

A LOWER BOUND FOR THE BOHR RADIUS OF THE BIDISC

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ABSTRACT. The Bohr radius K_2 of the bidisc is the largest r such that every holomorphic $f(z) = \sum c_\alpha z^\alpha$ on $\mathbb{D}^2$ with $|f| \leq 1$ satisfies $\sum |c_\alpha| r^{|\alpha|} \leq 1$, and its exact value is unknown. In an unrefereed solution posted on MathDB, Patel proved $K_2 > 0.30158758$. His argument combines three inequalities for the coefficients of degrees two and three, one of which comes from the unitary realization of Schur functions on the bidisc, with Knese's bound in the higher degrees. We present his argument in full detail, confirm that it is correct, and check its exact computation. Knese's bound in degree n carries the factor $\sqrt{n}$, and this factor can be replaced by the Sidon constant of $\{0, 1, \dots, n\}$, which is smaller than $\sqrt{n}$ for every $n \geq 5$ by a theorem of Shapiro. We prove the computer-assisted bounds $S(\{0, 1, \dots, 5\}) \leq 2.187$, consistent with the exact value found by Ye et al., and $S(\{0, 1, \dots, 6\}) \leq 2.363$. With them we obtain $K_2 > 0.301657$. The numerical steps use exact rational arithmetic and ball arithmetic.

1. INTRODUCTION

For $d \geq 1$ let $\mathbb{D}^d = \{z \in \mathbb{C}^d : |z_1|, \dots, |z_d| < 1\}$ be the unit polydisc and let the Schur class $\mathcal{S}(\mathbb{D}^d)$ be the set of holomorphic maps $f: \mathbb{D}^d \rightarrow \mathbb{D}$. For $f(z) = \sum_\alpha c_\alpha z^\alpha$ and $r > 0$ write $\|f_r\|_1 = \sum_\alpha |c_\alpha| r^{|\alpha|}$, where $|\alpha| = \alpha_1 + \dots + \alpha_d$. The Bohr radius of $\mathbb{D}^d$ is

$$K_d = \sup\{r > 0 : \|f_r\|_1 \leq 1 \text{ for all } f \in \mathcal{S}(\mathbb{D}^d)\}.$$

Bohr [9] proved that $K_1 \geq 1/6$. The sharp value $K_1 = 1/3$ was found independently by Riesz, Schur and Wiener, and Bohr's paper includes Wiener's proof (see [8]). Boas and Khavinson [8] introduced K_d for $d \geq 2$, proved $1/(3\sqrt{d}) < K_d < 2\sqrt{(\log d)/d}$ and asked for the exact value of K_d when $d > 1$. Defant, Frerick, Ortega-Cerdà, Ounaies and Seip [10] showed that K_d has order $\sqrt{(\log d)/d}$, and Bayart, Pellegrino and Seoane-Sepúlveda [6] proved $K_d \sim \sqrt{(\log d)/d}$ as $d \rightarrow \infty$. No exact value of K_d is known for any $d \geq 2$.

For the bidisc, functions of one variable show $K_2 \leq 1/3$. Knese [13, Corollary 1.2] proved $K_2 \geq 0.3006$: by his Theorem 1.1 and Andô's inequality, K_2 is at least the root $0.3006282\dots$ of $\sum_{k \geq 1} \sqrt{k} r^k = 1/2$ (his corrigendum [14] corrects Theorem 1.4 of [13]). Baran, Pikul, Woerdeman and Wojtylak [5, Theorem 6.4] proved $K_2 < 0.3177$, and Griego [11] lowered this to $K_2 < 0.3174541$.

Patel [17] posted two bounds in an unrefereed solution to a problem on MathDB. For the upper bound he exhibits a rational inner function, and the post describes an exact integer computation, with its outcome, that gives $K_2 < 0.302825279492$. His reproducibility package [18] contains two programs that carry it out and a Lean proof. For the lower bound, a rotation and the substitution $H = (1 - f)/(1 - f(0))$ reduce the problem to the coefficients of holomorphic functions H on $\mathbb{D}^2$ with $\operatorname{Re} H > 0$ and $H(0) = 1$. Patel proves three inequalities for the coefficients of H of degrees two and three. The first two follow from the one-variable Carathéodory inequality on slices, the second with the help of a positive semidefinite Toeplitz form. The third couples the two degrees and comes from the unitary realization of Schur functions on the bidisc [1, 4], which rests on

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Andô's inequality [3]. Using Knese's estimate [13, Lemma 2.1] in degrees $n \geq 4$, he then obtains $K_2 > 0.30158758$ by an exact rational computation.

