_0 , and of H_1 if $I_1 = \{a\}$. Let $\mathcal{G}$ consist of H_k , and of H_{k-1} if $I_k = \{g\}$. Then no member of $\mathcal{A}$ belongs to $\mathcal{G}$, and no member of $\mathcal{A}$ meets S_L .

Proof. (2) The neighbours of a outside D lie in $N[D] \setminus D \subseteq \Omega \setminus D$, and $O_D \cap (\Omega \setminus D) = P \cap (\Omega \setminus D)$ by Lemma 2.5(2). Since P is induced and $a \notin \{0, z\}$, the only sites of P adjacent to a are $\text{pred}(a)$ and $\text{succ}(a)$. By the choice of a , $\text{pred}(a) \notin D$, so it is the last site of O_0 . The site $\text{succ}(a)$ lies outside D exactly when $I_1 = \{a\}$, and then it belongs to O_1 . The argument for g is symmetric.

(1) $a \neq g$ because $|P \cap D| \geq 3$. If a and g were adjacent, they would be consecutive on P because P is induced. All sites of $P \cap D$ lie between a and g on P , so we would get $P \cap D = \{a, g\}$.

(3) If $I_1 = \{a\}$ then $k \geq 2$, since otherwise $P \cap D = \{a\}$. Likewise $k \geq 2$ if $I_k = \{g\}$. As the H_j are pairwise distinct (Lemma 2.6), a common member of $\mathcal{A}$ and $\mathcal{G}$ would need equal indices. The index 0 differs from $k \geq 1$, and from $k - 1$ when $I_k = \{g\}$. The index 1 (present only if $I_1 = \{a\}$) differs from k . It equals $k - 1$ only if $k = 2$, and if moreover $I_2 = \{g\}$ then $P \cap D = I_1 \cup I_2 = \{a, g\}$, which is impossible. The members of $\mathcal{A}$ are components H_j with $j \in \{0, 1\}$ and $j < k$, so they do not meet S_L by Lemma 2.6. $\square$

Definition 2.9 (interior design). Let $D = B_\rho(c)$ with $\rho \geq m + 1$, and let $a \neq g$ be sites of $S_\rho(c)$. An *interior design* for (a, g) in D is a set $\omega_1 = R \cup G \cup (\omega_g + c)$, where the *red set* R and the *green set* G are subsets of D such that:

- (D1) $a \in R$, $g \in G$, and R and G are connected;
- (D2) $R \setminus \{a\}$ and $G \setminus \{g\}$ lie in $B_{\rho-1}(c)$;
- (D3) $R \cap B_m(c) = \{c + me_1\}$ and $G \cap B_m(c) = \{c - me_1\}$, or the same with the two signs exchanged;
- (D4) no site of R equals or is adjacent to a site of G .

By (D3), $\omega_1 \cap B_m(c) = \omega_g + c$.

Theorem 2.10 (cf. [BBR, pp. 9–11]). Let $D = B_\rho(c)$ be an admissible ball with $\rho \geq m + 1$, and let ω_1 be an interior design for $(a(D), g(D))$ in D . Put $\omega'' = (\omega' \setminus D) \cup \omega_1$. Then c is s -pivotal for R_L in the state (ω'', α) . Moreover ω'' agrees with ω outside Ω , and $c \in \Omega$.

Proof. Write $a = a(D)$ and $g = g(D)$, use the notation of Lemma 2.6 and Proposition 2.8, and suppose that $R \cap B_m(c) = \{c + me_1\}$ (the other case is symmetric). Let C_+ and C_- be the components of $\omega_g + c$ that contain $c + me_1$ and $c - me_1$. They are distinct by Lemma 2.2(2).

Step 1: the enhanced configurations. The configuration ω'' agrees with ω' , hence with ω , outside Ω . By Lemma 2.4, $E(\omega'', \alpha) = \omega'' \cup Y = O_D \cup \omega_1$, because $O_D = (\omega' \cup Y) \setminus D = (\omega' \setminus D) \cup Y$. Also $c \notin \alpha$. Since $r_E < m$ and $\omega'' \cap B_m(c) = \omega_g + c$, the enhancement at c sees $\omega'' \cap B_{r_E}(c) = (\omega_g + c) \cap B_{r_E}(c)$. Hence $E(\omega'', \alpha \cup \{c\}) = O_D \cup \omega_1 \cup X$ with $X = E_0(\omega_g) + c \subseteq B_{m-1}(c)$.

Step 2: the components of ω_1 . A site of $R \setminus \{c + me_1\}$ lies outside $B_m(c)$, so its neighbours in $\omega_g + c$ lie in $S_m(c) \cap (\omega_g + c) = \{c \pm me_1\}$, and by (D4) it is not adjacent to $c - me_1 \in G$. Hence every site of $\omega_g + c$ that equals or is adjacent to a site of R is $c + me_1$ or a neighbour of $c + me_1$, and therefore lies in C_+ . In the same way, every site of $\omega_g + c$ that equals or is adjacent to a site of G lies in C_- . Together with (D4) and $C_+ \neq C_-$, this shows that $R \cup C_+$ and $G \cup C_-$ are distinct components of ω_1 . The other components of ω_1 are components of $\omega_g + c$.

Step 3: no crossing without activation. By (D2) and $\omega_g + c \subseteq B_m(c) \subseteq B_{\rho-1}(c)$, every site of ω_1 other than a and g lies in $\text{int } D$, so all its neighbours lie in D , which O_D avoids. Hence a and g are the only sites of ω_1 with neighbours in O_D , and these neighbours are given by Proposition

2.8(2). Let Γ_0 be the union of $R \cup C_+$ and the members of $\mathcal{A}$. It is connected, since $\text{pred}(a) \in H_0$ and, if $I_1 = \{a\}$, $\text{succ}(a) \in H_1$. No site outside Γ_0 is adjacent to Γ_0 : the members of $\mathcal{A}$ are components of O_D , they do not contain $\text{pred}(g)$ or $\text{succ}(g)$ by Proposition 2.8(3), and $R \cup C_+$ is a component of ω_1 whose only site with neighbours in O_D is a . So Γ_0 is the component of $\mathbf{0}$ in $E(\omega'', \alpha)$. It contains no site of S_L , because the members of $\mathcal{A}$ do not meet S_L (Proposition 2.8(3)) and $R \cup C_+ \subseteq D \subseteq \Omega$. Hence $E(\omega'', \alpha) \notin R_L$.

Step 4: a crossing with activation. By Lemma 2.2(3), $(\omega_g + c) \cup X$ contains a path from $c + me_1$ to $c - me_1$. The following walk lies in $E(\omega'', \alpha \cup \{c\})$: along P from $\mathbf{0}$ to $\text{pred}(a)$ (these sites lie in $O_0 \subseteq O_D$), then to a , along a path in R to $c + me_1$, through $(\omega_g + c) \cup X$ to $c - me_1$, along a path in G to g , then to $\text{succ}(g)$ and along P to $z \in S_L$ (these sites lie in $O_k \subseteq O_D$). Hence $E(\omega'', \alpha \cup \{c\}) \in R_L$. Together with Step 3 and $c \notin \alpha$, this shows that c is s-pivotal in (ω'', α) . $\square$

2.7. Corner pairs, the Corner Lemma, and the reduction

Definition 2.11. Let $c \in \mathbb{Z}^d$ and $\rho \geq 1$.

1. A *corner pair* of $S_\rho(c)$ is a set $\{c + \rho\pi, c + (\rho - 1)\pi + \delta\}$ with $\pi, \delta \in U$ orthogonal. If $\rho \geq 2$, the site $c + (\rho - 1)\pi + \delta$ has two nonzero coordinates relative to c , so the pair determines the corner $c + \rho\pi$ and hence π . We call π the *pole*, $c + \rho\pi$ the *apex* and $c + (\rho - 1)\pi + \delta$ the *partner*.
2. Two sites $x, y \in S_\rho(c)$ are *separated* if there are a coordinate i and an integer s with $|s| \leq \rho - 5$ lying strictly between $x_i - c_i$ and $y_i - c_i$.

A corner pair is never separated, and it admits no interior design. Indeed, by (D1)–(D3) the coloured set containing the apex $c + \rho\pi$ is connected, contains a site of $B_m(c)$ and has all its other sites in $B_{\rho-1}(c)$, so it contains a neighbour of the apex in $B_{\rho-1}(c)$. The only such neighbour is $c + (\rho - 1)\pi$, which is adjacent to the partner, contrary to (D4). This is the obstruction of [BBR, §2] in ℓ^1 form.

Corner Lemma CL(d). *For every integer $\rho_0 \geq 2$ there is an integer $K = K(d, \rho_0)$ with the following property. Let P be an induced path in $\mathbb{Z}^d$, ordered from one end to the other, that passes through a site v and has both end sites outside $B_K(v)$. Then there is a ball $B_\rho(c)$, with $c \in \mathbb{Z}^d$, $\rho \geq \rho_0$, $B_{\rho+1}(c) \subseteq B_K(v)$ and $|P \cap B_\rho(c)| \geq 3$, such that the first and the last site of P in $B_\rho(c)$ do not form a corner pair of $S_\rho(c)$.*

Reversing the order of P exchanges the first and the last site, so CL(d) does not depend on the orientation of P . If CL(d) holds with some K , it holds with every larger K . We prove CL(d) for every $d \geq 2$ in Section 4 (Theorem 4.8). Here we show that it implies Claim 5.

