

BOUNDS FOR THE SMALLEST LIMIT POINT OF THE AREAL MAHLER MEASURE OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

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ABSTRACT. Pritsker's areal analog of the Mahler measure replaces the integral of $\log |P|$ over the unit circle by its integral over the unit disc with respect to normalized area. For an algebraic number α of degree n whose minimal polynomial $P \in \mathbb{Z}[x]$ is primitive with positive leading coefficient, let $N_0(\alpha) = \|P\|_0^{1/n}$. Flammang proved that $N_0(\alpha) \geq 1.380047$ for every totally positive algebraic integer α outside an explicit finite set. We prove that all but finitely many totally positive algebraic numbers satisfy $N_0(\alpha) > 1.3830134$, and that infinitely many totally positive algebraic integers satisfy $N_0(\alpha) < 1.3842933$. So the smallest limit point of the values of N_0 on totally positive algebraic integers, and the essential minimum, lie between 1.3830134 and 1.3842933; we are not aware of an earlier upper bound. The lower bound comes from an auxiliary function with Flammang's polynomials and a logarithmic energy term, and the upper bound from a measure that satisfies a criterion of Smith, as in our two papers on the classical Mahler measure of totally positive algebraic integers. Both certificates are verified in ball arithmetic.

1. INTRODUCTION

For a nonzero polynomial $P(z) = a \prod_{i=1}^n (z - \alpha_i)$ with complex coefficients, Pritsker [9] defined

$$\|P\|_0 = \exp\left(\frac{1}{\pi} \int_D \log |P(z)| dA(z)\right),$$

where D is the unit disc and dA is the area measure, and he proved

$$(1.1) \quad \|P\|_0 = M(P) \exp\left(\frac{1}{2} \sum_{|\alpha_i| < 1} (|\alpha_i|^2 - 1)\right), \quad M(P) = |a| \prod_{i=1}^n \max(1, |\alpha_i|),$$

so that $e^{-n/2} M(P) \leq \|P\|_0 \leq M(P)$ [9, Theorem 1.1 and Corollary 1.2]. Flammang [4] calls $\|P\|_0$ the Pritsker measure of an algebraic integer α with minimal polynomial P , and $N_0(\alpha) = \|P\|_0^{1/\deg \alpha}$ its absolute Pritsker measure. We use the same definitions for every algebraic number α of degree n , with P its minimal polynomial in $\mathbb{Z}[x]$, primitive and with leading coefficient $a \geq 1$. We call α totally positive if all its conjugates α_i are real and positive. For such α , (1.1) gives

$$(1.2) \quad n \log N_0(\alpha) = \log a + \sum_{i=1}^n g(\alpha_i), \quad g(x) = \log^+ x + \frac{1}{2}(\min(1, x)^2 - 1) \quad (x > 0),$$

where $\log^+ x = \log \max(1, x)$. The function g is continuously differentiable on $(0, \infty)$, increasing, equal to $\log x$ for $x \geq 1$, and tends to $-\frac{1}{2}$ as $x \rightarrow 0$; moreover $\log^+ x - \frac{1}{2} \leq g(x) \leq \log^+ x$ and $g(x) \geq \log x$ for all $x > 0$.

Let $\mathcal{L}_A^+$ be the set of the values $N_0(\alpha)$, where α runs through the totally positive algebraic integers, let λ_A^+ be the smallest limit point of $\mathcal{L}_A^+$, and let

$$E_A^+ = \sup\{c : \text{only finitely many totally positive algebraic integers } \alpha \text{ satisfy } N_0(\alpha) \leq c\}$$

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be the essential minimum. The set $\mathcal{L}_A^+$ lies in $[1, \infty)$ [9, (1.3)], and Lemma 3.1 below shows that it has limit points, so λ_A^+ exists. Every neighbourhood of λ_A^+ contains infinitely many points of $\mathcal{L}_A^+$, so $E_A^+ \leq \lambda_A^+$. Flammang [4, Theorem 2] proved $N_0(\alpha) \geq 1.380047$ for every totally positive algebraic integer α whose minimal polynomial is not one of x , $x-1$, x^2-3x+1 , x^3-6x^2+5x-1 , $x^4-7x^3+13x^2-7x+1$ and $x^5-11x^4+29x^3-26x^2+9x-1$, so that $E_A^+ \geq 1.380047$ (Section 5 compares her printed table with this constant). The values of N_0 at the roots of the last four polynomials are the four smallest points of $\mathcal{L}_A^+$ other than 1, and Flammang conjectures that the next one is $1.3816081\dots$, attained at the roots of $x^5-13x^4+32x^3-27x^2+9x-1$. We are not aware of a published upper bound for λ_A^+ .