A second route uses Sidon constants. For a finite set $\Lambda \subset \mathbb{Z}$ the Sidon constant $S(\Lambda)$ is the least C with

$$\sum_{\lambda \in \Lambda} |a_\lambda| \leq C \max_{|w|=1} \left| \sum_{\lambda \in \Lambda} a_\lambda w^\lambda \right| \quad (a_\lambda \in \mathbb{C}).$$

Bernal-González, Cabana, García, Maestre, Muñoz-Fernández and Seoane-Sepúlveda [7, Theorem 3.3] showed that K_d is at least the root of $\sum_{k \geq 1} S(k, d) r^k = 1/2$, where $S(k, d)$ is the Sidon constant of the k -homogeneous polynomials in d variables. We write

$$S_k = S(\{0, 1, \dots, k\}) \quad (k \geq 1).$$

For $d = 2$, $S(k, 2) = S_k$ (Remark 2.4). Newman's bound $S_k \leq \sqrt{k}$ [20, Chapter 3] (see also [16, (1)]) turns this equation into Knese's, and Shapiro [20] showed that $S_k = \sqrt{k}$ exactly when $k \in \{1, 2, 4\}$. Neuwirth conjectured $S_3 = 5/3$ [15] and later proved $S_3 \geq 5/3$ [16]. Yu et al. proved $S_3 \leq 5/3$, and hence $S_3 = 5/3$, with a formalized proof [22], and Ye et al. determined $S_5 = 2.186529771892569 \dots$ in a computer-assisted preprint [21]. With these values and $S_k \leq \sqrt{k}$ for $k \geq 6$, the equation of [7] gives $K_2 \geq 0.301339$ (Corollary 2.5). With the two bounds of Proposition 4.2 in place of the value of Ye et al. and of Newman's bound for S_6 , it gives $K_2 \geq 0.301362$. Patel's coupling of degrees two and three goes beyond this.

In this paper we present Patel's argument with full details and improve its result. His post contains all the steps, and we have also checked his exact computation. Our improvement starts from the following observation. The slice argument behind Patel's inequalities also shows that the coefficients of degree n of H have absolute sum at most $2S_n$ (Lemma 2.3), which is the Carathéodory form of [7, Lemma 3.1]. Patel uses Knese's estimate $2\sqrt{n}$ in the degrees $n \geq 4$, and there $2S_n$ is smaller wherever $S_n < \sqrt{n}$. Since $S_4 = 2 = \sqrt{4}$, the first such degree is $n = 5$, and by Shapiro's theorem every larger degree qualifies. We prove $S_5 \leq 2.187$ and $S_6 \leq 2.363$ with computer-assisted certificates (Proposition 4.2). The exact value of S_5 found by Ye et al. [21] is consistent with the first bound and would serve equally well, but we use our own bound to keep the proof independent of that computation. With these bounds, an exact computation gives the following.

Theorem 1.1. $K_2 > 0.301657$.

The gain over Patel's bound, about $7 \cdot 10^{-5}$, comes from the terms of degrees five and six. Together with Patel's upper bound, which is likewise unrefereed, Theorem 1.1 gives $0.301657 < K_2 < 0.302825279492$.

In Section 2 we set up the normalization and prove the slice estimate, including Carathéodory's inequality on slices. In Section 3 we write out Patel's treatment of the coefficients of degrees two and three, including the Gram argument behind the cubic inequality, the unitary extension and the identities of the realization. Section 4 treats the Sidon constants and proves the bounds for S_5 and S_6 , Section 5 proves Theorem 1.1, and Section 6 lists the computer checks. Throughout, lower bounds are rounded down and upper bounds up.

2. THE CARATHÉODORY NORMALIZATION AND SLICES

Let $\mathcal{P}_2$ be the set of holomorphic functions H on $\mathbb{D}^2$ with $\operatorname{Re} H > 0$ and $H(0) = 1$. We write

$$H(z) = 1 + \sum_{n \geq 1} H_n(z), \quad H_n(z) = \sum_{j+k=n} h_{jk} z_1^j z_2^k,$$

and set $\|H_n\|_1 = \sum_{j+k=n} |h_{jk}|$ and $M_r(H) = \sum_{n \geq 1} \|H_n\|_1 r^n$.

Patel [17] uses the following reduction.

Lemma 2.1. *Let $0 < r < 1$. If $M_r(H) \leq 1$ for every $H \in \mathcal{P}_2$, then $K_2 \geq r$.*

Proof. Constant functions satisfy $\|f_r\|_1 \leq 1$. Let $f \in \mathcal{S}(\mathbb{D}^2)$ be nonconstant. By the maximum principle $|f| < 1$ on $\mathbb{D}^2$. Multiplying f by a unimodular constant does not change $\|f_r\|_1$, so we may assume that $t = f(0)$ lies in $[0, 1)$. Then $H = (1 - f)/(1 - t)$ lies in $\mathcal{P}_2$. Its coefficients of positive degree are those of f multiplied by $-1/(1 - t)$. Hence $\|f_r\|_1 = t + (1 - t)M_r(H) \leq 1$. $\square$

Remark 2.2. The normalization loses nothing: if $r < K_2$, then $M_r(H) \leq 1$ for every $H \in \mathcal{P}_2$. Indeed, for $0 < s < 1$ the function $z \mapsto H(sz)$ maps $\overline{\mathbb{D}}^2$ into a compact subset of the right half-plane, hence into a disc $\{w : |w - \rho| < \rho\}$ with $\rho \geq 1$. Then $f(z) = 1 - H(sz)/\rho$ lies in $\mathcal{S}(\mathbb{D}^2)$ with $f(0) = 1 - 1/\rho$, and $\|f_r\|_1 \leq 1$ becomes $\sum_n \|H_n\|_1 s^n r^n \leq 1$. Let $s \rightarrow 1$. Thus $K_2 = \sup\{r : M_r(H) \leq 1 \text{ for all } H \in \mathcal{P}_2\}$.