For the given E_0 let m be as in Lemma 2.2, and put

$$\rho_0 = \max(2d + 1, m + 13, 20), \quad K = K(d, \rho_0), \quad R = 3K + 100, \quad L_0 = 100R. \quad (2.2)$$

Since CL(d) asks for a ball $B_{\rho+1}(c) \subseteq B_K(v)$ with $\rho \geq \rho_0$, we have $K \geq \rho_0 + 1 > m + 1 > r_E + 1$.

Proposition 2.12. *Assume $CL(d)$ and define ρ_0 and K by (2.2). Let (ω, α) be a state in which v is p -pivotal for R_L , and suppose that $K + 10 < |v| < L - K - 10$ and $\alpha \cap B_{K+r_E}(v) = \emptyset$. Then there are a configuration ω'' that agrees with ω outside $B_K(v)$ and a site $c \in B_K(v)$ that is s -pivotal for R_L in the state (ω'', α) .*

Proof. The hypotheses (2.1) hold, with $K \geq m + 1$. Let ω' and P be as in Lemma 2.5. By Lemma 2.5(1),(4), P is an induced path through v with both ends outside $B_K(v)$. $CL(d)$ gives a ball $D = B_\rho(c)$ with $\rho \geq \rho_0$, $B_{\rho+1}(c) \subseteq \Omega$ and $|P \cap D| \geq 3$ whose ports do not form a corner pair. This ball is admissible, and its ports are distinct sites of $S_\rho(c)$ (Definition 2.7, Proposition 2.8). Since $\rho \geq \max(2d + 1, m + 13, 20)$, Corollary 3.3 below provides an interior design for $(a(D), g(D))$ in D . Theorem 2.10 now gives the claim. $\square$

Theorem 2.13. *Let $d \geq 2$ and assume $CL(d)$. Then Claim 5 holds for every essential enhancement E_0 of site percolation on $\mathbb{Z}^d$, with the constants R and L_0 of (2.2).*

Proof. Let $L \geq L_0$, let v be p -pivotal for R_L in the state (ω, α) , and suppose that $\alpha \cap B_R(v) = \emptyset$. We find a state (ω'', α) that agrees with (ω, α) outside $B_R(v)$ and a site $c \in B_R(v)$ that is s -pivotal in it. Recall that $r_E < K$.

Case 1: $K + 10 < |v| < L - K - 10$. Since $B_{K+r_E}(v) \subseteq B_R(v)$, Proposition 2.12 applies directly.

Case 2: $|v| \leq K + 10$ (cf. [BBR, p. 11]). Let w be a site of S_{K+19} nearest to v , so that $|v - w| = K + 19 - |v| \leq K + 19$. Let

$$\tilde{\omega} = (\omega \setminus B_{K+21}) \cup (B_{K+21} \setminus S_{K+19}) \cup \{w\}.$$

Thus every site of B_{K+21} is open in $\tilde{\omega}$, except the sites of S_{K+19} other than w . For $a \in \alpha$ we have $|a| \geq |a - v| - |v| > 2K + 90$, so $B_{r_E}(a)$ misses B_{K+21} . As in Lemma 2.4, there is a set Y' disjoint from B_{K+21} such that $E(\tilde{\omega}, \alpha) = \tilde{\omega} \cup Y'$ for every $\tilde{\omega}$ that agrees with ω outside B_{K+21} .

The configuration $E(\tilde{\omega}, \alpha)$ contains a path from 0 to S_L . Indeed, $E(\omega \cup \{v\}, \alpha)$ contains such a path Γ . Let x' be the last site of Γ in B_{K+21} , so that $|x'| = K + 21$. The part of Γ after x' lies in $(\omega \cup \{v\} \cup Y') \setminus B_{K+21} = E(\tilde{\omega}, \alpha) \setminus B_{K+21}$. Inside B_{K+21} , the configuration $\tilde{\omega}$ contains the connected set B_{K+18} , which contains 0 and an inner neighbour of w ; it contains w ; and it contains $S_{K+20} \cup S_{K+21}$, which is connected by Lemma 2.3 and contains x' and an outer neighbour of w . Concatenation gives the path. On the other hand, every path from 0 to S_L meets S_{K+19} , and w is the only site of S_{K+19} in $E(\tilde{\omega}, \alpha)$. Therefore w is p -pivotal for R_L in $(\tilde{\omega}, \alpha)$. Moreover $K + 10 < |w| = K + 19 < L - K - 10$ and $\alpha \cap B_{K+r_E}(w) \subseteq \alpha \cap B_{2K+19+r_E}(v) = \emptyset$. Proposition 2.12, applied at w , gives ω'' that agrees with $\tilde{\omega}$ outside $B_K(w)$ and a site $c \in B_K(w)$ that is s -pivotal in (ω'', α) . The configuration ω'' agrees with ω outside $B_{K+21} \cup B_K(w) \subseteq B_{2K+31}(v)$, and $c \in B_{2K+19}(v)$.

Case 3: $|v| \geq L - K - 10$ (cf. [BBR, pp. 11–12]). First, $|v| \leq L$. Otherwise take a path from 0 to S_L in $E(\omega \cup \{v\}, \alpha)$ and stop it at its first site on S_L . All its sites have norm at most L , so it avoids v . Since $\alpha \cap B_{r_E}(v) = \emptyset$, no enhancement sees v , so $E(\omega \cup \{v\}, \alpha) \setminus \{v\} \subseteq E(\omega \setminus \{v\}, \alpha)$, and v would not be p -pivotal.

Choose w with $|w| = L - K - 20$ and $|v - w| = |v| - |w| \leq K + 20$, by moving v towards the origin one coordinate step at a time. Put $\Omega' = B_{K+30}(w)$. Then $0 \notin \Omega'$, and $\alpha \cap B_{K+30+r_E}(w) \subseteq \alpha \cap B_{2K+50+r_E}(v) = \emptyset$. So no enhancement reads or writes in Ω' , and the proofs of Lemmas 2.4 and 2.5 apply verbatim with Ω' in place of Ω . They give a set Y disjoint from Ω' , a configuration ω' that agrees with ω outside Ω' , and an induced path P from 0 to S_L through v , meeting S_L only at its last site z , such that $P \subseteq E(\omega', \alpha)$ and $E(\omega', \alpha) \cap \Omega' = \omega' \cap \Omega' = P \cap \Omega'$. Let u be the first site of P in Ω' . Then $|u - w| = K + 30$. As $v \in B_{K+20}(w)$, the site u comes before v on P , so $u \neq z$ and $|u| < L$.

Following [BBR], we build a path P' from u to S_L through w . First go from u to w , reducing the distance to w by one at each step, and take all steps that reduce the absolute value of a coordinate before all steps that increase one. Let the last of these steps go from $w - e$ to w , where $e = \varepsilon e_i \in U$. Then step to $w + e$, and from there take steps that increase the absolute value of coordinates other than the i -th, until S_L is reached. The norm along the first segment is at most $\max(|u|, |w|) < L$. Since $|u| < L$ and $|u - w| = K + 30$, at least six of the steps from u to w increase the norm, so the last one does, and $|w + e| = |w| + 1$. Along the second segment the norm increases by one at each step, so P' meets S_L only at its last site, which is reached $K + 20$ steps after w . The distance to w decreases by one at each step of the first segment and increases by one at each step of the second, because each step after $w + e$ moves some coordinate x_j , $j \neq i$, away from 0 , starting from w_j . Hence $P' \setminus \{u\} \subseteq B_{K+29}(w)$. The path P' is induced: sites whose distances to w differ by at least two are not adjacent, and a site x of the first segment other than w satisfies $\varepsilon(x_i - w_i) \leq -1$, while every site x of the second segment other than w satisfies $\varepsilon(x_i - w_i) = 1$.

Let $\hat{\omega} = (\omega' \setminus \Omega') \cup P'$, so that $E(\hat{\omega}, \alpha) = \hat{\omega} \cup Y$. The part of P before u lies in $(\omega' \setminus \Omega') \cup Y$, and followed by P' it gives a path from 0 to S_L in $E(\hat{\omega}, \alpha)$. Suppose that Γ is a path from 0 to S_L in $E(\hat{\omega} \setminus \{w\}, \alpha)$. The sites of $P' \setminus \{u\}$ lie in $B_{K+29}(w)$, so all their neighbours lie in Ω' , where the open sites of $E(\hat{\omega} \setminus \{w\}, \alpha)$ are those of $P' \setminus \{w\}$. Since P' is induced, the second segment without w forms a component that does not contain 0 , and the first segment without u and w is attached to the rest only through u and contains no site of S_L . Hence Γ avoids $\Omega' \setminus \{u\}$. As $u \in P \cap \Omega' \subseteq \omega'$ and $v \in \Omega' \setminus \{u\}$, the path Γ lies in $(\omega' \setminus \{v\}) \cup Y = E(\omega' \setminus \{v\}, \alpha) \subseteq E(\omega \setminus \{v\}, \alpha)$, contradicting the p-pivotality of v in (ω, α) . Thus w is p-pivotal for R_L in $(\hat{\omega}, \alpha)$. Moreover $K + 10 < |w| < L - K - 10$ and $\alpha \cap B_{K+r_E}(w) = \emptyset$. Proposition 2.12, applied at w , gives ω'' that agrees with $\hat{\omega}$ outside $B_K(w)$ and a site $c \in B_K(w)$ that is s-pivotal in (ω'', α) . The configuration ω'' agrees with ω outside $\Omega' \subseteq B_{2K+50}(v)$, and $c \in B_{2K+20}(v)$.