Theorem 1.1. *Let $m_0 = \frac{45636208945290293391}{140737488355328000000} = 0.3242647675\dots$ and $v^* = 325189755077/10^{12}$.*

- (i) *For every $\varepsilon > 0$, all but finitely many totally positive algebraic numbers α satisfy $\log N_0(\alpha) \geq m_0 - \varepsilon$. In particular, all but finitely many totally positive algebraic numbers satisfy $N_0(\alpha) > 1.3830134$.*
- (ii) *There are infinitely many totally positive algebraic integers α with $N_0(\alpha) < 1.3842933$, and $\lambda_A^+ \leq e^{v^*}$.*

In particular

$$1.3830134 < e^{m_0} \leq E_A^+ \leq \lambda_A^+ \leq e^{v^*} < 1.3842933.$$

The interval of Theorem 1.1 has length less than 0.0013. Since $\|P\|_0$ includes the leading coefficient, the lower bound concerns all totally positive algebraic numbers, not only integers. The finitely many exceptions in (i) include 1, the roots of the last four polynomials above, the roots of Flammang's quintic, where $N_0 = 1.3816081\dots < e^{m_0}$, and numbers that are not integers, such as $1/2$, where $N_0(1/2) = 2e^{-3/8} = 1.37457\dots$. The theorem leaves open which totally positive algebraic integers have $N_0(\alpha) < e^{m_0}$, and whether the quintic gives the next point of $\mathcal{L}_A^+$, as Flammang conjectures.

The method. Both bounds are adaptations of our papers on the classical Mahler measure of totally positive algebraic integers, where $\log^+ x$ takes the place of $g(x)$: the lower bound of [15] and the upper bound of [16]. By (1.2), the main change in the arguments is that the conjugates are averaged against g instead of $\log^+$.

Lower bounds come from auxiliary functions. If $g(x) \geq m + \sum_j c_j \log |Q_j(x)|$ for all $x > 0$, with nonzero $Q_j \in \mathbb{Z}[x]$ and $c_j \geq 0$, then averaging over the conjugates of a totally positive algebraic integer α with $Q_j(\alpha) \neq 0$ gives $\log N_0(\alpha) \geq m$, because each $\prod_i Q_j(\alpha_i)$ is a nonzero integer. This is the method that Smyth [14] introduced for the Mahler measure and that Flammang [4, Section 3] used for N_0 . As in [15], we add a term $\lambda(2U_\nu - I(\nu))$, where U_ν and $I(\nu)$ are the logarithmic potential and energy of a reference measure ν ; its average over the conjugates is bounded below because discriminants are nonzero integers. This energy condition was introduced by Orloski, Talebizadeh Sardari and Smith [8] for the Schur–Siegel–Smyth trace problem.

Upper bounds come from measures. If a probability measure ν on an interval $[0, X]$ with $X > 4$ satisfies $\int \log |Q| d\nu \geq 0$ for every nonzero $Q \in \mathbb{Z}[x]$, then by a theorem of Smith [12, Theorem 1.5] it is a weak limit of the distributions of conjugates of distinct totally positive algebraic integers α_k , and then $N_0(\alpha_k) \rightarrow \exp \int g d\nu$; Lemma 3.1 turns this into an upper bound for λ_A^+ . A criterion of Smith [12, Proposition 5.15] reduces the infinitely many conditions to finitely many, which we check on a measure with a piecewise linear density.