Lemma 2.3. *Let $H \in \mathcal{P}_2$ and $n \geq 1$. Then $|H_n(\zeta)| \leq 2$ for $\zeta \in \mathbb{T}^2$, and $\|H_n\|_1 \leq 2S_n$.*

Proof. Fix $\zeta \in \mathbb{T}^2$. The function $\varphi(\lambda) = H(\lambda\zeta) = 1 + \sum_{m \geq 1} H_m(\zeta)\lambda^m$ is holomorphic on $\mathbb{D}$ with $\operatorname{Re} \varphi > 0$. For $0 < \rho < 1$,

$$H_n(\zeta)\rho^n = \frac{1}{\pi} \int_0^{2\pi} \operatorname{Re} \varphi(\rho e^{i\theta}) e^{-in\theta} d\theta,$$

so $|H_n(\zeta)|\rho^n \leq 2 \operatorname{Re} \varphi(0) = 2$. Letting $\rho \rightarrow 1$, we obtain Carathéodory's inequality $|H_n(\zeta)| \leq 2$. Now write $H_n(\zeta_1/\zeta_2) = \zeta_2^n q_n(\zeta_1/\zeta_2)$ with $q_n(w) = \sum_{j=0}^n h_{j,n-j} w^j$. As ζ runs over $\mathbb{T}^2$, the quotient ζ_1/ζ_2 runs over $\mathbb{T}$ and $|H_n(\zeta)| = |q_n(\zeta_1/\zeta_2)|$. Hence $\max_{\mathbb{T}} |q_n| \leq 2$, and by the definition of S_n ,

$$\|H_n\|_1 = \sum_{j=0}^n |h_{j,n-j}| \leq S_n \max_{\mathbb{T}} |q_n| \leq 2S_n. \quad \square$$

Remark 2.4. The same identification shows that $S(k, 2) = S_k$: on $\mathbb{T}^2$ a k -homogeneous polynomial in two variables is, up to the factor ζ_2^k , a polynomial in ζ_1/ζ_2 of degree at most k . With this identification, the next corollary is the case $d = 2$ of [7, Theorem 3.3].

Corollary 2.5. *If $0 < r < 1$ and $\sum_{k \geq 1} S_k r^k \leq 1/2$, then $K_2 \geq r$.*

Proof. By Lemma 2.3, $M_r(H) \leq 2 \sum_{k \geq 1} S_k r^k \leq 1$ for every $H \in \mathcal{P}_2$. Apply Lemma 2.1. $\square$

In [7] the corresponding estimate is proved for f itself. There Wiener's inequality takes the place of Carathéodory's: the Taylor coefficients $\hat{g}(k)$ of a holomorphic $g: \mathbb{D} \rightarrow \overline{\mathbb{D}}$ satisfy $|\hat{g}(k)| \leq 1 - |g(0)|^2$ for $k \geq 1$. Patel derives from Kneser's estimate [13, Lemma 2.1], by a limiting argument with $(1 - \varepsilon H)/(1 + \varepsilon H)$, the bound $\|H_n\|_1 \leq 2\sqrt{n}$ for all $H \in \mathcal{P}_2$ [17]. Lemma 2.3 with Newman's bound gives it directly.

3. THE COEFFICIENTS OF DEGREES TWO AND THREE

Throughout this section $H \in \mathcal{P}_2$, and

$$A_2 = |h_{20}| + |h_{02}|, \quad B_2 = |h_{11}|, \quad A_3 = |h_{30}| + |h_{03}|, \quad B_3 = |h_{21}| + |h_{12}|,$$

so that $\|H_2\|_1 = A_2 + B_2$ and $\|H_3\|_1 = A_3 + B_3$. The three inequalities and the way they are combined are due to Patel [17]. Inner products are linear in the first argument, $\langle x, y \rangle = y^* x$.

Lemma 3.1. $A_2^2 + B_2^2 \leq 4$.

