

A LOWER BOUND OF 0.6463 FOR THE BEARDWOOD–HALTON–HAMMERSLEY CONSTANT

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ABSTRACT. Let β be the constant of the Beardwood–Halton–Hammersley theorem: the shortest tour through n independent uniform random points in the unit square has length asymptotic to $\beta\sqrt{n}$ almost surely. Dong and Cao proved $\beta \geq 0.6421$. They show that $\beta \geq 5/8 + \sum_{k=3}^K C_k$ for every K , where C_k is an integral over configurations of k points in the plane that form a connected graph when points at distance at most 1 are joined, and they bound C_3 , C_4 and C_5 from below on a uniform grid of boxes. We bound the same three integrals more tightly and prove $\beta \geq 0.6463$. On a box of configurations we bound the area of the union of the unit disks centred at the points by its first-order Taylor polynomial plus a remainder controlled by the angular motion of the vertices of the union, and we average this bound over the box. Boxes on which the graph varies, boxes that cross the boundary of the symmetry domain and boxes in which two points can coincide receive separate bounds. An adaptive bisection chooses the boxes, and a checker in ball arithmetic recomputes the bound on every box to which the search program assigns a nonzero value. It certifies $C_3 \geq 0.0117130195$, $C_4 \geq 0.0063707166$ and $C_5 \geq 0.0032174604$.

1. INTRODUCTION

Let $X_1, X_2, \dots$ be independent uniform random points in $[0, 1]^2$, and let L_n be the length of the shortest closed tour through $X_1, \dots, X_n$. Beardwood, Halton and Hammersley [1] proved that $L_n/\sqrt{n}$ converges almost surely to a constant β . The value of β is not known. Numerical experiments suggest that β is about 0.71 [7, 2]. It is listed as constant C_{12} in the repository of optimization constants [9].

Every point of a tour is joined to two other points, so it contributes at least half the sum of the distances to its two nearest neighbours. Beardwood, Halton and Hammersley used this to prove $\beta \geq 5/8$ [1] (see also [8, Section 3.1]). Steinerberger [8] refined their argument and claimed $\beta \geq 5/8 + 19/5184$. Gaudio and Jaillet [5] found an error in this argument, corrected it to $\beta \geq 5/8 + 19/10368 > 0.6268$ and tightened it to $\beta \geq 0.6277$. Dong and Cao [3] proved $\beta \geq 0.6421$. With importance sampling they also obtained $\beta \geq 0.6536$ with probability at least $1 - 10^{-4}$ over the randomness of the sampling. Dong and Cao also proved $\beta \leq 0.8810$ [3].

The bound of Dong and Cao has the form $\beta \geq 5/8 + \sum_{k=3}^K C_k$ for every $K \geq 3$ (Theorem 2.1). The number C_k is the contribution of the connected components with k points of the graph that joins two points of a Poisson process when their distance is at most r , integrated over r . It is, up to the factor $\Gamma(k - \frac{1}{2})/(2k!)$, an integral over configurations $y = (y_1, \dots, y_k)$ of points in the plane, with $y_1 = 0$, of $g(G)A^{-(k-1/2)}$. Here G is the graph on $y_1, \dots, y_k$ that joins two points at distance at most 1, the number $g(G) \geq 0$ is a lower bound, exact for connected graphs with at most six vertices, for the number of paths needed to cover G beyond those that the degrees of its vertices force, and A is the area of the union of the unit disks centred at the points. Dong and Cao bound C_3 , C_4 and C_5 from below. They cover a bounded part of the

2020 *Mathematics Subject Classification.* 60D05, 90C27; 52A22, 65G40.

Key words and phrases. Euclidean traveling salesman problem, Beardwood–Halton–Hammersley theorem, random geometric graph, union of disks, interval arithmetic, computer-assisted proof.

space of configurations with boxes of side $1/1600$, $1/50$ and $1/10$, and on each box they use the minimum of g and an upper bound for A that is built from a spanning tree. Their values are $C_3 \geq 0.011135$, $C_4 \geq 0.005040$ and $C_5 \geq 0.001015$ [3, Section 3.3]. Monte Carlo estimates give $C_3 \approx 0.0117$, $C_4 \approx 0.0064$ and $C_5 \approx 0.0039$ (Section 5), so most of the room is in C_4 and C_5 .

Theorem 1.1. *We have $C_3 \geq 0.0117130195$, $C_4 \geq 0.0063707166$ and $C_5 \geq 0.0032174604$. Consequently*

$$\beta \geq \frac{5}{8} + C_3 + C_4 + C_5 > 0.6463.$$

The method. We use the reduction of Dong and Cao to a box in $\mathbb{R}^{2k-3}$ (Section 2) and subdivide the box adaptively. The new part is the bound on a single box Q with centre c . For y in Q we bound $A(y)$ from above by $A(c) + \nabla A(c) \cdot (y - c) + \varrho(y - c)$ (Lemma 3.1). The gradient of A is an integral over the exposed arcs of the circles, and its variation along a segment comes from the motion of the endpoints of these arcs, which are the vertices of the union. The remainder ϱ adds up, over these vertices, the displacement of the two centres times a bound for the angle swept by the vertex. Since $t \mapsto t^{-(k-1/2)}$ is convex, Jensen's inequality turns this into a lower bound for the average of the integrand over Q that loses only the average of ϱ and terms of second order in the widths of Q (Proposition 3.4). On boxes where some edges of the graph are undetermined we add, to the minimum of g , the excess of g on events whose probabilities we bound from below (Proposition 3.6). Boxes that cross the boundary of the symmetry domain, and boxes in which two points can coincide, receive cruder bounds (Propositions 3.7 and 3.8). A box is split while the estimated loss of its bound exceeds a threshold.