In both certificates the polynomials are x and polynomials from Flammang's table for N_0 [4, Table 2], and the reference measures have piecewise linear densities on rational nodes, so that their potentials and energies have closed forms. Section 2 proves the lower bound from an auxiliary function, Section 3 proves the criterion for the upper bound, Section 4 describes

the certificates and their verification and proves Theorem 1.1, and Section 5 collects numerical observations.

2. THE LOWER BOUND

For finite positive Borel measures θ, η on $\mathbb{C}$ with compact support we write $U_\eta(z) = \int \log |z - y| d\eta(y)$, $I(\theta, \eta) = \int U_\eta d\theta$ and $I(\theta) = I(\theta, \theta)$.

Lemma 2.1. *Let $Q_1, \dots, Q_J \in \mathbb{Z}[x]$ be nonzero, $c_1, \dots, c_J > 0$, $\lambda \geq 0$ and $m \in \mathbb{R}$. Let ν be a probability measure on $\mathbb{C}$ with compact support, and let $K > 0$ be such that $\nu(\{z : |z - y| < r\}) \leq Kr$ for all $y \in \mathbb{C}$ and $0 < r \leq 1$. Let Z be the set of positive real roots of the polynomials Q_j . Assume that*

$$(2.1) \quad F(x) = g(x) - m - \sum_{j=1}^J c_j \log |Q_j(x)| - \lambda(2U_\nu(x) - I(\nu)) \geq 0 \quad \text{for all } x \in (0, \infty) \setminus Z.$$

Then $\sum_j c_j \deg Q_j + 2\lambda \leq 1$, and every totally positive algebraic number α of degree $n \geq 1/(2K)$ that is not a root of one of the polynomials Q_j satisfies

$$(2.2) \quad \log N_0(\alpha) \geq m - \frac{\lambda(1 + \log(2Kn))}{n}.$$

Proof. This is [15, Lemma 2.2] with $\log^+ x$ replaced by $g(x)$ and $\log \Omega(\alpha)$ by $\log N_0(\alpha)$. The proof given there uses the left side of (2.1) only twice. First, since $g(x) = \log x$ for $x \geq 1$, we still have $F(x) = (1 - \sum_j c_j \deg Q_j - 2\lambda) \log x + O(1)$ as $x \rightarrow +\infty$, which forces the degree condition. Second, the identity $n \log \Omega(\alpha) = \log a + \sum_i \log^+ \alpha_i$ of [15, (2.5)] is replaced by (1.2). The rest of the argument is unchanged, and we recall it. Let $P = a \prod_{i=1}^n (x - \alpha_i)$ be the minimal polynomial of α . The resultants $\text{Res}(P, Q_j)$ and the discriminant of P are nonzero integers, so $\sum_i \log |Q_j(\alpha_i)| \geq -\deg(Q_j) \log a$ for each j and $\sum_{i \neq k} \log |\alpha_i - \alpha_k| \geq -(2n - 2) \log a$. Replacing each conjugate by the normalized arc length measure on the circle of radius $r \in (0, 1]$ around it, and comparing the resulting measure with ν by the inequality $I(\theta) \leq 2I(\theta, \eta) - I(\eta)$ for probability measures [15, Lemma 2.1], gives

$$\frac{1}{n} \sum_{i=1}^n (2U_\nu(\alpha_i) - I(\nu)) \geq \frac{\log r}{n} - \frac{(2n - 2) \log a}{n^2} - 2Kr$$

[15, (2.9)]. Averaging (2.1) over the conjugates and using these bounds, we get

$$\frac{1}{n} \sum_{i=1}^n g(\alpha_i) \geq m - \frac{\log a}{n} \left(\sum_j c_j \deg Q_j + 2\lambda \frac{n-1}{n} \right) - \lambda \left(2Kr + \frac{\log(1/r)}{n} \right),$$

and by (1.2)

$$\log N_0(\alpha) \geq m + \frac{\log a}{n} \left(1 - \sum_j c_j \deg Q_j - 2\lambda \frac{n-1}{n} \right) - \lambda \left(2Kr + \frac{\log(1/r)}{n} \right).$$

The middle term is nonnegative because $\log a \geq 0$ and $\sum_j c_j \deg Q_j + 2\lambda \leq 1$, and the choice $r = 1/(2Kn)$ gives (2.2). $\square$