Proof. Choose $\zeta_0 \in \mathbb{T}$ such that $h_{20}\zeta_0^2$ and h_{02} have the same argument, and put $a = h_{20}\zeta_0^2 + h_{02}$ and $b = h_{11}\zeta_0$. Then $|a| = A_2$, $|b| = B_2$ and $H_2(\pm\zeta_0, 1) = a \pm b$. By Lemma 2.3, $|a \pm b| \leq 2$, and therefore $|a|^2 + |b|^2 = \frac{1}{2}(|a + b|^2 + |a - b|^2) \leq 4$. $\square$

Lemma 3.2. *Let R be a positive semidefinite Hermitian Toeplitz matrix of size $k+1$, and let $q(\zeta) = \sum_{j=0}^k c_j \zeta^j$. Then $c^* R c \leq R_{00} \max_{\mathbb{T}} |q|^2$, where $c = (c_0, \dots, c_k)^\top$.*

Proof. Let $e_0, \dots, e_k$ be vectors in $\mathbb{C}^{k+1}$ with $\langle e_i, e_j \rangle = R_{ji}$, for example the columns of $R^{1/2}$. Since R is Toeplitz, the tuples $(e_0, \dots, e_{k-1})$ and $(e_1, \dots, e_k)$ have the same Gram matrix. Hence $e_j \mapsto e_{j+1}$ ($0 \leq j < k$) extends to a well-defined linear isometry from the span of $e_0, \dots, e_{k-1}$ onto the span of $e_1, \dots, e_k$, and then to a unitary operator Ω on the span of all the e_j . Then $e_j = \Omega^j e_0$, so $\sum_j c_j e_j = q(\Omega) e_0$. As Ω is unitary, $\|q(\Omega)\| \leq \max_{\mathbb{T}} |q|$, and

$$c^* R c = \left\| \sum_j c_j e_j \right\|^2 = \|q(\Omega) e_0\|^2 \leq \max_{\mathbb{T}} |q|^2 \|e_0\|^2 = R_{00} \max_{\mathbb{T}} |q|^2. \quad \square$$

Lemma 3.2 is a consequence of the Carathéodory–Toeplitz theorem. Patel’s post contains this argument for $k=3$, and Ye et al. prove it in the same way [21, Lemma 3.1].

Lemma 3.3. *Let $p(\zeta) = \sum_{j=0}^3 c_j \zeta^j$ satisfy $\max_{\mathbb{T}} |p| \leq 1$. Then*

$$35(|c_0| + |c_3|)^2 + 28(|c_1| + |c_2|)^2 \leq 44.$$

Consequently $35A_3^2 + 28B_3^2 \leq 176$.

Proof. Replacing $p(\zeta)$ by $p(e^{i\theta}\zeta)$ for a suitable θ , we may assume that c_1 and c_2 have the same argument. Choose $\xi \in \mathbb{T}$ with $|\xi c_0 + c_3| = |c_0| + |c_3|$, and let

$$u = (3, 4, -4, -3)^\top, \quad v = (0, 1, 1, 0)^\top, \quad w = (\xi, 0, 0, 1)^\top, \quad R = uu^* + 28vv^* + 35ww^*.$$

The matrix R is positive semidefinite. Since $|\xi| = 1$, it is also Toeplitz, with first row $(44, 12, -12, -9 + 35\xi)$. By Lemma 3.2, with $c = (c_0, \dots, c_3)^\top$,

$$|u^* c|^2 + 28|v^* c|^2 + 35|w^* c|^2 = c^* R c \leq 44.$$

Here $|v^* c| = |c_1 + c_2| = |c_1| + |c_2|$ and $|w^* c| = |\xi c_0 + c_3| = |c_0| + |c_3|$, which proves the first claim. For the second, apply it to $p = q_3/2$ from the proof of Lemma 2.3. Then $|c_0| + |c_3| = A_3/2$ and $|c_1| + |c_2| = B_3/2$. $\square$

Patel describes this argument as a weighted variant of the identity in the proof of $S_3 = 5/3$ by Yu et al. [22]. The proof above does not depend on [22], and it extends to a one-parameter family. For $-1 \leq t \leq 2$ and $u_t = (t, 1, -1, -t)^\top$, the matrix $u_t u_t^* + (t+1)vv^* + (2+t-t^2)ww^*$ is positive semidefinite and Toeplitz, with first row $(2+t, t, -t, -t^2 + (2+t-t^2)\xi)$, so the same argument shows

$$(2+t-t^2)(|c_0| + |c_3|)^2 + (t+1)(|c_1| + |c_2|)^2 \leq 2+t.$$

Lemma 3.3 is the case $t = 3/4$. The case $t = 1/2$ is the inequality $9(|c_0| + |c_3|)^2 + 6(|c_1| + |c_2|)^2 \leq 10$, from which Yu et al. obtain $S_3 \leq 5/3$ by the Cauchy–Schwarz inequality [22, (6)]. On its own, Lemma 3.3 gives only $\|H_3\|_1 \leq \sqrt{396/35} \approx 3.364$, more than $2S_3 = 10/3$. It is used together with the next inequality, which bounds A_3 in terms of B_2 .

Lemma 3.4. $A_3 + \frac{1}{2}B_2^2 \leq 2$.