In each case the modification lies in $B_R(v)$ and $c \in B_R(v)$, which is the conclusion of Claim 5. $\square$

Remark 2.14. In Case 2 we follow [BBR] with two precisions. First, [BBR] open the ball B_{r+20} . With ℓ^1 balls this does not suffice: no two sites of S_{r+20} are adjacent, so the outer neighbours of w can be cut off from the rest of the crossing. Opening B_{r+21} repairs this. Second, [BBR] take $w = (r + 19)e_1$, which can be at distance $2r + 29$ from v . Their main case at w needs $\alpha \cap B_{2r+2}(w) = \emptyset$, and $B_{2r+2}(w)$ is then only known to lie in $B_{4r+31}(v)$, which is not contained in

$B_R(v)$, $R = 3r + 100$, once $r > 69$. Taking w nearest to v repairs this. Case 3 follows [BBR, pp. 11–12], with r replaced by K .

3. The two-port theorem in every dimension

In this section the centre is 0. By translation, all statements hold for balls $B_r(c)$, with coordinates taken relative to c and terminals $c \pm me_1$. For a site x with $x_i \neq 0$, the *inward step* of x in coordinate i is $x - \text{sgn}(x_i)e_i$, and it lowers the layer by one.

3.1. Lemma 8 of [BBR] in every dimension

Lemma 3.1 ([BBR, Lemma 8]). *Let $d \geq 2$, $m \geq 1$ and $r \geq m + 10$. Let $a, g \in S_r$ be separated, that is, for some coordinate i there is an integer s with $|s| \leq r - 5$ strictly between a_i and g_i . Then there are a red path starting at a and a green path starting at g with the following properties: one of them ends at me_1 and the other at $-me_1$; all their sites other than a and g lie in B_{r-1} ; all their sites other than these two end sites lie outside B_m ; and no red site equals or is adjacent to a green site.*

[BBR] state and prove this lemma for general d . One step of their proof, the connectivity of the two halves of an annulus, is asserted without proof. Lemma 2.3 supplies it.

Proof. The statement is symmetric under exchanging a and g , so we may assume $a_i > s > g_i$. Put $w = (r - 2)e_i$ and

$$A_+ = (S_{r-1} \cup S_{r-2}) \cap \{x : x_i > s\}, \quad A_- = (S_{r-1} \cup S_{r-2}) \cap \{x : x_i < s\}.$$

Walks below can always be shortened to paths with the same ends and a subset of the sites, so it suffices to build walks.

The annulus. Since $s < r - 2$, Lemma 2.3 with $n = r - 2$ shows that A_+ is connected and contains w . Applying the same lemma to $-s$ and using the symmetry $x \mapsto -x$, the set A_- is connected and contains $-w$.

Entering the annulus. If $a_j \neq 0$ for some $j \neq i$, the inward step of a in coordinate j lies in S_{r-1} and has the same i -th coordinate as a , so it lies in A_+ . Otherwise $a = \pm re_i$, and $a_i > s \geq -(r - 5)$ forces $a = re_i$, whose neighbour $(r - 1)e_i$ lies in A_+ . In the same way g has a neighbour in A_- .

First stage. Join a by a red walk through A_+ to w , and continue it with $(r - 3)e_i$ and $w' = (r - 4)e_i$. Join g by a green walk through A_- to $-w$, and continue it with $-(r - 3)e_i$ and $-w'$. Every red site x so far satisfies $x_i \geq s + 1$, and every green site satisfies $x_i \leq s - 1$: this holds on $A_\pm$ by definition, at a and g by hypothesis, and at $\pm(r - 3)e_i$ and $\pm w'$ because $|s| \leq r - 5$. Hence no red site equals or is adjacent to a green site. Apart from a and g , all first-stage sites lie in layers $r - 1, \dots, r - 4$, and the only first-stage sites in layers $r - 3$ and $r - 4$ are $\pm(r - 3)e_i$ and $\pm w'$.

Second stage. All further sites lie in layers at most $r - 4$. Such a site can be adjacent to a first-stage site only if that site is $\pm w'$, or if it lies in layer $r - 4$ and is adjacent to $\pm(r - 3)e_i$. The

only site of layer $r - 4$ adjacent to $(r - 3)e_i$ is w' , and similarly for $-(r - 3)e_i$. So it suffices to join w' and $-w'$ to the terminals by a red and a green walk without red-green contacts among themselves. If $i = 1$, go along the axis from $w' = (r - 4)e_1$ to me_1 , and from $-w'$ to $-me_1$. If $i \neq 1$, we may assume $i = 2$ after permuting the coordinates $2, \dots, d$, which fixes $\pm me_1$. Join $w' = (r - 4)e_2$ to $(r - 4)e_1$ by the staircase

$$(r - 4)e_2, (r - 5)e_2, e_1 + (r - 5)e_2, e_1 + (r - 6)e_2, \dots, (r - 5)e_1, (r - 4)e_1,$$

whose sites x satisfy $x_1, x_2 \geq 0$ and $x_1 + x_2 \in \{r - 5, r - 4\}$, and then go along the axis to me_1 . The green walk is the reflection of the red one through the origin. In both cases every red second-stage site has nonnegative coordinates and every green one has nonpositive coordinates, and all of them have norm at least m . Consequently a red site x and a green site y satisfy $|x - y| = |x| + |y| \geq 2m \geq 2$.

Location. First-stage sites other than a and g lie in layers $r - 4, \dots, r - 1$, and $r - 4 \geq m + 6$. Second-stage sites lie in layers at most $r - 4$. The staircase lies in layers $r - 5$ and $r - 4$, which exceed m , and the axis segments meet B_m only at $\pm me_1$. $\square$

3.2. Escaping from a non-corner pair

Theorem 3.2 (escape). *Let $d \geq 2$ and $r \geq \max(2d + 1, 20)$. Let $a \neq g$ be sites of S_r that do not form a corner pair. Then there are $k \in \{0, 1, 2, 3\}$ and walks γ_a from a to a site a^* and γ_g from g to a site g^* such that:*

- (E1) $a^*, g^* \in S_{r-k}$, every site of γ_a and γ_g other than a and g lies in the layers $r - k, \dots, r - 1$, and layer $r - k$ is met only by a^* and g^* ;
- (E2) no site of γ_a equals or is adjacent to a site of γ_g ;
- (E3) a^* and g^* are separated in S_{r-k} .

For $k = 0$ the walks are $\gamma_a = (a)$ and $\gamma_g = (g)$. Together with Lemma 3.1, the theorem replaces Lemma 9 of [BBR], which is specific to $d = 3$, by a statement valid in every dimension. Its proof distinguishes five cases.

Notation. Put $\Delta = g - a$. Since $|a| = |g| = r$, the norm $|\Delta|$ is even and at least 2. We use the following bookkeeping rule throughout: when a site of one walk takes a step in coordinate j , its distance to a fixed site y of the other walk grows by one if the step moves its j -th coordinate away from y_j , and drops by one otherwise.

Step 0. If a and g are separated in S_r , take $k = 0$. From now on we assume that they are not. If $|\Delta_j| \geq 2$ for some j , the integers strictly between a_j and g_j form a nonempty interval that avoids $[-(r - 5), r - 5]$. Hence a_j and g_j have the same sign and $|a_j|, |g_j| \geq r - 5$. Since $r \geq 20$, each of a and g has at most one coordinate of absolute value at least $r - 7$.

Type II: every coordinate of a and of g has absolute value at most $r - 7$. By Step 0, $|\Delta_j| \leq 1$ for every j , and neither a nor g is a corner of S_r . Call $j \in \text{supp } \Delta$ *good* if a_j and g_j are both nonzero. They then have the same sign and differ by one.

Case II-A: a good j exists. Let X be the one of a, g with the smaller $|X_j|$ and Y the other, so that $Y_j = X_j + \text{sgn}(X_j)$. Put $X' = X - \text{sgn}(X_j)e_j$. Since Y is not a corner, it has a nonzero coordinate $i \neq j$, and we put $Y' = Y - \text{sgn}(Y_i)e_i$. When $|\Delta| = 2$ and $\text{supp } \Delta = \{j, l\}$, we choose $i \notin \{j, l\}$ if Y has such a nonzero coordinate, and $i = l$ otherwise. Let $\gamma_X = (X, X')$ and $\gamma_Y = (Y, Y')$, so $k = 1$.

The step from X to X' moves X_j away from Y_j , so $|X' - Y| = |\Delta| + 1$. The sites X' and Y' have j -th coordinates $X_j - \text{sgn } X_j$ and $X_j + \text{sgn } X_j$, so $|X' - Y'| \geq 2$. If $|\Delta| \geq 4$, then $|X - Y'| \geq |\Delta| - 1 \geq 3$. Let $|\Delta| = 2$ with $\text{supp } \Delta = \{j, l\}$. If $i \notin \{j, l\}$, then $X_i = Y_i \neq 0$, the step moves Y_i away from X_i , and $|X - Y'| = 3$. Otherwise $\text{supp } Y = \{j, l\}$ and $i = l$. If the step moved Y_l towards X_l , we would have $|X_l| = |Y_l| - 1$. Together with $|X_j| = |Y_j| - 1$ and $X = Y = 0$ outside $\{j, l\}$ this gives $|X| = |Y| - 2$, which is false. So again $|X - Y'| = 3$, and (E2) holds. Both X' and Y' lie in layer $r - 1$, which gives (E1). For (E3), the integer $s = X_j$ lies strictly between X'_j and $Y'_j = Y_j$, and $|s| \leq r - 7 \leq (r - 1) - 5$.