Verification. A search program in double precision chooses the boxes and scores the leaves. A checker that contains a verbatim copy of the subdivision visits the same leaves for the same number of initial boxes, and recomputes the bound on every leaf of nonzero float value in ball arithmetic with the Arb library [6] of FLINT [4] (Section 4.2). Programs written separately estimated $C_3, \dots, C_6$ by Monte Carlo integration and tested the components of the checker (Section 4.5).

Limits. The certified values are 99.9%, 99.4% and 82.3% of the Monte Carlo estimates of C_3 , C_4 and C_5 . Numerically, with $K = 5$ the framework cannot give more than about 0.6470 (Section 5).

Section 2 recalls the framework of Dong and Cao. Section 3 proves the bounds on a box. Section 4 describes the computation and proves Theorem 1.1. Section 5 records numerical observations on the limits of the method.

2. THE INTEGRALS C_k

For $a, b \in \mathbb{R}^2$ we write $|a|$ for the Euclidean norm and $B(a, \rho)$ for the closed disk with centre a and radius ρ . For $t \in \mathbb{R}$ let $u(t) = (\cos t, \sin t)$. For a finite graph G let $n_1(G)$ be the number of vertices of degree 1, let $p(G)$ be the least number of vertex-disjoint paths that cover the vertices of G , where a single vertex counts as a path, and let $\text{cc}(G)$ be the number of connected components, with $\text{cc}(G) = 0$ for the empty graph. For $S \subseteq V(G)$ let $G - S$ be the subgraph induced by $V(G) \setminus S$. For a connected graph G , Dong and Cao [3, Definition 5] put

$$D(G) = \frac{1}{2} \sum_v (2 - d_G(v))_+, \quad q(G) = \max \left\{ 1, \left\lceil \frac{n_1(G)}{2} \right\rceil, \max_{S \subseteq V(G)} (\text{cc}(G - S) - |S|) \right\},$$

where $d_G(v)$ is the degree of v , and $g(G) = q(G) - D(G)$. For a disconnected graph they put $g(G) = 0$. They prove $p(G) \geq q(G) \geq D(G)$ [3, Lemma 3], so $0 \leq g(G) \leq p(G) - D(G)$. If G is connected and has at least two vertices, then $D(G) = n_1(G)/2$. An enumeration of all graphs on

three to six labelled vertices shows that $p(G) = q(G)$ for every connected graph G with at most six vertices (for one or two vertices $p = q = 1$). Hence $g(G) = p(G) - n_1(G)/2$ for connected graphs with between two and six vertices, and this is the form in which our programs evaluate g . (The proof of [3, Theorem 3] uses only $0 \leq g \leq p - D$ and $g = 0$ on disconnected graphs, so Theorem 2.1 below also holds with g replaced by $p - D$.)

For $k \geq 3$ and $y = (y_1, \dots, y_k) \in (\mathbb{R}^2)^k$ let $G_1(y)$ be the graph on $\{1, \dots, k\}$ with an edge ij whenever $|y_i - y_j| \leq 1$, and let

$$A(y) = \text{area}\left(\bigcup_{i=1}^k B(y_i, 1)\right).$$

Theorem 2.1 (Dong and Cao [3, Theorem 3]). *For every integer $K \geq 3$,*

$$\beta \geq \frac{5}{8} + \sum_{k=3}^K C_k, \quad C_k = \frac{\Gamma(k - \frac{1}{2})}{2k!} \int_{(\mathbb{R}^2)^{k-1}} \frac{g(G_1(y))}{A(y)^{k-1/2}} dy_2 \cdots dy_k,$$

where $y_1 = 0$ in the integral.

Hence lower bounds for C_3 , C_4 and C_5 give a lower bound for β , with $K = 5$.

We now reduce the domain of integration. Write $s = k - \frac{1}{2}$ and $\varphi(t) = t^{-s}$ for $t > 0$. Following Dong and Cao [3, Section 3.3], we put $y_2 = (r_2, 0)$ and $y_j = (\xi_j, \eta_j)$ for $3 \leq j \leq k$, and we collect the coordinates in

$$x = (r_2, \xi_3, \eta_3, \dots, \xi_k, \eta_k) \in \mathbb{R}^{2k-3}.$$

Let D_k be the set of x with $r_2 \geq 0$, $\eta_3 \geq 0$ and $\xi_3 \leq \xi_4 \leq \dots \leq \xi_k$, and let X_k be the box of the x with $r_2 \in [0, k-2]$, $\eta_3 \in [0, k-2]$ and all other coordinates in $[-(k-2), k-2]$. Put

$$\kappa_k = \frac{2\pi \Gamma(k - \frac{1}{2}) (k-2)!}{k!}.$$

Lemma 2.2. *For $k \geq 3$,*

$$C_k = \kappa_k \int_{D_k \cap X_k} r_2 g(G_1(y)) \varphi(A(y)) dx.$$

Proof. Dong and Cao [3, Section 3.3] write y_2 in polar coordinates, use the invariance of the integrand under rotations about $y_1 = 0$ and under the reflection in the first axis, and obtain

$$C_k = \frac{2\pi \Gamma(k - \frac{1}{2})}{k!} \int_{\{r_2 \geq 0, \eta_3 \geq 0\}} r_2 g(G_1(y)) \varphi(A(y)) dx.$$