Corollary 2.2. *Under the assumptions of Lemma 2.1, for every $\varepsilon > 0$ only finitely many totally positive algebraic numbers α satisfy $\log N_0(\alpha) \leq m - \varepsilon$.*

Proof. Choose $n_0 \geq 1/(2K)$ such that $\lambda(1 + \log(2Kn))/n < \varepsilon$ for all $n \geq n_0$. If α is totally positive and $\log N_0(\alpha) \leq m - \varepsilon$, then either α is one of the finitely many roots of the polynomials Q_j , or α has degree less than n_0 by (2.2). In the second case the absolute logarithmic Weil

height $h(\alpha) = \frac{1}{n} \log M(P)$ [1, Section 1.6] satisfies $h(\alpha) \leq \log N_0(\alpha) + \frac{1}{2} \leq m + \frac{1}{2}$, because $M(P) \leq e^{n/2} \|P\|_0$, and by Northcott's theorem [1, Theorem 1.6.8] there are only finitely many such α . $\square$

Remark 2.3. For $\lambda = 0$, Lemma 2.1 is the classical argument, extended to algebraic numbers that need not be integers, and its conclusion is $\log N_0(\alpha) \geq m$ for every totally positive algebraic number α that is not a root of one of the Q_j : the measure ν then plays no role, and with ν uniform on $[0, 1]$ and $K = 2$ every $n \geq 1$ is admissible. Flammang's auxiliary functions for N_0 [4, Section 3] are of this kind.

3. THE CRITERION FOR THE UPPER BOUND

Let $X > 4$ be rational. Let $0 \leq e_0 < e_1 < \dots < e_N \leq X$ be rational nodes and let $G: \mathbb{R} \rightarrow [0, \infty)$ be the continuous function that is linear on each interval $[e_{k-1}, e_k]$, takes rational values $G(e_k)$ with $G(e_0) = G(e_N) = 0$, and vanishes outside $[e_0, e_N]$. Let $\mu = G(x) dx$, let $s = \mu(\mathbb{R}) > 0$ and $U(z) = \int \log |z - y| G(y) dy$. Let $Q_1, \dots, Q_J \in \mathbb{Z}[x]$ be distinct primitive polynomials that are irreducible over $\mathbb{Q}$, let $b_1, \dots, b_J \geq 0$, and put $h(z) = U(z) - \sum_{j: b_j > 0} b_j \log |Q_j(z)|$, with $h = +\infty$ at the roots of these Q_j .

Lemma 3.1. *Assume that*

- (C1) $\sum_j b_j \deg Q_j < s$;
- (C2) $\int \log |Q_j| d\mu \geq 0$ for every j with $b_j > 0$;
- (C3) $h(x) \geq 0$ for every $x \in [e_0, e_N]$.

Let $v = \frac{1}{s} \int g d\mu$. Then there are distinct totally positive algebraic integers $\alpha_1, \alpha_2, \dots$ with $N_0(\alpha_k) \rightarrow e^v$, and $E_A^+ \leq \lambda_A^+ \leq e^v$.

Proof. This is [16, Lemma 2.1] with $\log^+$ replaced by g , and we follow its proof. Steps 1 and 2 there do not involve $\log^+$: they show that $h \geq 0$ on $\mathbb{C}$ and that $\int \log |P| d\mu \geq 0$ for every nonzero $P \in \mathbb{Z}[x]$. By [12, Theorem 1.5], applied to μ/s on $\Sigma = [0, X]$, which has capacity $X/4 > 1$ [11, Example I.3.5], there are distinct monic irreducible polynomials $R_k \in \mathbb{Z}[x]$ with all roots in Σ whose root distributions converge weakly to μ/s . At most one of them is x ; for the others, a root α_k of R_k is a totally positive algebraic integer, and by (1.2)

$$\log N_0(\alpha_k) = \frac{1}{\deg R_k} \sum_{R_k(\rho)=0} g(\rho) \rightarrow \frac{1}{s} \int g d\mu = v,$$