Case II-B: no good j exists. Then every $j \in \text{supp } \Delta$ has $\{a_j, g_j\} = \{0, \pm 1\}$. If $|a_j| \leq 1$ also held for every $j \notin \text{supp } \Delta$, we would get $r = |a| \leq d$, contrary to $r \geq 2d + 1$. So some $j \notin \text{supp } \Delta$ has $|a_j| \geq 2$, and there $a_j = g_j =: x$. Put $\sigma = \text{sgn } x$ and $\gamma_a = (a, a - \sigma e_j, a - 2\sigma e_j)$. Let $\gamma_g = (g, g_1, g_2)$ consist of two successive inward steps of g in coordinates other than j , chosen as follows. If $|\Delta| = 2$, then $\text{supp } \Delta = \{j_1, j_2\}$, where (a, g) takes the values $(0, \pm 1)$ in one of these coordinates and $(\pm 1, 0)$ in the other, because $|a| = |g|$. We take both steps in coordinates outside $\text{supp } \Delta \cup \{j\}$, where g has total mass $r - |x| - 1 \geq 6$. If $|\Delta| \geq 4$, we take them in any coordinates other than j , where g has mass $r - |x| \geq 7$. Here $k = 2$.

Write $a_s = a - s\sigma e_j$. The steps of a move its j -th coordinate away from x , which is the j -th coordinate of every g_t , so $|a_s - g_t| = |a - g_t| + s$. If $|\Delta| = 2$, the steps of g are in coordinates where a and g agree and a never moves, so $|a - g_t| = 2 + t$. If $|\Delta| \geq 4$, then $|a - g_t| \geq |\Delta| - t \geq 2$. This proves (E2). Since $|x| \geq 2$, the sites a_1, g_1 lie in layer $r - 1$ and a_2, g_2 in layer $r - 2$, which gives (E1). For (E3), the j -th coordinates of a_2 and g_2 are $x - 2\sigma$ and x , and $s = x - \sigma$ satisfies $|s| = |x| - 1 \leq r - 8 \leq (r - 2) - 5$.

Type I: some coordinate p has $\max(|a_p|, |g_p|) \geq r - 6$. Then a_p and g_p have a common sign σ_p and $|a_p|, |g_p| \geq r - 7$: this follows from Step 0 if $|\Delta_p| \geq 2$, and directly if $|\Delta_p| \leq 1$. Write **down** $x = x - \sigma_p e_p$. Each down step below is applied to a site with $|x_p| \geq (r - 7) - 3 \geq 10$, so it is an inward step. The coordinates $j \neq p$ are called *small*. The small parts of a and g have norm at most $7 < r - 5$, so $|a_j - g_j| \leq 1$ for every small j , by Step 0. A small $j \in \text{supp } \Delta$ is *good* if a_j and g_j are both nonzero.

Case I-A: a good small j exists. Define X, Y and X' as in Case II-A, put $Y' = \text{down } Y$, and let $\gamma_X = (X, X')$ and $\gamma_Y = (Y, Y')$, so $k = 1$. As in Case II-A, $|X' - Y| = |\Delta| + 1$, and X' and Y' differ by two in coordinate j . The down step moves Y_p towards X_p only if $|X_p| < |Y_p|$. If $|\Delta| = 2$ and $\text{supp } \Delta = \{j, p\}$, then $|X_j| = |Y_j| - 1$ forces $|X_p| = |Y_p| + 1$. If $|\Delta| = 2$ and $\text{supp } \Delta = \{j, l\}$ with $l \neq p$, then $X_p = Y_p$. So $|X - Y'| = 3$ if $|\Delta| = 2$, and $|X - Y'| \geq |\Delta| - 1 \geq 3$ otherwise. This gives (E2), and (E1) is clear. (E3) holds in coordinate j with $s = X_j$, since $|s| \leq 7 \leq (r - 1) - 5$.

Case I-B1: no good small coordinate exists, but some small j has $a_j = g_j$ and $|a_j| \geq 2$. Let X be one of a, g with $|X_p| \geq |Y_p|$, where Y is the other, and put $\sigma_j = \text{sgn } a_j$. Let $\gamma_X = (X, X - \sigma_j e_j, X - 2\sigma_j e_j)$ and $\gamma_Y = (Y, \text{down } Y, \text{down}^2 Y)$, so $k = 2$. The walk γ_X moves only in coordinate j , where $X_j = Y_j$ and γ_Y never moves. The walk γ_Y moves only in coordinate p , away from X_p because $|X_p| \geq |Y_p|$, and γ_X never moves in p . By the bookkeeping rule, the s -th site of γ_X and the t -th site of γ_Y are at distance $|\Delta| + s + t \geq 2$. Both walks visit the layers $r, r-1, r-2$ in this order. (E3) holds in coordinate j with $s = a_j - \sigma_j$, since $|s| \leq 6 \leq (r-2) - 5$.

Case I-B2: all other cases. Every small coordinate takes values in $\{-1, 0, 1\}$ on a and g , and the two values agree whenever both are nonzero. Indeed, if $a_j \neq g_j$ then one of the values is 0 (as j is not good) and the other is ± 1 . If $a_j = g_j$, then $|a_j| \leq 1$ (as we are not in Case I-B1). Let Q be the number of small coordinates in which exactly one of a_j, g_j is nonzero. Then $Q \geq 1$, since otherwise the small parts agree, so $|a_p| = |g_p|$, $a_p = g_p$ and $a = g$. Choose $X \in \{a, g\}$ and a small i with $X_i =: \tau \neq 0 = Y_i$, where Y is the other port. Let

$$\gamma_X = (X, \text{down } X, \text{down}^2 X, X_3, \text{down } X_3, \text{down}^2 X_3), \quad X_3 = \text{down}^2 X + \tau e_i.$$

Its layers are $r, r-1, r-2, r-1, r-2, r-3$. The site X_3 is back in layer $r-1$ because $|2\tau| = |\tau| + 1$. We write its sites as $X_s = X - c_s \sigma_p e_p + \ell_s \tau e_i$ for $s = 0, \dots, 5$, with

$$(c_s, \ell_s) = (0, 0), (1, 0), (2, 0), (2, 1), (3, 1), (4, 1).$$

- If $Q \geq 2$, let $\gamma_Y = (Y, \text{down } Y, \text{down}^2 Y, \text{down}^3 Y)$, with layers $r, r-1, r-2, r-3$. The small parts of X_s and of any site of γ_Y differ by one in each of the $Q-1$ coordinates other than i where exactly one value is nonzero, and by $1 + \ell_s$ in coordinate i , which moves from τ to 2τ , away from $Y_i = 0$. Their distance is therefore at least $Q + \ell_s \geq 2$.
- If $Q = 1$, then the small part of X is that of Y plus τe_i , so $|X_p| = |Y_p| - 1$, that is, $X_p = Y_p - \sigma_p$. The site Y is not a corner of S_r : otherwise $Y = r\sigma_p e_p$ and $X = (r-1)\sigma_p e_p + \tau e_i$, and $\{X, Y\}$ would be a corner pair. (This is the only use of the hypothesis that $\{a, g\}$ is not a corner pair.) So Y has a small coordinate j with $Y_j \neq 0$, and then $j \neq i$, $|Y_j| = 1$ and $X_j = Y_j$. Let $\gamma_Y = (Y, Y_1, Y_2, Y_3)$ with $Y_t = Y - Y_j e_j - (t-1)\sigma_p e_p$ for $t \geq 1$. Its layers are $r, r-1, r-2, r-3$, and

$$|X_s - Y| = (1 + \ell_s) + (1 + c_s) \geq 2, \quad |X_s - Y_t| = (1 + \ell_s) + 1 + |t-2-c_s| \geq 2 \quad (t = 1, 2, 3).$$

In both sub-cases $k = 3$, (E2) holds, and (E1) follows from the lists of layers. (E3) holds in coordinate i : the end values are 2τ and 0, and $s = \tau$ satisfies $|s| = 1 \leq (r-3) - 5$.

Type I and Type II are complementary, Type II is split according to whether a good coordinate exists, and Type I into the three cases above. So the five cases are exhaustive, and Theorem 3.2 is proved. $\square$

3.3. The two-port theorem

Corollary 3.3 (two-port theorem). *Let $d \geq 2$, $m \geq 1$, $r \geq \max(2d+1, m+13, 20)$ and $c \in \mathbb{Z}^d$, and let $a \neq g$ be sites of $S_r(c)$ that do not form a corner pair. Then there are sets $R, G \subseteq B_r(c)$*

satisfying (D1)–(D4) of Definition 2.9 for $D = B_r(c)$. In particular (a, g) has an interior design in $B_r(c)$.

Proof. We may take $c = 0$. Theorem 3.2 gives k , γ_a and γ_g . Since $r - k \geq r - 3 \geq m + 10$, Lemma 3.1 applies to the separated pair (a^*, g^*) in S_{r-k} . It gives a walk λ_a from a^* and a walk λ_g from g^* that end at the two terminals $\pm me_1$. Let R be the set of sites of γ_a and λ_a , and G the set of sites of γ_g and λ_g .

(D1) is clear. (D2): the sites of γ_a, γ_g other than a, g lie in layers at most $r - 1$ by (E1), and the sites of λ_a, λ_g other than a^*, g^* lie in B_{r-k-1} . (D3): the sites of γ_a and γ_g lie in layers at least $r - 3 \geq m + 10$, and the sites of λ_a, λ_g other than $\pm me_1$ lie outside B_m . (D4): a contact between a red and a green site would be one of three kinds. A contact between γ_a and γ_g is excluded by (E2). A contact between λ_a and λ_g , including a^* and g^* , is excluded by Lemma 3.1. The remaining kind is between a site of $\gamma_a \setminus \{a^*\}$ and a site of $\lambda_g \setminus \{g^*\}$, or the same with the colours exchanged. The first lies in a layer at least $r - k + 1$ by (E1), the second in a layer at most $r - k - 1$, so they are neither equal nor adjacent. $\square$

Remark 3.4.