We undo the reflection, which gives the same integral over $\{r_2 \geq 0\}$ with the factor $\pi \Gamma(k - \frac{1}{2})/k!$. The integrand is symmetric in $y_3, \dots, y_k$, and the set where two of the ξ_j coincide has measure zero, so the integral over $\{r_2 \geq 0\}$ is $(k-2)!$ times the integral over the set where in addition $\xi_3 \leq \dots \leq \xi_k$. The reflection in the first axis fixes y_1, y_2 and every ξ_j , maps η_3 to $-\eta_3$ and preserves the integrand, so this integral is twice the integral over D_k . Finally, Dong and Cao show that $g(G_1(y)) > 0$ implies $|y_j| \leq k-2$ for all j [3, Section 3.3]: in a connected graph with k vertices in which some vertex is at graph distance $k-1$ from y_1 , a shortest path to that vertex contains all k vertices and any further edge would shorten it, so $G_1(y)$ is a path and $g = 0$. Hence every y_j is joined to y_1 by at most $k-2$ edges of length at most 1. Points of D_k with $|y_j| \leq k-2$ for all j lie in X_k . $\square$

3. BOUNDS ON A BOX

3.1. Notation. A *box* is a set $Q = \prod_{d=0}^{2k-4} [a_d, a_d + h_d] \subseteq X_k$ in the coordinates x , where $d = 0$ stands for r_2 . Its widths are $h_d > 0$, its volume is $v = \prod_d h_d$, and its centre c is the configuration with coordinates $a_d + h_d/2$. For $x \in Q$, the point y_1 is 0, the point y_2 ranges over the segment $[a_0, a_0 + h_0] \times \{0\}$, and y_j ranges over a rectangle R_j for $j \geq 3$. We put $R_1 = \{0\}$ and $R_2 = [a_0, a_0 + h_0] \times \{0\}$, write c_j for the centre of R_j , $\delta_j = y_j - c_j$ for the displacement and $\varepsilon_j = \max_Q |\delta_j|$ for half the diagonal of R_j .

For $i < j$ let d_{ij}^- and d_{ij}^+ be the least and the largest distance between a point of R_i and a point of R_j . The edge ij is *present* on Q if $d_{ij}^+ \leq 1$, *absent* if $d_{ij}^- > 1$, and *undetermined* otherwise. Let $g_{\min}(Q)$ and $g_{\max}(Q)$ be the least and the largest value of g over the graphs that contain the present edges and no absent edge ($g_{\min}(Q)$ is the quantity $\underline{g}(Q)$ of [3, Section 3.3]). Then $g_{\min}(Q) \leq g(G_1(y)) \leq g_{\max}(Q)$ for $x \in Q$.

We write $\mathbb{E}$ for the average over x uniform in Q . Under this measure the coordinates are independent and uniform, and $\mathbb{E}\delta_j = 0$.

3.2. The union area and its gradient. Let y be a configuration with distinct points. For each i let $E_i(y)$ be the set of $t \in \mathbb{R}/2\pi\mathbb{Z}$ such that $y_i + u(t)$ lies in no open disk $B^\circ(y_j, 1)$ with $j \neq i$. The boundary of the union of the disks consists of the arcs $y_i + u(E_i)$, and Green's theorem gives

$$(3.1) \quad A(y) = \frac{1}{2} \sum_{i=1}^k \int_{E_i(y)} (1 + \langle y_i, u(t) \rangle) dt.$$

Moving the disk $B(y_i, 1)$ with velocity w changes the area of the union at the rate $\int_{E_i} \langle w, u(t) \rangle dt$, the integral of the normal velocity over the exposed boundary of the moving disk. The right-hand side of (3.2) below depends continuously on y , since the endpoints of the arcs move continuously and arcs appear or vanish only with length zero. Hence A is continuously differentiable on configurations with distinct points, and

$$(3.2) \quad \nabla_{y_i} A(y) = \int_{E_i(y)} u(t) dt.$$

For a pair $i \neq j$ with $0 < |y_i - y_j| < 2$ the circles $\partial B(y_i, 1)$ and $\partial B(y_j, 1)$ meet in two points. Seen from y_i they lie in the directions $\theta_{ij}^\pm = \psi_{ij} \pm \alpha_{ij}$, where ψ_{ij} is the direction of $y_j - y_i$ and $\alpha_{ij} = \arccos(|y_i - y_j|/2)$. The endpoints of the arcs that form $E_i(y)$ are such directions. We call these intersection points the *vertices* of the pair ij .

3.3. The vertex remainder. For a box Q , a pair $i < j$ with $d_{ij}^- < 2$ and $w \geq 0$, put

$$(3.3) \quad T_{ij}(w) = \lambda_{ij}w + \mu_{ij} + \min\left(\frac{w}{d_{ij}^-}, \pi\right),$$

where $(\lambda_{ij}, \mu_{ij}) = (1/\sqrt{4 - (d_{ij}^+)^2}, 0)$ if $d_{ij}^+ < 2$, and $(\lambda_{ij}, \mu_{ij}) = (0, 2 \arcsin(\min(d_{ij}^+, 2)/2) - 2 \arcsin(d_{ij}^-/2))$ otherwise. (The second choice is also allowed when $d_{ij}^+ < 2$; the programs use the first one when $d_{ij}^+ < 1.98$.) A vertex of the pair ij is *covered* on Q if it exists at c and, along every segment from c to a configuration of Q , it lies in the interior of some disk $B(y_l, 1)$ with $l \notin \{i, j\}$ at all times at which it exists. Let $\nu_{ij} \in \{0, 1, 2\}$ be an upper bound for the number of vertices of the pair ij that are not covered on Q (Remark 3.2).