since g extends to a continuous function on Σ with $g(0) = -\frac{1}{2}$. For the bound on λ_A^+ , choose $X' \geq X$ with $\log(X'/4) > v$. The values $N_0(\alpha_k)$ might coincide for infinitely many k , so this alone does not give a limit point; to obtain one, we mix μ/s with the equilibrium measure ω of $[0, X']$. Since $U_\omega \geq \log(X'/4)$ on $\mathbb{C}$ [11, (I.1.4) and Example I.3.5], we have $\int \log |P| d\omega \geq 0$ for every nonzero $P \in \mathbb{Z}[x]$, as in Step 4 of [16, Lemma 2.1], and $w = \int g d\omega \geq \int \log x d\omega = U_\omega(0) \geq \log(X'/4) > v$, because $g(x) \geq \log x$. For $t \in [0, 1]$ the measure $(1-t)\mu/s + t\omega$ satisfies the same integrality condition, and Smith's theorem on $[0, X']$ gives distinct totally positive algebraic integers α_k with $\log N_0(\alpha_k) \rightarrow (1-t)v + tw$. So the closure of $\mathcal{L}_A^+$ contains the interval $[e^v, e^w]$, which is nondegenerate because $w > v$, every point of this interval is a limit point of $\mathcal{L}_A^+$, and $\lambda_A^+ \leq e^v$. $\square$

4. THE CERTIFICATES AND THEIR VERIFICATION

Flammang's table for N_0 [4, Table 2] lists 58 polynomials with integer coefficients, together with the weights of her auxiliary function; the first four are x , $x - 1$, $2x - 1$ and $x^2 - 3x + 1$. We refer to them by their numbers in her table. Both certificates list the coefficients of their

polynomials, so the proof does not depend on a transcription of the table (Section 5). Some of the polynomials have non-real roots.

4.1. The lower certificate. The lower certificate consists of

- the 4,660 rational nodes $0 = e_0 < e_1 < \dots < e_N = 24$; consecutive nodes are at most $1/100$ apart, and at most $7 \cdot 10^{-4}$ apart in $[r - 0.004, r + 0.004]$ for each of 189 positive roots r of polynomials of [4, Table 2] (see Section 5);
- rational values $G_k \geq 0$ of a continuous piecewise linear function G , which vanishes on $[0, 13/100]$ and at $e_N = 24$;
- the rational number m_0 ;
- the 34 polynomials of [4, Table 2] with numbers 1–8, 10, 12–18, 23–29, 31, 33, 35, 37, 38, 40, 44, 46, 50, 52 and 53, with positive dyadic weights c_j .

The weight of x is about 0.341, and that of $x - 1$ about 0.206. With $\rho = G(x) dx$, $s = \rho(\mathbb{R}) = 0.0955758\dots$, $\nu = \rho/s$ and $\lambda = s$, the function (2.1) becomes

$$(4.1) \quad F(x) = g(x) - m_0 - \sum_j c_j \log |Q_j(x)| - 2U_\rho(x) + \frac{I(\rho)}{s}.$$

A disc of radius r meets $\mathbb{R}$ in an interval of length at most $2r$, and $\max G/s < 2.47$, so ν satisfies the hypothesis of Lemma 2.1 with $K = 5$. The potential and the energy have the closed forms of [15, Lemma 3.1]. Let C_i be the jump of the slope of G at e_i , so that $G'' = \sum_i C_i \delta_{e_i}$ and $\sum_i C_i = \sum_i C_i e_i = 0$, and let $H(t) = \frac{t^2}{2} \log |t| - \frac{3}{4}t^2$ and $L(t) = \frac{t^4}{24} \log |t| - \frac{25}{288}t^4$, with $H(0) = L(0) = 0$. Then

$$U_\rho(x) = \sum_i C_i H(e_i - x), \quad I(\rho) = \sum_{i,k} C_i C_k L(e_i - e_k).$$

The degree sum is $\sum_j c_j \deg Q_j + 2s = 0.903938\dots < 1$.