1. The constants are not optimal. The hypothesis $r \geq 2d + 1$ is used only in Case II-B, where $r \geq d + 1$ suffices, and Lemma 3.1 holds for $r \geq m + 6$.
2. Corner pairs are the only exceptions, and they admit no interior design (Definition 2.11). For $d = 3$ they are exactly the pairs excluded in [BBR, Lemma 9], a corner of S_r and a site of S_r at distance 2 from it. The difficulty of [BBR, §5] for $d = 3$, and of Conjecture 7 for $d \geq 4$, is the case where the first sites of the two halves of the path in the fixed ball $B_r(v)$, that is, the ports of $B_r(v)$, form a corner pair. Section 4 shows instead that some other admissible ball avoids it.

4. Proof of the Corner Lemma

4.1. Ordered sets and corner-paired balls

Let $V \subseteq \mathbb{Z}^d$ and let $\prec$ be a strict linear order on V . For a ball B that meets V , let $a(B)$ and $g(B)$ be the $\prec$ -first and $\prec$ -last sites of $V \cap B$, and call

$$W(B) = \{z \in V : a(B) \preceq z \preceq g(B)\}$$

the *window* of B . It contains $V \cap B$, and it is an interval of $(V, \prec)$. A ball $B = B_r(c)$ with $r \geq 1$ is *corner-paired* if $|V \cap B| \geq 2$ and $\{a(B), g(B)\}$ is a corner pair of $S_r(c)$. For $r \geq 2$ its pole $\sigma_r(c)$ and its apex $c + r\sigma_r(c)$ are then well defined (Definition 2.11). Fix an integer $\rho_0 \geq 2$. We consider two properties of $(V, \prec)$:

- **(Deg)** every site of V has exactly two neighbours in V ;
- **(Cor)** every ball $B_\rho(c)$ with $\rho \geq \rho_0$ and $|V \cap B_\rho(c)| \geq 3$ is corner-paired.

If P is a path as in $\text{CL}(d)$, ordered from one end to the other, and the conclusion of $\text{CL}(d)$ fails for P , then (Cor) holds for the balls with $B_{\rho+1}(c) \subseteq B_K(v)$. Moreover each site of $P \cap B_K(v)$ has exactly two neighbours in P , namely its predecessor and its successor, because P is induced and its ends lie outside $B_K(v)$. The proof of $\text{CL}(d)$ shows that these two properties cannot hold on all of $\mathbb{Z}^d$ in a stationary way (Theorem 4.7).

4.2. The stationary limit

Let $\Xi = \{0, 1\}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ with the product topology, a compact metrizable space. We encode a pair $(V, \prec)$ by the indicator functions of V and of $\{(x, y) : x \prec y\}$. For $z \in \mathbb{Z}^d$ let T_z be the translation by z : $T_z(V, \prec) = (V + z, \prec_z)$, where $x \prec_z y$ if and only if $x - z \prec y - z$. Let $\mathcal{X}_{\rho_0} \subseteq \Xi$ be the set of pairs for which $\prec$ is a strict linear order on V and (Deg) and (Cor) hold.

Each condition in this definition involves finitely many coordinates. The order axioms ($x \prec y$ only for $x, y \in V$; irreflexivity; transitivity; totality on V) involve at most three sites at a time. (Deg) at x involves $V \cap N[x]$. (Cor) at (c, ρ) can be stated without reference to first sites: if $|V \cap B_\rho(c)| \geq 3$, there are $x_-, x_+ \in V$ forming a corner pair of $S_\rho(c)$ with $x_- \preceq z \preceq x_+$ for all $z \in V \cap B_\rho(c)$. This involves only the coordinates indexed by $B_\rho(c)$ and $B_\rho(c)^2$. Each condition therefore defines a clopen set, and $\mathcal{X}_{\rho_0}$ is closed and translation-invariant.

Proposition 4.1. *Let $d \geq 2$. If $\text{CL}(d)$ fails, then for some $\rho_0 \geq 2$ there is a translation-invariant Borel probability measure ν on Ξ with $\nu(\mathcal{X}_{\rho_0}) = 1$, under which V is almost surely infinite.*

Proof. Suppose that $\text{CL}(d)$ fails for ρ_0 . As $\text{CL}(d)$ is invariant under translations, for every K there is an induced path P_K through 0 , with both ends outside B_K and order $\prec_K$ from one end to the other, such that every ball $B_\rho(c)$ with $\rho \geq \rho_0$, $B_{\rho+1}(c) \subseteq B_K$ and $|P_K \cap B_\rho(c)| \geq 3$ is corner-paired for $(P_K, \prec_K)$.

Compactness. For each R the restriction of $(P_K, \prec_K)$ to B_R takes finitely many values. By a diagonal argument there is a sequence $K_j \rightarrow \infty$ along which the restriction to B_R is eventually constant, for every R . The eventual restrictions are consistent, and their union is a pair $(V_0, \prec_0)$ with $0 \in V_0$. Each order axiom involves at most three sites, all inside some B_R , so $\prec_0$ is a strict linear order on V_0 .

(Deg) and (Cor) for $(V_0, \prec_0)$. Let $x \in V_0$. For large j we have $|x| + 1 \leq K_j$ and $P_{K_j} \cap B_{|x|+1} = V_0 \cap B_{|x|+1}$. Since $x \in B_{K_j}$ is not an end of P_{K_j} , it has a predecessor and a successor, and as P_{K_j} is induced these are its only neighbours in P_{K_j} . Hence x has exactly two neighbours in V_0 . Next fix c and $\rho \geq \rho_0$ with $|V_0 \cap B_\rho(c)| \geq 3$. For large j we have $B_{\rho+1}(c) \subseteq B_{K_j}$, and the restrictions of $(P_{K_j}, \prec_{K_j})$ and $(V_0, \prec_0)$ to $B_\rho(c)$ coincide. By the choice of P_{K_j} the ball is corner-paired for P_{K_j} , hence for $(V_0, \prec_0)$. So $(V_0, \prec_0) \in \mathcal{X}_{\rho_0}$, and so are all its translates.

Averaging. Let $\mu_n = |B_n|^{-1} \sum_{x \in B_n} \delta_{T_x(V_0, \prec_0)}$, and let μ be a weak limit point of (μ_n) , along a subsequence (n_k) . This is the averaging argument of Krylov and Bogolyubov [KB], with ℓ^1 balls as averaging sets. For continuous f on Ξ and $e \in U$ we have

$$\left| \int f \circ T_e d\mu_n - \int f d\mu_n \right| \leq \|f\|_\infty \frac{|B_n \triangle (B_n - e)|}{|B_n|} \longrightarrow 0,$$

since $|B_n \triangle (B_n - e)| = O(n^{d-1})$. Hence μ is translation-invariant. As $\mathcal{X}_{\rho_0}$ is closed and $\mu_n(\mathcal{X}_{\rho_0}) = 1$ for all n , the portmanteau theorem gives $\mu(\mathcal{X}_{\rho_0}) = 1$.

Density. The event $\{0 \in V\}$ is clopen, and $\mu_n(0 \in V) = |V_0 \cap B_n|/|B_n|$. Hence $\mu(0 \in V) = \lim_k |V_0 \cap B_{n_k}|/|B_{n_k}|$. We bound this from below using (Deg) and (Cor) at the site $0 \in V_0$. Let $\rho \geq \rho_0$ and $|c| \leq \rho - 1$. Then $B_\rho(c)$ contains 0 and the two neighbours of 0 in V_0 , so it is corner-paired by (Cor). Its apex $c + \rho\sigma_\rho(c)$ is a site of V_0 , and it lies in $B_{2\rho-1}$. The map $c \mapsto (c + \rho\sigma_\rho(c), \sigma_\rho(c))$ is injective, and there are $2d$ possible poles, so $|V_0 \cap B_{2\rho-1}| \geq |B_{\rho-1}|/(2d)$. For $n \geq 2\rho_0 - 1$ put $\rho = \lfloor (n+1)/2 \rfloor \geq \rho_0$. Then $2\rho - 1 \leq n$, and

$$\frac{|V_0 \cap B_n|}{|B_n|} \geq \frac{|B_{\rho-1}|}{2d|B_n|} \longrightarrow \frac{2^{-d}}{2d} \quad (n \rightarrow \infty).$$

So $\mu(0 \in V) \geq 2^{-d}/(2d) > 0$, and in particular $\mu(V \neq \emptyset) > 0$.

Conditioning. Let $\nu = \mu(\cdot \mid V \neq \emptyset)$. The event $\{V \neq \emptyset\}$ is invariant, so ν is invariant and $\nu(\mathcal{X}_{\rho_0}) = 1$. On the event $\{0 < |V| < \infty\}$ let ℓ be the lexicographically least site of V . Then $\ell \circ T_z = \ell + z$, so by invariance $\nu(\ell = z) = \nu(\ell = 0)$ for all z . These numbers sum to $\nu(0 < |V| < \infty) \leq 1$, so they all vanish, and V is ν -almost surely infinite. $\square$

4.3. Mass transport

A *covariant partial map* assigns to each $\xi \in \Xi$ a map φ_ξ from a set $\text{dom } \varphi_\xi \subseteq \mathbb{Z}^d$ to $\mathbb{Z}^d$, measurably in ξ , such that $\text{dom } \varphi_{T_z \xi} = \text{dom } \varphi_\xi + z$ and $\varphi_{T_z \xi}(x + z) = \varphi_\xi(x) + z$ for all z and x .