Lemma 3.1. *Let Q be a box with $d_{ij}^- > 0$ for all $i < j$. Then for every $x \in Q$,*

$$|A(y) - A(c) - \langle \nabla A(c), y - c \rangle| \leq \varrho(\delta) := \sum_{i < j, d_{ij}^- < 2} \nu_{ij} (|\delta_i| + |\delta_j|) T_{ij}(|\delta_j - \delta_i|).$$

Here $\langle \nabla A(c), y - c \rangle = \sum_i \langle \nabla_{y_i} A(c), \delta_i \rangle$.

Proof. Let $y(t) = c + t(y - c)$ for $t \in [0, 1]$, so that $y_i(t) = c_i + t\delta_i$, and let $f(t) = A(y(t))$. All configurations $y(t)$ lie in Q , so their points are distinct. By (3.2), $f'(t) = \sum_i \langle \delta_i, \gamma_i(t) \rangle$ with $\gamma_i(t) = \int_{E_i(y(t))} u$. Since $f(1) - f(0) - f'(0) = \int_0^1 (f'(t) - f'(0)) dt$ and $|u| = 1$,

$$|f(1) - f(0) - f'(0)| \leq \sup_{t \in [0, 1]} \sum_i |\delta_i| |E_i(y(t)) \triangle E_i(y(0))|,$$

where $|\cdot|$ is the length of a subset of the circle. The set E_i is the complement of the union of the closed arcs $\{t : |t - \psi_{ij}| \leq \alpha_{ij}\}$ over the j with $|y_i - y_j| < 2$, up to finitely many points. Along the path these arcs move continuously, an arc appears or vanishes only with length zero at a tangency, and moving an endpoint that lies in the interior of another arc does not change the union. A covered vertex lies in the interior of another arc whenever it exists. Hence, if a direction lies in exactly one of $E_i(y(t))$ and $E_i(y(0))$, then at some time in $[0, t]$ an endpoint of an arc passes through it while lying in no open arc of another disk. Up to finitely many tangency directions, this endpoint is the direction $\theta_{ij}^\pm$ of a vertex that is not covered. So the symmetric difference lies, up to a null set, in the union of the ranges of these directions, and its length is at most the sum of their total variations over the times at which the vertices exist.

We bound this total variation. Write $z = c_j - c_i$ and $w = \delta_j - \delta_i$, so that $y_j(t) - y_i(t) = z + tw$ has length $\ell(t) \geq d_{ij}^- > 0$. Let $e(t)$ be its direction as a unit vector and $e^\perp(t)$ the rotation of $e(t)$ by $\pi/2$. Then $\psi'_{ij} = \langle e^\perp, w \rangle / \ell$ and $\ell' = \langle e, w \rangle$, and from $\cos \alpha_{ij} = \ell/2$ we get $\alpha'_{ij} = -\ell' / \sqrt{4 - \ell^2}$ where $\ell < 2$. So

$$|(\theta_{ij}^\pm)'| \leq \frac{|\langle e^\perp, w \rangle|}{\ell} + \frac{|\ell'|}{\sqrt{4 - \ell^2}}.$$

The first term integrates to the total variation of the direction of the vector $z + tw$, which moves on a line that avoids the origin, so its integral is less than π . It is also at most $|w|/d_{ij}^-$ pointwise. The second term is the absolute value of the derivative of $\arcsin(\ell/2)$. The function ℓ is convex, hence first decreasing and then increasing, and its values on the times where the vertex exists lie in $[d_{ij}^-, \min(d_{ij}^+, 2)]$. So the integral of the second term is at most $2 \arcsin(\min(d_{ij}^+, 2)/2) - 2 \arcsin(d_{ij}^-/2)$. If $d_{ij}^+ < 2$, the second term is also at most $|w|/\sqrt{4 - (d_{ij}^+)^2}$ pointwise. In both cases the total variation of $\theta_{ij}^\pm$ is at most $T_{ij}(|w|)$. The direction of the same vertex seen from y_j is $\psi_{ij} + \pi \mp \alpha_{ij}$, which satisfies the same bound. Every vertex that is not covered therefore contributes at most $(|\delta_i| + |\delta_j|) T_{ij}(|w|)$, there are at most ν_{ij} of them for the pair ij , and the lemma follows. $\square$

Remark 3.2. The programs treat a vertex of the pair ij as covered only if it exists at the centre and $|P - c_l| + \varepsilon_i + T_{ij}(\varepsilon_i + \varepsilon_j) + \varepsilon_l < 1$ for some $l \notin \{i, j\}$, where P is the vertex at the centre. Along the segment from c to any configuration of Q the vertex exists on an interval containing 0, because ℓ is convex, and on this interval it moves by at most $\varepsilon_i + T_{ij}(\varepsilon_i + \varepsilon_j)$ while y_l moves by at most ε_l . So the vertex is covered.