The verification follows [15, Section 3.3], with $\log^+$ replaced by g . It uses Arb ball arithmetic [7] through python-flint [10], with the certified root isolation of FLINT [3]. The program reads the data as exact rationals, checks the degree sum exactly, and computes $I(\rho)$ from the closed form. On $(0, x_{\text{lo}}]$ with $x_{\text{lo}} = 1/1000$ it uses four bounds: $g \geq -\frac{1}{2}$; the term $-c \log x$ of the polynomial x , where c is its weight; upper bounds for the other $|Q_j|$; and $U_\rho(x) \leq U_\rho(0)$, which holds because U_ρ decreases to the left of the support of G . It obtains $F > 1.532$ there. On $[x_{\text{hi}}, \infty)$ with $x_{\text{hi}} = 4357$ it uses $g(x) = \log x$, the bounds $|Q_j(x)| \leq \|Q_j\|_1 x^{\deg Q_j}$, where $\|Q_j\|_1$ is the sum of the absolute values of the coefficients, and $U_\rho(x) \leq s \log x$. This gives $F \geq \kappa \log x - b$ with $\kappa = 1 - \sum_j c_j \deg Q_j - 2s = 0.0960615\dots$ and an explicit constant b , and hence $F > 0.192$ there. On $[x_{\text{lo}}, x_{\text{hi}}]$ it encloses the positive roots of the Q_j in 230 windows. On a window it bounds g from below by its value at the left end, since g is increasing. The nodes, the ends of the windows and the point 1, where g'' jumps, cut the rest into 4,790 intervals. On such an interval $[p, q]$ the program bounds F from below by $\min(F(p), F(q)) - \frac{q-p}{4} \int_p^q (F'')_+$ [15, Lemma 3.3], and it bisects the interval if the bound is not positive. In the bound for $(F'')_+$ the term $g'' = 1$ on $(0, 1)$ is included, and $g'' = -1/x^2$ on $(1, \infty)$ is dropped. The program made 69,298 interval checks, and the smallest lower bound for F at an end of an interval was $2.96 \cdot 10^{-6}$. With 48 processes on an x86-64 server with 96 threads the check took 3,665 seconds.

4.2. The upper certificate. The upper certificate consists of

- $X = 16$;
- the 3,170 nodes $e_k = 31/200 + k/200$, from $e_0 = 31/200$ to $e_N = 16$;
- rational values of G , whose mass s differs from 1 by less than $2 \cdot 10^{-15}$;
- the rational number $v^* = 0.325189755077$;

- the 23 polynomials of [4, Table 2] with numbers 1–4, 6, 7, 8, 10–14, 22–26, 34, 36, 40, 41, 49 and 50, with positive weights b_j .

All of these polynomials are primitive and irreducible, and $\sum_j b_j \deg Q_j = 0.939524\dots < s$. The program uses the same libraries. It checks the data; (C1), in exact arithmetic; (C2), from the closed form of U at the roots of the Q_j , with the complex closed form of [16, Section 3.2] at the non-real roots; (C3), on all of $[e_0, e_N]$, as in [16, Section 3.2]; and the inequality $\frac{1}{s} \int g d\mu \leq v^*$. The last integral has a closed form on each linear piece of G : on $[e_{k-1}, e_k] \cap [0, 1]$ the integrand is a polynomial with rational coefficients, and on $[e_{k-1}, e_k] \cap [1, \infty)$ it involves $\int x^i \log x dx$ for $i \leq 1$. The smallest of the integrals in (C2) is about 10^{-7} . For (C3) the program used 138 root windows and 3,302 initial intervals and made 27,000 interval checks; the smallest lower bound for h at an end of an interval was $1.56 \cdot 10^{-6}$. With 40 processes the check took 55 seconds.

4.3. The programs and an independent check. The two programs are the first programs of [15] and [16] with the changes described above.