Proof. We use the transfer-function realization of Schur functions on the bidisc. The Schur–Agler class consists of the $g \in \mathcal{S}(\mathbb{D}^2)$ with $\|g(X_1, X_2)\| \leq 1$ for every pair of commuting strict contractions X_1, X_2 on a Hilbert space. By Andô’s inequality [3], every $g \in \mathcal{S}(\mathbb{D}^2)$ belongs to this class. By Agler’s theorem [1] (see also [4, 2]) there are a Hilbert space $\mathcal{K}$, complementary orthogonal projections P and Q on $\mathcal{K}$, and an isometry $V = \begin{pmatrix} \kappa & x^* \\ y & W \end{pmatrix}$ of $\mathbb{C} \oplus \mathcal{K}$, the colligation, such that

$$g(z) = \kappa + x^* Z (I - WZ)^{-1} y, \quad Z = z_1 P + z_2 Q \quad (z \in \mathbb{D}^2).$$

For $z \in \mathbb{D}^2$ we have $\|Z\| \leq \max_j |z_j| < 1$, so $I - WZ$ and the other operators $I - ZW$, $I - ZU$ inverted below are invertible. We may take V unitary, which matters if the realization is given with an isometric colligation, as in [13, Section 2]. Indeed, the operator $\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix}$ on $(\mathbb{C} \oplus \mathcal{K}) \oplus (\mathbb{C} \oplus \mathcal{K})$ is unitary and agrees with V on the first summand. We use it as the new colligation, with state space $\mathcal{K} \oplus (\mathbb{C} \oplus \mathcal{K})$ and projections $P \oplus I$ and $Q \oplus 0$. The new y is the old one, the new W and x^* agree with the old ones on $\mathcal{K}$, and the new projections map $\mathcal{K}$ into itself, so each term $x^*Z(WZ)^m y$ of the expansion of g is unchanged. From now on $\mathcal{K}$, V , x , y , W , P and Q denote the new data.

Apply this to $g = (H - 1)/(H + 1)$, which lies in $\mathcal{S}(\mathbb{D}^2)$ and vanishes at 0. Then $\kappa = 0$, and since V is unitary,

$$\|x\| = \|y\| = 1, \quad Wx = 0, \quad W^*y = 0, \quad xx^* + W^*W = I, \quad yy^* + WW^* = I.$$

Put $U = W + yx^*$. These identities imply $U^*U = UU^* = I$. Let $G = I - ZW$. Then $Gx = x$ and $x^*G^{-1}Zy = x^*Z(I - WZ)^{-1}y = g$. Since $I - ZU = G - Zyx^*$ is a rank-one perturbation of G and $1 - g \neq 0$,

$$x^*(I - ZU)^{-1}x = x^*G^{-1}x + \frac{(x^*G^{-1}Zy)(x^*G^{-1}x)}{1 - x^*G^{-1}Zy} = 1 + \frac{g}{1 - g} = \frac{1}{1 - g}.$$

Hence $x^*(I + ZU)(I - ZU)^{-1}x = 2x^*(I - ZU)^{-1}x - 1 = (1 + g)/(1 - g) = H$, and expanding the resolvent,

$$H_n(z) = 2x^*(ZU)^n x \quad (n \geq 1).$$

Relative to $\mathcal{K} = P\mathcal{K} \oplus Q\mathcal{K}$ write

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \end{pmatrix}, \quad Ux = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix},$$

so that $\|a\|^2 + \|b\|^2 = \|\tilde{a}\|^2 + \|\tilde{b}\|^2 = 1$. Collecting the monomials $z_1 z_2$, z_1^3 and z_2^3 in $2x^*(ZU)^n x$, we find

$$\frac{1}{2}h_{11} = a^*U_{12}\tilde{b} + b^*U_{21}\tilde{a}, \quad \frac{1}{2}h_{30} = a^*U_{11}^2\tilde{a}, \quad \frac{1}{2}h_{03} = b^*U_{22}^2\tilde{b}.$$

Put $\sigma = \|U_{11}^*a\|^2 + \|U_{22}^*b\|^2$ and $\tau = \|U_{11}\tilde{a}\|^2 + \|U_{22}\tilde{b}\|^2$. Since U^* is an isometry, $\|U_{11}^*a\|^2 + \|U_{12}^*a\|^2 = \|a\|^2$ and $\|U_{21}^*b\|^2 + \|U_{22}^*b\|^2 = \|b\|^2$, so $\|U_{12}^*a\|^2 + \|U_{21}^*b\|^2 = 1 - \sigma$. Since U is an isometry, $\|U_{12}\tilde{b}\|^2 + \|U_{21}\tilde{a}\|^2 = 1 - \tau$ in the same way. Applying the Cauchy–Schwarz inequality to $\frac{1}{2}h_{11} = \langle \tilde{b}, U_{12}^*a \rangle + \langle \tilde{a}, U_{21}^*b \rangle = \langle U_{12}\tilde{b}, a \rangle + \langle U_{21}\tilde{a}, b \rangle$ in the two groupings, we obtain

$$\frac{1}{4}B_2^2 \leq 1 - \sigma \quad \text{and} \quad \frac{1}{4}B_2^2 \leq 1 - \tau.$$