Since $|\delta_j - \delta_i| \leq |\delta_i| + |\delta_j|$ and T_{ij} is increasing, $\varrho(\delta) \leq \sum \nu_{ij} \chi_{ij} T_{ij}(\chi_{ij})$ with $\chi_{ij} = |\delta_i| + |\delta_j|$. In particular

$$(3.4) \quad \varrho(\delta) \leq R_{\max} := \sum_{i < j, d_{ij}^- < 2} \nu_{ij} (\varepsilon_i + \varepsilon_j) T_{ij}(\varepsilon_i + \varepsilon_j)$$

on Q . For the averages we use the moments of $|\delta_j|$. If R_j has half-sides σ and σ' , then $\mathbb{E}|\delta_j|^2 = (\sigma^2 + \sigma'^2)/3$, $\mathbb{E}|\delta_j| \leq (\mathbb{E}|\delta_j|^2)^{1/2}$ and $\mathbb{E}|\delta_j|^4 = \sigma^4/5 + 2\sigma^2\sigma'^2/9 + \sigma'^4/5$. Different points are independent, and $(a+b)^4 \leq 8(a^4 + b^4)$. This bounds $\mathbb{E}\chi_{ij}$, $\mathbb{E}\chi_{ij}^2$ and $\mathbb{E}\chi_{ij}^4$.

Lemma 3.3. *Let Q be as in Lemma 3.1. With the sums over the pairs with $d_{ij}^- < 2$,*

$$\begin{aligned} \mathbb{E}\varrho &\leq \sum \nu_{ij} \left(\lambda_{ij} \mathbb{E}\chi_{ij}^2 + \mu_{ij} \mathbb{E}\chi_{ij} + \min(\mathbb{E}\chi_{ij}^2/d_{ij}^-, \pi \mathbb{E}\chi_{ij}) \right), \\ (\mathbb{E}\varrho^2)^{1/2} &\leq \sum \nu_{ij} \left(3\lambda_{ij}^2 \mathbb{E}\chi_{ij}^4 + 3\mu_{ij}^2 \mathbb{E}\chi_{ij}^2 + 3 \min(\mathbb{E}\chi_{ij}^4/(d_{ij}^-)^2, \pi^2 \mathbb{E}\chi_{ij}^2) \right)^{1/2}. \end{aligned}$$

Proof. Each term of ϱ is at most $\nu_{ij}(\lambda_{ij}\chi_{ij}^2 + \mu_{ij}\chi_{ij} + \min(\chi_{ij}^2/d_{ij}^-, \pi\chi_{ij}))$, and $\mathbb{E} \min(a, b) \leq \min(\mathbb{E}a, \mathbb{E}b)$. For the second moment use $(a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$ for each term and Minkowski's inequality for the sum. $\square$

3.4. Interior boxes. In the coordinates x the displacement is $x - x_c$, with components Δ_d , and Lemma 3.1 gives $A(y) \leq A(c) + \langle b, \Delta \rangle + \varrho$, where $b_0 = \partial A / \partial r_2$ and b_d for $d \geq 1$ are the partial derivatives of A at c with respect to the coordinates ξ_j and η_j . Put

$$\tau = A(c) - \frac{1}{2} \sum_d |b_d| h_d, \quad S_b = \left(\sum_d b_d^2 h_d^2 / 12 \right)^{1/2}.$$

Proposition 3.4. *Let Q be as in Lemma 3.1, let $\bar{R} \geq \mathbb{E}\varrho$ and $\bar{S} \geq (\mathbb{E}\varrho^2)^{1/2}$, and suppose that $\tau > 0$. Then $\mathbb{E}[r_2\varphi(A(y))] \geq V(Q)$, where $\bar{r} = a_0 + h_0/2$ and*

$$V(Q) = \bar{r} \varphi(A(c) + \bar{R}) + \varphi'(A(c)) \frac{b_0 h_0^2}{12} - |\varphi'(A(c))| \frac{h_0}{2} \bar{R} - \frac{h_0}{4} \varphi''(\tau) (S_b + \bar{S})^2.$$

Proof. Put $Z = A(c) + \langle b, \Delta \rangle + \varrho$. Then $A(y) \leq Z$, and $Z \geq \tau$ because $\varrho \geq 0$. Since φ is decreasing and $r_2 \geq 0$ on Q , $r_2\varphi(A(y)) \geq r_2\varphi(Z) = \bar{r}\varphi(Z) + \Delta_0\varphi(Z)$. Since φ is convex and decreasing and $\mathbb{E}\langle b, \Delta \rangle = 0$, Jensen's inequality gives $\mathbb{E}\varphi(Z) \geq \varphi(A(c) + \mathbb{E}\varrho) \geq \varphi(A(c) + \bar{R})$. For the other part, Taylor's theorem gives $\varphi(Z) = \varphi(A(c)) + \varphi'(A(c))(Z - A(c)) + \omega$ with $|\omega| \leq \frac{1}{2}\varphi''(\tau)(Z - A(c))^2$, because Z and $A(c)$ are at least τ and φ'' is decreasing. Now $\mathbb{E}\Delta_0 = 0$, $\mathbb{E}[\Delta_0\langle b, \Delta \rangle] = b_0 h_0^2/12$ and $|\mathbb{E}[\Delta_0\varrho]| \leq \frac{h_0}{2}\mathbb{E}\varrho$, since $\varrho \geq 0$ and $|\Delta_0| \leq h_0/2$. Finally $|\mathbb{E}[\Delta_0\omega]| \leq \frac{h_0}{4}\varphi''(\tau)\mathbb{E}(Z - A(c))^2$, and $\mathbb{E}(Z - A(c))^2 \leq (S_b + (\mathbb{E}\varrho^2)^{1/2})^2$ by Minkowski's inequality, since $\mathbb{E}\langle b, \Delta \rangle^2 = S_b^2$. $\square$

If $g_{\min}(Q) = g_{\max}(Q)$, then $g(G_1(y)) = g_{\min}(Q)$ on Q , and the box contributes at least $v g_{\min}(Q) \max(V(Q), 0)$ to the integral of Lemma 2.2, provided that $Q \subseteq D_k$.