Lemma 4.2 (mass transport on $\mathbb{Z}^d$). *Let ν be a translation-invariant probability measure on Ξ .*

1. *If $F : \mathbb{Z}^d \times \mathbb{Z}^d \times \Xi \rightarrow [0, \infty]$ is measurable and $F(x + z, y + z, T_z \xi) = F(x, y, \xi)$ for all x, y, z and ξ , then $\mathbb{E}_\nu \sum_y F(0, y, \xi) = \mathbb{E}_\nu \sum_x F(x, 0, \xi)$.*
2. *If φ is a covariant partial map and φ_ξ is injective for ν -almost every ξ , then $\nu(0 \in \varphi_\xi(\text{dom } \varphi_\xi)) = \nu(0 \in \text{dom } \varphi_\xi)$.*

Proof. (1) For each y , covariance with $z = -y$ and invariance give $\mathbb{E} F(0, y, \xi) = \mathbb{E} F(-y, 0, T_{-y} \xi) = \mathbb{E} F(-y, 0, \xi)$. Sum over y . (2) Apply (1) to $F(x, y, \xi) = \mathbf{1}[x \in \text{dom } \varphi_\xi, \varphi_\xi(x) = y]$. The left side is $\nu(0 \in \text{dom } \varphi_\xi)$. The right side is the expected number of preimages of 0 , which is almost surely at most one and equals the indicator of $0 \in \varphi_\xi(\text{dom } \varphi_\xi)$. $\square$

Part (1) is the mass-transport principle of Benjamini, Lyons, Peres and Schramm [BLPS] (see also [LP, Chapter 8]) in its simplest case, the group $\mathbb{Z}^d$.

4.4. Shrinking towards the apex

The next lemma is deterministic and holds for every pair $(V, \prec)$. For $R \geq 2$ let $\mathcal{D}_R$ be the set of sites x such that $B_R(x)$ is corner-paired. For $R \geq r \geq 2$ and $x \in \mathcal{D}_R$ put

$$\Psi_{R \rightarrow r}(x) = x + (R - r)\sigma_R(x).$$

Since $\mathcal{D}_R$ and $\sigma_R(x)$ are determined by the restriction of $(V, \prec)$ to $B_R(x)$, the map $\Psi_{R \rightarrow r}$ is a covariant partial map with domain $\mathcal{D}_R$.

Lemma 4.3. *Let $R \geq r \geq 2$ and $x \in \mathcal{D}_R$, with pole σ , apex $C = x + R\sigma$ and partner $N = x + (R - 1)\sigma + \delta$. Put $y = \Psi_{R \rightarrow r}(x)$.*

1. $B_r(y) \subseteq B_R(x)$, and $B_r(y)$ contains C and N . Consequently $B_r(y)$ has the same first and last sites and the same window as $B_R(x)$, and it is corner-paired with pole σ and apex C .
2. $\Psi_{R \rightarrow r}$ is injective on $\mathcal{D}_R$.

Proof. (1) If $|z - y| \leq r$ then $|z - x| \leq r + (R - r) = R$. The vectors $C - y = r\sigma$ and $N - y = (r - 1)\sigma + \delta$ have norm r . The first site of $V \cap B_R(x)$ belongs to the smaller set $V \cap B_r(y)$, so it is also the first site of $V \cap B_r(y)$, and the same holds for the last site. Hence the windows coincide, and $\{C, N\} = \{y + r\sigma, y + (r - 1)\sigma + \delta\}$ is a corner pair of $\mathcal{S}_r(y)$ with pole σ .

(2) If $\Psi_{R \rightarrow r}(x) = y$, then by (1) the ball $B_r(y)$ is corner-paired with pole $\sigma_R(x)$. As $r \geq 2$, the pole of $B_r(y)$ is unique, so $\sigma_R(x) = \sigma_r(y)$ and $x = y - (R - r)\sigma_r(y)$ is determined by y . $\square$

For $r = 1$ the pole is not unique and injectivity can fail, which is why all radii below are at least 2.

Proposition 4.4. *Let ν be a translation-invariant probability measure on Ξ with $\nu(\mathcal{X}_{\rho_0}) = 1$ under which V is almost surely infinite. Then ν -almost surely, for every $y \in \mathbb{Z}^d$ and every $r \geq 2$, the ball $B_r(y)$ is corner-paired, and for every $R \geq r$ the ball $B_R(y - (R - r)\sigma_r(y))$ is corner-paired with the same window and the same apex $y + r\sigma_r(y)$ as $B_r(y)$.*

Proof. Fix $r \geq 2$, and for $R \geq r$ let $\mathcal{E}_R$ be the event $\{0 \in \Psi_{R \rightarrow r}(\mathcal{D}_R)\}$.

First let $R \geq \max(r, \rho_0)$, and let A_R be the set of sites x with $|V \cap B_R(x)| \geq 3$. By (Cor), $A_R \subseteq \mathcal{D}_R$ almost surely. Since V is almost surely infinite, $\beta_R := \nu(0 \in A_R)$ tends to 1 as $R \rightarrow \infty$. By Lemma 4.3(2) and Lemma 4.2(2), $\nu(\mathcal{E}_R) = \nu(0 \in \mathcal{D}_R) \geq \beta_R$.

Next, $\mathcal{E}_{R'} \subseteq \mathcal{E}_R$ whenever $R' \geq R \geq r$. Indeed, let $0 = \Psi_{R' \rightarrow r}(x)$ with $x \in \mathcal{D}_{R'}$ and pole σ . By Lemma 4.3(1), the site $x' = \Psi_{R' \rightarrow R}(x) = x + (R' - R)\sigma$ lies in $\mathcal{D}_R$ with pole σ , and $\Psi_{R \rightarrow r}(x') = x + (R' - r)\sigma = 0$.

Hence $\nu(\mathcal{E}_R) \geq \nu(\mathcal{E}_{R'}) \geq \beta_{R'}$ for every $R' \geq \max(R, \rho_0)$, so $\nu(\mathcal{E}_R) = 1$ for every $R \geq r$, and the events $\mathcal{E}_R$, $R \geq r$, hold simultaneously almost surely. On this event, fix $R \geq r$ and $x \in \mathcal{D}_R$ with $\Psi_{R \rightarrow r}(x) = 0$. By Lemma 4.3(1), $B_r(0)$ is corner-paired with pole $\sigma_R(x)$, and $\sigma_R(x) = \sigma_r(0)$ by uniqueness of the pole. Thus $x = -(R - r)\sigma_r(0)$, and $B_R(x)$ has the same window as $B_r(0)$ and the apex $x + R\sigma_r(0) = r\sigma_r(0)$. This proves the statement for $y = 0$ and the fixed r . By translation invariance it holds for each y almost surely, and a countable intersection over y and r completes the proof. $\square$

4.5. One pole per apex

Lemma 4.5. *Let $(V, \prec)$ be any pair, and let $B_1 = B_{r_1}(c_1)$ and $B_2 = B_{r_2}(c_2)$ be corner-paired balls with $r_1, r_2 \geq 2$, poles $\pi_1 \neq \pi_2$, apexes C_1, C_2 and partners $N_1 = C_1 - \pi_1 + \delta_1$, $N_2 = C_2 - \pi_2 + \delta_2$. Then neither of the following holds:*

- (perpendicular) $\pi_1 \perp \pi_2$ and $C_2 = C_1 + t(\pi_2 - \pi_1)$ for an integer t with $2 \leq t \leq \min(r_1, r_2)$;
- (antipodal) $\pi_2 = -\pi_1 =: -\sigma$ and $C_1 = C_2 + t\sigma$ for an integer t with $2 \leq t \leq \min(r_1, r_2) + 1$.

Proof. We first show that in either situation each ball contains both ports of the other.

In the perpendicular situation $c_1 = C_1 - r_1\pi_1$ and $c_2 = C_2 - r_2\pi_2$, so

$$\begin{aligned} C_2 - c_1 &= (r_1 - t)\pi_1 + t\pi_2, & N_2 - c_1 &= (r_1 - t)\pi_1 + (t - 1)\pi_2 + \delta_2, \\ C_1 - c_2 &= t\pi_1 + (r_2 - t)\pi_2, & N_1 - c_2 &= (t - 1)\pi_1 + (r_2 - t)\pi_2 + \delta_1. \end{aligned}$$

The first and third vectors have norms r_1 and r_2 . Since $\delta_2 \perp \pi_2$, either $\delta_2 = -\pi_1$, or $\delta_2 = \pi_1$, or δ_2 is orthogonal to both poles. The norm of $N_2 - c_1$ is then $|r_1 - t - 1| + t - 1$, r_1 or r_1 , and in all cases at most r_1 . The fourth vector is treated in the same way and has norm at most r_2 .

In the antipodal situation $c_1 = C_1 - r_1\sigma$, $c_2 = C_2 + r_2\sigma$, $N_1 = C_1 - \sigma + \delta_1$ and $N_2 = C_2 + \sigma + \delta_2$, so

$$C_2 - c_1 = (r_1 - t)\sigma, \quad N_2 - c_1 = (r_1 - t + 1)\sigma + \delta_2, \quad C_1 - c_2 = (t - r_2)\sigma, \quad N_1 - c_2 = (t - 1 - r_2)\sigma + \delta_1.$$

Since $2 \leq t \leq \min(r_1, r_2) + 1$, their norms $|r_1 - t|$, $r_1 - t + 2$, $|t - r_2|$ and $r_2 - t + 2$ are at most r_1 , r_1 , r_2 and r_2 .