A second program, written independently of the first two and based only on the description of the certificate formats and on Lemmas 2.1 and 3.1, checks both certificates by a different method. On the main range it bounds F and h from below by quadratic minorants. These come from Taylor's formula with a rigorous lower bound for the second derivative, using a Taylor model for the terms of distant nodes and chords for the concave terms of nearby nodes. The program cuts out the real roots of the polynomials by brackets of width 10^{-40} with a logarithmic lower bound, treats g exactly on each side of 1, and bisects an interval until the bound is positive. For the upper certificate it uses the method of the second program of [16]. It confirms both certificates, with 39,833 interval tests and 230 root brackets for the lower certificate and 9,839 tests and 138 brackets for (C3). Appendix A describes runs of all the programs on deliberately modified certificates.

Proof of Theorem 1.1. For the lower certificate the function (4.1) is nonnegative on $(0, \infty)$ outside the roots of the Q_j , and it is the function (2.1) with $\nu = \rho/s$, $\lambda = s$, $K = 5$ and $m = m_0$. Corollary 2.2 gives the first statement of (i), and with $\varepsilon = m_0 - \log 1.3830134 > 0$ the second. It also gives $E_A^+ \geq e^{m_0 - \varepsilon}$ for every $\varepsilon > 0$, so $E_A^+ \geq e^{m_0}$. For the upper certificate, (C1)–(C3) hold and $v \leq v^*$, so Lemma 3.1 gives distinct totally positive algebraic integers α_k with $\log N_0(\alpha_k) \rightarrow v$, and $\lambda_A^+ \leq e^v \leq e^{v^*}$; in its proof we may take $X' = X = 16$, since $\log 4 > v^*$. Since $e^{v^*} < 1.3842933$, all but finitely many of the α_k satisfy $N_0(\alpha_k) < 1.3842933$. Finally $1.3830134 < e^{m_0}$ and $E_A^+ \leq \lambda_A^+$. $\square$

5. REMARKS

The statements of this section are numerical observations, computed in floating point; the proof does not use them.

The search. For the lower bound we first minimized $\int g d\theta$ over probability measures θ whose densities are combinations of hat functions on the nodes of the lower certificate. The constraints were $\int \log |Q| d\theta \geq 0$ for the 58 polynomials of [4, Table 2] and $I(\theta) \geq 0$, a second-order cone constraint. We used an interior-point solver [5] through a modeling language [2]. The value corresponds to $N_0 \approx 1.3831041$, and to $N_0 \approx 1.38008$ without the energy constraint. As in [15], every bound that Lemma 2.1 gives with these polynomials is at most this value. We then fixed the reference measure ν to the computed optimal measure and solved the linear program for m , the weights $c_j \geq 0$ and $\lambda \geq 0$. In it, (2.1) is imposed on about $4.7 \cdot 10^5$ points of $(0, 80]$, which include many points near the roots of the Q_j , and cutting planes are added at the local minima of F . Its value corresponds to $N_0 \approx 1.3830176$, and m_0 is this value minus $3 \cdot 10^{-6}$, after rounding of the data to exact rationals. The nodes come from a preliminary run of these

two steps with the nodes $k/100$ on $[0, 20]$: to the points $k/100$ of $[0, 24]$ we added the points at distance $k/2000$, $0 \leq k \leq 8$, from each of the 189 positive roots of the polynomials that received weight more than 10^{-6} in its linear program, rounded to ten decimal places, and we then removed nodes until consecutive nodes were at least $2 \cdot 10^{-4}$ apart.

For the upper bound we minimized $\int g d\mu$ over measures of mass 1 whose densities are combinations of hat functions with spacing $1/200$ on $[0, 16]$, and over weights $b_j \geq 0$ for the same 58 polynomials. The constraints were $\sum_j b_j \deg Q_j \leq 1 - 10^{-6}$, $\int \log |Q_j| d\mu \geq 10^{-7}$, and $h \geq 3 \cdot 10^{-6}$ on a grid; cutting planes refine the grid until h is positive on a check grid of spacing $2 \cdot 10^{-5}$ near the support of the density. We used the dual simplex method of HiGHS [6] through SciPy. The value corresponds to $N_0 \approx 1.3842933$, and v^* exceeds the exact value of $\frac{1}{s} \int g d\mu$ for the rounded data by about 10^{-9} . In both searches only a part of the 58 polynomials received positive weight.