3.5. Boxes with undetermined edges. For a box Q as in Lemma 3.1 put

$$\begin{aligned} f^- &= a_0 \varphi \left(A(c) + \frac{1}{2} \sum_d |b_d| h_d + R_{\max} \right), \\ f^+ &= (a_0 + h_0) \varphi \left(\max \left(\pi, A(c) - \frac{1}{2} \sum_d |b_d| h_d - R_{\max} \right) \right). \end{aligned}$$

By Lemma 3.1 and (3.4), and since the union contains a unit disk, $f^- \leq r_2\varphi(A(y)) \leq f^+$ on Q . For $0 \leq q \leq 1$ let

$$\Phi(q) = \max(qf^-, V^* - (1-q)f^+),$$

where $V^* = \max(V(Q), 0)$ if $\tau > 0$ and $V^* = 0$ otherwise.

Lemma 3.5. *If $F \subseteq Q$ is measurable and $\Pr(F) \geq q$ for the uniform measure on Q , then $\frac{1}{v} \int_F r_2 \varphi(A(y)) dx \geq \Phi(q)$.*

Proof. The first expression is at least $\Pr(F)f^-$. Also $\int_F = \int_Q - \int_{Q \setminus F}$, the first integral is at least vV^* by Proposition 3.4 and since the integrand is nonnegative, and the second is at most $v(1 - q)f^+$. $\square$

Proposition 3.6. *Let $Q \subseteq D_k$ be as in Lemma 3.1. Let $\mathcal{T}$ be a set of one or two undetermined edges, and for each assignment σ of the states present and absent to the edges in $\mathcal{T}$ let $F_\sigma \subseteq Q$ be the event that the edges have these states, q_σ a lower bound for $\Pr(F_\sigma)$, and g_σ the least value of g over the graphs that contain the present edges of Q , avoid its absent edges and give the edges in $\mathcal{T}$ the states σ . Then*

$$\int_Q r_2 g(G_1(y)) \varphi(A(y)) dx \geq v g_{\min}(Q) V^* + v \sum_{\sigma} (g_{\sigma} - g_{\min}(Q)) \Phi(q_{\sigma}).$$

Proof. The events F_σ partition Q , and $g(G_1(y)) \geq g_{\min}(Q) + \sum_{\sigma} (g_{\sigma} - g_{\min}(Q)) \mathbf{1}_{F_\sigma}(x)$ on Q . Integrate against $r_2 \varphi(A(y))$ and use Proposition 3.4 and Lemma 3.5. $\square$

The programs evaluate the right-hand side for every single undetermined edge and every pair of them, and use the largest value. The edge ij is present exactly when $|U_i - U_j| \leq 1$, where U_i and U_j are independent and uniform on R_i and R_j . The difference $U_i - U_j$ has independent coordinates; each has a trapezoidal density with explicit distribution function, except the second coordinate for the pair $(1, 2)$, which is 0. We cut the range of the first coordinate into eight strips of equal width, inscribe in the unit disk and circumscribe about it the union of rectangles over these strips, and obtain lower bounds for $\Pr(|U_i - U_j| \leq 1)$ and $\Pr(|U_i - U_j| > 1)$ as sums of products of exact differences of distribution functions. For two edges without a common vertex the events are independent, and we multiply the bounds. For two edges with a common vertex we use the Fréchet bound $\Pr(F \cap F') \geq \Pr(F) + \Pr(F') - 1$.

3.6. Boxes at the boundary of the domain. The box Q lies outside D_k , up to a null set, if $\min_Q \xi_j > \max_Q \xi_{j+1}$ for some j . Such boxes contribute nothing. The pair $(j, j + 1)$ *straddles* Q if $\min_Q \xi_j \leq \max_Q \xi_{j+1}$ and $\max_Q \xi_j > \min_Q \xi_{j+1}$.

Proposition 3.7. *Let Q be as in Lemma 3.1 with $a_0 \geq 0$ and $\min_Q \eta_3 \geq 0$, and suppose that exactly one pair $(j, j + 1)$ straddles Q and that no other pair violates the order. Let $q = \Pr(\xi_j \leq \xi_{j+1})$ for the uniform measure on Q . Then*

$$\int_{Q \cap D_k} r_2 g(G_1(y)) \varphi(A(y)) dx \geq v g_{\min}(Q) \Phi(q).$$

Proof. Here $Q \cap D_k$ is the event $\xi_j \leq \xi_{j+1}$, and $g \geq g_{\min}(Q)$. Apply Lemma 3.5. $\square$

The probability q is the value at 0 of the trapezoidal distribution function of $\xi_j - \xi_{j+1}$. Boxes with two or more straddling pairs contribute nothing.

3.7. Boxes in which two points can coincide. If $d_{ij}^- = 0$ for some pair, Lemma 3.1 does not apply. Put $\varepsilon = \max_j \varepsilon_j$.

Proposition 3.8. *Let $Q \subseteq D_k$ be a box. Then*

$$\int_Q r_2 g(G_1(y)) \varphi(A(y)) dx \geq v g_{\min}(Q) a_0 \varphi((1 + \varepsilon)^2 A(c/(1 + \varepsilon))),$$

where $c/(1 + \varepsilon)$ is the configuration with points $c_j/(1 + \varepsilon)$.