So $C_2, N_2 \in B_1$ and $C_1, N_1 \in B_2$. The ports of B_2 are sites of $V \cap B_1 \subseteq W(B_1)$, and $W(B_1)$ is an interval of $(V, \prec)$, so $W(B_2) \subseteq W(B_1)$. By symmetry $W(B_1) = W(B_2)$, so the two balls have the same first and last sites, and $\{C_1, N_1\} = \{C_2, N_2\}$. Since $t \neq 0$ we have $C_1 \neq C_2$, hence $C_1 = N_2$ and $C_2 = N_1$. In the perpendicular situation, $C_2 - C_1 = N_1 - C_1 = -\pi_1 + \delta_1$ has π_1 -coefficient -1 , while $t(\pi_2 - \pi_1)$ has π_1 -coefficient $-t$, so $t = 1$, a contradiction. In the antipodal situation, $C_1 - C_2 = N_2 - C_2 = \sigma + \delta_2$ would equal $t\sigma$, which is impossible because $\delta_2 \perp \sigma$. $\square$

The hypothesis $t \geq 2$ cannot be dropped in the perpendicular situation. For $t = 1$ the two balls can have the same port pair $\{C_1, C_2\}$. For $t = 0$ a single site can be the apex for both poles, with the common port pair $\{C, C - \pi_1 - \pi_2\}$. So Lemma 4.5 alone does not exclude two poles at one apex, and the next lemma uses stationarity.

Lemma 4.6. *Let ν be as in Proposition 4.4. Then ν -almost surely no site is the apex of two corner-paired balls, of radii at least 2, with different poles.*

Proof. We work on the full-measure event of Proposition 4.4. Let C be the apex of a corner-paired ball $B_r(y)$ with pole π and $r \geq 2$. Then for every $r' \geq 2$ the ball $B_{r'}(C - r'\pi)$ is corner-paired with pole π and apex C : for $r' \geq r$ this is Proposition 4.4, since $y - (r' - r)\pi = C - r'\pi$, and for $2 \leq r' < r$ it is Lemma 4.3(1). The set $A(\pi)$ of apexes with pole π therefore does not depend on the radius, and it is measurable.

Fix poles $\pi_1 \neq \pi_2$. If $\pi_1 \perp \pi_2$ put $u = \pi_2 - \pi_1$, and if $\pi_2 = -\pi_1 =: -\sigma$ put $u = \sigma$. For a line $\ell = q + \mathbb{Z}u$ and $k = 1, 2$ let $J_k(\ell) = \{t \in \mathbb{Z} : q + tu \in A(\pi_k)\}$. We claim that for all $t_1 \in J_1(\ell)$ and $t_2 \in J_2(\ell)$,

$$t_2 \leq t_1 + 1 \text{ in the perpendicular case,} \quad t_1 \leq t_2 + 1 \text{ in the antipodal case.}$$

In the perpendicular case, suppose $t = t_2 - t_1 \geq 2$ and put $C_k = q + t_k u$. Then $C_2 = C_1 + t(\pi_2 - \pi_1)$, and the balls $B_t(C_k - t\pi_k)$ are corner-paired with poles π_k and apexes C_k . This contradicts the perpendicular case of Lemma 4.5 with $r_1 = r_2 = t$. In the antipodal case, $t = t_1 - t_2 \geq 2$ gives $C_1 = C_2 + t\sigma$, and the balls of radius t contradict the antipodal case of Lemma 4.5. The constraint is one-sided, and on a single line it can be met, for instance by $J_1 = [-1, \infty)$ and $J_2 = (-\infty, 0]$ in the perpendicular case. Translation invariance excludes it.

We treat the perpendicular case. The antipodal case is the same with J_1 and J_2 exchanged. Let $\ell_0 = \mathbb{Z}u$, and let $\mathcal{E}$ be the event that $J_1(\ell_0)$ and $J_2(\ell_0)$ are both nonempty. On $\mathcal{E}$ the claim gives $\min J_1(\ell_0) \geq t_2 - 1$ for any $t_2 \in J_2(\ell_0)$, so $J^* = \min J_1(\ell_0)$ is a finite integer. By covariance $A(\pi_k)$ is shifted by u under T_u , so $J_k(\ell_0) \circ T_u = J_k(\ell_0) + 1$. Hence $\mathcal{E}$ is T_u -invariant and $J^* \circ T_u = J^* + 1$. By invariance of ν the numbers $\nu(\mathcal{E} \cap \{J^* = j\})$, $j \in \mathbb{Z}$, are all equal. They sum to $\nu(\mathcal{E}) \leq 1$, so they vanish and $\nu(\mathcal{E}) = 0$. By translation invariance, almost surely no line $q + \mathbb{Z}u$ has both J_1 and J_2 nonempty. A site in $A(\pi_1) \cap A(\pi_2)$ would lie on such a line, with $0 \in J_1 \cap J_2$ when q is that site. A union over the finitely many pairs (π_1, π_2) completes the proof. $\square$

4.6. Counting, and the proof of the Corner Lemma

Theorem 4.7. *Let $d \geq 2$ and $\rho_0 \geq 2$. There is no translation-invariant probability measure ν on Ξ with $\nu(\mathcal{X}_{\rho_0}) = 1$ under which V is almost surely nonempty.*

Proof. Suppose that ν is such a measure. The last step of the proof of Proposition 4.1 shows that V is ν -almost surely infinite, so Proposition 4.4 and Lemma 4.6 apply. Fix $r \geq 2$. On the full-measure event of Proposition 4.4 and Lemma 4.6, every site c is the centre of a corner-paired ball $B_r(c)$, whose apex $\mathcal{C}(c) = c + r\sigma_r(c)$ is one of its two ports and hence a site of V . The map $\mathcal{C}$ is injective. Indeed, if $\mathcal{C}(c_1) = \mathcal{C}(c_2)$ with $c_1 \neq c_2$, then $\sigma_r(c_1) \neq \sigma_r(c_2)$, because $c_k = \mathcal{C}(c_k) - r\sigma_r(c_k)$, and the common apex would carry two poles, contrary to Lemma 4.6. Defined on $\mathcal{D}_r$ by the same formula, $\mathcal{C}$ is a covariant partial map, and almost surely $\mathcal{D}_r = \mathbb{Z}^d$ and $\mathcal{C}$ is injective. Lemma 4.2(2) then gives $\nu(0 \in \mathcal{C}(\mathcal{D}_r)) = \nu(0 \in \mathcal{D}_r) = 1$. As $\mathcal{C}(\mathcal{D}_r) \subseteq V$, we get $\nu(0 \in V) = 1$, and by translation invariance $V = \mathbb{Z}^d$ almost surely. Then every site of V has $2d \geq 4$ neighbours in V , contradicting (Deg). $\square$

Theorem 4.8 (the Corner Lemma). *CL(d) holds for every $d \geq 2$.*

Proof. If CL(d) failed for some ρ_0 , Proposition 4.1 would give a measure that Theorem 4.7 excludes. $\square$

4.7. Proof of Theorem 1.1

Let $d \geq 2$ and let E_0 be an essential enhancement of site percolation on $\mathbb{Z}^d$. Let m be as in Lemma 2.2 and $\rho_0 = \max(2d + 1, m + 13, 20)$. By Theorem 4.8, CL(d) holds. By Theorem 2.13,

Claim 5 holds for E_0 , with $R = 3K(d, \rho_0) + 100$ and $L_0 = 100R$. By Theorem 2.1, the conclusion of Conjecture 1 holds for E_0 . $\square$

Formal verification. Claim 5 for every essential enhancement of site percolation on $\mathbb{Z}^d$ and every $d \geq 2$ has been formalized in Lean 4 [dMU] with the Mathlib library [Mat], including the gadget (Lemma 2.2), the deletion step (Lemma 2.5), the two-port theorem (Corollary 3.3) and the Corner Lemma (Theorem 4.8). Theorem 1.1 is derived from it in the same development, with the probabilistic step of Theorem 2.1 as an explicit hypothesis. The proofs use only Lean’s standard axioms `propext`, `Classical.choice` and `Quot.sound`, and every proof is checked by Lean’s kernel. The formal proof of the Corner Lemma does not follow Sections 4.2–4.6: it replaces the stationary limit and mass transport by a finitary counting argument in a large cube, which uses Lemma 4.5 at a single radius and the density bound from the proof of Proposition 4.1.

5. Remarks

5.1. What the proof of Section 4 uses

The degree condition (Deg) enters the proof of the Corner Lemma (Theorem 4.8) in exactly two places: at the single site $\mathbf{0}$ in Proposition 4.1, to make the limit measure nontrivial (without it, $V_0 = \{\mathbf{0}\}$ would satisfy (Cor) vacuously, and the averages would converge to the point mass at $V = \emptyset$), and in the last line of the proof of Theorem 4.7. Proposition 4.4, Lemma 4.6 and the injectivity of the apex map use only translation invariance, (Cor), and the infinitude of V . So Sections 4.4–4.6 prove the following: *if ν is translation-invariant, V is almost surely nonempty, and almost surely $\prec$ is a strict linear order on V satisfying (Cor), then $V = \mathbb{Z}^d$ almost surely.* The degree condition cannot be omitted from Theorem 4.7, as the following example shows.