Finally $\frac{1}{2}|h_{30}| = |\langle U_{11}\tilde{a}, U_{11}^*a \rangle| \leq \frac{1}{2}(\|U_{11}^*a\|^2 + \|U_{11}\tilde{a}\|^2)$, and likewise for h_{03} , so $A_3 \leq \sigma + \tau \leq 2 - \frac{1}{2}B_2^2$. $\square$

Remark 3.5. Lemmas 3.1 and 3.4 are sharp. Equality holds in both for $H = (1 + z_1)/(1 - z_1)$, where $A_2 = A_3 = 2$ and $B_2 = 0$, and for $H = (1 + z_1 z_2)/(1 - z_1 z_2)$, where $A_2 = A_3 = 0$ and $B_2 = 2$.

Lemma 3.6. *Let $\alpha, \beta > 0$, $\gamma \geq 0$ and $r > 0$, and put*

$$\mathcal{B}(r) = 4\alpha + 176\beta + 2\gamma + \frac{1}{4\alpha} + \frac{1}{4\alpha + 2\gamma} + \frac{(r - \gamma)^2}{140\beta} + \frac{r^2}{112\beta}.$$

Then $\|H_2\|_1 + r\|H_3\|_1 \leq \mathcal{B}(r)$.

Proof. Expanding the squares verifies the identity

$$\begin{aligned} \mathcal{B}(r) - (A_2 + B_2 + r(A_3 + B_3)) &= \alpha(4 - A_2^2 - B_2^2) + \beta(176 - 35A_3^2 - 28B_3^2) + \gamma(2 - A_3 - \tfrac{1}{2}B_2^2) \\ &\quad + \alpha\left(A_2 - \frac{1}{2\alpha}\right)^2 + \left(\alpha + \frac{\gamma}{2}\right)\left(B_2 - \frac{1}{2\alpha + \gamma}\right)^2 \\ &\quad + 35\beta\left(A_3 - \frac{r - \gamma}{70\beta}\right)^2 + 28\beta\left(B_3 - \frac{r}{56\beta}\right)^2. \end{aligned}$$

By Lemmas 3.1, 3.3 and 3.4, each term on the right is nonnegative. $\square$

Thus $\mathcal{B}(r)$ is the Lagrangian bound, with multipliers α , β and γ , for the maximum of $A_2 + B_2 + r(A_3 + B_3)$ under the three inequalities of Lemmas 3.1, 3.3 and 3.4.

4. SIDON CONSTANTS AND CERTIFICATES FOR S_5 AND S_6

Table 1 lists what is known about S_k for small k . The proof of Theorem 1.1 uses upper bounds for S_1 and for S_k with $k \geq 4$. Degrees two and three are covered by Lemma 3.6.

TABLE 1. Sidon constants $S_k = S(\{0, 1, \dots, k\})$.

k	S_k	source
1	1	elementary
2	$\sqrt{2}$	Newman (see [15])
3	$5/3$	Yu et al. [22]; lower bound Neuwirth [16]
4	2	Newman's bound; equality by Shapiro [20]
5	$2.186529771892569\dots; \leq 2.187$	Ye et al. [21]; Proposition 4.2
6	≤ 2.363	Proposition 4.2
$k \geq 7$	$\leq \sqrt{k}$	Newman [20, Chapter 3]

Ye et al. [21, Theorem 1.1] show that S_5 is the root in (2.18, 2.19) of $675X^4 - 1026X^3 - 936X^2 - 414X + 677$. Proposition 4.2 below gives the weaker bound $S_5 \leq 2.187$, which suffices for Theorem 1.1, together with $S_6 \leq 2.363$. Its proof rests on the following lemma.

Lemma 4.1. *Let $k \geq 1$. For $w \in \mathbb{C}^{k+1}$ put*

$$\nu(w) = \max \left\{ \operatorname{Re} \sum_{j=0}^k \bar{w}_j c_j : \max_{\mathbb{T}} \left| \sum_{j=0}^k c_j z^j \right| \leq 1 \right\}.$$

- (a) $S_k = \max\{\nu(w) : w_0 = w_1 = 1, |w_j| = 1 \ (2 \leq j \leq k)\}$.
- (b) ν is convex.
- (c) Let $y \in \mathbb{C}^k$, and let $T(y)$ be the Hermitian Toeplitz matrix with diagonal entries 1 and entries $T(y)_{jl} = y_{l-j}$ for $l > j$. If $T(y)$ is positive definite, then $\nu(w)^2 \leq w^* T(y)^{-1} w$.
- (d) For $0 < \delta \leq \pi/4$, the arc $\{e^{i\varphi} : |\varphi - \varphi_0| \leq \delta\}$ lies in the triangle with vertices $e^{i(\varphi_0 - \delta)}$, $e^{i(\varphi_0 + \delta)}$ and $e^{i\varphi_0} / \cos \delta$.