Proof. For $x \in Q$ we have $B(y_j, 1) \subseteq B(c_j, 1 + \varepsilon)$, so $A(y)$ is at most the area of the union of the disks $B(c_j, 1 + \varepsilon)$, which is $(1 + \varepsilon)^2 A(c/(1 + \varepsilon))$ by scaling. Moreover $r_2 \geq a_0$ and $g \geq g_{\min}(Q)$. $\square$

The programs use Proposition 3.8 for every box with $d_{ij}^- = 0$ for some pair, provided that no pair straddles it. If a pair straddles such a box, the box receives 0.

4. THE COMPUTATION

4.1. The boxes. For $k = 3, 4, 5$ the search program subdivides X_k . It first bisects X_k breadth first, always at the midpoint of the widest coordinate, into N initial boxes, where N is a power of two. Then it processes every box Q recursively as follows. It drops Q if Q lies outside D_k or $g_{\max}(Q) = 0$. It bisects Q at the midpoint of its widest coordinate if this width exceeds a parameter W_0 . Otherwise it computes the bound that applies: Proposition 3.8 if two points can coincide and no pair straddles Q ; 0 if two points can coincide and a pair straddles Q , or if two or more pairs straddle Q ; Proposition 3.7 if exactly one pair straddles Q ; Proposition 3.6 if $g_{\min}(Q) < g_{\max}(Q)$; and $v g_{\min}(Q) \max(V(Q), 0)$ otherwise. It also computes an estimate of the loss of this bound, namely $\kappa_k v g_{\max}(Q) \bar{r} \varphi(A(c))$ minus κ_k times the bound. If the estimate exceeds a parameter tol and the relevant width exceeds a floor W_f , it bisects Q at the midpoint of the coordinate that realizes the relevant width. The relevant width is the largest width for interior boxes and for boxes in which two points can coincide; the larger of the widths of ξ_j and ξ_{j+1} , over the pairs $(j, j+1)$ that straddle Q , for straddling boxes; and the largest width among the coordinates of the points that meet an undetermined edge for the other boxes. A box that is neither dropped nor split is a *leaf*. Since every split is a bisection, the leaves and the dropped boxes partition X_k . All box corners are dyadic rationals, which are exact in double precision. The program uses double precision throughout and is not part of the proof.

The leaves do not depend on N as long as the initial bisection splits only boxes whose widest coordinate is wider than W_0 , that is, for $N \leq 2^{13}$, 2^{18} and 2^{26} when $k = 3, 4, 5$ and W_0 is as in Table 1. Indeed, the recursion splits such boxes at the same coordinate, and a box that is dropped has only dropped sub-boxes. The checker takes N as a parameter. The search program takes N to be the least power of two that is at least 64 times the number of threads; for $k = 3$ this is at most 2^{13} only on at most 128 threads.

4.2. The checker. The checker contains a verbatim copy of the recursion of the search program, so for the same N , or any N in the range of Section 4.1, it visits the same leaves; the float decisions of this copy need not be correct. For every leaf it takes from the copy only the corners (and, for the skip below, the float value) and computes the following in ball arithmetic [6], at 128 bits of precision.

- (1) It checks that the centre and the widths of the leaf are exact in double precision, that $a_0 \geq 0$ and $\min \eta_3 \geq 0$, and which pairs violate or straddle the order.
- (2) It classifies the edges from the exact squared distances $(d_{ij}^\pm)^2$ between the dyadic rectangles.
- (3) It computes $g_{\min}$, $g_{\max}$ and the values g_σ from its own table of $2g = 2p - n_1$, where p is computed by enumerating the orders of the vertices.
- (4) It computes $A(c)$ and $\nabla A(c)$ from (3.1) and (3.2). The endpoints $\psi_{ij} \pm \alpha_{ij}$ of the arcs covered by the other disks are balls; it replaces each by an exact double, computes the exposed set of these doubles exactly on a circle of length equal to the double nearest to 2π , integrates over this set in ball arithmetic, and adds an explicit bound for the effect of the replacement and of the change of circle.
- (5) It computes ν_{ij} (dropping a vertex only if the test of Remark 3.2 holds for the whole ball), $R_{\max}$, and the bounds of Lemma 3.3.

- (6) It evaluates the bound of Propositions 3.4–3.8 that applies, with the probabilities of Propositions 3.6 and 3.7 from exact distribution functions and with $V^* = 0$ unless $\tau > 0$ is certified, and takes the lower endpoint, or 0 if this is negative.
- (7) It adds the lower endpoints with rigorous rounding and multiplies the sum by κ_k in ball arithmetic.

Leaves whose float value is zero are skipped, which can only lower the result. The checker also prints a second value from a variant of Proposition 3.4, which we do not use. The checker prints $\lfloor 10^{10}x \rfloor / 10^{10}$ for the lower endpoint x of the product. It can split the initial boxes into contiguous ranges and treat each range separately. A range prints the exact integer $\lfloor 10^{30}x \rfloor$ for the lower endpoint x of its sum, and a separate step adds these integers after checking that the ranges come from the same parameters and the same build of the checker and do not overlap.