Example 5.1 (the parity double sweep). Let $V = \mathbb{Z}^d$ and $A = \{z : z_2 + \dots + z_d \text{ even}\}$. Let $\prec$ list A first, by decreasing z_1 , and then the complement A^c , by increasing z_1 . Within a hyperplane $\{z_1 = \text{const}\}$, ties are broken by the lexicographic order of $(z_2, \dots, z_d)$. Then every ball $B_r(c)$ with $r \geq 1$ is corner-paired with pole e_1 (for $r = 1$, e_1 is one of the two admissible poles). Indeed, $c + re_1$ is the only site of $B_r(c)$ with first coordinate $c_1 + r$, and the sites with first coordinate $c_1 + r - 1$ are $c + (r - 1)e_1$ and the $2(d - 1)$ sites $c + (r - 1)e_1 \pm e_j$, $j \geq 2$, whose parity class is opposite to that of $c + (r - 1)e_1$ and $c + re_1$. If $c_2 + \dots + c_d$ is even, then $c + re_1 \in A$ is the first site, and the sites of A^c with the largest first coordinate are the $c + (r - 1)e_1 \pm e_j$, so one of them is the last site. If $c_2 + \dots + c_d$ is odd, these $2(d - 1)$ sites are the sites of A with the largest first coordinate, so one of them is first, and $c + re_1 \in A^c$ is last. In both cases the ports form a corner pair with pole e_1 .

The order is invariant under translations by vectors u with $u_2 + \dots + u_d$ even. Translation by e_2 exchanges A and A^c , and gives a second pair $(V, \prec')$ with the same property. The average $\nu = \frac{1}{2}(\delta_{(V, \prec)} + \delta_{(V, \prec')})$ is translation-invariant, satisfies (Cor) for every $\rho_0 \geq 2$ (indeed at every radius $r \geq 1$), and satisfies the conclusions of Proposition 4.4 and Lemma 4.6 (every apex has pole e_1), as well as $V = \mathbb{Z}^d$. Only (Deg) fails, since every site has $2d$ neighbours in V .

The argument uses the closeness of the two ports of a corner pair. For patterns in which the first and last sites of a ball lie at opposite poles of its sphere, the analogue of Lemma 4.3 fails, because

a ball shrunk towards one pole no longer contains the opposite port.

5.2. Constants

The constants $\rho_0 = \max(2d + 1, m + 13, 20)$, $R = 3K + 100$ and $L_0 = 100R$ are explicit in terms of m and $K = K(d, \rho_0)$, and m is explicit in terms of the gadget of [BBR, §4]. The proof of Section 4 obtains K from a compactness argument and gives no explicit value. Claim 5 only asks for constants that depend on E_0 and d , so this suffices for Theorem 1.1. The finitary proof used in the formalization (Section 4.7) gives an explicit but very large K . A suitable K is also computable in principle. Call a pair $(V, \prec)$ with $V \subseteq B_K$ and $0 \in V$ a K -window if (Deg) holds at every site of V whose neighbours all lie in B_K , and (Cor) holds for every ball $B_\rho(c)$ with $\rho \geq \rho_0$ and $B_{\rho+1}(c) \subseteq B_K$. If CL(d) failed at K for a path P through $v = 0$, then $P \cap B_K$ with its path order would be a K -window. Conversely, if K -windows existed for every K , a diagonal limit would be a pair on $\mathbb{Z}^d$ with $0 \in V$, (Deg) and (Cor), and the averaging step of Proposition 4.1 would produce a measure excluded by Theorem 4.7. So K -windows cease to exist for some finite K , and any such K works in CL(d). For each K the existence of a K -window is a finite question. The values relevant here, with $\rho_0 \geq 20$, are far beyond exhaustive search, and the cross-checks in Appendix A only treat $d = 2$ and $\rho_0 \leq 4$.

5.3. Other lattices

The proof uses the geometry of $\mathbb{Z}^d$ with ℓ^1 balls in several places: the ℓ^1 form of the gadget (Lemma 2.2), the boundary cases of Theorem 2.13, the two-layer connectivity (Lemma 2.3) and Lemma 3.1, the case analysis of Theorem 3.2, and the shape of corner pairs in Section 4 (Lemmas 4.3 and 4.5). For bond percolation on $\mathbb{Z}^d$ and site percolation on the triangular lattice, Conjecture 1 is [BBR, Theorem 3], whose proof is outlined in [BBR, §6]. Beyond these cases and Theorem 1.1, the general case of Conjecture 1 was left open in [BBR]. The structure of the present proof (deletion, recentring, a two-port theorem for a family of balls, and a stationarity argument showing that the exceptional port pairs cannot occur in every admissible ball) may apply more widely, but each step would have to be redone for the lattice at hand.

Appendix A. Computational cross-checks

The computations below are cross-checks, and no step of any proof depends on them. The searches in A.1.1 and A.2.4 do not use the proofs. The programs of A.1.2–A.1.5 follow the constructions of the proofs and check their output against the statements, and those of A.2.1–A.2.3 test the statements directly.

A.1. The two-port theorem.

1. *Exhaustive escape search.* For all non-separated pairs on S_r , up to the symmetries of $\mathbb{Z}^d$, escape walks with $k \leq 4$ and length at most 7 were searched for, without using the case analysis of Theorem 3.2. The pairs were: $d = 2$, $r = 20$ (20 pairs); $d = 3$, $r = 20$ (222 pairs); $d = 4$, $r = 12, 16, 20, 25$ (328, 682, 1244, 2309 pairs); $d = 5$, $r = 13$ (953 pairs). Some of these radii are below the threshold of Theorem 3.2. In every run the only pairs without an escape were the corner pairs, in both orders.

2. *Implementation of the construction.* The case analysis of Theorem 3.2 was applied to every non-separated, non-corner pair up to symmetry. In Case II-A the program took the first step of Y not adjacent to X , and in Case II-B the first pair of steps of g that passed the check of (E1)–(E3), instead of following the rule of the proof (A.1.3 covers every choice the proof allows). The resulting walks were checked against (E1)–(E3): $d = 2, \dots, 9$ at $r = 20$; $d = 10$ at $r = 21 = 2d + 1$; $d = 4$ at $r = 25, 31, 40$; $d = 5$ at $r = 31$; $d = 3$ at $r = 41$. No pair failed. For example, there are 1,071,277 such pairs for $d = 9$ and 3,705,055 for $d = 10$.
3. *Complete designs.* Every escape that the proof of Theorem 3.2 allows (every admissible choice of coordinates and of the port X) was completed by an implementation of Lemma 3.1 at radius $r - k$, for every separating coordinate and both extreme admissible values of s . The resulting sets were checked against (D1)–(D4), with $m = r - 13$, so that Lemma 3.1 runs at radius exactly $m + 10$ when $k = 3$. This was done exhaustively for $d = 2$ ($r = 20$), $d = 3$ (each r from 20 to 40), $d = 4$ (each r from 20 to 30, and $r = 31$), $d = 5$ ($r = 20, 26$) and $d = 6, 7$ ($r = 20$), and on samples of pairs and escapes for $d = 8, 9$ ($r = 20$) and $d = 10, 11, 12$ ($r = 2d + 1$). No design failed.
4. *Lemma 3.1 alone.* Exhaustively up to symmetry for $d = 2$ ($r = 20$) and $d = 3$ ($r = 20, \dots, 23$), and on random samples for $d = 4, 5, 6$, with no failures.
5. *Theorem 2.10 on random paths.* Random induced walks in a box played the role of P . Random balls $B_\rho(c)$ with $\rho \in [\rho_0, \rho_0 + 4]$ and random centres (almost all of them off the walk), meeting the walk in at least three sites and with non-corner ports, had their intersection with the walk replaced by a design. The gadget was modelled by its two terminals, with an axis segment playing the role of X . For 8734, 4500 and 3000 balls in $d = 3, 4, 5$, the two ends of the walk were disconnected without X and connected with X , in every case.

A.2. The Corner Lemma.

1. Lemma 4.3 was checked on all geometric cases for $d = 2, \dots, 5$ and $R \leq 9$, and on random pairs $(V, \prec)$ in boxes (6564 corner-paired balls), with no failures for $r \geq 2$. For $r = 1$ the map Ψ has collisions, as expected.
2. Lemma 4.5 was checked for all ordered pairs of signed poles and all partners, $d = 2, \dots, 5$, $r_1, r_2 \in [2, 9]$ and all t in the stated ranges: 1,760,576 cases, none of which failed.
3. Example 5.1 was checked for $d = 2, 3, 4$: every tested ball is corner-paired with pole e_1 , the conclusions of Proposition 4.4 and Lemma 4.6 hold, and the averaged measure is invariant on random translations.
4. K -windows (Section 5.2) were searched for by a SAT solver for $d = 2$, with a separate checker for every satisfying assignment. The largest K admitting a K -window and the smallest K admitting none were $(4, 5)$ for $\rho_0 = 2$, $(7, 8)$ for $\rho_0 = 3$ and $(9, 10)$ for $\rho_0 = 4$. For $\rho_0 = 2$ an independent backtracking search confirmed that a 4-window exists. The non-existence of 5-, 8- and 10-windows rests on the solver's unsatisfiability answers, which were not checked independently. Subject to this, CL(2) holds with $K = 5, 8, 10$ for $\rho_0 = 2, 3, 4$.

Data availability

The Lean formalization described in Section 4.7 is archived, with instructions for checking it, at <https://doi.org/10.5281/zenodo.23588127>.

Use of AI tools

Generative AI tools were used to find and check the proofs, to write the Lean formalization, and to draft the text. The author is responsible for all statements.

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