Proof. (a) Write $q(z) = \sum_j c_j z^j$. If $|w_j| = 1$ for all j , then $\nu(w) \leq \max\{\sum_j |c_j| : \max_{\mathbb{T}} |q| \leq 1\} = S_k$. Equality holds when c attains this maximum and $w_j = c_j/|c_j|$ (any unimodular number if $c_j = 0$). The substitution $c_j \mapsto c_j e^{i(\psi + j\theta)}$ replaces $q(z)$ by $e^{i\psi} q(e^{i\theta} z)$ and w_j by $w_j e^{-i(\psi + j\theta)}$ without changing ν , and a choice of ψ and θ makes $w_0 = w_1 = 1$. (b) ν is a supremum of

real-linear functions of w . (c) By Lemma 3.2, $c^*T(y)c \leq \max_{\mathbb{T}} |q|^2$ for $c = (c_0, \dots, c_k)^\top$. Hence, by the Cauchy–Schwarz inequality,

$$\operatorname{Re} w^*c \leq (w^*T(y)^{-1}w)^{1/2} (c^*T(y)c)^{1/2} \leq (w^*T(y)^{-1}w)^{1/2} \max_{\mathbb{T}} |q|.$$

(d) After a rotation, $\varphi_0 = 0$. The triangle is $\{u \in \mathbb{C} : \operatorname{Re} u \geq \cos \delta, \operatorname{Re}(ue^{-i\delta}) \leq 1, \operatorname{Re}(ue^{i\delta}) \leq 1\}$, and $u = e^{i\varphi}$ with $|\varphi| \leq \delta$ satisfies $\cos \varphi \geq \cos \delta$ and $\cos(\varphi \mp \delta) \leq 1$. $\square$

Parts (a) and (c) are variants of the reduction to phase functionals and of the Toeplitz domination bound used by Ye et al. [21, (14) and Proposition 3.2], who normalize the phases differently [21, (17)]. Proposition 4.2 also differs from their proof in the covering of the phase space.

Proposition 4.2. $S_5 \leq 2.187$ and $S_6 \leq 2.363$.

Proof (computer-assisted). Let (k, b_k) be $(5, 2.187)$ or $(6, 2.363)$. By Lemma 4.1(a) it suffices to show that $\nu(1, 1, e^{i\varphi_2}, \dots, e^{i\varphi_k}) \leq b_k$ for $(\varphi_2, \dots, \varphi_k) \in [-\pi, \pi]^{k-1}$. The certificate for k covers $[-\pi, \pi]^{k-1}$ by boxes, obtained from a uniform grid of 8^{k-1} boxes by repeated bisection. Each box has half-widths at most $\pi/8$. By Lemma 4.1(d), the vectors $(1, 1, e^{i\varphi_2}, \dots, e^{i\varphi_k})$ with φ in the box lie in a product of $k-1$ triangles. By Lemma 4.1(b), the maximum of ν over this product is attained at one of its 3^{k-1} vertices. For each vertex v that occurs, the certificate gives $y \in \mathbb{C}^k$ with $T(y)$ positive definite and $v^*T(y)^{-1}v < b_k^2$, so that $\nu(v) < b_k$ by Lemma 4.1(c). A vertex with a coordinate -1 occurs with two labels, phase $-\pi$ and phase π , and has a certificate for each. Table 2 gives the sizes of the two certificates, and Section 6 describes the check. $\square$

TABLE 2. The certificates of Proposition 4.2.

	$k = 5$	$k = 6$
bound b_k	2.187	2.363
boxes	8,912	61,118
vertex certificates	269,769	9,893,956
smallest margin $b_k^2 - v^*T(y)^{-1}v$	$1.49 \cdot 10^{-5}$	$8.48 \cdot 10^{-8}$

For $k = 6$ the certificate is split into 48 files, and a vertex that occurs in several files has a certificate in each, so the count 9,893,956 includes repetitions. There are 4,333,585 distinct vertex labels. A numerical search, which is not part of the proof, gives $S_6 \approx 2.36256$, so the bound 2.363 is close to the true value.

5. PROOF OF THEOREM 1.1

Let $N = 40$. Let $s_4 = 2$, $s_5 = 2.187$, $s_6 = 2.363$, and $s_n = \lceil 10^8 \sqrt{n} \rceil / 10^8$ for $7 \leq n \leq N$. By Table 1 and Proposition 4.2, $S_n \leq s_n$ for $4 \leq n \leq N$, and $S_n \leq \sqrt{n} \leq n$ for all n . Let $H \in \mathcal{P}_2$, let $0 < r < 1$, and fix $\alpha, \beta > 0$ and $\gamma \geq 0$. Lemma 2.3 bounds $\|H_n\|_1$ by 2 for $n = 1$, by $2s_n$ for $4 \leq n \leq N$ and by $2n$ for $n > N$, and Lemma 3.6 bounds $\|H_2\|_1 + r\|H_3\|_1$. Summing the series $\sum_{n>N} nr^n = r^{N+1}(N+1-Nr)/(1-r)^2$, we obtain a majorant of the form that Patel uses [17]:

$$M_r(H) \leq E(r) := 2r + r^2 \mathcal{B}(r) + 2 \sum_{n=4}^N s_n r^n + \frac{2r^{N+1}(N+1-Nr)}{(1-r)^2}.$$