4.3. Proof of Theorem 1.1. For $k = 3, 4, 5$ the checker, run with the parameters of Table 1, prints the stated lower bounds. Every leaf Q that it evaluates satisfies the hypotheses of the proposition that it applies, by steps (1)–(5) and, for Proposition 3.4, the certified test $\tau > 0$ in step (6). So the value it adds is at most $\int_{Q \cap D_k} r_2 g \varphi(A) dx$, and the other leaves and the dropped boxes contribute nonnegative amounts. The leaves and the dropped boxes partition X_k , so Lemma 2.2 gives the bounds for C_3 , C_4 and C_5 , and Theorem 2.1 with $K = 5$ gives $\beta \geq 5/8 + 0.0117130195 + 0.0063707166 + 0.0032174604 = 0.6463011965 > 0.6463$.

4.4. Runs. Table 1 lists the runs. For $k = 4$ and $k = 5$ the initial boxes were split into ranges of equal length, which ran one after another. In all runs no leaf value falls below 0.999 times the float value of the search program. A rerun of the checker on another machine of the same type reproduced the certified values and leaf counts for $k = 3, 4, 5$; for $k = 5$ it used $N = 2^{21}$ without ranges, which illustrates that the leaves do not depend on N .

TABLE 1. The checker runs, on a machine with 192 threads. Times are wall-clock times, summed over the ranges.

	$k = 3$	$k = 4$	$k = 5$
$W_0; W_f$	0.1; $2 \cdot 10^{-4}$	0.4; $5 \cdot 10^{-4}$	0.6; $5 \cdot 10^{-4}$
tol; N	10^{-12} ; 4,096	10^{-13} ; 262,144	10^{-12} ; 4,194,304
Ranges of initial boxes	1	16	32
Leaves	12,835,507	1,992,539,492	3,584,498,511
evaluated in ball arithmetic	11,563,336	1,751,885,102	2,343,722,715
of these, with coinciding points possible	3,786	17,657,522	116,635,320
Float sum of the leaf bounds	0.011713020	0.006370717	0.003217460
Certified lower bound for C_k	0.0117130195	0.0063707166	0.0032174604
Time	5 s	1,115 s	3,417 s

4.5. Further checks. We tested the checker in three ways.

Against the search program. The checker compares every rigorous leaf value with the float value of the search program (Section 4.4); raising all float values by 1% is caught by this comparison. A self-test checks the union areas, the areas of unions of enlarged disks and the distribution functions of the checker against the float code, its union areas and the probability bounds of Section 3.5 against Monte Carlo estimates, and its table of g against that of the search program.

With deliberate errors. To see that each ingredient affects the result, we made shorter runs of the checker, with coarser parameters, after introducing one of the following errors: (i) g raised

by $\frac{1}{2}$ on every connected graph; (ii) $\Gamma(k + \frac{1}{2})$ in place of $\Gamma(k - \frac{1}{2})$ in κ_k ; (iii) the remainder ϱ set to zero; (iv) undetermined edges treated as present; (v) all pairs of edges treated as independent; (vi) f^- in place of f^+ ; (vii) the order constraint ignored; (viii) the disks of Proposition 3.8 not enlarged. Each error raised the certified value.

Against separately written programs. Programs written separately from the checker estimated $C_3, \dots, C_6$ by Monte Carlo integration (Section 5) and tested the components of the checker against their own implementations; Table 2 lists the components and the numbers of test inputs. They also compared the bounds of 5,998, 2,626 and 1,842 leaves of the runs of Table 1 for $k = 3, 4, 5$ with the integrals over these leaves, computed by Gauss–Legendre quadrature or Monte Carlo integration. None of these tests found a violation.

TABLE 2. Components of the checker tested against separately written programs.

Component	Test inputs
Table of g	all edge sets, $k \leq 6$
$g_{\min}, g_{\max}$ and g_σ	about 12,000 random states of the edges
Union areas	16,000 configurations
Unions of enlarged disks	12,000 configurations
Upper bound of Lemma 3.1 with $R_{\max}$; Lemma 3.3	about 4,400 boxes
Probability bounds of Section 3.5	1,200 pairs of rectangles
Distribution functions	200,000 cases
The function Φ	80,000 cases

5. LIMITS OF THE METHOD

We record numerical observations, not used in the proofs.

Two Monte Carlo programs estimate C_k by sampling configurations along random trees, in the spirit of the importance sampling of Dong and Cao [3, Section 3.4], with exact union areas and exact values of p . One of them gives $C_3 = 0.0117147(7)$, $C_4 = 0.0064059(3)$, $C_5 = 0.0039065(3)$ and $C_6 = 0.0026084(4)$, where the digits in parentheses are the standard error in the last digit, and the other agrees to within about 1.2 standard errors. So, numerically, the framework with $K = 5$ cannot give more than $5/8 + C_3 + C_4 + C_5 \approx 0.6470$, and with $K = 6$ not more than about 0.6496. The sampling bounds of Dong and Cao for $k \leq 10$ [3, Section 3.4] suggest that the framework with $K = 10$ gives more than 0.6536.

Most of the remaining loss is in C_5 , on boxes with undetermined edges. For $k = 6$ the integral is nine-dimensional, and its mass is spread over many boxes with small integrands; we did not certify C_6 .

DATA AVAILABILITY

The search program, the checker, the program that compares p and q , and the logs of the checker runs are archived at doi:10.5281/zenodo.23524862 and available at <https://github.com/tejasvi/bhh-constant-bound>. The checker needs a C compiler with OpenMP and FLINT version 3. The archive also contains the separately written programs used for the tests in Section 4.5 and the Monte Carlo program that gives the values of $C_3, \dots, C_6$ quoted in Section 5. These programs are not part of the proof.

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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