The gap. Integrating (2.1) against a measure θ of the kind considered in the search, as in [15, Section 4.1], shows that no auxiliary function of Lemma 2.1 with the 58 polynomials gives more than the infimum of the measure problem, which is at most about 1.3831041. The lower bound of Theorem 1.1 lies $9.1 \cdot 10^{-5}$ below this value, a loss due to the fixed reference measure, the discretization and the margin, and the remaining $1.2 \cdot 10^{-3}$ separates it from the upper bound. More polynomials, and a finer discretization in the search for the upper bound, should narrow the second part.

Flammang's table. Row 39 of [4, Table 2] is printed with the terms $+59x^{10} + 60x^9$, whose signs break the alternation of the coefficients seen in the other rows; we use it as printed. With the weights printed in the table, the minimum of her auxiliary function gives $N_0 \geq 1.3800359$, slightly below her constant 1.380047, as the rounding of the printed weights allows. Re-optimizing the weights of the same polynomials by linear programming gives 1.3800500. Every polynomial of the certificates agrees with the table.¹

Smyth's sequence. Smyth's totally real algebraic integers β_n , defined by $\beta_0 = 1$ and $\beta_n - 1/\beta_n = \beta_{n-1}$ with $\beta_n > 0$ [13], give totally positive algebraic integers β_n^2 . We computed $N_0(\beta_n^2)$ from the 2^n conjugates of β_n . The values for $n = 19, \dots, 22$ are 1.4096307898, 1.4096308214, 1.4096308365 and 1.4096308437, and the increments roughly halve, which suggests the limit 1.40963085.... This is well above the upper bound of Theorem 1.1.

APPENDIX A. TESTS OF THE VERIFICATION PROGRAMS

To test that the programs detect violations, we ran them on the modified certificates of Table 1 and on four further narrow modifications, two for each certificate; all the programs reject all of them. Two further modifications still state true bounds (m_0 raised by $2 \cdot 10^{-6}$, less than the margin of $3 \cdot 10^{-6}$ left by the search, and v^* raised by 10^{-6} , which only weakens the claim), and the programs accept them.

DATA AVAILABILITY

The two certificates, the verification programs with their output, the modified certificates with the output of the programs on them, and the programs that produced the certificates are archived at doi:10.5281/zenodo.23537180 and available at <https://github.com/tejasvi/areal-mahler-measure-v2>. The SHA-256 checksums of the lower and the upper certificate are

5295faaacff1340587ef77ae593b69432cecaf793ed7c5f599ae38bc48cac5e8 and
cb2c7960028f60e3bcce2c9f76f08cadfe00eaa131fda60b6b18ffe7591080a3.

¹Two independent transcriptions of the table agree, and the wrong last row of a transcription used in some steps of the searches has weight 0 in both certificates.

TABLE 1. Modified certificates. Each narrow modification changes the density by a tent of mass about 10^{-7} and width $4 \cdot 10^{-6}$ at a point where the inequality is nearly tight; the inequality still holds at the nodes of the tent and at nearby points but fails on a short interval.

certificate	modifications
lower	$G_N \neq 0$; a negative G_k ; two nodes swapped; degree sum ≥ 1 ; x_{10} inside the support of G ; G multiplied by 0.99 or 1.01; the weight of x multiplied by 1.001; $x^2 - 3x + 1$ replaced by $x^2 - 3x + 2$; the heaviest, the second heaviest or the polynomial $x - 1$ removed; m_0 raised by 10^{-5} or $4 \cdot 10^{-6}$; a narrow notch removed from G at $x = 6129/25000$, where $g'' = 1$
upper	$X = 4$; $G(e_N) \neq 0$; G multiplied by 0.5, 0.99 or 1.01; $\sum_j b_j \deg Q_j > s$; two nodes swapped; a non-primitive or a reducible weighted polynomial; the heaviest or the second heaviest polynomial removed; v^* lowered by 10^{-6} or $2 \cdot 10^{-9}$; the heaviest weight multiplied by 0.99 or 1.01; $x^2 - 3x + 1$ replaced by $x^2 - 3x - 1$; a narrow bump added to G at $x = 12.50711$, with v^* raised by 10^{-6}

USE OF AI TOOLS

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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