We take

$$r_* = \frac{301657}{10^6}, \quad \alpha = \frac{165392627}{5 \cdot 10^8}, \quad \beta = \frac{62421603}{25 \cdot 10^9}, \quad \gamma = \frac{25496997}{25 \cdot 10^7}.$$

All quantities in $E(r_*)$ are then rational, and an exact computation shows $E(r_*) < 0.99999548 < 1$. Since α , β and γ are fixed, E is continuous on $(0, 1)$, so there is $r' > r_*$ with $E(r') < 1$. By Lemma 2.1, as in Patel's argument, $K_2 \geq r' > r_*$, which proves Theorem 1.1.

With Patel's parameters $\alpha = 264631/800000$, $\beta = 249627/10^8$ and $\gamma = 10197021/10^8$, with $N = 20$ and with $s_n = \lceil 10^8 \sqrt{n} \rceil / 10^8$ for all $4 \leq n \leq 20$, the function E is the rational majorant that Patel evaluates in [17]. Its value at his radius $r_P = 15079379/(5 \cdot 10^7)$ is less than 0.99999997857, below his bound $1 - 1/(5 \cdot 10^7)$ (Table 3).

6. VERIFICATION

Four programs check the computations, and a fifth checks Patel's computation. All of them are in the archive listed under Data availability, together with the certificates, the programs that produced them and the output of all runs.

The first program checks the certificate for S_5 in ball arithmetic, using Arb [12] through python-flint [19] at a precision of 128 bits. The certificate stores the entries of each y as decimal numerals, which Python parses into the nearest binary64 numbers. These binary64 numbers are the certificate, and the program reads them exactly into balls. It rebuilds the boxes from the uniform grid and the recorded bisections, so the boxes cover $[-\pi, \pi]^4$ by construction. For each box it forms the 81 vertices of Lemma 4.1(d) from rational multiples of π . For each vertex label it checks that the leading principal minors of $T(y)$ are positive and that $v^*T(y)^{-1}v < 2.187^2$, solving the linear system in ball arithmetic. The run takes about 30 seconds and 0.6 GB of memory on one core of an x86-64 server.

The certificate for S_6 has about 3.4 GB and is split into 48 files. Each file contains some cells of the initial grid, their subtrees and the vertex certificates they need. The second program makes the checks of the first program for one such file and lists the cells it has covered. The third reads these lists and checks that the 48 files together cover each of the 8^5 cells exactly once. It trusts the lists it reads, so they must come from runs of the second program on the 48 files. The logs in the archive record such a run. Each of the 48 runs of the second program takes under a minute and about 0.7 GB of memory on one core.

The fourth program evaluates $E(r_*)$ of Section 5 in exact rational arithmetic and compares it with 1. The fifth checks, in exact rational arithmetic, that $\mathcal{B}(r_P)$ equals the fraction displayed in [17], that his table of rational upper bounds for $\sqrt{n}$ consists of the integers $\lceil 10^8 \sqrt{n} \rceil$, that the two parts of his majorant lie below the two rational bounds he displays, and that the sum of these bounds is below $1 - 1/(5 \cdot 10^7)$. Table 3 lists the results, with the exact values rounded up.

TABLE 3. Results of the programs.

run	result
certificate for S_5 , bound 2.187	accepted
certificate for S_6 , bound 2.363 (48 files and their coverage)	accepted
$E(r_*)$ with $s_5 = 2.187$ and $s_6 = 2.363$	< 0.99999548
$E(r_P)$ in Patel's computation [17]	< 0.99999997857

As consistency checks, the first program rejects the certificate for S_5 with the bound 2.186, which is less than S_5 [21], and six altered copies of it, each with an entry removed or changed. The second rejects a file of the certificate for S_6 with the bound 2.362, and the third rejects the lists when one file is missing or counted twice. The fourth finds $E(r_* + 10^{-6}) \geq 1$ with the same α , β , γ and s_n , and $E(r_*) \geq 1$ when $s_6 = 2.363$ is replaced by $\lceil 10^8 \sqrt{6} \rceil / 10^8$.

DATA AVAILABILITY

The certificates of Proposition 4.2, the programs of Section 6, the programs that produced the certificates, the controls and the output of all runs are archived at doi:10.5281/zenodo.23538814. The repository <https://github.com/tejasvi/bidisc-bohr-radius> holds the same files except the 48 files of the certificate for S_6 . The certificate for S_5 has the SHA-256 checksum

ce3e9dd41a0dbd9c3a8ea49b04aa7aaf21f0fe557291a2084c7739461a460a14,

and the archive lists the checksums of the other files.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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